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SPHERICAL FUNCTIONS OF HERMITIAN AND SYMMETRIC FORMS III

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Introduction. In the previous papers [2] and [3], we have introduced and studied spherical functions and a spherical transform on the space of nondegenerate hermitian, or symmetric, matrices over a $\mathfrak{p}$-adic number field. In [2], we have shown the injectivity of the spherical transform, and in [3] we have closely studied the case of matrices of size 2. In this paper, making use of the results in [3], we shall show the functional equations for spherical functions and determine their possible poles.

Let k be a $\mathfrak{P}$-adic number field with $\mathfrak{P}$ not lying over 2, $\mathcal{O}$ the ring of integers of k and Π a prime element of k. Let X be the space of nondegenerate symmetric matrices of size n with entries in k. Then $K = GL_n(\mathcal{O})$ acts on X by $k \cdot x = kx^tk$, $k \in K$, $x \in X$. For $x \in X$, a character $\chi = (\chi_1, \cdots, \chi_n)$ of $(k^*/k^{*2})^n$ and $s = (s_1, \cdots, s_n) \in \boldsymbol{C}^n$, consider the following integral:

$$(*) \qquad L(x; \chi; s) = \int_{K'} \prod_{i=1}^{n} |d_i(k \cdot x)|^{s_i} \chi_i(d_i(k \cdot x)) dk ,$$

where dk is the Haar measure on K normalized by $\int_K dk = 1$, $d_i(k \cdot x)$ is the determinant of the upper left i by i block of $k \cdot x$, and $K' = \{k \in K: \prod_{i=1}^n d_i(k \cdot x) \neq 0\}$.

The right hand side of $(*)$ is absolutely convergent for $\mathrm{Re}(s_1), \cdots, \mathrm{Re}(s_{n-1}) \geqq 0$, and has an analytic continuation to a rational function in $q^{s_1}, \cdots, q^{s_n}$ (cf. [1]). Thus we may regard $L(x; \chi; s)$ as an element in $C^\infty(K \backslash X)$, the space of all K-invariant complex-valued functions on X. We call $L(x; \chi; s)$ a spherical function on X.

We introduce a new variable $z = (z_1, \cdots, z_n)$ which is related to s as follows:

$$\begin{cases} s_i = -z_i + z_{i+1} - \dfrac{1}{2} & (1 \leqq i \leqq n-1) \\ s_n = -z_n + \dfrac{n-1}{4} . \end{cases}$$

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We shall show the following theorem:

THEOREM. (1) *Let* $\mathfrak{S}_n$ *be the symmetric group on n letters. For each* $\sigma \in \mathfrak{S}_n$, *there exists a matrix* $C(\sigma, z)$ *in* $GL_{4^n}(\mathbf{C}(q^{z_1}, \cdots, q^{z_n}))$ *such that*

$$L(x; \chi; \sigma(z)) = \sum_{\chi'} C(\sigma, z)_{\chi,\chi'} L(x; \chi'; z)\ , \quad \textit{for each} \quad x \in X\ ,$$

where χ' *ranges over the character group* $((k^*/k^{*2})^n)^\wedge$ *of* $(k^*/k^{*2})^n$.

(2) *The function*

$$\prod_{\substack{1 \le i,j \le n \\ i \ne j}} (q^{2z_i} - q^{2z_j - 1}) \cdot L(x; \chi; z)$$

is a polynomial in $q^{\pm z_1}, \cdots, q^{\pm z_n}$.

A formula for $C(\sigma, z)$ is given for the transpositions $(\alpha\ \alpha + 1)$, $1 \leqq \alpha \leqq n - 1$. One can calculate $C(\sigma, z)$ for arbitrary $\sigma \in \mathfrak{S}_n$ by using the cocycle property $C(\sigma\tau, z) = C(\sigma, \tau(z))C(\tau, z)$.

In this paper, we shall consider also the hermitian cases and show similar theorems. To prove the theorems, we need the explicit forms of spherical functions of size 2 given in [3]. The functional equations, possible poles and zeros of spherical functions are related to the image of the spherical transform: for example, in the unramified hermitian case, the functional equations imply that the image is contained in

$$\prod_{1 \le i < j \le n} \frac{q^{2z_j} - q^{2z_i - 1}}{q^{2z_j} + q^{2z_i}} \mathbf{C}[q^{\pm 2z_1}, \cdots, q^{\pm 2z_n}]^{\mathfrak{S}_n}$$

(cf. Remarks at the end of § 2 and § 4).

In the real case, analogous functional equations for spherical functions were given by Oshima and Sekiguchi [5, § 4, Proposition 4.6 and Theorem 4.10].

1. Prelminaries. We shall use the same notation as in [2]. We denote by k_0 a $\mathfrak{p}$-adic number field with $\mathfrak{p}$ not lying over 2. As before, we shall consider the following three cases:

(U) the unramified hermitian case (k is an unramified extension of k_0 of degree 2),

(R) the ramified hermitian case (k is a ramified extension of k_0 of degree 2),

(S) the symmetric case ($k = k_0$).

Denote by $\mathcal{O}$ and $\mathfrak{P} = (\Pi)$ the ring of integers of k and the maximal ideal of k, respectively, where Π is a fixed prime element such as $\Pi \in k_0$ in Case (U) and $\Pi^2 \in k_0$ in Case (R). For the hermitian cases, let $*$ be the nontrivial k_0-automorphism of k.

For a positive integer n, let $G = G_n = GL_n(k)$ and $K = K_n = GL_n(\mathcal{O})$.

For a matrix $A=(a_{ij})\in M_{m,n}(k)$, A^* denotes the matrix $(a_{ji}^*)\in M_{n,m}(k)$ in Cases (U) and (R), and A^* denotes the transposed matrix of A in Case (S). For a positive integer n, let $X=X_n=\{A\in G\colon A^*=A\}$ and $X(\mathcal{O})=X_n(\mathcal{O})=X\cap M_n(\mathcal{O})$. For each case, the group G acts on X by $g\cdot x=gxg^*$ ($x\in X$, $g\in G$). For each $x\in X$ and an integer i, $1\leqq i\leqq n$, let $x_{(i)}$ be the upper left i by i block of x and $d_i(x)$ the determinant of $x_{(i)}$.

Denote by $\mathscr{H}(G,K)$ the Hecke algebra of G with respect to K. Let $C^\infty(K\backslash X)$ be the space of all K-invariant complex-valued functions on X and $S(K\backslash X)$ the subspace of $C^\infty(K\backslash X)$ consisting of all compactly supported functions in $C^\infty(K\backslash X)$. By the convolution product, $C^\infty(K\backslash X)$ and $S(K\backslash X)$ become $\mathscr{H}(G,K)$-modules. We are interested in the $\mathscr{H}(G,K)$-module structure of $S(K\backslash X)$.

Now we recall the spherical functions and the spherical transform on $S(K\backslash X)$:
in Case (U),

$$\zeta(x;s)=\zeta(x;s_1,\cdots,s_n)=\int_{K'}\prod_{i=1}^{n}|d_i(k\cdot x)|^{s_i}dk\ ,$$

$$F\colon S(K\backslash X)\to \boldsymbol{C}(q^{z_1},\cdots,q^{z_n})\ ,$$

$$F(f)(z)=\int_X f(x)\cdot\zeta(x^{-1};z)dx\ ,\quad f\in S(K\backslash X)\ ;$$

in Cases (R) and (S),

$$\begin{aligned}L(x;\chi;s)&=L(x;\chi_1,\cdots,\chi_n;s_1,\cdots,s_n)\\&=\int_{K'}\prod_{i=1}^{n}|d_i(k\cdot x)|^{s_i}\chi_i(d_i(k\cdot x))dk\ ,\end{aligned}$$

$$F=(F_\chi)\colon S(K\backslash X)\to\bigoplus_\chi \boldsymbol{C}(q^{z_1},\cdots,q^{z_n})\ ,$$

$$F_\chi(f)(z)=\int_X f(x)\cdot L(x^{-1};\chi;z)dx\ ,\quad f\in S(K\backslash X)\ ,$$

where $x\in X$, $s\in\boldsymbol{C}^n$, $K'=\{k\in K\colon \prod_{i=1}^n d_i(k\cdot x)\neq 0\}$, χ_i is a character of k^*/k^{*2} for which $\chi_i(\Pi)=1$, dk is the Haar measure on K normalized by $\int_K dk=1$, dx is the G-invariant measure on X normalized by $\int_{K\cdot 1_n}dx=1$, and the variable z is related to s by the following formula:

(1.1) in Cases (R) with $\left(\frac{-1}{\mathfrak{p}}\right)=1$ and (S),

$$\begin{cases}s_i=-z_i+z_{i+1}-\dfrac{1}{2}\quad(1\leqq i\leqq n-1)\\[2mm] s_n=-z_n+\dfrac{n-1}{4}\end{cases}$$

in Cases (U) and (R) with $\left(\frac{-1}{\mathfrak{p}}\right)=-1$,

$$\begin{cases} s_i = -z_i + z_{i+1} - \dfrac{1}{2} - \dfrac{\pi\sqrt{-1}}{2\log q} & (1 \leqq i \leqq n-1) \\ s_n = -z_n + \dfrac{n-1}{4} - \dfrac{\pi\sqrt{-1}}{2\log q} \,. \end{cases}$$

In Cases (R) and (S), the number of the ways of the choice for χ is 2^n, and hence $\bigoplus_\chi$ means the direct product of 2^n copies. The integrals $\zeta(x; s)$ and $L(x; \chi; s)$ are absolutely convergent for $\mathrm{Re}(s_1), \cdots, \mathrm{Re}(s_{n-1}) \geqq 0$, and have analytic continuations to rational functions in $q^{s_1}, \cdots, q^{s_n}$ (cf. [1]).

Now we introduce the following integrals: in Case (U),

$$\Phi(s; f) = \Phi(s_1, \cdots, s_n; f) = \int_{X'} \prod_{i=1}^n |d_i(x)|^{s_i} f(x)dx \;;$$

in Cases (R) and (S),

$$\begin{aligned} \Phi(\chi; s; f) &= \Phi(\chi_1, \cdots, \chi_n; s_1, \cdots, s_n; f) \\ &= \int_{X'} \prod_{i=1}^n |d_i(x)|^{s_i} \chi_i(d_i(x)) f(x)dx \,, \end{aligned}$$

where $f \in S(K\backslash X)$, $s \in \boldsymbol{C}^n$, $X' = \{x \in X \colon \prod_{i=1}^n d_i(x) \neq 0\}$ and $\chi = (\chi_1, \cdots, \chi_n)$ is a character of $(k^*/k^{*2})^n$.

