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Spherical Fuzzy Soft Rough Average Aggregation Operators and Their Applications to Multi-Criteria Decision Making

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ABSTRACT The objective of this article is to explore and generalize the notions of soft set and rough set along with spherical fuzzy set and to introduce the novel concept called spherical fuzzy soft rough set that is free from all those complications faced by many modern concepts like intuitionistic fuzzy soft rough set, Pythagorean fuzzy soft rough set, and q-rung orthopair fuzzy soft rough set. Since aggregation operators are the fundamental tools to translate the complete information into a distinct number, so some spherical fuzzy soft rough new average aggregation operators are introduced, such as spherical fuzzy soft rough weighted average, spherical fuzzy soft rough ordered weighted average, and spherical fuzzy soft rough hybrid average aggregation operators. Also, the basic characteristics of these introduced operators have been elaborated in detail. Furthermore, a multi-criteria decision-making (MCDM) technique has been developed and a descriptive example is given to support newly presented work. At the end of this article, a comparative study of the introduced technique has been established that shows how our work is more superior and efficient compared to the picture fuzzy soft set.

INDEX TERMS Spherical fuzzy soft rough set, aggregation operators, multi-criteria decision making.

I. INTRODUCTION

Fuzzy set theory is the extension of the crisp set theory introduced by Zadeh [1] and the idea of the fuzzy set was presented that considers only positive grade. Atanassov [2] introduced the notion of an intuitionistic fuzzy set (IFS) as an extension of fuzzy set in which we consider the positive grade as well as negative grade with the condition that the sum (positive grade, negative grade) is less than or equal to 1. IFS has attained more importance since its appearance and it has been widely used in decision-making problems, such as some intuitionistic fuzzy frank power aggregation operators are conceived by Zhang *et al.* [3]. Also, Seikh and Mandal [4] introduced some intuitionistic fuzzy Dombi aggregation operators and applied them to MCDM problems. Moreover, Zeng *et al.* [5] conceived the MCDM based on intuitionistic fuzzy hybrid confident aggregation operator and social network analysis. Note that in IFS, the restriction *sum* $(\mathfrak{f}, \mathfrak{h}) \in [0, 1]$ limits the possibility of positive grade "f" and negative grade "$\mathfrak{h}$". To avoid this condition, Yager [6] presented the design of Pythagorean fuzzy set (PyFS) as a generalization of IFS in which we use the necessary condition given as $0 \leq \mathfrak{f}^2 + \mathfrak{h}^2 \leq 1$.

PyFS is more strong apparatus and it provides more space for decision-makers (DMs) to tackle the data in fuzzy set theory. After the invention of PyFS, many researchers have tried to develop new aggregation operators (AOs) based on PyFS, therefore some Pythagorean Dombi fuzzy AOs are introduced by Akram *et al.* [7]. Moreover, some confidence level-based PyF aggregation operators and their application to DM problems have been discussed in [8]. Also, an algorithm for MADM using the interactive Archimedean norm operations under the notion of PyFS has been developed by Wang and Garg [9]. Moreover, Wu *et al.* [10] presented the

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MAGDM flexibility based on the information fusion technique and hesitant Pythagorean fuzzy sets. PyFS is a limited notion noticed by researchers because when DMs provide 0.8 as positive grade (PG) and 0.7 as negative grade (NG) then $sum\ (0.8^2,\ 0.7^2) \notin [0,\ 1]$, so in this situation, PyFS is unsuccessful to solve this fact. To lower this difficulty, Yager [11] established the concept of q-rung orthopair fuzzy set (q-ROFS) as a generalization of IFS and PyFS. Later, Xing *et al.* [12] presented some point-weighted AOs for q-ROFSs and provide their application for MCDM problems. Furthermore, some AOs based on q-ROFSs and their application to MADM problems have been presented by Liu and Wang [13]. Notice that all the above theories discuss only PG and NG but the abstinence grade (AG) is ignored by all the above-given concepts while in many real-life situations we have to discuss the AG. This issue was pointed out by Cuong and Kreinovich [14] and he developed a new idea known as picture fuzzy set (PFS) in literature. PFS is a stronger toll in handling the vagueness and hesitancy of MCDM problems. Note that PFS uses the condition that $sum\ (\mathfrak{f},\ \mathfrak{g},\ \mathfrak{h}) \in [0,\ 1]$ and this characteristic distinguishes PFS from all other theories to make it more general. Moreover, based on PFS, some PF Hamacher AOs are discussed by Wei [15]. Notice that PFS is a restricted formation because when DMs come up with 0.5 as PG, 0.4 as NG, and 0.2 as AG, then the main condition of PFS fails to hold because $0.5+0.2+0.4 \notin [0,\ 1]$. To control this issue, the notion of a spherical fuzzy set (SFS) has been presented by Mahmood *et al.* [16]. SFS attains more attention from the researchers since its appearance. Also, some theories for spherical fuzzy numbers and the MCDM approach have been developed by Deli and Cagman [17]. Furthermore, some cosine similarity measures based on SFS and their application to DM problems have been introduced by Rafiq *et al.* [18]. Also, some symmetric-based AOs for SF data are proposed by Ashraf *et al.* [19].

Soft set ($S_{ft}S$) theory introduced by Molodtsov [20] is a common mathematical structure to handle the uncertain and vague data that other traditional tolls cannot handle. The notion of $S_{ft}S$ has been proved useful in different fields like DM [21] data analysis [22] and optimization [23]. Maji *et al.* [24] introduced the study of compound structure concerning both FS and $S_{ft}S$ and the idea of fuzzy $S_{ft}S$ $(FS_{ft}S)$ was initiated as a fuzzy generalization of classical $S_{ft}S$. After the invention of $FS_{ft}S$, many researchers have started working on the hybrid structure and many extended versions have been invented, such as IF soft set [25] $(IFS_{ft}S)$, PyF soft set [26] $(PyFS_{ft}S)$ and q-ROF soft set [27] $(q-ROFS_{ft}S)$. Moreover, some Bonferroni mean aggregation operators based on $IFS_{ft}S$ have been developed by Garg and Arora [28]. Also, Arora and Garg [29] established several robust AOs for $IFS_{ft}Nsnn$ and presented their application to MCDM problems. Naeem *et al.* [30] developed TOPSIS and VIKOR methods based on $PyFS_{ft}Snnn$ and provided its application to MCGDM problems. Furthermore, some $q-ROFS_{ft}$ average and geometric AOs are discussed in [27] and [31], respectively.

A rough set (RS) initiated by Pawlak [34], [35] is another apparatus to deal the vagueness and it seems to be a well-suited mathematical model for vagueness and uncertainty. RS is an extension of classical set theory (CST) in which the basic tool is relation that is representative of information systems. It is noticed that equivalence relation in Pawlak RS theory is restricted in many practical fields so many authors have extended the Pawlak RS theory by using non-equivalence relation and similarity relations are given in [36] and [37]. In recent literature, many new theories have been established by the combination of FS theories along with $S_{ft}S$ and RS to develop $S_{ft}FRS$ [38], IF soft rough set $(IFS_{ft}RS)$ [39], PyF soft rough set $(PyFS_{ft}RSn)$ [40], q-rung orthopair fuzzy soft rough set [41], Neutrosophic N-Soft Set [42].

No doubt intuitionistic fuzzy set, Pythagorean fuzzy set, and q-rung orthopair fuzzy have their importance in their structures, but these theories can consider only two types of aspects like yes or no, on the other hand human opinion can never be restricted to yes or no type of aspects, for example, consider the occurrence of voting where one can vote in favor of someone or vote against someone or refuse to vote or abstinence to vote. In this situation, all prevailing theories fail to handle this situation. Also, note that nevertheless picture fuzzy set can cope with these circumstances but it has limitations in its structure. But the spherical fuzzy set is a more dominant structure for coping with these types of situations.

Also, note that the prevailing theories have to face some sort of hurdles either in their structures or their necessary conditions, given as follows

1. When a decision-maker has to handle the data involving positive grade, abstinence grade, and negative grade then prevailing theories like $IFS_{ft}RS$ [39], $PyFS_{ft}RS$ [40] and $q-ROFS_{ft}RS$ [41] can never handle such type of information and has deficiencies in their structure.
2. Although the idea of the picture fuzzy soft rough set $(PFS_{ft}RS)$ CAN consider the phenomenon of voting but it has limited conditions in the form of lower and upper approximation operators like $0 \leq \underline{f_{\jmath}}(¢_i) + \underline{\mathfrak{g}_{\jmath}}(¢_i) + \underline{\mathfrak{h}_{\jmath}}(¢_i) \leq 1$ and $0 \leq \overline{f_{\jmath}}(¢_i) + \overline{\mathfrak{g}_{\jmath}}(¢_i) + \overline{\mathfrak{h}_{\jmath}}(¢_i) \leq 1$ but when decision-makers take their information in the form of spherical fuzzy soft rough set $(SFS_{ft}RS)$ consisting of lower and upper approximations like $\left\{\begin{matrix}(0.5, 0.6, 0.3),\\ (0.6, 0.4, 0.3)\end{matrix}\right\}$ then note that the sum of lower and upper approximation values exceeds [0, 1] that is $0 \nleq 0.5+0.6+0.3 \nleq 1$ and $0 \nleq 0.6+0.4+0.3 \nleq 1$ that can never be handled by $PFS_{ft}RS$ that restrict the notion of $PFS_{ft}RS$.

So motivated by the dominant feature of spherical fuzzy set, in this article, we have combined soft set, rough set, and spherical fuzzy set to present spherical fuzzy soft rough set. Hence the main contribution of this study is to invent a beneficiary decision-making strategy under the environment of the spherical fuzzy soft rough set.

Moreover, as AOs are the structural tools to deal the fuzzy information, so based on the invented work, some new spherical fuzzy soft rough weighted average ($SFS_{ft}RWA$), spherical fuzzy soft rough ordered weighted average $(SFS_{ft}ROWA)$, and spherical fuzzy soft rough hybrid average ($SFS_{ft}RHA$) aggregation operators have been proposed. Also, the basic characteristics of these invented operators are debated. Furthermore, a critical analysis of the established work has been given through MCDM and an illustrative example is given to support our work.

For more convenience, we have illustrated the presented work by frame diagram given in Figure 1.

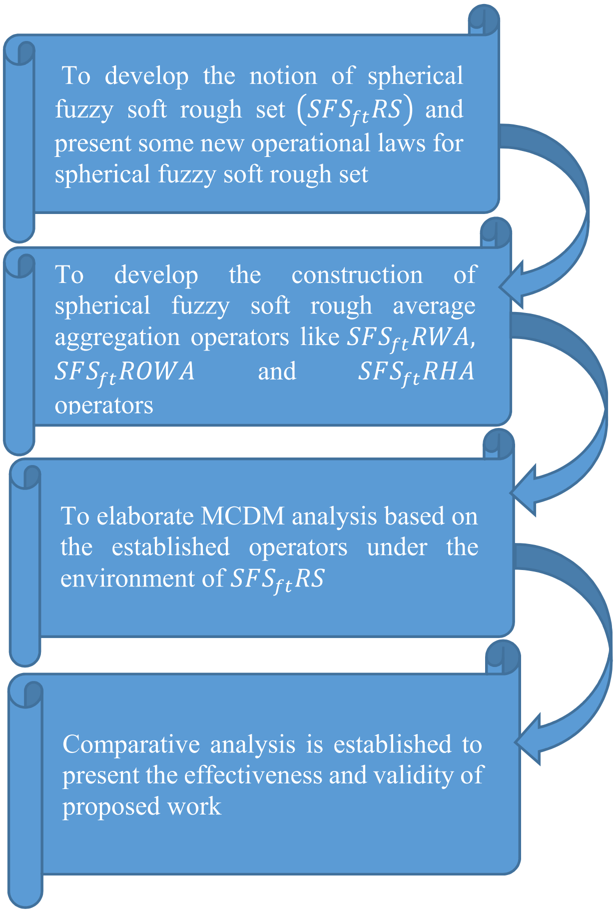

FIGURE 1. **Frame diagram for proposed work.**

The remainder of the article is categorized as follows: In the second phase of this article we have over-viewed some basic definitions of FS, $S_{ft}S$, SFS, $SFS_{ft}S$ and RS. Section 3 presents the basic interpretation of $SFS_{ft}RS$ and some operational laws for $SFS_{ft}RNs$. Section 4 carry out the principal structure of $SFS_{ft}RWA$, $SFS_{ft}ROWA$ and $SFS_{ft}RHA$ aggregation operators and their properties. In section 5, an algorithm along with an illustrative example is given to show how these operators work. In section 6, we have given a comparative analysis of established work. Finally, section 7 summarizes the conclusion remarks.

II. PRELIMINARIES

In this phase of the article, we present some basic notions of fuzzy set, soft set, spherical fuzzy set, spherical fuzzy soft set, and rough set that will help us in further discussion.

Definition 1 [1]: Fuzzy set on a general set $\check{Z}$ is an expression of the form:

$$F = \left\{\check{z},\ f\left(\check{z}\right) :\ \check{z} \in \check{Z}\right\} \tag{1}$$

where $f:\ \check{Z} \to [0,\ 1]$ denotes the PG.

Definition 2 [20]: Let $\check{Z}$ be a universal set and E be parameters set, the pair $(B,\ {}^\circ C)$ is called soft set over $\check{Z}$, if $B\ :\ {}^\circ C\ \to\ P\left(\check{Z}\right)$ is a set-valued mapping where $P\left(\check{Z}\right)$ denote power set of $\check{Z}$.

Definition 3 [16]: Let $\check{Z}$ be a universal set. Spherical fuzzy set over $\check{Z}$ is of the shape

$$SF = \left\{\left\langle\check{z},\ f(\check{z}),\ \mathfrak{g}(\check{z}),\ \mathfrak{h}(\check{z})\right\rangle : \check{z} \in \check{Z}\right\} \tag{2}$$

where $f:\ \check{Z} \to [0,\ 1]$ is the PG, $\mathfrak{g}:\check{Z} \to [0,\ 1]$ is the AG and $\mathfrak{h}:\check{Z} \to [0,\ 1]$ is NG using condition $0 \leq \left(f\left(\check{z}\right)\right)^2 + \left(\mathfrak{g}\left(\check{z}\right)\right)^2 + \left(\mathfrak{h}(\check{z})\right)^2 \leq 1$.