Since f has compact support in X, $\{d_n(x) \colon x \in \mathrm{Supp}(f)\}$ is compact in k^*. Hence the integrals $\Phi(s; f)$ and $\Phi(\chi; s; f)$ are absolutely convergent for $\mathrm{Re}(s_1), \cdots, \mathrm{Re}(s_{n-1}) \geqq 0$, and have analytic continuations to rational functions in $q^{s_1}, \cdots, q^{s_n}$ (cf. [1]), more precisely in $q^{2s_1}, \cdots, q^{2s_n}$ in Cases (U) and (R). For each $x \in X$, let ch_x be the characteristic function of $K \cdot x$ and $v(K \cdot x) = \int_{K \cdot x} dy$. Then it is easy to see that $\Phi(s; \mathrm{ch}_x) = v(K \cdot x) \cdot \zeta(x; s)$ and $\Phi(\chi; s; \mathrm{ch}_x) = v(K \cdot x) \cdot L(x; \chi; s)$. For each $f \in S(K\backslash X)$, let $f^\vee \in S(K\backslash X)$ be determined by $f^\vee(x) = f(x^{-1})$ for every $x \in X$. Then we have $\Phi(z; f) = F(f^\vee)(z)$ and $\Phi(\chi; z; f) = F_\chi(f^\vee)(z)$, where the variable z is related to s by (1.1).

We shall determine the functional equations, possible poles and zeros of $\Phi(s; f)$ and $\Phi(\chi; s; f)$ (cf. Theorems in the beginning of § 2–§ 4). The symmetric group $\mathfrak{S}_n$ on n letters acts on $\{z_1, \cdots, z_n\}$ by $\sigma(z_j) = z_{\sigma(j)}$, $1 \leqq j \leqq n$, $\sigma \in \mathfrak{S}_n$.

The following lemma enables us to reduce the proof of the functional equations for arbitrary size n to the case $n = 2$. In Case (S), we shall decompose $\Phi(\chi; z; f)$ and give a similar identity for each summand in § 4.

(1.2) LEMMA. *Let α be an integer with $1 \leqq \alpha \leqq n-1$ and, for each $x \in X'$, let $\tilde{x}$ be the lower right 2 by 2 block of $x^{-1}_{(\alpha+1)}$. When* $\mathrm{Re}(s_1), \cdots, \mathrm{Re}(s_{n-1}) \geqq 0$, *the following identities hold for every f in* $S(K\backslash X)$: *in Case* (U),

$$\Phi(s; f) = \int_{X'} \prod_{i\neq\alpha,\alpha\pm1} |d_i(x)|^{s_i} \prod_{j=\alpha\pm1} |d_j(x)|^{s_\alpha/2+s_j} \cdot f(x)\zeta_s(\tilde{x}; s_\alpha, -s_\alpha/2)dx \ ;$$

in Case (R),

$$\Phi(\chi; s; f) = \int_{X'} \prod_{i\neq\alpha,\alpha\pm1} |d_i(x)|^{s_i}\chi_i(d_i(x)) \prod_{j=\alpha\pm1} |d_j(x)|^{s_\alpha/2+s_j}\chi_j(d_j(x)) \times \chi_\alpha(d_{\alpha+1}(x))f(x)L_s(\tilde{x}; \chi_\alpha, 1; s_\alpha, -s_\alpha/2)dx \ ,$$

where the suffix s in ζ_s (;) *and* L_s(; ;) *means that they are written in the variable s.*

PROOF. Assume that $\mathrm{Re}(s_1), \cdots, \mathrm{Re}(s_{n-1}) \geqq 0$ and fix an α with $1 \leqq \alpha \leqq n-1$. For each $x \in X$, let $x^\wedge$ be the (2, 2)-entry of $\tilde{x}$. In each case we have $d_\alpha(x) = d_{\alpha+1}(x)x^\wedge$. Hence we have, in Case (U),

$$\Phi(s; f) = \int_{X'} \prod_{i\neq\alpha,\alpha+1} |d_i(x)|^{s_i} \cdot |d_{\alpha+1}(x)|^{s_\alpha+s_{\alpha+1}}|x^\wedge|^{s_\alpha}f(x)dx \ ;$$

and in Case (R),

$$\Phi(\chi; s; f) = \int_{X'} \prod_{i\neq\alpha,\alpha+1} |d_i(x)|^{s_i}\chi_i(d_i(x))|d_{\alpha+1}(x)|^{s_\alpha+s_{\alpha+1}}(\chi_\alpha\chi_{\alpha+1})(d_{\alpha+1}(x)) \times |x^\wedge|^{s_\alpha}\chi_\alpha(x^\wedge)f(x)dx \ .$$

Define an action of $K_2 = GL_2(\mathcal{O})$ on X through the embedding

$$K_2 \to K, \ k \mapsto \begin{pmatrix} 1_{\alpha-1} & & 0 \\ & k & \\ 0 & & 1_{n-\alpha-1} \end{pmatrix},$$

where 1_m denotes the identity matrix of size m. Then we obtain, in Case (U),

$$\begin{aligned}\Phi(s; f) &= \int_{X'} \prod_{i\neq\alpha,\alpha+1} |d_i(x)|^{s_i} \cdot |d_{\alpha+1}(x)|^{s_\alpha+s_{\alpha+1}}|x^\wedge|^{s_\alpha}\int_{K_2} f(k^{-1}\cdot x)dx \\ &= \int_X\int_{K_2'} \prod_{i\neq\alpha,\alpha+1} |d_i(k\cdot z)|^{s_i} \cdot |d_{\alpha+1}(k\cdot x)|^{s_\alpha+s_{\alpha+1}}|(k\cdot x)^\wedge|^{s_\alpha}f(x)dkdx \\ &= \int_{X'} \prod_{i\neq\alpha,\alpha+1} |d_i(x)|^{s_i} \cdot |d_{\alpha+1}(x)|^{s_\alpha+s_{\alpha+1}}f(x)\left\{\int_{K_2'} |(k\cdot x)^\wedge|^{s_\alpha}dk\right\}dx \ ,\end{aligned}$$

where $K_2' = \{k \in K_2 \colon (k\cdot x)^\wedge \neq 0\}$. Since we have

$$\begin{aligned}(k\cdot x)^\wedge &= \text{the } (\alpha+1, \alpha+1)\text{-entry of } \begin{pmatrix} 1_{\alpha-1} & \\ & k \end{pmatrix} \cdot x_{(\alpha+1)}^{-1} \\ &= \text{the (2, 2)-entry of } k\cdot\tilde{x} = d_1\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}k\cdot\tilde{x}\right) ,\end{aligned}$$

we see that

$$\begin{aligned}\int_{K_2'} |(k\cdot x)^\wedge|^{s_\alpha}dk &= \zeta_s(\tilde{x}; s_\alpha, 0) \\ &= |d_{\alpha-1}(x)|^{s_\alpha/2}|d_{\alpha+1}(x)|^{-s_\alpha/2}\zeta_s(\tilde{x}; s_\alpha, -s_\alpha/2) \ ,\end{aligned}$$

and this completes the proof in Case (U). In Case (R), we obtain in a similar manner,

$$\begin{aligned}\Phi(\chi; s; f) &= \int_{X'} \prod_{i\neq\alpha,\alpha+1} |d_i(x)|^{s_i}\chi_i(d_i(x))\cdot|d_{\alpha+1}(x)|^{s_\alpha+s_{\alpha+1}} \\ &\qquad \times(\chi_\alpha\chi_{\alpha+1})(d_{\alpha+1}(x))\cdot f(x)\left\{\int_{K_2'}|(k\cdot x)^\wedge|^{s_\alpha}\chi_\alpha((k\cdot x)^\wedge)dk\right\}dx \\ &= \int_{X'} \prod_{i\neq\alpha,\alpha+1} |d_i(x)|^{s_i}\chi_i(d_i(x))\cdot|d_{\alpha+1}(x)|^{s_\alpha+s_{\alpha+1}}(\chi_\alpha\chi_{\alpha+1})(d_{\alpha+1}(x)) \\ &\qquad \times f(x)L_s(\tilde{x}; \chi_\alpha, 1; s_\alpha, 0)dx \\ &= \int_{X'} \prod_{i\neq\alpha,\alpha\pm1} |d_i(x)|^{s_i}\chi_i(d_i(x)) \prod_{j=\alpha\pm1} |d_j(x)|^{s_\alpha/2+s_j}\chi_j(d_j(x)) \\ &\qquad \times\chi_\alpha(d_{\alpha+1}(x))f(x)L_s(\tilde{x}; \chi_\alpha, 1; s_\alpha, -s_\alpha/2)dx\,. \end{aligned}$$

q.e.d.

Now we establish Lemma (1.5), which will be used to determine possible poles of the spherical functions.