Definition 4 [32]: For a fixed set $\check{Z}$, ${}^\circ C$ a parameter set and $Y \subseteq {}^\circ C$, the pair $(M,\ Y)$ is called spherical fuzzy soft set over $\check{Z}$, where M is the mapping form Y to $SFS^{(\check{Z})}$, where $SFS^{(\check{Z})}$ is the family of all SFS over $\check{Z}$ given as

$$M\left(\mathfrak{S}_{\jmath}\right) = \left\{\check{z}_{i},\ f_{\jmath}\left(\check{z}_{i}\right),\ \mathfrak{g}_{\jmath}\left(\check{z}_{i}\right),\ \mathfrak{h}_{\jmath}\left(\check{z}_{i}\right) | \check{z} \in \check{Z}\right\} \tag{3}$$

with condition that $0 \leq \left(f_{\jmath}\left(\check{z}_{i}\right)\right)^2 + \left(\mathfrak{g}_{\jmath}\left(\check{z}_{i}\right)\right)^2 + \left(\mathfrak{h}_{\jmath}\left(\check{z}_{i}\right)\right)^2 \leq 1$ where $\mathfrak{S}_{\jmath}$ is the notation for parameters.

Definition 5 [33]: Let $SF_{\mathfrak{S}_{i\jmath}} = (f_{i\jmath},\ \mathfrak{g}_{i\jmath},\ \mathfrak{h}_{i\jmath})$, $SF'_{\mathfrak{S}_{i\jmath}} = (f'_{i\jmath},\ \mathfrak{g}'_{i\jmath},\ \mathfrak{h}'_{i\jmath})$ be two $SFS_{ft}Ns$ where $\mathfrak{S}_{i\jmath}$ represent the parameters for $i = 1,\ 2, \ldots,\ n$ and $\jmath = 1,\ 2, \ldots, m$ and $L > 0$. Some fundamental laws for $SFS_{ft}Ns$ are defined by:

1. $SF_{\mathfrak{S}_{i\jmath}} \subseteq SF'_{\mathfrak{S}_{i\jmath}}$ iff $f_{i\jmath} \leq f'_{i\jmath},\ \mathfrak{g}_{i\jmath} \leq \mathfrak{g}'_{i\jmath}$ and $\mathfrak{h}_{i\jmath} \geq \mathfrak{h}'_{i\jmath}$.
2. $SF_{\mathfrak{S}_{i\jmath}} = SF'_{\mathfrak{S}_{i\jmath}}$ iff $SF_{\mathfrak{S}_{i\jmath}} \subseteq SF'_{\mathfrak{S}_{i\jmath}}$ and $SF'_{\mathfrak{S}_{i\jmath}} \subseteq SF_{\mathfrak{S}_{i\jmath}}$.
3. $SF_{\mathfrak{S}_{i\jmath}} \cup SF'_{\mathfrak{S}_{i\jmath}} = \left\langle max\left(f_{i\jmath},\ f'_{i\jmath}\right),\ min\left(\mathfrak{g}_{i\jmath},\ \mathfrak{g}'_{i\jmath}\right),\ min\left(\mathfrak{h}_{i\jmath},\ \mathfrak{h}'_{i\jmath}\right)\right\rangle$.
4. $SF_{\mathfrak{S}_{i\jmath}} \bigcap SF'_{\mathfrak{S}_{i\jmath}} = \left\langle min\left(f_{i\jmath},\ f'_{i\jmath}\right),\ min\left(\mathfrak{g}_{i\jmath},\ \mathfrak{g}'_{i\jmath}\right),\ max\left(\mathfrak{h}_{i\jmath},\ \mathfrak{h}'_{i\jmath}\right)\right\rangle$.
5. $SF_{\mathfrak{S}_{i\jmath}}{}^{c} = (\mathfrak{h}_{i\jmath},\ \mathfrak{g}_{i\jmath},\ f_{i\jmath})$.
6. $SF_{\mathfrak{S}_{i\jmath}} \oplus SF'_{\mathfrak{S}_{i\jmath}} = \left(\sqrt{\left(f_{i\jmath}\right)^2 + \left(f'_{i\jmath}\right)^2 - \left(f_{i\jmath}\right)^2\left(f'_{i\jmath}\right)^2},\ \mathfrak{g}_{i\jmath}\mathfrak{g}'_{i\jmath},\ \mathfrak{h}_{i\jmath}\mathfrak{h}'_{i\jmath}\right)$.
7. $SF_{\mathfrak{S}_{i\jmath}} \otimes SF'_{\mathfrak{S}_{i\jmath}} = \left(f_{i\jmath}f'_{i\jmath},\ \mathfrak{g}_{i\jmath}\mathfrak{g}'_{i\jmath},\ \sqrt{\left(\mathfrak{h}_{i\jmath}\right)^2 + \left(\mathfrak{h}'_{i\jmath}\right)^2 - \left(\mathfrak{h}_{i\jmath}\right)^2\left(\mathfrak{h}'_{i\jmath}\right)^2}\right)$.
8. $LSF_{\mathfrak{S}_{i\jmath}} = (\sqrt{1 - \left(1 - f_{i\jmath}{}^2\right)^L},\ \mathfrak{g}_{i\jmath}{}^L,\ \mathfrak{h}_{i\jmath}{}^L)$.

TABLE 1. SFS_{ft} **relation from** $\check{Z}$ **to** $°C$.

$\mathfrak{R}_e$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	...	$\mathfrak{S}_n$
$¢_1$	$\begin{pmatrix} f(¢_1,\mathfrak{S}_1),\mathfrak{g}(¢_1,\mathfrak{S}_1), \\ \mathfrak{h}(¢_1,\mathfrak{S}_1) \end{pmatrix}$	$\begin{pmatrix} f(¢_1,\mathfrak{S}_2),\mathfrak{g}(¢_1,\mathfrak{S}_2), \\ \mathfrak{h}(¢_1,\mathfrak{S}_2) \end{pmatrix}$	...	$\begin{pmatrix} f(¢_1,\mathfrak{S}_n),\mathfrak{g}(¢_1,\mathfrak{S}_n), \\ \mathfrak{h}(¢_1,\mathfrak{S}_n) \end{pmatrix}$
$¢_2$	$\begin{pmatrix} f(¢_2,\mathfrak{S}_1),\mathfrak{g}(¢_2,\mathfrak{S}_1), \\ \mathfrak{h}(¢_2,\mathfrak{S}_1) \end{pmatrix}$	$\begin{pmatrix} f(¢_2,\mathfrak{S}_2),\mathfrak{g}(¢_2,\mathfrak{S}_2), \\ \mathfrak{h}(¢_2,\mathfrak{S}_2) \end{pmatrix}$	...	$\begin{pmatrix} f(¢_2,\mathfrak{S}_n),\mathfrak{g}(¢_2,\mathfrak{S}_n), \\ \mathfrak{h}(¢_2,\mathfrak{S}_n) \end{pmatrix}$
...	⋮	⋮	⋱	⋮
$¢_m$	$\begin{pmatrix} f(¢_m,\mathfrak{S}_1),\mathfrak{g}(¢_m,\mathfrak{S}_1), \\ \mathfrak{h}(¢_m,\mathfrak{S}_1) \end{pmatrix}$	$\begin{pmatrix} f(¢_m,\mathfrak{S}_2),\mathfrak{g}(¢_m,\mathfrak{S}_2), \\ \mathfrak{h}(¢_m,\mathfrak{S}_2) \end{pmatrix}$	...	$\begin{pmatrix} f(¢_m,\mathfrak{S}_n),\mathfrak{g}(¢_m,\mathfrak{S}_n), \\ \mathfrak{h}(¢_m,\mathfrak{S}_n) \end{pmatrix}$

9. $SF_{\mathfrak{S}_{i\jmath}}{}^{L} = (f_{i\jmath}{}^{L}, \mathfrak{g}_{i\jmath}{}^{L}, \sqrt{1-(1-\mathfrak{h}_{i\jmath}{}^{2})^{L}})$.

Definition 6 [36]: For a universal set $\check{Z}$ and $\mathfrak{R}_e \subseteq \check{Z} \times \check{Z}$ an arbitrary relation on $\check{Z}$. Let $\mathfrak{R}_e{}^*$ is a set-valued map $\mathfrak{R}_e{}^* : \check{Z} \to P(\check{Z})$ defined as $\mathfrak{R}_e{}^*(\check{z}) = \{y \in \check{Z} : (\check{z}, y) \in \mathfrak{R}_e \text{ and } \check{z} \in \check{Z}\}$, then the pair $(\check{Z}, \mathfrak{R}_e)$ is crisp approximation space. Now let $\mathfrak{G} \subseteq \check{Z}$, then the lower (LR) and upper (UR) approximations of $\mathfrak{G}$ w.r.t $(\check{Z}, \mathfrak{R}_e)$ are given by

$$\underline{\mathfrak{R}_e}(\mathfrak{G}) = \left\{\left\langle\left(\check{z} \in \check{Z} : \mathfrak{R}_e{}^*(\check{z}) \subseteq \mathfrak{G}\right)\right\rangle\right\} \quad (4)$$

$$\overline{\mathfrak{R}_e}(\mathfrak{G}) = \left\{\left\langle\left(\check{z} \in \check{Z} : \mathfrak{R}_e{}^*(\check{z}) \cap \mathfrak{G} \neq \emptyset\right)\right\rangle\right\} \quad (5)$$

The pair $(\underline{\mathfrak{R}_e}(\mathfrak{G}), \overline{\mathfrak{R}_e}(\mathfrak{G}))$ is a rough set (RS), where $\underline{\mathfrak{R}_e}(\mathfrak{G}) \neq \overline{\mathfrak{R}_e}(\mathfrak{G})$. Also, $\underline{\mathfrak{R}_e}(\mathfrak{G}), \overline{\mathfrak{R}_e}(\mathfrak{G}) : P(\check{Z}) \to P(\check{Z})$ is called lower and upper approximation operators according to $(\check{Z}, \mathfrak{R}_e)$.

III. SPHERICAL FUZZY SOFT ROUGH SET

Notion of $S_{ft}S$ is the generalization of CST that is free from all issues faced by some contemporary theories. We note from the existing theories that $S_{ft}S$ and RS are influential mathematical apparatuses to deal with the vagueness and uncertain data. Motivated from the combined structure of the soft rough set, this section is devoted to presenting the hybrid structure of SFS, $S_{ft}S$ and RS to get a new idea called $SFS_{ft}RS$. Moreover, some basic operational laws for this developed structure has been introduced. We use these fundamental operations to discuss some new AOs and their fundamental properties in detail.

Definition 7: Suppose $(°F, °C)$ denote the $SFS_{ft}S$ over $\check{Z}$. Any subset $\mathfrak{R}_e$ of $\check{Z} \times °C$ is called a SFS_{ft} relation from $\check{Z}$ to $°C$ and defined by

$$\mathfrak{R}_e = \left\{\left\{(¢_i, \mathfrak{S}_i), f(¢_i, \mathfrak{S}_i), \mathfrak{g}(¢_i, \mathfrak{S}_i), \mathfrak{h}(¢_i, \mathfrak{S}_i)\right\} \mid (¢_i, \mathfrak{S}_i) \in \check{Z} \times °C\right\},$$

where $f : \check{Z} \times °C \to [0, 1]$, $\mathfrak{g} : \check{Z} \times °C \to [0, 1]$ *and* $\mathfrak{h} : \check{Z} \times °C \to [0, 1]$ represent the PG, AG, and NG respectively with condition $0 \le f(¢_i, \mathfrak{S}_i)^2 + \mathfrak{g}(¢_i, \mathfrak{S}_i)^2 + \mathfrak{h}(¢_i, \mathfrak{S}_i)^2 \le 1$ for all $(¢_i, \mathfrak{S}_i) \in \check{Z} \times °C$.

If $\check{Z} = \{¢_1, ¢_2, ¢_3, \ldots, ¢_m\}$ and $°C = \{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \ldots, \mathfrak{S}_n\}$, then SFS_{ft} relation $\mathfrak{R}_e$ from $\check{Z}$ to $°C$ is given in Table 1.

Now we can define the definition of $SFS_{ft}RS$ as follows:

Definition 8: For a universal set $\check{Z}$, $°C$ being parameter set and $(°F, °C)$ be a $SFS_{ft}S$. Suppose $\mathfrak{R}_e$ be arbitrary SFS_{ft} relation from $\check{Z}$ to $°C$. Then the triplet $(\check{Z}, °C, \mathfrak{R}_e)$ is called SFS_{ft} approximation space. Now for any element $\mathfrak{G} \in SFS^{°C}$, then LR and UR approximation space of $\mathfrak{G}$ w.r.t approximation space $(\check{Z}, °C, \mathfrak{R}_e)$ are presented and given as

$$\underline{\mathfrak{R}_e}(\mathfrak{G}) = \left\{\left(¢_i, \underline{f_\jmath}(¢_i), \underline{\mathfrak{g}_\jmath}(¢_i), \underline{\mathfrak{h}_\jmath}(¢_i)\right) | ¢_i \in \check{Z}\right\} \quad (6)$$

$$\overline{\mathfrak{R}_e}(\mathfrak{G}) = \left\{\left(¢_i, \overline{f_\jmath}(¢_i), \overline{\mathfrak{g}_\jmath}(¢_i), \overline{\mathfrak{h}_\jmath}(¢_i)\right) | ¢_i \in \check{Z}\right\} \quad (7)$$

where

$$\underline{f_\jmath}(¢_i) = \wedge_{\mathfrak{S}_\jmath \in °C}\left\{f_{\mathfrak{R}_e}(¢_i, \mathfrak{S}_\jmath) \wedge f_{\mathfrak{G}}(\mathfrak{S}_\jmath)\right\},$$
$$\underline{\mathfrak{g}_\jmath}(¢_i) = \vee_{\mathfrak{S}_\jmath \in °C}\left\{\mathfrak{g}_{\mathfrak{R}_e}(¢_i, \mathfrak{S}_\jmath) \vee \mathfrak{g}_{\mathfrak{G}}(\mathfrak{S}_\jmath)\right\} \text{ and}$$
$$\underline{\mathfrak{h}_\jmath}(¢_i) = \vee_{\mathfrak{S}_\jmath \in °C}\left\{\mathfrak{h}_{\mathfrak{R}_e}(¢_i, \mathfrak{S}_\jmath) \vee \mathfrak{h}_{\mathfrak{G}}(\mathfrak{S}_\jmath)\right\}$$
$$\overline{f_\jmath}(¢_i) = \vee_{\mathfrak{S}_\jmath \in °C}\left\{f_{\mathfrak{R}_e}(¢_i, \mathfrak{S}_\jmath) \vee f_{\mathfrak{G}}(\mathfrak{S}_\jmath)\right\},$$
$$\overline{\mathfrak{g}_\jmath}(¢_i) = \wedge_{\mathfrak{S}_\jmath \in °C}\left\{\mathfrak{g}_{\mathfrak{R}_e}(¢_i, \mathfrak{S}_\jmath) \wedge \mathfrak{g}_{\mathfrak{G}}(\mathfrak{S}_\jmath)\right\} \text{ and}$$
$$\overline{\mathfrak{h}_\jmath}(¢_i) = \wedge_{\mathfrak{S}_\jmath \in °C}\left\{\mathfrak{h}_{\mathfrak{R}_e}(¢_i, \mathfrak{S}_\jmath) \wedge \mathfrak{h}_{\mathfrak{G}}(\mathfrak{S}_\jmath)\right\}$$

Such that

$$0 \le \left(\underline{f_\jmath}(¢_i)\right)^2 + \left(\underline{\mathfrak{g}_\jmath}(¢_i)\right)^2 + \left(\underline{\mathfrak{h}_\jmath}(¢_i)\right)^2 \le 1 \text{ and}$$
$$0 \le \left(\overline{f_\jmath}(¢_i)\right)^2 + \left(\overline{\mathfrak{g}_\jmath}(¢_i)\right)^2 + \left(\overline{\mathfrak{h}_\jmath}(¢_i)\right)^2 \le 1$$

It is clear that $\underline{\mathfrak{R}_e}(\mathfrak{G})$ *and* $\overline{\mathfrak{R}_e}(\mathfrak{G})$ are two *SFSs* in $\check{Z}$ and the operators $\underline{\mathfrak{R}_e}(\mathfrak{G}), \overline{\mathfrak{R}_e}(\mathfrak{G}) : SFS_{ft}{}^{°C} \to SFS_{ft}{}^{\check{Z}}$ are respectively called LR and UR $SFS_{ft}R$ approximation operators. Hence spherical fuzzy soft rough set $(SFS_{ft}RS)$ is a pair $\mathfrak{R}_e(\mathfrak{G}) = \left(\underline{\mathfrak{R}_e}(\mathfrak{G}), \overline{\mathfrak{R}_e}(\mathfrak{G})\right) = \left(¢_i, \left(\underline{f_\jmath}(¢_i), \underline{\mathfrak{g}_\jmath}(¢_i), \underline{\mathfrak{h}_\jmath}(¢_i)\right), \left(\overline{f_\jmath}(¢_i), \overline{\mathfrak{g}_\jmath}(¢_i), \overline{\mathfrak{h}_\jmath}(¢_i)\right)\right)$. We note that decision-makers can give $SFS_{ft}RS$ in the form of lower and upper approximation operators that satisfy the necessary condition $0 \le \left(\underline{f_\jmath}(¢_i)\right)^2 + \left(\underline{\mathfrak{g}_\jmath}(¢_i)\right)^2 + \left(\underline{\mathfrak{h}_\jmath}(¢_i)\right)^2 \le 1$ and $0 \le \left(\overline{f_\jmath}(¢_i)\right)^2 + \left(\overline{\mathfrak{g}_\jmath}(¢_i)\right)^2 + \left(\overline{\mathfrak{h}_\jmath}(¢_i)\right)^2 \le 1$ in which all three aspects can be involved like positive grade, abstinence grade, and negative grade.

For simplicity, we write $\mathfrak{R}_e(\mathfrak{G}) = \left(\underline{\mathfrak{R}_e}(\mathfrak{G}), \overline{\mathfrak{R}_e}(\mathfrak{G})\right) = \left(\text{¢}_i, \left(\underline{f_j}(\text{¢}_i), \underline{\mathfrak{g}_j}(\text{¢}_i), \underline{\mathfrak{h}_j}(\text{¢}_i)\right), \left(\overline{f_j}(\text{¢}_i), \overline{\mathfrak{g}_j}(\text{¢}_i), \overline{\mathfrak{h}_j}(\text{¢}_i)\right)\right)$ as $\mathfrak{R}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right) = \left(\left(\underline{f_{ij}}, \underline{\mathfrak{g}_{ij}}, \underline{\mathfrak{h}_{ij}}\right), \left(\overline{f_{ij}}, \overline{\mathfrak{g}_{ij}}, \overline{\mathfrak{h}_{ij}}\right)\right)$ and call this expression a spherical fuzzy soft rough number $(SFS_{ft}SRN)$.

Remark 1:

a) If we ignore the AG in Eqs. (6) and (7), then the developed $SFS_{ft}R$ approximation operators reduce to Pythagorean fuzzy soft rough approximation operators.

b) If only one soft parameter $\mathfrak{S}_1$ i.e. $(n = 1)$ is used, then the developed $SFS_{ft}R$ approximation operators will degenerate into spherical fuzzy rough approximation operators.

Example 1: Suppose Mr. X wants to buy the best mobile phone brands from a set of five alternatives $\check{Z} = \{\text{¢}_1 = APPLE, \text{¢}_2 = ACER, \text{¢}_3 = VIVO, \text{¢}_4 = HTC, \text{¢}_5 = LG\}$ under consideration having the set of parameters as $°C = \{\mathfrak{S}_1 = Long - lasting\ battry, \mathfrak{S}_2 = Crystle - clear\ display, \mathfrak{S}_3 = Reasonable\ price, \mathfrak{S}_4 = Plenty\ of\ storage\ space\}$. Let Mr. X presents the attractiveness of mobile phone brands in the form of SFS_{ft} relation $\mathfrak{R}_e$ from $\check{Z}$ to $°C$ as presented in Table 2.

TABLE 2. ***SFS$_{ft}$* relation from Ž to °C.**

$\mathfrak{R}_e$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
¢_1	(0.1, 0.2, 0.4)	(0.5, 0.1, 0.6)	(0.6, 0.4, 0.6)	(0.3, 0.2, 0.4)
¢_2	(0.3, 0.5, 0.2)	(0.3, 0.5, 0.4)	(0.5, 0.2, 0.3)	(0.2, 0.1, 0.3)
¢_3	(0.4, 0.1, 0.5)	(0.2, 0.6, 0.5)	(0.4, 0.5, 0.5)	(0.1, 0.5, 0.1)
¢_4	(0.2, 0.4, 0.3)	(0.1, 0.3, 0.6)	(0.1, 0.2, 0.4)	(0.4, 0.6, 0.5)
¢_5	(0.5, 0.3, 0.1)	(0.6, 0.1, 0.5)	(0.3, 0.6, 0.2)	(0.5, 0.4, 0.2)

Now suppose that Mr. X established the decision object in the form of spherical fuzzy subset over a parameter ser $°C$ as given below

$$\mathfrak{G} = \left\{\begin{array}{l}(\mathfrak{S}_1, 0.4, 0.5, 0.2), (\mathfrak{S}_2, 0.3, 0.4, 0.5), \\ (\mathfrak{S}_3, 0.5, 0.3, 0.2), (\mathfrak{S}_4, 0.4, 0.2, 0.4)\end{array}\right\}.$$

Now we use Eqs. (6) and (7) to get:

$$\begin{aligned}
&\underline{f_1}(\text{¢}_1) = 0.1, \underline{f_2}(\text{¢}_2) = 0.2, \underline{f_3}(\text{¢}_3) = 0.1, \\
&\underline{f_4}(\text{¢}_4) = 0.1, \underline{f_5}(\text{¢}_5) = 0.3 \\
&\underline{\mathfrak{g}_1}(\text{¢}_1) = 0.5, \underline{\mathfrak{g}_2}(\text{¢}_2) = 0.5, \underline{\mathfrak{g}_3}(\text{¢}_3) = 0.6, \\
&\underline{\mathfrak{g}_4}(\text{¢}_4) = 0.6, \underline{\mathfrak{g}_5}(\text{¢}_5) = 0.6 \\
&\underline{\mathfrak{h}_1}(\text{¢}_1) = 0.6, \underline{\mathfrak{h}_2}(\text{¢}_2) = 0.6, \underline{\mathfrak{h}_3}(\text{¢}_3) = 0.5, \\
&\underline{\mathfrak{h}_4}(\text{¢}_4) = 0.6, \underline{\mathfrak{h}_5}(\text{¢}_5) = 0.5 \ and \\
&\overline{f_1}(\text{¢}_1) = 0.6, \overline{f_2}(\text{¢}_2) = 0.6, \overline{f_3}(\text{¢}_3) = 0.5, \\
&\overline{f_4}(\text{¢}_4) = 0.5, \overline{f_5}(\text{¢}_5) = 0.6 \\
&\overline{\mathfrak{g}_1}(\text{¢}_1) = 0.1, \overline{\mathfrak{g}_2}(\text{¢}_2) = 0.1, \overline{\mathfrak{g}_3}(\text{¢}_3) = 0.1, \\
&\overline{\mathfrak{g}_4}(\text{¢}_4) = 0.2, \overline{\mathfrak{g}_5}(\text{¢}_5) = 0.1 \\
&\overline{\mathfrak{h}_1}(\text{¢}_1) = 0.2, \overline{\mathfrak{h}_2}(\text{¢}_2) = 0.2, \overline{\mathfrak{h}_3}(\text{¢}_3) = 0.1, \\
&\overline{\mathfrak{h}_4}(\text{¢}_4) = 0.2, \overline{\mathfrak{h}_5}(\text{¢}_5) = 0.1.
\end{aligned}$$

Now lower and upper $SFS_{ft}R$ approximation operators are given as

$$\underline{\mathfrak{R}_e}(\mathfrak{G}) = \left\{\begin{array}{l}(\text{¢}_1, 0.1, 0.5, 0.6), \\ (\text{¢}_2, 0.2, 0.5, 0.6), \\ (\text{¢}_3, 0.1, 0.6, 0.5), \\ (\text{¢}_4, 0.1, 0.6, 0.6), \\ (\text{¢}_5, 0.3, 0.6, 0.5)\end{array}\right\}$$

$$\overline{\mathfrak{R}_e}(\mathfrak{G}) = \left\{\begin{array}{l}(\text{¢}_1, 0.6, 0.1, 0.2), \\ (\text{¢}_2, 0.6, 0.1, 0.2), \\ (\text{¢}_3, 0.5, 0.1, 0.1), \\ (\text{¢}_4, 0.5, 0.2, 0.2), \\ (\text{¢}_5, 0.6, 0.1, 0.1)\end{array}\right\}.$$

Hence

$$\mathfrak{R}_e(\mathfrak{G}) = \left\{\begin{array}{l}(\text{¢}_1, (0.1, 0.5, 0.6), (0.6, 0.1, 0.2)), \\ (\text{¢}_2, (0.2, 0.5, 0.6), (0.6, 0.1, 0.2)), \\ (\text{¢}_3, (0.1, 0.6, 0.5), (0.5, 0.1, 0.1)), \\ (\text{¢}_4, (0.1, 0.6, 0.6), (0.5, 0.2, 0.2)), \\ (\text{¢}_5, (0.3, 0.6, 0.5), (0.6, 0.1, 0.1))\end{array}\right\}.$$

Definition 9: For two $SFS_{ft}RNs$ $\mathfrak{R}_{e\mathfrak{S}_j}(\mathfrak{G}_1) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_1), \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_1)\right)$ for $(j = 1, 2)$. The following operations can be defined:

1.

$$\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1) \cup \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_1) = \left\{\begin{array}{l}\left(\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \cup \underline{\mathfrak{R}_{e\mathfrak{S}_2}}(\mathfrak{G}_1)\right), \\ \left(\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \cup \overline{\mathfrak{R}_{e\mathfrak{S}_2}}(\mathfrak{G}_1)\right)\end{array}\right\};$$

2.

$$\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1) \bigcap \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_1) = \left\{\begin{array}{l}\left(\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \bigcap \underline{\mathfrak{R}_{e\mathfrak{S}_2}}(\mathfrak{G}_1)\right), \\ \left(\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \bigcap \overline{\mathfrak{R}_{e\mathfrak{S}_2}}(\mathfrak{G}_1)\right)\end{array}\right\};$$

3.

$$\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1) \oplus \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_1) = \left\{\begin{array}{l}\left(\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \oplus \underline{\mathfrak{R}_{e\mathfrak{S}_2}}(\mathfrak{G}_1)\right), \\ \left(\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \oplus \overline{\mathfrak{R}_{e\mathfrak{S}_2}}(\mathfrak{G}_1)\right)\end{array}\right\};$$

4.

$$\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1) \otimes \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_1) = \left\{\begin{array}{l}\left(\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \otimes \underline{\mathfrak{R}_{e\mathfrak{S}_2}}(\mathfrak{G}\ 1)\right), \\ \left(\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \otimes \overline{\mathfrak{R}_{e\mathfrak{S}_2}}(\mathfrak{G}_1)\right)\end{array}\right\};$$

5.

$$\begin{aligned}
&\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1) \subseteq \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_1) \\
&\quad = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \subseteq \underline{\mathfrak{R}_{e\mathfrak{S}_2}}(\mathfrak{G}_1)\right) \\
&\quad\quad and \left(\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \subseteq \overline{\mathfrak{R}_{e\mathfrak{S}_2}}(\mathfrak{G}_1)\right);
\end{aligned}$$

6.

$$\tau\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1)\right) = \left(\tau\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1), \tau\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1)\right) \text{ for } \tau \geq 1;$$

7.

$$\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1)\right)^{\tau} = \left(\left(\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1)\right)^{\tau}, \left(\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1)\right)^{\tau}\right)$$

for $\tau \geq 1$.

8.

$$\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1)\right)^{c} = \left(\left(\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1)\right)^{c}, \left(\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1)\right)^{c}\right)$$

where $\left(\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1)\right)^{c}$ *and* $\left(\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1)\right)^{c}$ are the complement of $SFS_{ft}R$ approximation operators $\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1)$ *and* $\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1)$.