Let α be an integer with $1 \leqq \alpha \leqq n-1$, and define the following domains:

(1.3) $\mathscr{D}_0 = \{s = (s_1, \cdots, s_n)\colon \operatorname{Re}(s_i) \geqq 0 \ (1 \leqq i \leqq n-1)\}$,

$$\mathscr{D}_{\alpha,1} = \begin{cases} \left\{s\in \boldsymbol{C}^n\colon \begin{array}{l}\operatorname{Re}(s_i)\geqq 0\ (3\leqq i\leqq n-1),\ \operatorname{Re}(s_1)\leqq -1 \\ \operatorname{Re}(s_1+s_2+1/2)\geqq 0\end{array}\right\} & \text{if } \alpha=1 \\ \left\{s\in \boldsymbol{C}^n\colon \begin{array}{l}\operatorname{Re}(s_i)\geqq 0\ (1\leqq i\leqq n-1,\ i\neq\alpha,\ \alpha\pm1), \\ \operatorname{Re}(s_\alpha)\leqq -1\ \operatorname{Re}(s_\alpha+s_j+1/2)\geqq 0\ (j=\alpha\pm1)\end{array}\right\} & \text{if } 2\leqq\alpha\leqq n-2 \\ \left\{s\in \boldsymbol{C}^n\colon \begin{array}{l}\operatorname{Re}(s_i)\geqq 0\ (1\leqq i\leqq n-3),\ \operatorname{Re}(s_{n-1})\leqq -1 \\ \operatorname{Re}(s_{n-1}+s_{n-2}+1/2)\geqq 0\end{array}\right\} & \text{if } \alpha=n-1, \end{cases}$$

$$\mathscr{D}_{\alpha,2} = \begin{cases} \left\{s\in \boldsymbol{C}^n\colon \begin{array}{l}\operatorname{Re}(s_i)\geqq 0\ (3\leqq i\leqq n-1),\ -1\leqq\operatorname{Re}(s_1)\leqq 0 \\ \operatorname{Re}(s_1/2+s_2)\geqq 0\end{array}\right\} & \text{if } \alpha=1 \\ \left\{s\in \boldsymbol{C}^n\colon \begin{array}{l}\operatorname{Re}(s_i)\geqq 0\ (1\leqq i\leqq n-1,\ i\neq\alpha,\ \alpha\pm1), \\ -1\leqq\operatorname{Re}(s_\alpha)\leqq 0,\ \operatorname{Re}(s_\alpha/2+s_j)\geqq 0\ (j=\alpha\pm1)\end{array}\right\} & \text{if } 2\leqq\alpha\leqq n-2 \\ \left\{s\in \boldsymbol{C}^n\colon \begin{array}{l}\operatorname{Re}(s_i)\geqq 0\ (1\leqq i\leqq n-3), \\ -1\leqq\operatorname{Re}(s_{n-1})\leqq 0\ \operatorname{Re}(s_{n-1}/2+s_{n-2})\geqq 0\end{array}\right\} & \text{if } \alpha=n-1, \end{cases}$$

$$\mathscr{D}_\alpha = \mathscr{D}_0\cup\mathscr{D}_{\alpha,1}\cup\mathscr{D}_{\alpha,2} \quad\text{and}\quad \mathscr{C} = \bigcup_{\alpha=1}^{n-1}\mathscr{D}_\alpha\,.$$

Let $\sigma_\alpha = (\alpha\ \alpha+1) \in \mathfrak{S}_n$. Then σ_α acts on $\{z_1, \cdots, z_n\}$ as the transposition of z_α and $z_{\alpha+1}$, and so σ_α acts on $\{s_1, \cdots, s_n\}$ as follows: in Cases (R) with $\left(\frac{-1}{\mathfrak{p}}\right) = 1$ and (S),

$$\sigma_\alpha(s_j) = \begin{cases} -s_\alpha - 1 & \text{if } j=\alpha \\ s_\alpha + s_j + \dfrac{1}{2} & \text{if } j = \alpha\pm1 \\ s_j & \text{otherwise}\,; \end{cases}$$

in Cases (U) and (R) with $\left(\frac{-1}{\mathfrak{p}}\right) = -1$,

$$\sigma_\alpha(s_j) = \begin{cases} -s_\alpha - 1 - \dfrac{\pi\sqrt{-1}}{\log q} & \text{if } j = \alpha \\ s_\alpha + s_j + \dfrac{1}{2} + \dfrac{\pi\sqrt{-1}}{2\log q} & \text{if } j = \alpha \pm 1 \\ s_j & \text{otherwise}\,. \end{cases}$$

In particular, we see that $\sigma_\alpha(\mathscr{D}_0) = \mathscr{D}_{\alpha,1}$ and $\sigma_\alpha(\mathscr{D}_{\alpha,2}) = \mathscr{D}_{\alpha,2}$.

(1.4) LEMMA. *Let $\mathscr{D} = \mathscr{C} \cup \bigcup_{1\le i\le j\le n-2} \sigma_i\sigma_{i+1}\cdots\sigma_j(\mathscr{C})$. Then $\mathscr{D}$ is connected and the convex hull of $\mathscr{D}$ is equal to C^n.*

PROOF. Since $\mathscr{D}_\alpha$ is connected and contains $\mathscr{D}_0$ for any α, we see that $\mathscr{C}$ is connected. For each j, since $\sigma_j(\mathscr{D}_j) = \mathscr{D}_j$, we have $\mathscr{C} \cap \sigma_j(\mathscr{C}) \neq \emptyset$, and so $\sigma_i\sigma_{i+1}\cdots\sigma_j(\mathscr{C}) \cap \sigma_i\sigma_{i+1}\cdots\sigma_{j-1}(\mathscr{C}) \neq \emptyset$, for every i with $i < j$. Hence we see that $\mathscr{D}$ is connected.

Let

$$t = \begin{cases} 0 & \text{in Cases (R) with } \left(\dfrac{-1}{\mathfrak{p}}\right) = 1 \text{ and (S)} \\ \dfrac{\pi\sqrt{-1}}{2\log q} & \text{in Cases (U) and (R) with } \left(\dfrac{-1}{\mathfrak{p}}\right) = -1\,. \end{cases}$$

For any $a \in C$ with $\operatorname{Re}(a) \ge 1/2$, we see that $P = (0, \cdots, 0, a, -a-1/2-t, 0) \in \mathscr{D}_{n-1,1}$. And so we get, for $2 \le i \le n-2$,

$$\sigma_i\sigma_{i+1}\cdots\sigma_{n-2}(P) = \Big(\cdots, 0, \overset{i-1}{a + \frac{n-i-1}{2} + (n-i-1)t}\,, \\ \overset{i}{-a - \frac{n-i}{2} - (n-i)t}, 0, \cdots\Big) \in \mathscr{D}\,,$$

and $\sigma_1\sigma_2\cdots\sigma_{n-2}(P) = (-a-(n-1)/2-(n-1)t, 0\cdots) \in \mathscr{D}$. Hence we see that, for any $a_i \in C$ with $\operatorname{Re}(a_i) \ge 1/2$ $(1 \le i \le n-1)$ and $b \in C$,

$$\frac{1}{n}\Big(-a_1 + a_2 - 1 - 2t, -a_2 + a_3 - 1 - 2t, \cdots, \\ -a_{n-2} + a_{n-1} - 1 - 2t, -a_{n-1} - \frac{1}{2} - t, b\Big)$$

is contained in the convex hull $\bar{\mathscr{D}}$ of $\mathscr{D}$, and so for any $b_i \in C$ with $\operatorname{Re}(b_i) \ge 1/n$ $(1 \le i \le n-1)$ and $b_n \in C$, $(-b_1, \cdots, -b_n)$ belongs to $\bar{\mathscr{D}}$. Since $(c_1, \cdots, c_n) \in \mathscr{D}_0$ for any $c_i \in C$ with $\operatorname{Re}(c_i) \ge 0$, we have $\bar{\mathscr{D}} = C^n$. q.e.d.

For any integer m, let $C(q^{mz_1}, \cdots, q^{mz_n})$ be the rational function field,

$C[q^{\pm mz_1}, \cdots, q^{\pm mz_n}]$ the polynomial ring in $q^{\pm mz_1}, \cdots, q^{\pm mz_n}$, and $C[q^{\pm mz_1}, \cdots, q^{\pm mz_n}]^{\mathfrak{S}_n}$ the subring of $C[q^{\pm mz_1}, \cdots, q^{\pm mz_n}]$ consisting of all polynomials which are invariant under the action of $\mathfrak{S}_n$.

(1.5) LEMMA. *Let $\mathscr{D}$ be as in (1.3) and $f(z) \in C(q^{mz_1}, \cdots, q^{mz_n})$. If $f(z)$ is holomorphic in $\mathscr{D}$ for the variable s, then $f(z)$ is contained in $C[q^{\pm mz_1}, \cdots, q^{\pm mz_n}]$.*

PROOF. By (1.4), it is known (cf. [4, Theorem 2.5.10]) that $f(z)$ is holomorphic in C^n for the variable s, and so we see that $f(z)$ is holomorphic in C^n for the variable z by (1.1). Since $f(z)$ is assumed to be a rational function in $q^{mz_1}, \cdots, q^{mz_n}$, we have $f(z) \in C[q^{\pm mz_1}, \cdots, q^{\pm mz_n}]$.
q.e.d.

2. The unramified hermitian case.

THEOREM. *For any $f \in S(K\backslash X)$,*

$$\prod_{1 \leq i < j \leq n} \frac{q^{2z_j} + q^{2z_i}}{q^{2z_j} - q^{2z_i - 1}} \cdot \Phi(z; f)$$

belongs to $C[q^{\pm 2z_1}, \cdots, q^{\pm 2z_n}]$ and is $\mathfrak{S}_n$-invariant.

(2.1) LEMMA. *For any $x \in X$ and $s \in C$,*

$$\frac{1 - q^{-2s-1}}{1 + q^{-2s-2}} \zeta_s(x; s, 0)$$

is a polynomial in q^{2s} and q^{-2s}, and satisfies the following identity:

$$\frac{1 - q^{-2s-1}}{1 + q^{-2s-2}} \zeta_s\left(x; s, -\frac{s}{2}\right) = \frac{-1 + q^{-2s-1}}{q^{-1} + q^{-2s-1}} \zeta_s\left(x; -s-1, \frac{s+1}{2} + \frac{\pi\sqrt{-1}}{2\log q}\right).$$

PROOF. By [3, §2, Theorem 1], we see that

$$\frac{q^{2z_2} + q^{2z_1}}{q^{2z_2} - q^{2z_1 - 1}} \zeta(x; z) \in C[q^{2z_1}, q^{2z_2}]^{\mathfrak{S}_2}.$$

Transforming the variable z into s and letting $s_1 = s$ and $s_2 = -s/2$, we get

$$\frac{1 - q^{-2s-1}}{1 + q^{-2s-2}} \zeta_s\left(x; s, -\frac{s}{2}\right) = \frac{-1 + q^{-2s-1}}{q^{-1} + q^{-2s-1}} \zeta_s\left(x; -s-1-\frac{\pi\sqrt{-1}}{\log q}, \frac{s+1}{2} + \frac{\pi\sqrt{-1}}{2\log q}\right)$$
$$= \frac{-1 + q^{-2s-1}}{q^{-1} + q^{-2s-1}} \zeta_s\left(x; -s-1, \frac{s+1}{2} + \frac{\pi\sqrt{-1}}{2\log q}\right). \quad \text{q.e.d.}$$