9.

$$\mathfrak{R}_e(\mathfrak{G}_1) = \mathfrak{R}_e(\mathfrak{G}_2) \text{ iff } \underline{\mathfrak{R}_e}(\mathfrak{G}_1) = \underline{\mathfrak{R}_e}(\mathfrak{G}_2) \text{ and } \overline{\mathfrak{R}_e}(\mathfrak{G}_1) = \overline{\mathfrak{R}_e}(\mathfrak{G}_2).$$

Definition 10: For a $SFS_{ft}RN$ $\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1), \overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1)\right) = \left(\left(\underline{\mathfrak{f}_{11}}, \underline{\mathfrak{g}_{11}}, \underline{\mathfrak{h}_{11}}\right), \left(\overline{\mathfrak{f}_{11}}, \overline{\mathfrak{g}_{11}}, \overline{\mathfrak{h}_{11}}\right)\right)$, then the score function for $\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1)$ can be defined as

$$Sc\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1)\right) = \frac{1}{3}\left(\begin{array}{c} 2 + \left(\underline{\mathfrak{f}_{11}}^2 + \overline{\mathfrak{f}_{11}}^2\right) \\ -\left(\underline{\mathfrak{g}_{11}}^2 + \overline{\mathfrak{g}_{11}}^2\right) - \left(\underline{\mathfrak{h}_{11}}^2 + \overline{\mathfrak{h}_{11}}^2\right) \end{array}\right).$$

Note that $Sc\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1)\right) \in [-1, 1]$.

IV. SPHERICAL FUZZY SOFT ROUGH AVERAGE ($SFS_{ft}RA$) AGGREGATION OPERATORS

This phase of the article deals with the notions of spherical fuzzy soft rough weighted average, spherical fuzzy soft rough ordered weighted average, and spherical fuzzy soft rough hybrid average aggregation operators. Moreover, the properties of these developed AOs are discussed in detail. The overall discussion is given below.

A. SPHERICAL FUZZY SOFT ROUGH WEIGHTED AVERAGE ($SFS_{ft}RWA$) OPERATOR

Definition 11: For a family of $SFS_{ft}RNs$ $\mathfrak{R}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right)$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$ and let $\mathscr{w} = (\mathscr{w}_1, \mathscr{w}_2, \mathscr{w}_3, \ldots, \mathscr{w}_m)^{\check{Z}}$, $(\widetilde{\hbar} = \left(\widetilde{\hbar}_1, \widetilde{\hbar}_2, \widetilde{\hbar}_3, \ldots, \widetilde{\hbar}_n\right)^{\check{Z}}$ be the weight vectors (WVs) of $¢_i$ experts and $\mathfrak{S}_j$ parameters using the condition that $\sum_{i=1}^{m} \mathscr{w}_i = 1$, $\sum_{j=1}^{n} \widetilde{\hbar}_i = 1$ *and* $0 \leq \mathscr{w}_i, \widetilde{\hbar}_j \leq 1$ respectively. Then $SFS_{ft}RWA$ aggregation operator is defined as:

$$SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) = \left(\begin{array}{c} \oplus_{j=1}^{n} \widetilde{\hbar}_j \left(\oplus_{i=1}^{m} \mathscr{w}_i \underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right), \\ \oplus_{j=1}^{n} \widetilde{\hbar}_j \left(\oplus_{i=1}^{m} \mathscr{w}_i \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right) \end{array}\right).$$

Theorem 1: Let $\mathfrak{R}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right)$ be a family of $SFS_{ft}RNs$. Also, suppose that $\mathscr{w} = (\mathscr{w}_1, \mathscr{w}_2, \mathscr{w}_3, \ldots, \mathscr{w}_m)^{\check{Z}}$, $\widetilde{\hbar} = \left(\widetilde{\hbar}_1, \widetilde{\hbar}_2, \widetilde{\hbar}_3, \ldots, \widetilde{\hbar}_n\right)^{\check{Z}}$ are the WVs of $¢_i$ experts and $\mathfrak{S}_j$ parameters using the situation that $\sum_{i=1}^{m} \mathscr{w}_i = 1$, $\sum_{j=1}^{n} \widetilde{\hbar}_i = 1$ *and* $0 \leq \mathscr{w}_i, \widetilde{\hbar}_j \leq 1$ respectively. Then $SFS_{ft}RWA$ aggregation operator is given as:

$$SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) = \left(\begin{array}{c} \oplus_{j=1}^{n} \widetilde{\hbar}_j \left(\oplus_{i=1}^{m} \mathscr{w}_i \underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right), \\ \oplus_{j=1}^{n} \widetilde{\hbar}_j \left(\oplus_{i=1}^{m} \mathscr{w}_i \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right) \end{array}\right)$$

$$= \left(\begin{array}{c} \left\{\begin{array}{c} \sqrt{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \left(\underline{\mathfrak{f}_{ij}}\right)^2\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j} \end{array}\right\}, \\ \left\{\begin{array}{c} \sqrt{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \left(\overline{\mathfrak{f}_{ij}}\right)^2\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j} \end{array}\right\} \end{array}\right).$$

Proof: We use the mathematical induction method to prove the theorem as follows.

By using operational laws:

$$\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) \oplus \underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_2) = \left(\begin{array}{c} \left\{\left(\underline{\mathfrak{f}_{11}}, \underline{\mathfrak{g}_{11}}, \underline{\mathfrak{h}_{11}}\right) \oplus \left(\underline{\mathfrak{f}_{12}}, \underline{\mathfrak{g}_{12}}, \underline{\mathfrak{h}_{12}}\right)\right\}, \\ \left\{\left(\left(\overline{\mathfrak{f}_{11}}, \overline{\mathfrak{g}_{11}}, \overline{\mathfrak{h}_{11}}\right) \oplus \left(\overline{\mathfrak{f}_{12}}, \overline{\mathfrak{g}_{12}}, \overline{\mathfrak{h}_{12}}\right)\right)\right\} \end{array}\right)$$

$$= \left(\begin{array}{c} \left\{\begin{array}{c} \sqrt{\left(\underline{\mathfrak{f}_{11}}\right)^2 + \left(\underline{\mathfrak{f}_{12}}\right)^2 - \left(\underline{\mathfrak{f}_{11}}\right)^2\left(\underline{\mathfrak{f}_{12}}\right)^2}, \\ \left(\underline{\mathfrak{g}_{11}}\right)\left(\underline{\mathfrak{g}_{12}}\right), \left(\underline{\mathfrak{h}_{11}}\right)\left(\underline{\mathfrak{h}_{12}}\right) \end{array}\right\}, \\ \left\{\begin{array}{c} \sqrt{\left(\overline{\mathfrak{f}_{11}}\right)^2 + \left(\overline{\mathfrak{f}_{12}}\right)^2 - \left(\overline{\mathfrak{f}_{11}}\right)^2\left(\overline{\mathfrak{f}_{12}}\right)^2}, \\ \left(\overline{\mathfrak{g}_{11}}\right)\left(\overline{\mathfrak{g}_{12}}\right), \left(\overline{\mathfrak{h}_{11}}\right)\left(\overline{\mathfrak{h}_{12}}\right) \end{array}\right\} \end{array}\right)$$

and

$$\tau\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) = \left(\begin{array}{c} \left\{\sqrt{1 - \left(1 - \left(\underline{\mathfrak{f}_{11}}\right)^2\right)^{\tau}}, \underline{\mathfrak{g}_{11}}^{\tau}, \underline{\mathfrak{h}_{11}}^{\tau}\right\}, \\ \left\{\sqrt{1 - \left(1 - \left(\overline{\mathfrak{f}_{11}}\right)^2\right)^{\tau}}, \overline{\mathfrak{g}_{11}}^{\tau}, \overline{\mathfrak{h}_{11}}^{\tau}\right\} \end{array}\right) \text{ for } \tau \geq 1.$$

Suppose that outcome is valid for $n = 2$ and $m = 2$,

$$SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i}),\ \mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i})\right)$$

$$=\left(\begin{array}{c}\oplus_{j=1}^{2}\widetilde{\hbar}_{j}\left(\oplus_{i=1}^{2}w_{i}\underline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i})\right),\\ \oplus_{j=1}^{2}\widetilde{\hbar}_{j}\left(\oplus_{i=1}^{2}w_{i}\overline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i})\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\left\{\begin{array}{c}\widetilde{\hbar}_{1}\left(w_{1}\underline{\mathfrak{R}_{e\mathfrak{S}_{1}}}(\mathfrak{G}_{1})\oplus w_{2}\underline{\mathfrak{R}_{e\mathfrak{S}_{1}}}(\mathfrak{G}_{2})\right)\\ \oplus\widetilde{\hbar}_{2}\left(w_{1}\underline{\mathfrak{R}_{e\mathfrak{S}_{2}}}(\mathfrak{G}_{1})\oplus w_{2}\underline{\mathfrak{R}_{e\mathfrak{S}_{2}}}(\mathfrak{G}_{2})\right)\end{array}\right\},\\ \left\{\begin{array}{c}\widetilde{\hbar}_{1}\left(w_{1}\overline{\mathfrak{R}_{e\mathfrak{S}_{1}}}(\mathfrak{G}_{1})\oplus w_{2}\overline{\mathfrak{R}_{e\mathfrak{S}_{1}}}(\mathfrak{G}_{2})\right)\\ \oplus\widetilde{\hbar}_{2}\left(w_{1}\overline{\mathfrak{R}_{e\mathfrak{S}_{2}}}(\mathfrak{G}_{1})\oplus w_{2}\overline{\mathfrak{R}_{e\mathfrak{S}_{2}}}(\mathfrak{G}_{2})\right)\end{array}\right\}\end{array}\right)$$

and

$$SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i}),\ \mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i})\right)$$

$$=\left(\begin{array}{c}\left\{\begin{array}{c}\sqrt{1-\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\left(1-\underline{f_{ij}}^{2}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}}},\\ \prod_{j=1}^{2}\left(\prod_{i=1}^{2}\underline{\mathfrak{g}_{ij}}^{w_{i}}\right)^{\widetilde{\hbar}_{j}},\ \prod_{j=1}^{2}\left(\prod_{i=1}^{2}\underline{\mathfrak{h}_{ij}}^{w_{i}}\right)^{\widetilde{\hbar}_{j}}\end{array}\right\},\\ \left\{\begin{array}{c}\sqrt{1-\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\left(1-\overline{f_{ij}}^{2}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}}},\\ \prod_{j=1}^{2}\left(\prod_{i=1}^{2}\overline{\mathfrak{g}_{ij}}^{w_{i}}\right)^{\widetilde{\hbar}_{j}},\ \prod_{j=1}^{2}\left(\prod_{i=1}^{2}\overline{\mathfrak{h}_{ij}}^{w_{i}}\right)^{\widetilde{\hbar}_{j}}\end{array}\right\}\end{array}\right)$$

Hence the statement is valid for $n = 2$ *and* $m = 2$.

Next, suppose that statement is valid for $n = K_1$ and $m = K_2$.

$$SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_{1}}(\mathfrak{G}_{1}),\ldots,\mathfrak{R}_{e\mathfrak{S}_{K_1}}(\mathfrak{G}_{K_2})\right)$$

$$=\left(\begin{array}{c}\left\{\begin{array}{c}\sqrt{1-\prod_{j=1}^{K_1}\left(\prod_{i=1}^{K_2}\left(1-\left(\underline{f_{ij}}\right)^{2}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}}},\\ \prod_{j=1}^{K_1}\left(\prod_{i=1}^{K_2}\left(\underline{\mathfrak{g}_{ij}}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}},\ \prod_{j=1}^{K_1}\left(\prod_{i=1}^{K_2}\left(\underline{\mathfrak{h}_{ij}}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}}\end{array}\right\},\\ \left\{\begin{array}{c}\sqrt{1-\prod_{j=1}^{K_1}\left(\prod_{i=1}^{K_2}\left(1-\left(\overline{f_{ij}}\right)^{2}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}}},\\ \prod_{j=1}^{K_1}\left(\prod_{i=1}^{K_2}\left(\overline{\mathfrak{g}_{ij}}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}},\ \prod_{j=1}^{K_1}\left(\prod_{i=1}^{K_2}\left(\overline{\mathfrak{h}_{ij}}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}}\end{array}\right\}\end{array}\right)$$

Further, we have to prove that statement is valid for $n = K_1 + 1$ and $m = K_2 + 1$.

$$SFS_{ft}RWA\left(\begin{array}{c}\mathfrak{R}_{e\mathfrak{S}_{1}}(\mathfrak{G}_{1}),\ldots,\mathfrak{R}_{e\mathfrak{S}_{K_1}}(\mathfrak{G}_{K_2}),\\ \mathfrak{R}_{e\mathfrak{S}_{K_1+1}}(\mathfrak{G}_{K_2+1})\end{array}\right)$$

TABLE 3. Tabular presentation of $\mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i}) = (\underline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i}), \overline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i}))$.

$\mathfrak{R}_e$	$\mathfrak{S}_1$	$\mathfrak{S}_2$
¢$_1$	{(0.5, 0.6, 0.3), (0.6, 0.2, 0.1)}	{(0.6, 0.3, 0.5), (0.2, 0.6, 0.1)}
¢$_2$	{(0.4, 0.1, 0.2), (0.3, 0.3, 0.4)}	{(0.3, 0.1, 0.6), (0.5, 0.3, 0.2)}
¢$_3$	{(0.3, 0.3, 0.1), (0.2, 0.5, 0.3)}	{(0.1, 0.1, 0.5), (0.6, 0.1, 0.1)}

$$=\left(\begin{array}{c}\left\{\begin{array}{c}\oplus_{j=1}^{K_1}\widetilde{\hbar}_{j}\left(\oplus_{i=1}^{K_2}w_{i}\underline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i})\right)\\ \oplus\widetilde{\hbar}_{K_1+1}\left(w_{K_2+1}\underline{\mathfrak{R}_{e\mathfrak{S}_{K_1+1}}}(\mathfrak{G}_{K_2+1})\right)\end{array}\right\},\\ \left\{\begin{array}{c}\oplus_{j=1}^{K_1}\widetilde{\hbar}_{j}\left(\oplus_{i=1}^{K_2}w_{i}\overline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i})\right)\\ \oplus\widetilde{\hbar}_{K_1+1}\left(w_{K_2+1}\overline{\mathfrak{R}_{e\mathfrak{S}_{K_1+1}}}(\mathfrak{G}_{K_2+1})\right)\end{array}\right\}\end{array}\right)$$

$$=\left(\begin{array}{c}\left\{\begin{array}{c}\sqrt{1-\prod_{j=1}^{K_1+1}\left(\prod_{i=1}^{K_2+1}\left(1-\left(\underline{f_{ij}}\right)^{2}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}}},\\ \prod_{j=1}^{K_1+1}\left(\prod_{i=1}^{K_2+1}\left(\underline{\mathfrak{g}_{ij}}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}},\ \prod_{j=1}^{K_1+1}\left(\prod_{i=1}^{K_2+1}\left(\underline{\mathfrak{h}_{ij}}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}}\end{array}\right\},\\ \left\{\begin{array}{c}\sqrt{1-\prod_{j=1}^{K_1+1}\left(\prod_{i=1}^{K_2+1}\left(1-\left(\overline{f_{ij}}\right)^{2}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}}},\\ \prod_{j=1}^{K_1+1}\left(\prod_{i=1}^{K_2+1}\left(\overline{\mathfrak{g}_{ij}}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}},\ \prod_{j=1}^{K_1+1}\left(\prod_{i=1}^{K_2+1}\left(\overline{\mathfrak{h}_{ij}}\right)^{w_{i}}\right)^{\widetilde{\hbar}_{j}}\end{array}\right\}\end{array}\right)$$

Hence the result is true for $n = K_1 + 1$ and $m = K_2 + 1$. Hence it is true for all $m,\ n \geq 1$.

As it is clear that $\underline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i})$ *and* $\overline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i})$ are $SFS_{ft}Ns$. So, by using the definition 9, we have that $\oplus_{j=1}^{n}\widetilde{\hbar}_{j}\left(\oplus_{i=1}^{m}w_{i}\underline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i})\right)$ and $\oplus_{j=1}^{n}\widetilde{\hbar}_{j}\left(\oplus_{i=1}^{m}w_{i}\overline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i})\right)$ are also $SFS_{ft}Ns$. Hence, $SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_{1}}(\mathfrak{G}_{1}),\ldots,\mathfrak{R}_{e\mathfrak{S}_{n}}(\mathfrak{G}_{m})\right)$ is also $SFS_{ft}RN$ in approximation space $\left(\check{Z},\ {}^{\circ}C,\ \mathfrak{R}_e\right)$.

Example 2: Suppose $\check{Z} = \{¢_1, ¢_2, ¢_3\}$, $\mathfrak{G} = \{\mathfrak{S}_1, \mathfrak{S}_2\} \subseteq {}^{\circ}C$ denote alternatives set and set of parameter respectively having WVs $w_i = \{0.35, 0.26, 0.39\}$ for $¢_i = (i = 1, 2, 3)$ and $\widetilde{\hbar} = \{0.65, 0.35\}$ for $(j = 1, 2)$. The data given in Table 3. consists of $SFS_{ft}RNs$

$$SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_{1}}(\mathfrak{G}_{1}),\ldots,\mathfrak{R}_{e\mathfrak{S}_{n}}(\mathfrak{G}_{m})\right)$$

$$=\left(\begin{array}{c}\oplus_{j=1}^{2}\widetilde{\hbar}_{j}\left(\oplus_{i=1}^{3}w_{i}\underline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i})\right),\\ \oplus_{j=1}^{2}\widetilde{\hbar}_{j}\left(\oplus_{i=1}^{3}w_{i}\overline{\mathfrak{R}_{e\mathfrak{S}_{j}}}(\mathfrak{G}_{i})\right)\end{array}\right)$$

$$
= \left[\begin{array}{c}
\left\{\begin{array}{c}
\sqrt{1-\left(\begin{array}{c}\left[\begin{array}{c}\left(1-0.5^2\right)^{0.35}\left(1-0.4^2\right)^{0.26}\\ \left(1-0.3^2\right)^{0.39}\end{array}\right]^{0.65}\\ \left[\begin{array}{c}\left(1-0.6^2\right)^{0.35}\left(1-0.3^2\right)^{0.26}\\ \left(1-0.1^2\right)^{0.39}\end{array}\right]^{0.35}\end{array}\right)},\\
\left\{\begin{array}{c}\left[\left(0.6^{0.35}\right)\left(0.1^{0.26}\right)\left(0.3^{0.39}\right)\right]^{0.65}\\ \left[\left(0.3^{0.35}\right)\left(0.1^{0.26}\right)\left(0.1^{0.39}\right)\right]^{0.35}\end{array}\right\},\\
\left\{\begin{array}{c}\left[\left(0.3^{0.35}\right)\left(0.2^{0.26}\right)\left(0.1^{0.39}\right)\right]^{0.65}\\ \left[\left(0.5^{0.35}\right)\left(0.6^{0.26}\right)\left(0.5^{0.39}\right)\right]^{0.35}\end{array}\right\}
\end{array}\right\},\\
\left\{\begin{array}{c}
\sqrt{1-\left(\begin{array}{c}\left[\begin{array}{c}\left(1-0.6^2\right)^{0.35}\left(1-0.3^2\right)^{0.26}\\ \left(1-0.2^2\right)^{0.39}\end{array}\right]^{0.65}\\ \left[\begin{array}{c}\left(1-0.2^2\right)^{0.35}\left(1-0.5^2\right)^{0.26}\\ \left(1-0.6^2\right)^{0.39}\end{array}\right]^{0.35}\end{array}\right)},\\
\left\{\begin{array}{c}\left[\left(0.2^{0.35}\right)\left(0.3^{0.26}\right)\left(0.5^{0.39}\right)\right]^{0.65}\\ \left[\left(0.6^{0.35}\right)\left(0.3^{0.26}\right)\left(0.1^{0.39}\right)\right]^{0.35}\end{array}\right\},\\
\left\{\begin{array}{c}\left[\left(0.1^{0.35}\right)\left(0.4^{0.26}\right)\left(0.3^{0.39}\right)\right]^{0.65}\\ \left[\left(0.1^{0.35}\right)\left(0.2^{0.26}\right)\left(0.1^{0.39}\right)\right]^{0.35}\end{array}\right\}
\end{array}\right\}
\end{array}\right]
$$

$$
= \left\{\begin{array}{c}(0.4093,\ 0.2272,\ 0.2577),\\ (0.4443,\ 0.2917,\ 0.1778)\end{array}\right\}.
$$

Next we will elaborate that $SFS_{ft}RWA$ operator has the following properties.

Theorem 2: Suppose $\mathfrak{N}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{N}_{e\mathfrak{S}_j}(\mathfrak{G}_i)}, \overline{\mathfrak{N}_{e\mathfrak{S}_j}(\mathfrak{G}_i)}\right)$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ is a set of $SFS_{ft}RNs$. Also assume that $(\mathscr{w} = (\mathscr{w}_1, \mathscr{w}_2, \mathscr{w}_3, \ldots, \mathscr{w}_m)^{\check{Z}}$, $\widetilde{\hbar} = \left(\widetilde{\hbar}_1, \widetilde{\hbar}_2, \widetilde{\hbar}_3, \ldots, \widetilde{\hbar}_n\right)^{\check{Z}}$ are WVs of ¢$_i$ experts and $\mathfrak{S}_j$ parameters with the situation that $\sum_{i=1}^{m}\mathscr{w}_i = 1$, $\sum_{j=1}^{n}\widetilde{\hbar}_i = 1$ and $0 \leq \mathscr{w}_i$, $\widetilde{\hbar}_j \leq 1$ respectively. Then $SFS_{ft}RWA$ operator holds the following properties:

i. (Idempotency): Let $\mathfrak{N}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \mathfrak{N}'_{e\mathfrak{S}}(\mathfrak{G})$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where

$$
\begin{aligned}
\mathfrak{N}'_{e\mathfrak{S}}(\mathfrak{G}) &= \left\{\underline{\mathfrak{N}'_{e\mathfrak{S}}}(\mathfrak{G}), \overline{\mathfrak{N}'_{e\mathfrak{S}}}(\mathfrak{G})\right\}\\
&= \left\{\left(\underline{\mathfrak{p}}, \underline{\mathfrak{q}}, \underline{\mathfrak{r}}\right), \left(\overline{\mathfrak{p}}, \overline{\mathfrak{q}}, \overline{\mathfrak{r}}\right)\right\}
\end{aligned}
$$

then $SFS_{ft}RWA\left(\mathfrak{N}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \ldots, \mathfrak{N}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) = \mathfrak{N}'_{e\mathfrak{S}}(\mathfrak{G})$.

Proof: If $\mathfrak{N}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \mathfrak{N}'_{e\mathfrak{S}}(\mathfrak{G})$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ where $\mathfrak{N}'_{e\mathfrak{S}}(\mathfrak{G}) = \left\{\underline{\mathfrak{N}'_{e\mathfrak{S}}}(\mathfrak{G}), \overline{\mathfrak{N}'_{e\mathfrak{S}}}(\mathfrak{G})\right\} = \left\{\left(\underline{\mathfrak{p}}, \underline{\mathfrak{q}}, \underline{\mathfrak{r}}\right), \left(\overline{\mathfrak{p}}, \overline{\mathfrak{q}}, \overline{\mathfrak{r}}\right)\right\}$ then

$$
SFS_{ft}RWA\left(\mathfrak{N}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \ldots, \mathfrak{N}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right)
$$

$$
= \left(\begin{array}{c}
\left\{\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{\mathfrak{f}_{ij}}\right)^2\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}\end{array}\right\},\\
\left\{\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{\mathfrak{f}_{ij}}\right)^2\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}\end{array}\right\}
\end{array}\right)
$$

For all i, $j\mathfrak{N}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \mathfrak{N}'_{e\mathfrak{S}}(\mathfrak{G}) = \left\{\underline{\mathfrak{N}'_{e\mathfrak{S}}}(\mathfrak{G}), \overline{\mathfrak{N}'_{e\mathfrak{S}}}(\mathfrak{G})\right\} = \left\{\left(\underline{\mathfrak{p}}, \underline{\mathfrak{q}}, \underline{\mathfrak{r}}\right), \left(\overline{\mathfrak{p}}, \overline{\mathfrak{q}}, \overline{\mathfrak{r}}\right)\right\}$. Therefore

$$
= \left(\begin{array}{c}
\left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{\mathfrak{p}_{ij}}\right)^2\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{q}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{r}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}\end{array}\right)\\
\left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{\mathfrak{p}_{ij}}\right)^2\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{q}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{r}_{ij}}\right)^{\mathscr{w}_i}\right)^{\widetilde{\hbar}_j}\end{array}\right)
\end{array}\right)
$$

$$
= \left(\begin{array}{c}\left(\sqrt{1-\left(1-\left(\underline{\mathfrak{p}}\right)^2\right)}, \underline{\mathfrak{q}}, \underline{\mathfrak{r}}\right),\\ \left(\sqrt{1-\left(1-\left(\overline{\mathfrak{p}}\right)^2\right)}, \overline{\mathfrak{q}}, \overline{\mathfrak{r}}\right)\end{array}\right)
$$

$$
= \left\{\begin{array}{c}\left(\underline{\mathfrak{p}}, \underline{\mathfrak{q}}, \underline{\mathfrak{r}}\right),\\ \left(\overline{\mathfrak{p}}, \overline{\mathfrak{q}}, \overline{\mathfrak{r}}\right)\end{array}\right\} = \mathfrak{N}'_{e\mathfrak{S}}(\mathfrak{G}).
$$

Hence $SFS_{ft}RWA\left(\mathfrak{N}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \ldots, \mathfrak{N}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) = \mathfrak{N}'_{e\mathfrak{S}}(\mathfrak{G})$.

ii. (Boundedness): If

$$
\left(\mathfrak{N}_{e\mathfrak{S}_j}(\mathfrak{G}_i)\right)^{-} = \left(\begin{array}{c}min_j min_i \underline{\mathfrak{N}_{e\mathfrak{S}_j}}(\mathfrak{G}_i),\\ max_j max_i \overline{\mathfrak{N}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\end{array}\right)
$$

And

$$
\left(\mathfrak{N}_{e\mathfrak{S}_j}(\mathfrak{G}_i)\right)^{+} = \left(\begin{array}{c}max_j max_i \underline{\mathfrak{N}_{e\mathfrak{S}_j}}(\mathfrak{G}_i),\\ min_j min_i \overline{\mathfrak{N}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\end{array}\right),
$$

Then

$$\left(\mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i})\right)^{-} \leq SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) \leq \left(\mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i})\right)^{+}.$$

Proof: As

$$\left(\mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i})\right)^{-} = \left\{ \begin{pmatrix} min_j min_i \left\{\underline{f_{ij}}\right\}, \; max_j max_i \left\{\underline{\mathfrak{g}_{ij}}\right\}, \\ max_j max_i \left\{\underline{\mathfrak{h}_{ij}}\right\} \end{pmatrix}, \begin{pmatrix} min_j min_i \left\{\overline{f_{ij}}\right\}, \; max_j max_i \left\{\overline{\mathfrak{g}_{ij}}\right\}, \\ max_j max_i \left\{\overline{\mathfrak{h}_{ij}}\right\} \end{pmatrix} \right\}$$

And

$$\left(\mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i})\right)^{+} = \left\{ \begin{pmatrix} max_j max_i \left\{\underline{f_{ij}}\right\}, \; min_j min_i \left\{\underline{\mathfrak{g}_{ij}}\right\}, \\ min_j min_i \left\{\underline{\mathfrak{h}_{ij}}\right\} \end{pmatrix}, \begin{pmatrix} max_j max_i \left\{\overline{f_{ij}}\right\}, \; min_j min_i \left\{\overline{\mathfrak{g}_{ij}}\right\}, \\ min_j min_i \left\{\overline{\mathfrak{h}_{ij}}\right\} \end{pmatrix} \right\}$$

Now we have to show that

$$\left(\mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i})\right)^{-} \leq SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) \leq \left(\mathfrak{R}_{e\mathfrak{S}_{j}}(\mathfrak{G}_{i})\right)^{+}.$$