PROOF OF THEOREM. Let α be an integer with $1 \leqq \alpha \leqq n-1$ and for each $x \in X'$ let $\tilde{x}$ be the lower right 2 by 2 block of $x_{(\alpha+1)}^{-1}$. By (1.2), we have the following identity for $s \in \mathscr{D}_0$:

$$(2.2) \qquad \frac{q^{2z_\alpha+1}+q^{2z_\alpha}}{q^{2z_\alpha+1}-q^{2z_\alpha-1}}\Phi(z;f)=\int_{X'}\prod_{i\neq\alpha,\alpha\pm1}|d_i(x)|^{s_i}\prod_{j=\alpha\pm1}|d_j(x)|^{s_\alpha/2+s_j}$$
$$\times f(x)\frac{1-q^{-2s_\alpha-1}}{1+q^{-2s_\alpha-2}}\zeta(\tilde{x};s_\alpha,-s_\alpha/2)dx\,.$$

The right hand side of (2.2) is absolutely convergent in $\mathscr{D}_{\alpha,1}=\sigma_\alpha(\mathscr{D}_0)$, since the integrand is σ_α-invariant by (2.1). We see that

$$\prod_{i\neq\alpha,\alpha\pm1}|d_i(x)|^{s_i}\prod_{j=\alpha\pm1}|d_j(x)|^{s_\alpha/2+s_j}$$

is bounded for $s\in\mathscr{D}_{\alpha,2}$ and

$$\frac{1-q^{-2s_\alpha-1}}{1+q^{-2s_\alpha-2}}\zeta(\tilde{x};s_\alpha,-s_\alpha/2)$$

is a polynomial in $q^{\pm s_\alpha}$. Since f has compact support, the right hand side of (2.2) is absolutely convergent in $\mathscr{D}_{\alpha,2}$. Thus we see that the right hand side of (2.2) is absolutely convergent in $\mathscr{D}_\alpha$, and so

$$\frac{q^{2z_\alpha+1}+q^{2z_\alpha}}{q^{2z_\alpha+1}-q^{2z_\alpha-1}}\Phi(z;f)$$

is holomorphic in $\mathscr{D}_\alpha$. Since the integrand is σ_α-invariant and $\sigma_\alpha(\mathscr{D}_\alpha)=\mathscr{D}_\alpha$, we see that

$$\frac{q^{2z_\alpha+1}+q^{2z_\alpha}}{q^{2z_\alpha+1}-q^{2z_\alpha-1}}\Phi(z;f)$$

is σ_α-invariant. On the other hand

$$\prod_{1\leq i<j\leq n}\frac{q^{2z_j}+q^{2z_i}}{q^{2z_j}-q^{2z_i-1}}\cdot\frac{q^{2z_\alpha+1}-q^{2z_\alpha-1}}{q^{2z_\alpha+1}+q^{2z_\alpha}}$$

is holomorphic in $\mathscr{D}_\alpha$ and σ_α-invariant. Thus

$$\prod_{1\leq i<j\leq n}\frac{q^{2z_j}+q^{2z_i}}{q^{2z_j}-q^{2z_i-1}}\cdot\Phi(z;f)$$

is holomorphic in $\mathscr{C}(=\bigcup_{\alpha=1}^{n-1}\mathscr{D}_\alpha)$ and $\mathfrak{S}_n$-invariant, and hence is holomorphic in $\bigcup_{\sigma\in\mathfrak{S}_n}\sigma(\mathscr{C})$. Now the result follows from (1.5). q.e.d.

REMARK. The theorem implies that the image of the spherical transform F is contained in

$$\prod_{1\leq i<j\leq n}\frac{q^{2z_j}-q^{2z_i-1}}{q^{2z_j}+q^{2z_i}}\cdot C[q^{\pm 2z_1},\cdots,q^{\pm 2z_n}]^{\mathfrak{S}_n}\,.$$

This observation together with [2, §3, Theorem] and [3, §2] leads us to the conjecture that, in Case (U), the spherical transform

$$F: S(K\backslash X) \to \prod_{1\leq i<j\leq n} \frac{q^{2z_j} - q^{2z_i - 1}}{q^{2z_j} + q^{2z_i}} \cdot C[q^{\pm 2z_1}, \cdots, q^{\pm 2z_n}]^{\mathfrak{S}_n}$$

is an $\mathscr{H}(G, K)$-module isomorphism.

3. The ramified hermitian case. Let χ^* be the character of k^*/k^{*2} determined by $\chi^*(\Pi) = 1$ and $\chi^*(\varepsilon) = \left(\frac{\varepsilon}{\mathfrak{p}}\right)$ for any unit ε in $\mathcal{O}$. Since every character χ of k^*/k^{*2} is assumed to satisfy $\chi^*(\Pi) = 1$, we see that $\chi = 1$ or $\chi = \chi^*$

THEOREM. *Let $f \in S(K\backslash X)$ and let $\chi = (\chi_1, \cdots \chi_n)$ be a character of $(k^*/k^{*2})^n$.*

(i) *In case $\chi_i = \chi^*$ for $1 \leqq i \leqq n-1$.*

$$\{\prod_{1\leq i<j\leq n} (q^{2z_j} - q^{2z_i - 1})\}^{-1} \cdot \Phi(\chi; z; f)$$

belongs to $C[q^{\pm 2z_1}, \cdots, q^{\pm 2z_n}]$ and is $\mathfrak{S}_n$-invariant.

(ii) *In general,*

$$\prod_{1\leq i<j\leq n} \left(q^{2z_j} - \left(\frac{-1}{\mathfrak{p}}\right) q^{2z_i}\right) \cdot \Phi(\chi; z; f)$$

belongs to $C[q^{\pm 2z_1}, \cdots, q^{\pm 2z_n}]$, and for each $\sigma \in \mathfrak{S}_n$, there exists an $C_{\sigma,\chi}(z)$ in $C(q^{2z_1}, \cdots, q^{2z_n})$ and a character χ_σ of $(k^*/k^{*2})^n$ such that

$$\Phi(\chi; \sigma(z); f) = C_{\sigma,\chi}(z) \cdot \Phi(\chi_\sigma; z; f) .$$

In particular, for $\sigma = (\alpha\ \alpha+1) \in \mathfrak{S}_n$, we have the following: in case $\chi_\alpha = 1$,

$$C_{\sigma,\chi}(z) = -\left(\frac{-1}{\mathfrak{p}}\right) \quad \text{and} \quad (\chi_\sigma)_j = \begin{cases} \chi^*\chi_j & \text{if } j = \alpha \pm 1 \\ \chi_j & \text{otherwise ;} \end{cases}$$

in case $\chi_\alpha = \chi^*$,

$$C_{\sigma,\chi}(z) = \frac{q^{2z_\alpha} - q^{2z_{\alpha+1}-1}}{q^{2z_{\alpha+1}} - q^{2z_\alpha - 1}} \quad \text{and} \quad \chi_\sigma = \chi .$$

(3.1) LEMMA. *For any $x \in X_2$, a character χ of k^*/k^{*2} and $s \in C$,*

$$\left(1 - \left(\frac{-1}{\mathfrak{p}}\right) q^{-2s-1}\right)^{-1} L_s\left(x; \chi^*, \chi; s, -\frac{s}{2}\right) \quad \text{and} \quad (1 - q^{-2s-1}) L_s\left(x; 1, \chi; s, -\frac{s}{2}\right)$$

are polynomials in q^{2s} and q^{-2s}, and satisfy the following identities:

$$\frac{L_s\left(x; \chi^*, \chi; s, -\frac{s}{2}\right)}{1 - \left(\frac{-1}{\mathfrak{p}}\right) q^{-2s-2}}$$

$$= \begin{cases} \dfrac{L_s\left(x; \chi^*, \chi; -s-1, \frac{s+1}{2}\right)}{q^{-2s-1} - q^{-1}} & if \quad \left(\frac{-1}{\mathfrak{p}}\right)=1 \\ \dfrac{L_s\left(x; \chi^*, \chi; -s-1-\frac{\pi\sqrt{-1}}{\log q}, \frac{s+1}{2}+\frac{\pi\sqrt{-1}}{2\log q}\right)}{-q^{2s-1} - q^{-1}} & if \quad \left(\frac{-1}{\mathfrak{p}}\right)=-1 , \end{cases}$$

$$(1 - q^{-2s-1})L_s\left(x; 1, \chi; s, -\frac{s}{2}\right)$$

$$= \begin{cases} (q^{-2s-1} - 1)L_s\left(x; 1, \chi^*\chi; -s-1, \frac{s+1}{2}\right) & if \left(\frac{-1}{\mathfrak{p}}\right) = 1 \\ (1 - q^{-2s-1})L_s\left(x; 1, \chi^*\chi; -s-1, \frac{s+1}{2}+\frac{\pi\sqrt{-1}}{2\log q}\right) & if \left(\frac{-1}{\mathfrak{p}}\right) = -1 . \end{cases}$$

PROOF. By [3, §3, Theorem 1], we see that

$$\frac{L(x; \chi^*, \chi; z)}{q^{2z_2} - q^{2z_1-1}} \in C[q^{\pm 2z_1}, q^{\pm 2z_2}]^{\mathfrak{S}_2}$$

and

$$\left(q^{2z_2} - \left(\frac{-1}{\mathfrak{p}}\right)q^{2z_1}\right)L(x; 1, \chi; z) \in C[q^{\pm 2z_1}, q^{\pm 2z_2}]$$

and it satisfies the following identity:

$$\left(q^{2z_2} - \left(\frac{-1}{\mathfrak{p}}\right)q^{2z_1}\right)L(x; 1, \chi; z_1, z_2) = \left(q^{2z_1} - \left(\frac{-1}{\mathfrak{p}}\right)q^{2z_2}\right)L(x; 1, \chi^*\chi; z_2, z_1) .$$

Transforming the variable z into s and letting $s_1 = s$ and $s_2 = -s/2$, we obtain the result. q.e.d.

(3.2) PROPOSITION. *Let $f \in S(K\backslash X)$ and let $\chi = (\chi_1, \cdots, \chi_n)$ be a character of $(k^*/k^{*2})^n$. For an integer α with $1 \leqq \alpha \leqq n-1$, let $\sigma_\alpha = (\alpha\, \alpha+1) \in \mathfrak{S}_n$ and let $\mathscr{D}_\alpha$ be the domain defined in* (1.3). *Then*

(i) *in case $\chi_\alpha = \chi^*$,*

$$\frac{\Phi(\chi; z; f)}{q^{2z_{\alpha+1}} - q^{2z_\alpha - 1}}$$

is holomorphic in $\mathscr{D}_\alpha$ and σ_α-invariant;

(ii) *in case $\chi_\alpha = 1$,*

$$\left(q^{2z_{\alpha+1}} - \left(\frac{-1}{\mathfrak{p}}\right)q^{2z_\alpha}\right)\Phi(\chi; z; f)$$

is holomorphic in $\mathscr{D}_\alpha$ and satisfies the following identity:

$$\sigma_\alpha\left(\left(q^{2z_{\alpha+1}} - \left(\frac{-1}{\mathfrak{p}}\right)q^{2z_\alpha}\right)\Phi(\chi; z; f)\right) = \left(q^{2z_{\alpha+1}} - \left(\frac{-1}{\mathfrak{p}}\right)q^{2z_\alpha}\right)\Phi(\chi'; z; f) ,$$

where

$$\chi'_j = \begin{cases} \chi^*\chi_j & \text{if } j = \alpha \pm 1 \\ \chi_j & \text{otherwise} . \end{cases}$$

PROOF. Let α be an integer with $1 \leqq \alpha \leqq n-1$ and for each $x \in X'$ let $\tilde{x}$ be the lower right 2 by 2 block of $x^{-1}_{(\alpha+1)}$. Let $\chi_\alpha = \chi^*$. By (1.2), we have the following identity for $s \in \mathscr{D}_0$:

$$(3.3) \quad \frac{\Phi(\chi; z; f)}{q^{2z_\alpha+1} - q^{2z_\alpha-1}} = \int_{X'} \prod_{i \neq \alpha, \alpha\pm 1} |d_i(x)|^{s_i}\chi_i(d_i(x)) \\ \times \prod_{j=\alpha\pm 1} |d_j(x)|^{s_\alpha/2+s_j}\chi_j(d_j(x))\cdot\chi^*(d_{\alpha+1}(x))\frac{f(x)L(\tilde{x}; \chi^*, 1; s_\alpha, -s_\alpha/2)}{q^{2z_\alpha+1}\left(1-\left(\frac{-1}{\mathfrak{p}}\right)q^{-2s_\alpha-2}\right)}dx .$$

The right hand side of (3.3) is absolutely convergent in $\mathscr{D}_{\alpha,1} = \sigma_\alpha(\mathscr{D}_0)$, since the integrand is σ_α-invariant by (3.1). We see that

$$\prod_{i \neq \alpha, \alpha\pm 1} |d_i(x)|^{s_i}\chi_i(d_i(x)) \cdot \prod_{j=\alpha\pm 1} |d_j(x)|^{s_\alpha/2+s_j}\chi_j(x)\cdot\chi^*(d_{\alpha+1}(x))q^{-2z_\alpha+1}$$

is bounded for $s \in \mathscr{D}_{\alpha,2}$ and

$$\frac{L(\tilde{x}; \chi^*, 1; s_\alpha, -s_\alpha/2)}{1-\left(\frac{-1}{\mathfrak{p}}\right)q^{-2s_\alpha-2}}$$

is a polynomial in $q^{\pm s_\alpha}$. Since f has compact support, the right hand side of (3.3) is absolutely convergent in $\mathscr{D}_\alpha$ and so

$$\frac{\Phi(\chi; z; f)}{q^{2z_\alpha+1} - q^{2z_\alpha-1}}$$

is holomorphic in $\mathscr{D}_\alpha$. Since the integrand is σ_α-invariant and $\sigma_\alpha(\mathscr{D}_\alpha) = \mathscr{D}_\alpha$, we see that

$$\frac{\Phi(\chi; z; f)}{q^{2z_\alpha+1} - q^{2z_\alpha-1}}$$

is σ_α-invariant, and this completes (i).

Let $\chi_\alpha = 1$. By (1.2), we have the following identity for $s \in \mathscr{D}_0$:

$$(3.4) \quad \left(q^{2z_\alpha+1} - \left(\frac{-1}{\mathfrak{p}}\right)q^{2z_\alpha}\right)\Phi(\chi; z; f) = \int_{X'} \prod_{i \neq \alpha, \alpha\pm 1} |d_i(x)|^{s_i}\chi_i(d_i(x)) \\ \times \prod_{j=\alpha\pm 1} |d_j(x)|^{s_\alpha/2+s_j}\chi_j(d_j(x))\cdot f(x)q^{2z_\alpha+1}(1-q^{-2s_\alpha-1})L(\tilde{x}; 1, 1; s_\alpha, -s_\alpha/2)dx$$

Denote by $A(\chi, z)$ the above integrand. Then we have by (3.1),

$$(3.5) \quad A(\chi, z) = A(\chi', \sigma_\alpha(z)) , \quad \text{where} \quad \chi'_j = \begin{cases} \chi^*\chi_j & \text{if } j = \alpha \pm 1 \\ \chi_j & \text{otherwise} . \end{cases}$$

Considering the identity (3.4) for χ', we see that the right hand side of

(3.4) is absolutely convergent in $\mathscr{D}_{\alpha,1} = \sigma_\alpha(\mathscr{D}_0)$. By the same argument as in case $\chi_\alpha = \chi^*$, we see that the right hand side of (3.4) is absolutely convergent in $\mathscr{D}_\alpha$ and so

$$\left(q^{2z_{\alpha+1}} - \left(\frac{-1}{\mathfrak{p}}\right)q^{2z_\alpha}\right)\Phi(\chi'; z; f)$$

are holomorphic in $\mathscr{D}_\alpha$, we have by (3.5),

$$\sigma_\alpha\left(\left(q^{2z_{\alpha+1}} - \left(\frac{-1}{\mathfrak{p}}\right)q^{2z_\alpha}\right)\Phi(\chi; z; f)\right) = \left(q^{2z_{\alpha+1}} - \left(\frac{-1}{\mathfrak{p}}\right)q^{2z_\alpha}\right)\Phi(\chi'; z; f) . \qquad \text{q.e.d.}$$

PROOF OF THEOREM. Let $\chi_i = \chi^*$ for $1 \leqq i \leqq n-1$. Since

$$(q^{2z_{\alpha+1}} - q^{2z_\alpha - 1})\{\prod_{1\leqq i<j\leqq n}(q^{2z_j} - q^{2z_i - 1})\}^{-1}$$

is holomorphic in $\mathscr{D}_\alpha$ and σ_α-invariant, we have by (3.2),

$$\Phi(\chi; z; f)\{\prod_{1\leqq i<j\leqq n}(q^{2z_j} - q^{2z_i-1})\}^{-1}$$

is holomorphic in $\mathscr{C}(=\bigcup_{\alpha=1}^{n-1}\mathscr{D}_\alpha)$ and $\mathfrak{S}_n$-invariant. Hence it is holomorphic in $\bigcup_{\sigma\in\mathfrak{S}_n}\sigma(\mathscr{C})$, and the result follows from (1.5) in this case.

Now we consider the general case. Let $\varepsilon = \left(\frac{-1}{\mathfrak{p}}\right)$,

$$G(z) = \prod_{1\leqq i<j\leqq n}(q^{2z_j} + \varepsilon q^{2z_i}) \quad \text{and} \quad K_{ij} = \frac{q^{2z_i} - q^{2z_j-1}}{q^{2z_j} - q^{2z_i-1}} \quad (1 \leqq i, j \leqq n) .$$

Then by (3.2), $G(z)\Phi(\chi; z; f)$ is holomorphic in $\mathscr{C}(=\bigcup_{\alpha=1}\mathscr{D}_\alpha)$, and $K_{\alpha,\alpha+1}\Phi(\chi; z; f)$ is holomorphic in $\mathscr{D}_\alpha$ is $\chi_\alpha = \chi^*$. On the other hand, K_{ij} is holomorphic in $\mathscr{C}$ unless $j \neq i \pm 1$. For each i with $1 \leqq i \leqq n-2$, we obtain the following identity for $s \in \mathscr{C} \cap \sigma_i(\mathscr{C})$

$$(3.6) \qquad \sigma_i(G(z)\Phi(\chi; z; f)) = \begin{cases} \varepsilon K_{i,i+1}G(z)\Phi(\chi; z; f) & \text{if } \chi_i = \chi^* \\ G(z)\Phi(\chi'; z; f) & \text{if } \chi_i = 1 , \end{cases}$$

where

$$\chi'_j = \begin{cases} \chi^*\chi_j & \text{if } j = i \pm 1 \\ \chi_j & \text{otherwise} . \end{cases}$$

Hence we see that $G(z)\Phi(\chi; z; f)$ is holomorphic in $\mathscr{C} \cup \bigcup_{i=1}^{n-2}\sigma_i(\mathscr{C})$. For any i, j with $1 \leqq i < j \leqq n-2$, using (3.6) repeatedly, we can express $\sigma_j\sigma_{j-1}\cdots\sigma_i(G(z)\Phi(\chi; z; f))$ as a product of $\pm G(z)\Phi(\tilde{\chi}; z; f)$ for a suitable $\tilde{\chi}$ and some of $K_{j,j+1}$ or $\sigma_j\sigma_{j-1}\cdots\sigma_{l+1}(K_{l,l+1}) = K_{l,j+1}$, $i \leqq l \leqq j$. The factor $K_{j,j+1}$ appears only if $\tilde{\chi}_j = \chi^*$, and then $K_{j,j+1}\Phi(\tilde{\chi}; z; f)$ is holomorphic in $\mathscr{D}_j$. Hence we see that the product is holomorphic in $\mathscr{C}$, and so $G(z)\Phi(\chi; z; f)$ is holomorphic in $\mathscr{C} \cup \{\bigcup_{1\leqq i\leqq j\leqq n-2}\sigma_i\sigma_{i+1}\cdots\sigma_j(\mathscr{C})\}$. Thus we obtain $G(z)\Phi(\chi; z; f) \in C[q^{\pm 2z_1}, \cdots, q^{\pm 2z_n}]$, by (1.5). We get easily the formula for $\Phi(\chi; \sigma_\alpha(z); f)$ by (3.2). q.e.d.

4. The symmetric case.

THEOREM 1. *Let $f \in S(K\backslash X)$ and let $\chi = (\chi_1, \cdots, \chi_n)$ be a character of $(k^*/k^{*2})^n$. Then*

$$\prod_{\substack{1 \leq i,j \leq n \\ i \neq j}} (q^{2z_i} - q^{2z_j - 1}) \cdot \Phi(\chi; z; f)$$

belongs to $C[q^{\pm z_1}, \cdots, q^{\pm z_n}]$, and for each $\sigma \in \mathfrak{S}_n$, there exists a matrix $C(\sigma; z)$, independent of f, in $GL_{4^n}(C(q^{\pm z_1}, \cdots, q^{\pm z_n}))$ such that

$$\Phi(\chi; \sigma(z); f) = \sum_{\chi'} C(\sigma; z)_{\chi, \chi'} \Phi(\chi'; z; f) ,$$

where χ' ranges over the character group $((k^*/k^{*2})^n)^\wedge$.