As for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

$$\begin{aligned}
& min_j min_i \left\{\underline{f_{ij}}\right\} \leq \left\{\underline{f_{ij}}\right\} \leq max_j max_i \left\{\underline{f_{ij}}\right\} \\
\Leftrightarrow\ & 1 - max_j max_i \left\{\underline{f_{ij}}^2\right\} \leq 1 - \underline{f_{ij}}^2 \leq 1 - min_j min_i \left\{\underline{f_{ij}}^2\right\} \\
\Leftrightarrow\ & \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - max_j max_i \left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\widetilde{\hbar}_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\widetilde{\hbar}_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - min_j min_i \left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\widetilde{\hbar}_j} \\
\Leftrightarrow\ & \left(\left(1 - max_j max_i \left(\underline{f_{ij}}\right)^2\right)^{\sum_{i=1}^{m} w_i}\right)^{\sum_{j=1}^{n} \widetilde{\hbar}_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\widetilde{\hbar}_j} \leq \left(\left(1 - min_j min_i \left(\underline{f_{ij}}\right)^2\right)^{\sum_{i=1}^{m} w_i}\right)^{\sum_{j=1}^{n} \widetilde{\hbar}_j} \\
\Leftrightarrow\ & 1 - max_j max_i \left(\underline{f_{ij}}\right)^2 \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\widetilde{\hbar}_j} \leq 1 - min_j min_i \left(\underline{f_{ij}}\right)^2 \\
\Leftrightarrow\ & 1 - \left(1 - min_j min_i \left(\underline{f_{ij}}\right)^2\right) \leq 1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\widetilde{\hbar}_j} \leq 1 - \left(1 - max_j max_i \left(\underline{f_{ij}}\right)^2\right).
\end{aligned}$$

Hence

$$min_j min_i \left(\underline{f_{ij}}\right) \leq \sqrt{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_j}} \leq max_j max_i \left(\underline{f_{ij}}\right) \tag{8}$$

Now for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
& min_j min_i \left(\underline{\mathfrak{g}_{ij}}\right) \leq \left(\underline{\mathfrak{g}_{ij}}\right) \leq max_j max_i \left(\underline{\mathfrak{g}_{ij}}\right) \\
\Leftrightarrow\ & \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(min_j min_i \left(\underline{\mathfrak{g}_{ij}}\right)\right)^{w_i}\right)^{\widetilde{\hbar}_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\left(\underline{\mathfrak{g}_{ij}}\right)\right)^{w_i}\right)^{\widetilde{\hbar}_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(max_j max_i \left(\underline{\mathfrak{g}_{ij}}\right)\right)^{w_i}\right)^{\widetilde{\hbar}_j} \\
\Leftrightarrow\ & \left(\left(min_j min_i \left(\underline{\mathfrak{g}_{ij}}\right)\right)^{\sum_{i=1}^{m} w_i}\right)^{\sum_{j=1}^{n} \widetilde{\hbar}_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{w_i}\right)^{\widetilde{\hbar}_j} \leq \left(\left(max_j max_i \left(\underline{\mathfrak{g}_{ij}}\right)\right)^{\sum_{i=1}^{m} w_i}\right)^{\sum_{j=1}^{n} \widetilde{\hbar}_j} \\
\Rightarrow\ & min_j min_i \left(\underline{\mathfrak{g}_{ij}}\right) \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{w_i}\right)^{\widetilde{\hbar}_j} \leq max_j max_i \left(\underline{\mathfrak{g}_{ij}}\right)
\end{aligned} \tag{9}$$

Also for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, we get

$$\begin{aligned}
& min_j min_i \left(\underline{\mathfrak{h}_{ij}}\right) \leq \left(\underline{\mathfrak{h}_{ij}}\right) \\
&\qquad \leq max_j max_i \left(\underline{\mathfrak{h}_{ij}}\right) \\
&\Leftrightarrow \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(min_j min_i \left(\underline{\mathfrak{h}_{ij}}\right)\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \\
&\qquad \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \\
&\qquad \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(max_j max_i \left(\underline{\mathfrak{h}_{ij}}\right)\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \\
&\Leftrightarrow \left(\left(min_j min_i \left(\underline{\mathfrak{h}_{ij}}\right)\right)^{\sum_{i=1}^{m}\mathscr{w}_i}\right)^{\sum_{j=1}^{n}\tilde{\hbar}_j} \\
&\qquad \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \\
&\qquad \leq \left(\left(max_j max_i \left(\underline{\mathfrak{h}_{ij}}\right)\right)^{\sum_{i=1}^{m}\mathscr{w}_i}\right)^{\sum_{j=1}^{n}\tilde{\hbar}_j} \\
&\Rightarrow min_j min_i \left(\underline{\mathfrak{h}_{ij}}\right) \\
&\qquad \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \\
&\qquad \leq max_j max_i \left(\underline{\mathfrak{h}_{ij}}\right)
\end{aligned} \tag{10}$$

Similarly, we can prove that

$$min_j min_i \left(\overline{f_{ij}}\right) \leq \sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{f_{ij}}\right)^2\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j}} \leq max_j max_i \left(\overline{f_{ij}}\right) \tag{11}$$

$$min_j min_i \left(\overline{\mathfrak{g}_{ij}}\right) \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{ij}}\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \leq max_j max_i \left(\underline{\mathfrak{g}_{ij}}\right) \tag{12}$$

$$min_j min_i \left(\overline{\mathfrak{h}_{ij}}\right) \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{ij}}\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \leq max_j max_i \left(\overline{\mathfrak{h}_{ij}}\right) \tag{13}$$

Therefore from Eqs. (8), (9), (10), (11), (12), and (13), it is clear that

$$\begin{aligned}
& min_j min_i \left(\underline{f_{ij}}\right) \leq \sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f_{ij}}\right)^2\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j}} \\
&\qquad \leq max_j max_i \left(\underline{f_{ij}}\right); \\
& min_j min_i \left(\underline{\mathfrak{g}_{ij}}\right) \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \\
&\qquad \leq max_j max_i \left(\underline{\mathfrak{g}_{ij}}\right); \\
& min_j min_i \left(\underline{\mathfrak{h}_{ij}}\right) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\underline{\mathfrak{h}_{ij}}\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \\
&\qquad \leq max_j max_i \left(\underline{\mathfrak{h}_{ij}}\right)
\end{aligned}$$

And

$$\begin{aligned}
& min_j min_i \left(\overline{f_{ij}}\right) \leq \sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{f_{ij}}\right)^2\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j}} \\
&\qquad \leq max_j max_i \left(\overline{f_{ij}}\right); \\
& min_j min_i \left(\overline{\mathfrak{g}_{ij}}\right) \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{ij}}\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \\
&\qquad \leq max_j max_i \left(\underline{\mathfrak{g}_{ij}}\right); \\
& min_j min_i \left(\overline{\mathfrak{h}_{ij}}\right) \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{ij}}\right)^{\mathscr{w}_i}\right)^{\tilde{\hbar}_j} \\
&\qquad \leq max_j max_i \left(\overline{\mathfrak{h}_{ij}}\right)
\end{aligned}$$

That implies that

$$\begin{aligned}
& \left(\mathfrak{R}_{e\mathfrak{S}_j}\left(\mathfrak{G}_i\right)\right)^{-} \\
&\quad \leq SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_1}\left(\mathfrak{G}_1\right), \mathfrak{R}_{e\mathfrak{S}_2}\left(\mathfrak{G}_2\right), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}\left(\mathfrak{G}_m\right)\right) \\
&\quad \leq \left(\mathfrak{R}_{e\mathfrak{S}_j}\left(\mathfrak{G}_i\right)\right)^{+}.
\end{aligned}$$

iii. (Monotonicity): Let

$$\mathfrak{R}'_{e\mathfrak{S}_j}\left(\mathfrak{G}_i\right) = \begin{pmatrix} \underline{\mathfrak{R}'_{e\mathfrak{S}_j}}\left(\mathfrak{G}_i\right), \\ \overline{\mathfrak{R}'_{e\mathfrak{S}_j}}\left(\mathfrak{G}_i\right) \end{pmatrix}$$

be any other collection of $SFS_{ft}RNs$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that $\underline{\mathfrak{R}'_{e\mathfrak{S}_j}}\left(\mathfrak{G}_i\right) \leq \underline{\mathfrak{R}_{e\mathfrak{S}_j}}\left(\mathfrak{G}_i\right)$, and $\overline{\mathfrak{R}'_{e\mathfrak{S}_j}}\left(\mathfrak{G}_i\right) \leq \overline{\mathfrak{R}_{e\mathfrak{S}_j}}\left(\mathfrak{G}_i\right)$.

Then

$$\begin{aligned}
& SFS_{ft}RWA\begin{pmatrix} \mathfrak{R}'_{e\mathfrak{S}_1}\left(\mathfrak{G}_1\right), \\ \mathfrak{R}'_{e\mathfrak{S}_2}\left(\mathfrak{G}_2\right), \ldots, \mathfrak{R}'_{e\mathfrak{S}_n}\left(\mathfrak{G}_m\right) \end{pmatrix} \\
&\qquad \leq SFS_{ft}RWA\begin{pmatrix} \mathfrak{R}_{e\mathfrak{S}_1}\left(\mathfrak{G}_1\right), \\ \mathfrak{R}_{e\mathfrak{S}_2}\left(\mathfrak{G}_2\right), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}\left(\mathfrak{G}_m\right) \end{pmatrix}.
\end{aligned}$$

Proof. As $\mathfrak{R}'_{e\mathfrak{S}_j}\left(\mathfrak{G}_i\right) = \left(\underline{\mathfrak{R}'_{e\mathfrak{S}_j}}\left(\mathfrak{G}_i\right), \overline{\mathfrak{R}'_{e\mathfrak{S}_j}}\left(\mathfrak{G}_i\right)\right) = \left\{\left(\underline{\mathfrak{p}_{ij}}, \underline{\mathfrak{q}_{ij}}, \underline{\mathfrak{r}_{ij}}\right), \left(\overline{\mathfrak{p}_{ij}}, \overline{\mathfrak{q}_{ij}}, \overline{\mathfrak{r}_{ij}}\right)\right\}$ and

$$\mathfrak{R}_{e\mathfrak{S}_j}\left(\mathfrak{G}_i\right) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}\left(\mathfrak{G}_i\right), \overline{\mathfrak{R}_{e\mathfrak{S}_j}}\left(\mathfrak{G}_i\right)\right)$$

$$= \left(\left(\underline{f_{i\jmath}},\ \underline{\mathfrak{g}_{i\jmath}},\ \underline{\mathfrak{h}_{i\jmath}}\right),\ \left(\overline{f_{i\jmath}},\ \overline{\mathfrak{g}_{i\jmath}},\ \overline{\mathfrak{h}_{i\jmath}}\right)\right).$$

Now we have to show that for $\underline{\mathfrak{R}'_{e\mathfrak{S}_\jmath}}(\mathfrak{S}_i) \leq \underline{\mathfrak{R}_{e\mathfrak{S}_\jmath}}(\mathfrak{S}_i)$, and $\overline{\mathfrak{R}'_{e\mathfrak{S}_\jmath}}(\mathfrak{S}_i) \leq \overline{\mathfrak{R}_{e\mathfrak{S}_\jmath}}(\mathfrak{S}_i)$ for $i = 1, 2, \ldots, m$ and $\jmath = 1, 2, \ldots, n$, so

$$\underline{\mathfrak{p}_{i\jmath}} \leq \underline{f_{i\jmath}} \Rightarrow 1 - \underline{f_{i\jmath}} \leq 1 - \underline{\mathfrak{p}_{i\jmath}}$$
$$\Rightarrow 1 - \underline{f_{i\jmath}}^2 \leq 1 - \underline{\mathfrak{p}_{i\jmath}}^2$$
$$\Rightarrow \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f_{i\jmath}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}$$
$$\leq \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{\mathfrak{p}_{i\jmath}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}$$
$$\Rightarrow 1-\prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{\mathfrak{p}_{i\jmath}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}$$
$$\leq 1-\prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f_{i\jmath}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}$$
$$\Rightarrow \sqrt{1-\prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{\mathfrak{p}_{i\jmath}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}}$$
$$\leq \sqrt{1-\prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f_{i\jmath}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}} \quad (14)$$

And

$$\underline{\mathfrak{q}_{i\jmath}} \geq \underline{\mathfrak{g}_{i\jmath}} \Rightarrow \prod_{i=1}^{m}\left(\underline{\mathfrak{q}_{i\jmath}}\right)^{w_i} \geq \prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{i\jmath}}\right)^{w_i}$$
$$\Rightarrow \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{q}_{i\jmath}}\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}$$
$$\geq \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{i\jmath}}\right)^{w_i}\right)^{\tilde{\hbar}_\jmath} \quad (15)$$

And

$$\underline{\mathfrak{r}_{i\jmath}} \geq \underline{\mathfrak{h}_{i\jmath}} \Rightarrow \prod_{i=1}^{m}\left(\underline{\mathfrak{r}_{i\jmath}}\right)^{w_i} \geq \prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{i\jmath}}\right)^{w_i}$$
$$\Rightarrow \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{r}_{i\jmath}}\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}$$
$$\geq \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{i\jmath}}\right)^{w_i}\right)^{\tilde{\hbar}_\jmath} \quad (16)$$

Similarly

$$\sqrt{1-\prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{\mathfrak{p}_{i\jmath}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}}$$
$$\leq \sqrt{1-\prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{f_{i\jmath}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}} \quad (17)$$

And

$$\overline{\mathfrak{q}_{i\jmath}} \geq \overline{\mathfrak{g}_{i\jmath}} \Rightarrow \prod_{i=1}^{m}\left(\overline{\mathfrak{q}_{i\jmath}}\right)^{w_i} \geq \prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{i\jmath}}\right)^{w_i}$$
$$\Rightarrow \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{q}_{i\jmath}}\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}$$
$$\geq \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{i\jmath}}\right)^{w_i}\right)^{\tilde{\hbar}_\jmath} \quad (18)$$

And

$$\overline{\mathfrak{r}_{i\jmath}} \geq \overline{\mathfrak{h}_{i\jmath}} \Rightarrow \prod_{i=1}^{m}\left(\overline{\mathfrak{r}_{i\jmath}}\right)^{w_i} \geq \prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{i\jmath}}\right)^{w_i}$$
$$\Rightarrow \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{r}_{i\jmath}}\right)^{w_i}\right)^{\tilde{\hbar}_\jmath}$$
$$\geq \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{i\jmath}}\right)^{w_i}\right)^{\tilde{\hbar}_\jmath} \quad (19)$$

Therefore from Eq. (14), (15), (16), (17), (18) and (19), we get $\underline{\mathfrak{R}'_{e\mathfrak{S}_\jmath}}(\mathfrak{S}_i) \leq \underline{\mathfrak{R}_{e\mathfrak{S}_\jmath}}(\mathfrak{S}_i)$ and $\overline{\mathfrak{R}'_{e\mathfrak{S}_\jmath}}(\mathfrak{S}_i) \geq \overline{\mathfrak{R}_{e\mathfrak{S}_\jmath}}(\mathfrak{S}_i)$