In Theorem 1, taking f to be the characteristic function for $K \cdot x$ ($x \in X$), we obtain Theorem in the introduction.

To prove Theorem 1, we need to decompose $\Phi(\chi; z; f)$. We shall give the functional equations for each summand of the decomposition in Theorem 2.

Let P be the subgroup of G consisting of all lower triangular matrices and, for $u = (u_1, \cdots, u_n) \in (k^*/k^{*2})^n$, let $X_u = \{x \in X: d_i(x) \equiv u_1 \cdots u_i (\mathrm{mod}\, k^{*2}), 1 \leqq i \leqq n\}$. Then X' can be expressed as

$$X' = \bigcup_u X_u \quad \text{(disjoint union)} ,$$

where u ranges over all representatives of $(k^*/k^{*2})^n$. For $p \in P$ and $x \in X$, we have $(p \cdot x)_{(i)} = p_{(i)} \cdot x_{(i)}$, and so $d_i(p \cdot x) \equiv d_i(x)$ $(\mathrm{mod}\, k^{*2})$. Now we define

$$\Phi_u(z; f) = \Phi_u(s; f) = \int_{X_u} \prod_{i=1}^{n} |d_i(x)|^{s_i} \cdot f(x) dx . \tag{4.1}$$

Then, for $\chi \in ((k^*/k^{*2})^n)^\wedge$,

$$\begin{cases} \Phi(\chi; z; f) = \sum_{u \in (k^*/k^{*2})^n} A_{\chi u} \cdot \Phi_u(z; f) \\ A_{\chi u} = \prod_{i=1}^{n} \chi_i(u_1 \cdots u_i) . \end{cases} \tag{4.2}$$

Note that $\Phi_u(s; f)$ is absolutely convergent in $\mathscr{D}_0$ and has an analytic continuation to a rational function in $q^{s_1}, \cdots, q^{s_n}$ (cf. [1]).

To begin with, we consider the case $n = 2$. For a character χ of k^*/k^{*2} for which $\chi(\Pi) = 1$, let $\tilde{\chi}$ be the character of k^*/k^{*2} defined by $\tilde{\chi}(\Pi) = -1$ and $\tilde{\chi}(\varepsilon) = \chi(\varepsilon)$ for $\varepsilon \in \mathcal{O}^*$.

(4.3) LEMMA. *Let $n = 2$ and $f \in S(K\backslash X)$. Arrange $u \in (k^*/k^{*2})^2$ in the following order*: $(1, 1)$, (δ, δ), (Π, Π), $(\Pi\delta, \Pi\delta)$, $(1, \delta)$, $(\delta, 1)$, $(\Pi, \Pi\delta)$, $(\Pi\delta, \delta)$, $(1, \Pi)$, $(\Pi, 1)$, $(\delta, \Pi\delta)$, $(\Pi\delta, \delta)$, $(1, \Pi\delta)$, $(\Pi\delta, 1)$, (δ, Π) *and* (Π, δ). *Let*

$$\alpha = \frac{(1 - 6q^{-1} + q^{-2})q^{2z_1+2z_2} + (1 + q^{-1})q^{4z_1} + (q^{-1} + q^{-2})q^{4z_2}}{2(q^{2z_1} - q^{2z_2-1})(q^{2z_2} - q^{2z_1-1})},$$

$$\beta = \frac{(1 - q^{-1})(q^{2z_1} + q^{2z_2-1})(q^{2z_2} - q^{2z_1})}{2(q^{2z_1} - q^{2z_2-1})(q^{2z_2} - q^{2z_1-1})},$$

$$\gamma = \frac{(1 - q^{-1})q^{z_1+z_2-1/2}(q^{2z_2} - q^{2z_1})}{(q^{2z_1} - q^{2z_2-1})(q^{2z_2} - q^{2z_1-1})},$$

$$a = \frac{(1 - q^{-1})(q^{2z_2} + q^{2z_1})}{2(q^{2z_2} - q^{2z_1-1})}, \quad b = \frac{(1 + q^{-1})(q^{2z_2} - q^{2z_1})}{2(q^{2z_2} - q^{2z_1-1})},$$

$$c = \frac{(1 - q^{-1})q^{z_1+z_2}}{q^{2z_2} - q^{2z_1-1}}, \quad d = \frac{q^{-1/2}(q^{2z_2} - q^{2z_1})}{q^{2z_2} - q^{2z_1-1}}.$$

Then

$$\Phi_u(z_2, z_1; f) = \sum_{v \in (k^*/k^{*2})^2} M(z_1, z_2)_{uv} \cdot \Phi_v(z_1, z_2; f),$$

where if $\left(\frac{-1}{\mathfrak{p}}\right) = 1$,

$$M(z_1, z_2) = \begin{pmatrix} \alpha & \beta & \gamma & \gamma \\ \beta & \alpha & \gamma & \gamma \\ \gamma & \gamma & \alpha & \beta \\ \gamma & \gamma & \beta & \alpha \end{pmatrix} \perp \begin{pmatrix} a & b \\ b & a \end{pmatrix} \perp \begin{pmatrix} a & b \\ b & a \end{pmatrix} \perp \begin{pmatrix} c & d \\ d & c \end{pmatrix} \perp \begin{pmatrix} c & d \\ d & c \end{pmatrix} \perp \begin{pmatrix} c & d \\ d & c \end{pmatrix} \perp \begin{pmatrix} c & d \\ d & c \end{pmatrix},$$

and if $\left(\frac{-1}{\mathfrak{p}}\right) = -1$,

$$M(z_1, z_2) = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \perp \begin{pmatrix} a & b \\ b & a \end{pmatrix} \perp \begin{pmatrix} \alpha & \beta & \gamma & \gamma \\ \beta & \alpha & \gamma & \gamma \\ \gamma & \gamma & \alpha & \beta \\ \gamma & \gamma & \beta & \alpha \end{pmatrix} \perp \begin{pmatrix} c & d \\ d & c \end{pmatrix} \perp \begin{pmatrix} c & d \\ d & c \end{pmatrix} \perp \begin{pmatrix} c & d \\ d & c \end{pmatrix} \perp \begin{pmatrix} c & d \\ d & c \end{pmatrix}.$$

REMARK. As is well-known, there are seven G-orbits in X which are represented by

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \Pi & 0 \\ 0 & \Pi\delta \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \Pi \end{pmatrix}, \begin{pmatrix} \delta & 0 \\ 0 & \Pi\delta \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \Pi\delta \end{pmatrix}, \begin{pmatrix} \delta & 0 \\ 0 & \Pi \end{pmatrix}\right\}$$

in case $\left(\frac{-1}{\mathfrak{p}}\right) = 1$,

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \Pi & 0 \\ 0 & \Pi \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \Pi \end{pmatrix}, \begin{pmatrix} \delta & 0 \\ 0 & \Pi\delta \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \Pi\delta \end{pmatrix}, \begin{pmatrix} \delta & 0 \\ 0 & \Pi \end{pmatrix}\right\}$$

in case $\left(\frac{-1}{\mathfrak{p}}\right) = -1$;

and

$$G\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}\ni\begin{pmatrix}\Pi&0\\0&\Pi\end{pmatrix}\quad\text{if }\left(\frac{-1}{\mathfrak{p}}\right)=1\,,$$
$$G\cdot\begin{pmatrix}1&0\\0&\delta\end{pmatrix}\ni\begin{pmatrix}\Pi&0\\0&\Pi\delta\end{pmatrix}\quad\text{if }\left(\frac{-1}{\mathfrak{p}}\right)=-1\,.$$

Note that the above decomposition of the matrix $M(z_1, z_2)$ corresponds to the G-orbit decomposition of X.

PROOF. Arrange $\chi=(\chi_1,\chi_2)$ in the following order: $(1, 1)$, $(1, \chi^*)$, $(\chi^*, 1)$, (χ^*, χ^*); $(1, \tilde{1})$, $(1, \widetilde{\chi^*})$, $(\chi^*, \tilde{1})$, $(\chi^*, \widetilde{\chi^*})$; $(\tilde{1}, 1)$, $(\tilde{1}, \chi^*)$, $(\widetilde{\chi^*}, 1)$, $(\widetilde{\chi^*}, \chi^*)$; $(\tilde{1}, \tilde{1})$, $(\tilde{1}, \widetilde{\chi^*})$, $(\widetilde{\chi^*}, \tilde{1})$ and $(\widetilde{\chi^*}, \widetilde{\chi^*})$. Arrange $u\in(k^*/k^{*2})^2$ as above. Then, for the characteristic function $f=\mathrm{ch}_{K\cdot x}$, $x\in X$, we have

$$v(K\cdot x)L(x;\chi;z)=\sum_{u\in(k^*/k^{*2})^2}A_{\chi u}\cdot\Phi_u(z;f)\,,\tag{4.4}$$

where $A_{\chi u}$ $(=\pm1)$ is as given in (4.2). Let $M(x;z)\in M_{16}(C(q^{z_1}, q^{z_2}))$ for which

$$L(x;\chi;z_2,z_1)=\sum_{\chi'\in((k^*/k^{*2})^2)^\wedge}M(x;z)_{\chi,\chi'}\cdot L(x;\chi';z_1,z_2)\,.$$

Let $\chi_i=1$ or χ^* and $z_i'=z_i-\pi\sqrt{-1}/\log q$, $i=1, 2$. Then

$$\begin{cases}L(x;\chi_1,\tilde{\chi}_2;z_1,z_2)=L(x;\chi_1,\chi_2\colon z_1',z_2')\\ L(x;\tilde{\chi}_1,\chi_2\colon z_1,z_2)=L(x;\chi_1,\chi_2\colon z_1',z_2)\\ L(x;\tilde{\chi}_1,\tilde{\chi}_2\colon z_1,z_2)=L(x;\chi_1,\chi_2\colon z_1,z_2')\,.\end{cases}\tag{4.5}$$

Recall the explicit formula for $L(x;\chi\colon z)$ given in [3, §4, Theorem 1]. It is easy to see that $M(x;z)$ has the form

$$\left(\begin{array}{cc|cc}M_1&0&&\\0&M_2&&0\\\hline&&0&M_4\\0&&M_3&0\end{array}\right),$$

where $M_i=M_i(x;z)\in M_4(C(q^{z_1}, q^{z_2}))$ is diagonal, $1\leqq i\leqq4$. Let $x=\langle\Pi^{\lambda_1}\varepsilon_1\rangle\perp\langle\Pi^{\lambda_2}\varepsilon_2\rangle$. We obtain the following functional equation:

$$L(x;\chi_1,\chi_2\colon z_2,z_1)=f(x;\chi;z)\cdot L(x;\chi_1,\chi_2\colon z_1,z_2)\,,$$

where

$$f(x;\chi;z)=\begin{cases}\dfrac{q^{z_1}+\chi_1(\varepsilon_1\varepsilon_2)q^{z_2-1/2}}{q^{z_2}+\chi_1(\varepsilon_1\varepsilon_2)q^{z_1-1/2}} & \text{if } 2\nmid\lambda_1+\lambda_2\\[2ex] \dfrac{q^{z_2}-q^{z_1-1/2}}{q^{z_1}-q^{z_2-1/2}}\cdot\dfrac{q^{z_1}+\chi^*(-\varepsilon_1\varepsilon_2)q^{z_2-1/2}}{q^{z_2}+\chi^*(-\varepsilon_1\varepsilon_2)q^{z_1-1/2}} & \text{if } 2\mid\lambda_1+\lambda_2 \text{ and } \chi_1=1\\[2ex] \dfrac{q^{2z_1}-q^{2z_2-1}}{q^{2z_2}-q^{2z_1-1}} & \text{if } 2\mid\lambda_1+\lambda_2\text{ and }\chi_1=\chi^*\,.\end{cases}$$

This immediately implies that

$$M_1(x; z) = \begin{cases} \dfrac{q^{z_1} + q^{z_2-1/2}}{q^{z_2} + q^{z_1-1/2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \perp \dfrac{q^{z_1} + \varepsilon q^{z_2-1/2}}{q^{z_2} + \varepsilon q^{z_1-1/2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \text{if } 2\nmid\lambda_1+\lambda_2 \\ \dfrac{(q^{z_1} + \varepsilon' q^{z_2-1/2})(q^{z_2} - q^{z_1-1/2})}{(q^{z_2} + \varepsilon' q^{z_1-1/2})(q^{z_1} - q^{z_2-1/2})}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \perp \dfrac{q^{2z_1} - q^{2z_2-1}}{q^{2z_2} - q^{2z_1-1}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \text{if } 2|\lambda_1+\lambda_2 , \end{cases}$$

where $\varepsilon = \chi^*(\det x)$ and $\varepsilon' = \left(\dfrac{-1}{\mathfrak{p}}\right)\chi^*(\det x)$.

The matrices $M_i(x; z)$, $i = 2, 3, 4$, can be easily obtained from the above formula for $M_1(x; z)$ and the following relations, which are immediate consequences of (4.5):

$$M_2(x; z_1, z_2) = M_1(x; z_1', z_2') ,$$
$$M_3(x; z_1, z_2) = M_1(x; z_1', z_2) \quad \text{and}$$
$$M_4(x; z_1, z_2) = M_1(x; z_1, z_2') .$$

Let $X = \cup_{l=1}^{7} X^{(l)}$ be the G-orbit decomposition, where the indices (l) in that order correspond to the representatives given in the remark above. It is easy to see that $M(x; z)$ depends only on $\det x \pmod{k^{*2}}$, and hence is determined by the G-orbit to which x belongs. We write $M(l; z) = M(x; z)$ if $x \in X^{(l)}$.

For $u = (u_1, u_2) \in (k^*/k^{*2})^2$, define the number $l(u)$, $1 \leqq l(u) \leqq 7$, by $\langle u_1\rangle \perp \langle u_2\rangle \in X^{(l(u))}$. Let $x \in X^{(l)}$. Then $\Phi_u(z_2, z_1; \mathrm{ch}_{K\cdot x}) = 0$ unless $l(u) = l$. If $l(u) = l$, we obtain

$$\begin{aligned}\Phi_u(z_2, z_1; \mathrm{ch}_{K\cdot x}) &= \sum_{v \in (k^*/k^{*2})^2} (A^{-1}M(l; z)A)_{uv}\cdot\Phi_v(z_1, z_2\colon \mathrm{ch}_{K\cdot x}) \\ &= \sum_{\substack{v \in (k^*/k^{*2})^2 \\ l(v)=l}} (A^{-1}M(l; z)A)_{uv}\cdot\Phi_v(z_1, z_2\colon \mathrm{ch}_{K\cdot x})\end{aligned}$$

where $A = (A_{\chi u})$ (cf. (4.4)). Since every $f \in S(K\backslash X)$ is a finite C-linear combination of $\mathrm{ch}_{K\cdot x}$, $x \in X$, we obtain

$$\Phi_u(z_2, z_1; f) = \sum_{\substack{v \in (k^*/k^{*2})^2 \\ l(v)=l(u)}} (A^{-1}M(l; z)A)_{uv}\cdot\Phi_v(z_1, z_2\colon f) .$$

Thus we get

$$M(z_1, z_2)_{uv} = \begin{cases}(A^{-1}M(l; z)A)_{uv} & \text{if } l(u) = l(v) = l \\ 0 & \text{otherwise} .\end{cases}$$

Therefore we can establish the lemma by elementary calculation. q.e.d.

(4.6) LEMMA. *Let $f \in S(K\backslash X)$, $u = (u_1, \cdots, u_n) \in (k^*/k^{*2})^n$ and let α be an integer with $1 \leqq \alpha \leqq n-1$. For each $x \in X'$, let $\tilde{x}$ be the lower 2 by 2 block of $x_{(\alpha+1)}^{-1}$ and $Y_u = \{x \in X\colon d_i(x) \equiv u_1 \cdots u_i \pmod{k^{*2}}$, for $1 \leqq i \leqq n$, $i \neq \alpha\}$. Let*

$$F_\alpha(x;z;f)=\prod_{i\neq\alpha,\alpha\pm1}|d_i(x)|^{s_i}\cdot\prod_{j=\alpha\pm1}|d_j(x)|^{s_\alpha/2+s_j}\cdot f(x)\ .$$

When $\mathrm{Re}(s_1),\cdots,\mathrm{Re}(s_{n-1})\geqq 0$, *the following identity holds*:

$$\Phi_u(z;f)=\int_{Y_u}F_\alpha(x;z;f)\cdot\Phi_v(s_\alpha,-s_{\alpha/2};a_{\tilde{x}}\,\mathrm{ch}_{K_2\cdot\tilde{x}})dx\ ,$$

where $v=(u_{\alpha+1},u_\alpha)\in(k^*/k^{*2})^2$ *and* $a_{\tilde{x}}$ *is a positive constant depending on the* K_2*-orbit containing* $\tilde{x}$.

PROOF. Assume that $\mathrm{Re}(s_1),\cdots,\mathrm{Re}(s_{n-1})\geqq 0$ and fix an α with $1\leqq\alpha\leqq n-1$. For each $x\in X'$, let $d_1^*(\tilde{x})$ be the (2, 2)-entry of $\tilde{x}$. By the same action of K_2 on X as in the proof of (1.2), we obtain

$$\begin{aligned}\Phi_u(z;f)&=\int_{X_u}\prod_{i\neq\alpha,\alpha+1}|d_i(x)|^{s_i}\cdot|d_{\alpha+1}(x)|^{s_\alpha+s_{\alpha+1}}\cdot|d_1^*(\tilde{x})|^{s_\alpha}\int_{K_2}f(k^{-1}\cdot x)dkdx\\&=\int_{Y_u}\prod_{i\neq\alpha,\alpha+1}|d_i(x)|^{s_i}\cdot|d_{\alpha+1}(x)|^{s_\alpha+s_{\alpha+1}}\cdot f(x)\int_{K_2'}|d_1^*(k\cdot\tilde{x})|^{s_\alpha}dkdx\ ,\end{aligned}$$

where $K_2'=\{k\in K_2\colon d_1^*(k\cdot\tilde{x})\equiv u_{\alpha+1}(k^{*2})\}$. Now

$$\begin{aligned}\int_{K_2'}|d_1^*(k\cdot\tilde{x})|^{s_\alpha}dk&=a\int_{K_2\cdot\tilde{x}\cap\{y:d_1^*(y)\equiv u_{\alpha+1}(k^{*2})\}}|d_1^*(y)|^{s_\alpha}dy\\&=\int_{X_{(u_{\alpha+1},u_\alpha)}}|d_1(y)|^{s_\alpha}\cdot a\cdot\mathrm{ch}_{K_2\cdot\tilde{x}}(y)dy=\Phi_{(u_{\alpha+1},u_\alpha)}(s_\alpha,0;a\cdot\mathrm{ch}_{K_2\cdot\tilde{x}})\ ,\end{aligned}$$

and the constant $a=a_{\tilde{x}}$ is given by

$$\int_{\{k\in K_2:k\cdot\tilde{x}=\tilde{x}\}}dk$$

for a suitable Haar measure on $\{k\in K_2\colon k\cdot\tilde{x}=\tilde{x}\}$. It is easy to see that $a_{\tilde{x}}$ depends only on the K_2-orbit containing $\tilde{x}$. Thus we obtain the required identity. q.e.d.

For $u=(u_1,\cdots,u_n)\in(k^*/k^{*2})^n$ and an integer α, $1\leqq\alpha\leqq n-1$, we say that u is of type 2 at α if $2|v_\Pi(u_\alpha u_{\alpha+1})$ and $\chi^*(-u_\alpha u_{\alpha+1})=1$, and u is of type 1 at α, otherwise. For $n=2$, it is easy to see that $\Phi_u(z;f)\in\boldsymbol{C}[q^{\pm z_1},q^{\pm z_2}]$ if u is of type 1 at 1, and $(q^{2z_2}-q^{2z_1-1})\Phi_u(z;f)\in\boldsymbol{C}[q^{\pm z_1},q^{\pm z_2}]$ if u is of type 2 at 1; there is no common factor in $\{\Phi_u(z;f)\colon f\in S(K\backslash X)\}$ if u is of type 1 at 1, nor in $\{(q^{2z_2}-q^{2z_1-1})\Phi_u(z;f)\colon f\in S(K\backslash X)\}$ if u is of type 2 at 1.

THEOREM 2. *Let* $f\in S(K\backslash X)$, $u=(u_1,\cdots,u_n)\in(k^*/k^{*2})^n$ *and*

$$G(z)=\prod_{\substack{1\leqq i,j\leqq n\\ i\neq j}}(q^{2z_i}-q^{2z_j-1})\ .$$

(i) *The function* $G(z)\Phi_u(z;f)$ *belongs to* $\boldsymbol{C}[q^{\pm z_1},\cdots,q^{\pm z_n}]$.