Therefore

$$SFS_{ft}RWA\left(\mathfrak{R}'_{e\mathfrak{S}_1}(\mathfrak{S}_1),\ \mathfrak{R}'_{e\mathfrak{S}_2}(\mathfrak{S}_2), \ldots, \mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{S}_m)\right)$$
$$\leq SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{S}_1),\ \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{S}_2), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{S}_m)\right).$$

iv. (Shift Invariance): If $\mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{S}) = \left\{\underline{\mathfrak{R}'_{e\mathfrak{S}}}(\mathfrak{S}), \overline{\mathfrak{R}'_{e\mathfrak{S}}}(\mathfrak{S})\right\} = \left\{\left(\underline{\mathfrak{p}},\ \underline{\mathfrak{q}},\ \underline{\mathfrak{r}}\right),\ \left(\overline{\mathfrak{p}},\ \overline{\mathfrak{q}},\ \overline{\mathfrak{r}}\right)\right\}$ is another family of $SFS_{ft}RNs$, then

$$SFS_{ft}RWA\begin{pmatrix}\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{S}_1) \oplus \mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{S}),\ \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{S}_2) \\ \oplus \mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{S}), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{S}_m) \oplus \mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{S})\end{pmatrix}$$
$$= SFS_{ft}RWA\begin{pmatrix}\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{S}_1),\ \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{S}_2), \ldots, \\ \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{S}_m)\end{pmatrix} \oplus \mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{S}).$$

Proof: Let $\mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{S}) = \left\{\underline{\mathfrak{R}'_{e\mathfrak{S}}}(\mathfrak{S}), \overline{\mathfrak{R}'_{e\mathfrak{S}}}(\mathfrak{S})\right\} = \left\{\left(\underline{\mathfrak{p}},\ \underline{\mathfrak{q}},\ \underline{\mathfrak{r}}\right),\ \left(\overline{\mathfrak{p}},\ \overline{\mathfrak{q}},\ \overline{\mathfrak{r}}\right)\right\}$ is any $SFS_{ft}RN$ and $\mathfrak{R}_{e\mathfrak{S}_\jmath}(\mathfrak{S}_i) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_\jmath}}(\mathfrak{S}_i), \overline{\mathfrak{R}_{e\mathfrak{S}_\jmath}}(\mathfrak{S}_i)\right) = \left(\left(\underline{f_{i\jmath}},\ \underline{\mathfrak{g}_{i\jmath}},\ \underline{\mathfrak{h}_{i\jmath}}\right),\ \left(\overline{f_{i\jmath}},\ \overline{\mathfrak{g}_{i\jmath}},\ \overline{\mathfrak{h}_{i\jmath}}\right)\right)$ be family of $SFS_{ft}RNs$, then

$$\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{S}_1) \oplus \mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{S}) = \begin{pmatrix}\left\{\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{S}_1) \oplus \underline{\mathfrak{R}'_{e\mathfrak{S}}}(\mathfrak{S})\right\}, \\ \left\{\overline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{S}_1) \oplus \overline{\mathfrak{R}'_{e\mathfrak{S}}}(\mathfrak{S})\right\}\end{pmatrix}$$

As

$$\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{S}_1) \oplus \underline{\mathfrak{R}'_{e\mathfrak{S}}}(\mathfrak{S})$$
$$= \left\{\left(\begin{matrix}\sqrt{1-\left(1-\underline{f_{11}}^2\right)\left(1-\underline{\mathfrak{p}}^2\right)}, \\ \underline{\mathfrak{g}_{11}}\underline{\mathfrak{q}},\ \underline{\mathfrak{h}_{11}}\underline{\mathfrak{r}}\end{matrix}\right)\right\}$$

Therefore,

$$
\begin{aligned}
&SFS_{ft}RWA\begin{pmatrix}\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1)\oplus\mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{G}),\\ \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_2)\\ \oplus\mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{G}),\ldots,\\ \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\oplus\mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{G})\end{pmatrix}\\
&=\left\{\begin{array}{c}\oplus_{j=1}^{n}\widetilde{\hbar}_j\left(\oplus_{i=1}^{m}w_i\left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\oplus\mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{G})\right)\right),\\ \oplus_{j=1}^{n}\widetilde{\hbar}_j\left(\oplus_{i=1}^{m}w_i\left(\overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\oplus\mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{G})\right)\right)\end{array}\right\}\\
&=\left(\begin{array}{c}\left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\begin{array}{c}\prod_{i=1}^{m}\left(1-\left(\underline{f_{ij}}\right)^2\right)^{w_i}\\ \left(1-\left(\underline{\mathfrak{p}}\right)^2\right)^{w_i}\end{array}\right)^{\tilde{\hbar}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{w_i}\left(\underline{\mathfrak{q}}^{w_i}\right)\right)^{\tilde{\hbar}_j},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{w_i}\left(\underline{\mathfrak{r}}^{w_i}\right)\right)^{\tilde{\hbar}_j}\end{array}\right),\\ \left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\begin{array}{c}\prod_{i=1}^{m}\left(1-\left(\overline{f_{ij}}\right)^2\right)^{w_i}\\ \left(1-\left(\overline{\mathfrak{p}}\right)^2\right)^{w_i}\end{array}\right)^{\tilde{\hbar}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{ij}}\right)^{w_i}\left(\overline{\mathfrak{q}}^{w_i}\right)\right)^{\tilde{\hbar}_j},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{ij}}\right)^{w_i}\left(\overline{\mathfrak{r}}^{w_i}\right)\right)^{\tilde{\hbar}_j}\end{array}\right)\end{array}\right)\\
&=\left(\begin{array}{c}\left(\begin{array}{c}\sqrt{1-\left(1-\left(\underline{\mathfrak{p}}\right)^2\right)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_j}},\\ \underline{\mathfrak{q}}\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j},\\ \underline{\mathfrak{r}}\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j}\end{array}\right),\\ \left(\begin{array}{c}\sqrt{1-\left(1-\left(\overline{\mathfrak{p}}\right)^2\right)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{f_{ij}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_j}},\\ \overline{\mathfrak{q}}\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j},\\ \overline{\mathfrak{r}}\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j}\end{array}\right)\end{array}\right)\\
&=\left(\begin{array}{c}\left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j}\end{array}\right)\\ \oplus\left(\underline{\mathfrak{p}},\ \underline{\mathfrak{q}},\ \underline{\mathfrak{r}}\right)\\ \left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{f_{ij}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j}\end{array}\right)\\ \oplus\left(\overline{\mathfrak{p}},\ \overline{\mathfrak{q}},\ \overline{\mathfrak{r}}\right)\end{array}\right)\\
&=\left(\begin{array}{c}\left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j}\end{array}\right)\\ \left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{f_{ij}}\right)^2\right)^{w_i}\right)^{\tilde{\hbar}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{ij}}\right)^{w_i}\right)^{\tilde{\hbar}_j}\end{array}\right)\end{array}\right)\\
&\quad\oplus\left(\left(\underline{\mathfrak{p}},\ \underline{\mathfrak{q}},\ \underline{\mathfrak{r}}\right),\ \left(\overline{\mathfrak{p}},\ \overline{\mathfrak{q}},\ \overline{\mathfrak{r}}\right)\right)\\
&=SFS_{ft}RWA\begin{pmatrix}\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1),\ \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_2),\\ \ldots,\mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\end{pmatrix}\oplus\mathfrak{R}'_{e\mathfrak{S}}(\mathfrak{G}).
\end{aligned}
$$

Hence the required result is proved.

v. (Homogeneity): For any $\mathfrak{k}\geq 0$

$$
\begin{aligned}
&SFS_{ft}RWA\left(\mathfrak{k}\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1),\ \mathfrak{k}\mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_2),\ldots,\mathfrak{k}\mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right)\\
&\leq\mathfrak{k}SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1),\mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_2),\ldots,\mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right).
\end{aligned}
$$

Proof: Let $\mathfrak{k}\geq 0$ and $\mathfrak{R}_{e\mathfrak{S}_j}(\mathfrak{G}_i)=\left\{\mathfrak{k}\left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right),\ \mathfrak{k}\left(\overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right)\right\}$ be a collection of $SFS_{ft}RNs$.

As

$$\mathfrak{k}\underline{\mathfrak{R}_{e\mathfrak{S}_1}}(\mathfrak{G}_1) = \left(\begin{array}{c}\left(\sqrt{\left(1-\left(1-\underline{f_{11}}^2\right)^{\mathfrak{k}}\right)}, \underline{\mathfrak{g}_{11}}^{\mathfrak{k}}, \underline{\mathfrak{h}_{11}}^{\mathfrak{k}}\right),\\ \left(\sqrt{\left(1-\left(1-\overline{f_{11}}^2\right)^{\mathfrak{k}}\right)}, \overline{\mathfrak{g}_{11}}^{\mathfrak{k}}, \overline{\mathfrak{h}_{11}}^{\mathfrak{k}}\right)\end{array}\right).$$

Now

$$SFS_{ft}RWA\left(\mathfrak{k}\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \mathfrak{k}\mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \mathfrak{k}\mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right)$$

$$= \left(\begin{array}{c}\left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f_{ij}}\right)^2\right)^{\mathfrak{k}w_i}\right)^{\tilde{h}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{\mathfrak{k}w_i}\right)^{\tilde{h}_j},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{\mathfrak{k}w_i}\right)^{\tilde{h}_j}\end{array}\right),\\ \left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{f_{ij}}\right)^2\right)^{\mathfrak{k}w_i}\right)^{\tilde{h}_j}},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{ij}}\right)^{\mathfrak{k}w_i}\right)^{\tilde{h}_j},\\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{ij}}\right)^{\mathfrak{k}w_i}\right)^{\tilde{h}_j}\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{c}\left(\begin{array}{c}\sqrt{1-\left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f_{ij}}\right)^2\right)^{w_i}\right)^{\tilde{h}_j}\right)^{\mathfrak{k}}},\\ \left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}_{ij}}\right)^{w_i}\right)^{\tilde{h}_j}\right)^{\mathfrak{k}},\\ \left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}_{ij}}\right)^{w_i}\right)^{\tilde{h}_j}\right)^{\mathfrak{k}}\end{array}\right),\\ \left(\begin{array}{c}\sqrt{1-\left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{f_{ij}}\right)^2\right)^{w_i}\right)^{\tilde{h}_j}\right)^{\mathfrak{k}}},\\ \left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}_{ij}}\right)^{w_i}\right)^{\tilde{h}_j}\right)^{\mathfrak{k}},\\ \left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}_{ij}}\right)^{w_i}\right)^{\tilde{h}_j}\right)^{\mathfrak{k}}\end{array}\right)\end{array}\right)$$

TABLE 4. Tabular representation of $\mathfrak{R}_{e\nabla\mathfrak{S}_j}(\mathfrak{G}_i) = (\underline{\mathfrak{R}_{e\nabla\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_{e\nabla\mathfrak{S}_j}}(\mathfrak{G}_i))$.

$\mathfrak{R}_e$	$\mathfrak{S}_1$	$\mathfrak{S}_2$
$\mathfrak{c}_1$	$\{(0.4, 0.1, 0.2), (0.3, 0.3, 0.4)\}$	$\{(0.1, 0.1, 0.5), (0.6, 0.1, 0.1)\}$
$\mathfrak{c}_2$	$\{(0.5, 0.6, 0.3), (0.6, 0.2, 0.1)\}$	$\{(0.3, 0.1, 0.6), (0.5, 0.3, 0.2)\}$
$\mathfrak{c}_3$	$\{(0.3, 0.3, 0.1), (0.2, 0.5, 0.3)\}$	$\{(0.6, 0.3, 0.5), (0.2, 0.6, 0.1)\}$

$$= \mathfrak{k}SFS_{ft}RWA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \mathfrak{R}_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right).$$

Thus prove is completed.

Remark 2:

a) If we ignore the AG, then the developed $SFS_{ft}RWA$ operator will degenerate into Pythagorean fuzzy soft rough weighted average ($P_yFS_{ft}RWA$) operator.
b) If only one soft parameter $\mathfrak{S}_1$ i.e. ($n = 1$) is used, then the developed $SFS_{ft}RWA$ will degenerate into a spherical fuzzy rough weighted average (SFRWA) operator.

B. SPHERICAL FUZZY SOFT ROUGH ORDERED WEIGHTED AVERAGE ($SFS_{ft}ROWA$) OPERATOR

In this subsection, we will introduce a spherical fuzzy soft rough ordered weighted average aggregation operator and its fundamental characteristics.

Definition 12: Suppose $\mathfrak{R}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right)$ is a family of $SFS_{ft}RNs$. Also, assume that $w = (w_1, w_2, w_3, \ldots, w_m)^{\check{Z}}$, $\tilde{h} = \left(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \ldots, \tilde{h}_n\right)^{\check{Z}}$ are the WVs of $\mathfrak{c}_i$ experts and $\mathfrak{S}_j$ parameters using the condition that $\sum_{i=1}^{m} w_i = 1$, $\sum_{j=1}^{n}\tilde{h}_i = 1$ and $0 \leq w_i, \tilde{h}_j \leq 1$ respectively.