(ii) *For each* $\sigma \in \mathfrak{S}_n$, *there exists a matrix* $B(\sigma; z)$ *in* $GL_{4^n}(C(q^{z_1}, \cdots, q^{z_n}))$ *such that*

$$\Phi_u(\sigma(z); f) = \sum_{v \in (k^*/k^{*2})^n} B(\sigma: z)_{uv} \cdot \Phi_v(z; f) .$$

(iii) *If* u *is of type* 1 *at* α, *let* $u' \in (k^*/k^{*2})^n$ *such that* $u \neq u'$, $u'_j = u_j$ *for* $j \neq \alpha$, $\alpha + 1$ *and*

$$\begin{pmatrix} u'_\alpha & 0 \\ 0 & u'_{\alpha+1} \end{pmatrix} \in G_2 \cdot \begin{pmatrix} u_\alpha & 0 \\ 0 & u_{\alpha+1} \end{pmatrix} .$$

Then, for $\sigma = (\alpha\ \alpha + 1) \in \mathfrak{S}_n$,

$$B(\sigma; z)_{uv} = 0 \quad \textit{unless} \quad v = u \textit{ or } u' ,$$

$$B(\sigma; z)_{uu} = \frac{a_\alpha}{q^{2z_{\alpha+1}} - q^{2z_\alpha - 1}} , \quad B(\sigma; z)_{uu'} = \frac{b_\alpha}{q^{2z_{\alpha+1}} - q^{2z_\alpha - 1}} ,$$

where

$$a_\alpha = \begin{cases} (1 - q^{-1}) q^{z_\alpha + z_{\alpha+1}} & \textit{if } 2 \nmid v_\Pi(u_\alpha u_{\alpha+1}) \\ \dfrac{1 - q^{-1}}{2} (q^{z_\alpha} + q^{z_{\alpha+1}}) & \textit{if } 2 | v_\Pi(u_\alpha u_{\alpha+1}) , \end{cases}$$

$$b_\alpha = \begin{cases} q^{-1/2} (q^{z_{\alpha+1}} - q^{z_\alpha}) & \textit{if } 2 \nmid v_\Pi(u_\alpha u_{\alpha+1}) \\ \dfrac{1 + q^{-1}}{2} (q^{z_{\alpha+1}} - q^{z_\alpha}) & \textit{if } 2 | v_\Pi(u_\alpha u_{\alpha+1}) . \end{cases}$$

(iv) *If* u *is of type* 2 *at* α, *let* $u = u^{(1)}$, *and* $u^{(2)}$, $u^{(3)}$, $u^{(4)} \in (k^*/k^{*2})^n$ *be defined as follows*:

$$u_j^{(i)} = u_j \quad \textit{if } j \neq \alpha, \alpha + 1$$

$$(u_\alpha^{(i)}, u_{\alpha+1}^{(i)}) = \begin{cases} (\delta u_\alpha, \delta u_{\alpha+1}) & \textit{if } i = 2 \\ (\Pi u_\alpha, \Pi u_{\alpha+1}) & \textit{if } i = 3 \\ (\Pi \delta u_\alpha, \Pi \delta u_{\alpha+1}) & \textit{if } i = 4 . \end{cases}$$

Then, for $\sigma = (\alpha\ \alpha + 1) \in \mathfrak{S}_n$,

$$B(\sigma; z)_{uv} = 0 \quad \textit{unless } v = u^{(i)},\ 1 \leqq i \leqq 4 ,$$

$$B(\sigma; z)_{uu^{(i)}} = \frac{c_{\alpha,i}}{(q^{2z_\alpha} - q^{2z_{\alpha+1} - 1})(q^{2z_{\alpha+1}} - q^{2z_\alpha - 1})} ,$$

where

$$c_{\alpha,1} = \frac{1}{2} \{ (1 - 6q^{-1} + q^{-2}) q^{2z_\alpha + 2z_{\alpha+1}} + (1 + q^{-1}) q^{2z_\alpha} + (q^{-1} + q^{-2}) q^{2z_{\alpha+1}} \} ,$$

$$c_{\alpha,2} = \frac{1}{2} (1 - q^{-1}) (q^{2z_\alpha} + q^{2z_{\alpha+1} - 1}) (q^{2z_{\alpha+1}} - q^{2z_\alpha}) ,$$

$$c_{\alpha,3} = c_{\alpha,4} = (1 - q^{-1}) q^{z_\alpha + z_{\alpha+1}} (q^{2z_{\alpha+1}} - q^{2z_\alpha}) .$$

PROOF. Let u be of type 1 at α. By (4.5), we have the following identity for $s \in \mathscr{D}_0$:

$$(4.7) \qquad (q^{2z\alpha} - q^{2z\alpha+1^{-1}})\Phi_u(z; f) = \int_{Y_u} F_\alpha(x; z; f)\cdot(q^{2z\alpha} - q^{2z\alpha+1^{-1}})\cdot\Phi_v(s_\alpha, -s_\alpha/2, a\tilde{x}\cdot \mathrm{ch}_{K_2\cdot\tilde{x}})dx ,$$

in the same notation as in (4.6). Since $F_\alpha(x; z; f)$ is σ_α-invariant, we see, by (4.3), that the right hand side of (4.7) is absolutely convergent in $\mathscr{D}_{\alpha,1} = \sigma_\alpha(\mathscr{D}_0)$. Since $F_\alpha(x; z; f)$ has compact support and

$$(q^{2z\alpha} - q^{2z\alpha+1^{-1}})\cdot\Phi_v(s_\alpha, -s_\alpha/2, a\tilde{x}\cdot \mathrm{ch}_{K_2\cdot\tilde{x}})$$

is a polynomial in $q^{\pm s_\alpha}$, the right hand side of (4.7) is absolutely convergent in $\mathscr{D}_{\alpha,2}$, and so (4.7) holds for $s \in \mathscr{D}_\alpha$. Hence $(q^{2z\alpha} - q^{2z\alpha+1^{-1}})\Phi_u(z; f)$ is holomorphic in $\mathscr{D}_\alpha$, and by (4.3) we have

$$(4.8) \qquad \sigma_\alpha((q^{2z\alpha} - q^{2z\alpha+1^{-1}})\Phi_u(z; f)) = a_\alpha\Phi_u(z; f) + b_\alpha\Phi_{u'}(z; f) ,$$

where a_α and b_α are defined as in Theorem 2, (iii), and this establishes (iii).

Let u be of type 2 at α. In a similar manner,

$$(q^{2z\alpha} - q^{2z\alpha+1^{-1}})(q^{2z\alpha+1} - q^{2z\alpha^{-1}})\cdot\Phi_u(z; f)$$

is holomorphic in $\mathscr{D}_\alpha$ and

$$(4.9) \qquad \sigma_\alpha((q^{2z\alpha} - q^{2z\alpha+1^{-1}})(q^{2z\alpha+1} - q^{2z\alpha^{-1}})\cdot\Phi_u(z; f)) = \sum_{i=1}^{4} c_{\alpha,i}\cdot\Phi_{u^{(i)}}(z; f) ,$$

where $c_{\alpha,i}$ are defined as in Theorem 2, (iv), and this establishes (iv).

The assertion (ii) follows from (iii) and (iv). Finally, we prove the first assertion. From the above argument, we see that $G(z)\Phi_u(z; f)$ is holomorphic in $\mathscr{C}(= \cup_{i=1}^{n-1}\mathscr{D}_i)$. By (4.8) and (4.9), we obtain the following identity for $s \in \mathscr{C} \cap \sigma_i(\mathscr{C})$:

$$(4.10) \qquad \sigma_i(G(z)\Phi_u(z; f)) = \frac{G(z)}{(q^{2z_i} - q^{2z_{i+1}^{-1}})(q^{2z_{i+1}} - q^{2z_i^{-1}})}\sum_{l=1}^{4}\varphi_l(z)\Phi_{u^{(l)}}(z; f) ,$$

$u^{(l)} \in (k^*/k^{*2})^n$ and $\varphi_l(z)$ is a polynomial in $C[q^{\pm z_1}, \cdots, q^{\pm z_n}]$. It is easy to see that $(q^{2z_i} - q^{2z_j^{-1}})^{-1}$ is holomorphic in $\mathscr{C}$ unless $j \neq i \pm 1$, and $\{(q^{2z_i} - q^{2z_{i+1}^{-1}})(q^{2z_{i+1}} - q^{2z_i^{-1}})\}^{-1}$ is holomorphic in $\mathscr{D}_k$ for $k \neq i$. Hence we see that the right hand side of (4.10) is holomorphic in $\cup_{k\neq i}\mathscr{D}_k$. The left hand side of (4.10) is holomorphic in $\sigma_i(\mathscr{C})$, which contains $\mathscr{D}_i$. Hence we see that $\sigma_i(G(z)\Phi_u(z; f))$ is holomorphic in $\mathscr{C}$. Therefore we see that $G(z)\Phi_u(z; f)$ is holomorphic in $\mathscr{C} \cup \cup_{i=1}^{n-1}\sigma_i(\mathscr{C})$. For any i, j with $1 \leqq i < j \leqq n-2$, using (4.10) repeatedly, we can express $\sigma_j\sigma_{j-1}\cdots\sigma_i$ $(G(z)\Phi_u(z; f))$ as a sum of terms of the form

$$\{\text{a function holomorphic in } \mathscr{C}\} \times \sigma_j(G(z)\Phi_w(z; f)), \quad w \in (k^*/k^{*2})^n .$$

Consequently, we see that $G(z)\Phi_u(z; f)$ is holomorphic in $\mathscr{C} \cup \{\bigcup_{1 \leq i \leq j \leq n-2} \sigma_i\sigma_{i+1} \cdots \sigma_j(\mathscr{C})\}$, and so belongs to $C[q^{\pm z_1}, \cdots, q^{\pm z_n}]$ by (1.5). q.e.d.

PROOF OF THEOREM 1. Recall (4.2). We see that the result easily follows from Theorem 2, in particular, the matrix $C(\sigma, z)$ is given by $A \cdot B(\sigma; z) \cdot A^{-1}$, where $A = (A_{\chi u})$. q.e.d.

REMARK. Contrary to Case (U), our result does not provide enough information to formulate a precise conjecture on the image of the spherical transform.

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