Then $SFS_{ft}ROWA$ aggregation operator is given as

$$SFS_{ft}ROWA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) = \left(\begin{array}{c}\oplus_{j=1}^{n}\tilde{h}_j\left(\oplus_{i=1}^{m} w_i\underline{\mathfrak{R}_{e\nabla\mathfrak{S}_j}}(\mathfrak{G}_i)\right),\\ \oplus_{j=1}^{n}\tilde{h}_j\left(\oplus_{i=1}^{m} w_i\overline{\mathfrak{R}_{e\nabla\mathfrak{S}_j}}(\mathfrak{G}_i)\right)\end{array}\right).$$

Theorem 3: Let $\mathfrak{R}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right)$ be a family of $SFS_{ft}RNs$. Also, suppose that $w = (w_1, w_2, w_3, \ldots, w_m)^{\check{Z}}$, $\tilde{h} = \left(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \ldots, \tilde{h}_n\right)^{\check{Z}}$ are WVs of $\mathfrak{c}_i$ experts and $\mathfrak{S}_j$ parameters using the condition that $\sum_{i=1}^{m} w_i = 1$, $\sum_{j=1}^{n}\tilde{h}_i = 1$ and $0 \leq w_i, \tilde{h}_j \leq 1$ respectively. Then $SFS_{ft}ROWA$ aggregation operator is given as:

$$SFS_{ft}ROWA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1), \ldots, \mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) = \left(\begin{array}{c}\oplus_{j=1}^{n}\tilde{h}_j\left(\oplus_{i=1}^{m} w_i\underline{\mathfrak{R}_{e\nabla\mathfrak{S}_j}}(\mathfrak{G}_i)\right),\\ \oplus_{j=1}^{n}\tilde{h}_j\left(\oplus_{i=1}^{m} w_i\overline{\mathfrak{R}_{e\nabla\mathfrak{S}_j}}(\mathfrak{G}_i)\right)\end{array}\right)$$

$$= \left(\begin{array}{c} \left\{ \begin{array}{c} \sqrt{1-\prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f_{\nabla i\jmath}}\right)^2\right)^{w_i}\right)^{\tilde{h}_\jmath}}, \\ \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{g_{\nabla i\jmath}}\right)^{w_i}\right)^{\tilde{h}_\jmath}, \\ \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{h_{\nabla i\jmath}}\right)^{w_i}\right)^{\tilde{h}_\jmath} \end{array} \right\}, \\ \left\{ \begin{array}{c} \sqrt{1-\prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{f_{\nabla i\jmath}}\right)^2\right)^{w_i}\right)^{\tilde{h}_\jmath}}, \\ \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{g_{\nabla i\jmath}}\right)^{w_i}\right)^{\tilde{h}_\jmath}, \\ \prod_{\jmath=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{h_{\nabla i\jmath}}\right)^{w_i}\right)^{\tilde{h}_\jmath} \end{array} \right\} \end{array} \right)$$

where $\Re_{e\nabla\mathfrak{S}_\jmath}(\mathfrak{G}_i) = \left(\underline{\Re_{e\nabla\mathfrak{S}_\jmath}}(\mathfrak{G}_i), \overline{\Re_{e\nabla\mathfrak{S}_\jmath}}(\mathfrak{G}_i)\right)$ represents the largest value of the permutation from ith row and $\jmath$th column of the family $i\times\jmath$

$$SFS_{ft}RNs\ \Re_{e\mathfrak{S}_\jmath}(\mathfrak{G}_i) = \left(\underline{\Re_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i), \overline{\Re_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i)\right).$$

Example 3: Consider the above Table 3 in Example (2) for the family of $SFS_{ft}RNs\ \Re_{e\mathfrak{S}_\jmath}(\mathfrak{G}_i) = \left(\underline{\Re_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i), \overline{\Re_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i)\right)$. Then new order of tabular presentation of $\Re_{e\mathfrak{S}_\jmath}(\mathfrak{G}_i) = \left(\underline{\Re_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i), \overline{\Re_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i)\right)$ through score function is given in Table 4.

On the basis of the definition 12, we have:

$$SFS_{ft}ROWA\left(\Re_{e\mathfrak{S}_1}(\mathfrak{G}_1), \ldots, \Re_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) = \{(0.4082,\ 0.2077,\ 0.2517), (0.4195, 0.3063, 0.1928)\}.$$

From the analysis of the above theorem, we observe that $SFS_{ft}ROWA$ operator has the following properties:

i. (Idempotency): Let $\Re_{e\nabla\mathfrak{S}_\jmath}(\mathfrak{G}_i) = \Re'_{e\mathfrak{S}}(\mathfrak{G})$ for all $i = 1, 2, \ldots, m$ and $\jmath = 1, 2, \ldots, n$ where $\Re'_{e\mathfrak{S}}(\mathfrak{G}) = \left\{\underline{\Re'_{e\mathfrak{S}}}(\mathfrak{G}), \overline{\Re'_{e\mathfrak{S}}}(\mathfrak{G})\right\} = \left\{\left(\underline{p}, \underline{q}, \underline{r}\right), \left(\overline{p}, \overline{q}, \overline{r}\right)\right\}$, then $SFS_{ft}ROWA(\Re_{e\mathfrak{S}_1}(\mathfrak{G}_1), \Re_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \Re_{e\mathfrak{S}_n}(\mathfrak{G}_m)) = \Re'_{e\mathfrak{S}}(\mathfrak{G})$.

ii. (Boundedness): If

$$\left(\Re_{e\nabla\mathfrak{S}_\jmath}(\mathfrak{G}_i)\right)^- = \left(min_\jmath min_i \underline{\Re_{e\nabla\mathfrak{S}_\jmath}}(\mathfrak{G}_i), max_\jmath max_i \overline{\Re_{e\nabla\mathfrak{S}_\jmath}}(\mathfrak{G}_i)\right) \text{ and}$$

$$\left(\Re_{e\nabla\mathfrak{S}_\jmath}(\mathfrak{G}_i)\right)^+ = \left(max_\jmath max_i \underline{\Re_{e\nabla\mathfrak{S}_\jmath}}(\mathfrak{G}_i), min_\jmath min_i \overline{\Re_{e\nabla\mathfrak{S}_\jmath}}(\mathfrak{G}_i)\right), \text{ then}$$

$$\left(\Re_{e\nabla\mathfrak{S}_\jmath}(\mathfrak{G}_i)\right)^- \leq SFS_{ft}ROWA\left(\Re_{e\mathfrak{S}_1}(\mathfrak{G}_1), \Re_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \Re_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) \leq \left(\Re_{e\nabla\mathfrak{S}_\jmath}(\mathfrak{G}_i)\right)^+.$$

iii. (Monotonicity): Let $\Re'_{e\mathfrak{S}_\jmath}(\mathfrak{G}_i) = \left(\underline{\Re'_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i), \overline{\Re'_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i)\right)$ be any other collection of $SFS_{ft}RNs$ for all $i = 1, 2, \ldots, m$ and $\jmath = 1, 2, \ldots, n$ such that $\underline{\Re'_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i) \leq \underline{\Re_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i)$, and $\overline{\Re'_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i) \leq \overline{\Re_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i)$, then

$$SFS_{ft}ROWA\left(\Re'_{e\mathfrak{S}_1}(\mathfrak{G}_1), \Re'_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \Re'_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) \leq SFS_{ft}ROWA\left(\begin{array}{c}\Re_{e\mathfrak{S}_1}(\mathfrak{G}_1), \Re_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \\ \Re_{e\mathfrak{S}_n}(\mathfrak{G}_m)\end{array}\right).$$

iv. (Shift Invariance): If $\Re'_{e\mathfrak{S}}(\mathfrak{G}) = \left\{\underline{\Re'_{e\mathfrak{S}}}(\mathfrak{G}), \overline{\Re'_{e\mathfrak{S}}}(\mathfrak{G})\right\} = \left\{\left(\underline{p}, \underline{q}, \underline{r}\right), \left(\overline{p}, \overline{q}, \overline{r}\right)\right\}$ is another collection of $SFS_{ft}RNs$, then

$$SFS_{ft}ROWA\left(\begin{array}{c}\Re_{e\mathfrak{S}_1}(\mathfrak{G}_1) \oplus \Re'_{e\mathfrak{S}}(\mathfrak{G}), \Re_{e\mathfrak{S}_2}(\mathfrak{G}_2) \\ \oplus\Re'_{e\mathfrak{S}}(\mathfrak{G}), \ldots, \Re_{e\mathfrak{S}_n}(\mathfrak{G}_m) \oplus \Re'_{e\mathfrak{S}}(\mathfrak{G})\end{array}\right) = SFS_{ft}ROWA\left(\Re_{e\mathfrak{S}_1}(\mathfrak{G}_1), \Re_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \Re_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) \oplus \Re'_{e\mathfrak{S}}(\mathfrak{G}).$$

v. (Homogeneity): For any $\mathfrak{k} \geq 0$,

$$SFS_{ft}ROWA\left(\mathfrak{k}\Re_{e\mathfrak{S}_1}(\mathfrak{G}_1), \mathfrak{k}\Re_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \mathfrak{k}\Re_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) \leq \mathfrak{k}SFS_{ft}ROWA\left(\begin{array}{c}\Re_{e\mathfrak{S}_1}(\mathfrak{G}_1), \Re_{e\mathfrak{S}_2}(\mathfrak{G}_2), \ldots, \\ \Re_{e\mathfrak{S}_n}(\mathfrak{G}_m)\end{array}\right).$$

Remark 3:

a) If we ignore the AG, then the developed $SFS_{ft}ROWA$ operator will degenerate into Pythagorean fuzzy soft rough ordered weighted average ($P_yFS_{ft}ROWA$) operator.

b) If only one soft parameter $\mathfrak{S}_1$ i.e. ($n = 1$) is used, then the developed $SFS_{ft}ROWA$ will degenerate into a spherical fuzzy rough ordered weighted average (SFROWA) operator.

C. SPHERICAL FUZZY SOFT ROUGH HYBRID AVERAGE ($SFS_{ft}RHA$) AGGREGATION OPERATOR

Definition 13: Suppose $\Re_{e\mathfrak{S}_\jmath}(\mathfrak{G}_i) = \left(\underline{\Re_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i), \overline{\Re_{e\mathfrak{S}_\jmath}}(\mathfrak{G}_i)\right)$ is a family of $SFS_{ft}RNs$. Let $\wp = (\wp_1, \wp_2, \wp_3, \ldots, \wp_m)^{\check{Z}}$, $\hbar = \left(\hbar_1, \hbar_2, \hbar_3, \ldots, \hbar_n\right)^{\check{Z}}$ denote WVs of $\mathfrak{e}_i$ experts and $\mathfrak{S}_\jmath$ parameters respectively using the condition that $\sum_{i=1}^{m}\wp_i = 1$, $\sum_{\jmath=1}^{n}\hbar_\jmath = 1$ and $0 \leq \wp_i, \hbar_i \leq 1$.

Also, assume that $w = (w_1, w_2, w_3, \ldots, w_m)^{\check{Z}}$, $\tilde{h} = \left(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \ldots, \tilde{h}_n\right)^{\check{Z}}$ are the associated WVs of $\mathfrak{e}_i$ experts and $\mathfrak{S}_\jmath$ parameters using the condition that $\sum_{i=1}^{m} w_i =$

1, $\sum_{j=1}^{n}\tilde{\hbar}_i = 1$ and $0 \le \omega_i, \tilde{\hbar}_j \le 1$ respectively. Then, $SFS_{ft}RHA$ aggregation operator is given as

$$SFS_{ft}RHA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1),\ldots,\mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) = \begin{pmatrix} \oplus_{j=1}^{n}\tilde{\hbar}_j\left(\oplus_{i=1}^{m}\omega_i\underline{\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}}(\mathfrak{G}_i)\right), \\ \oplus_{j=1}^{n}\tilde{\hbar}_j\left(\oplus_{i=1}^{m}\omega_i\overline{\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}}(\mathfrak{G}_i)\right) \end{pmatrix}$$

Theorem 4: Suppose $\mathfrak{R}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right)$ is a family of $SFS_{ft}RNs$. Let $\mathscr{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \ldots, \mathscr{P}_m)^{\check{Z}}$ and $\mathscr{f} = \left(\mathscr{f}_1, \mathscr{f}_2, \mathscr{f}_3, \ldots, \mathscr{f}_n\right)^{\check{Z}}$ are the WVs of $¢_i$ experts and $\mathfrak{S}_j$ parameters respectively using the condition that $\sum_{i=1}^{m}\mathscr{P}_i = 1$, $\sum_{j=1}^{n}\mathscr{f}_j = 1$ and $0 \le \mathscr{P}_i, \mathscr{f}_j \le 1$. Also, assume that $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_m)^{\check{Z}}$, $\tilde{\hbar} = \left(\tilde{\hbar}_1, \tilde{\hbar}_2, \tilde{\hbar}_3, \ldots, \tilde{\hbar}_n\right)^{\check{Z}}$ are the associated WVs of $¢_i$ experts and $\mathfrak{S}_j$ parameters using the condition that $\sum_{i=1}^{m}\omega_i = 1$, $\sum_{j=1}^{n}\tilde{\hbar}_i = 1$, and $0 \le \omega_i, \tilde{\hbar}_j \le 1$, respectively. Then $SFS_{ft}RHA$ aggregation operator is given as;

$$SFS_{ft}RHA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1),\ldots,\mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) = \begin{pmatrix} \oplus_{j=1}^{n}\tilde{\hbar}_j\left(\oplus_{i=1}^{m}\omega_i\underline{\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}}(\mathfrak{G}_i)\right), \\ \oplus_{j=1}^{n}\tilde{\hbar}_j\left(\oplus_{i=1}^{m}\omega_i\overline{\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}}(\mathfrak{G}_i)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left\{ \begin{array}{c} \sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\underline{f^{*}{}_{\nabla ij}}\right)^2\right)^{\omega_i}\right)^{\tilde{\hbar}_j}}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{g}^{*}{}_{\nabla ij}}\right)^{\omega_i}\right)^{\tilde{\hbar}_j}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\mathfrak{h}^{*}{}_{\nabla ij}}\right)^{\omega_i}\right)^{\tilde{\hbar}_j} \end{array} \right\}, \\ \left\{ \begin{array}{c} \sqrt{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\left(\overline{f^{*}{}_{\nabla ij}}\right)^2\right)^{\omega_i}\right)^{\tilde{\hbar}_j}}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{g}^{*}{}_{\nabla ij}}\right)^{\omega_i}\right)^{\tilde{\hbar}_j}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\mathfrak{h}^{*}{}_{\nabla ij}}\right)^{\omega_i}\right)^{\tilde{\hbar}_j} \end{array} \right\} \end{pmatrix}$$

where $\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}}(\mathfrak{G}_i)\right)$

$$= \left(n\mathscr{P}_i\mathscr{f}_j\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i), n\mathscr{P}_i\mathscr{f}_j\overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right)$$

present the largest value of the permutation from ith row and jth column of the family $i\times j$ $SFS_{ft}RNs$ $\mathfrak{R}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right)$ and "n" is the balancing coefficient.

Example 4: Consider the above Table 3 of Example (2) for the family of $SFS_{ft}RNs$ $\mathfrak{R}_{e\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right)$ with $\mathscr{P} = (0.33, 0.36, 0.31)^{\check{Z}}$, $\mathscr{f} = (0.47, 0.53)^{\check{Z}}$ be the WVs of $¢_i$ experts and $\mathfrak{S}_j$ parameters. Also, consider that $\omega = (0.37, 0.28, 0.35)^{\check{Z}}$, $\tilde{\hbar} = (0.41, 0.59)^{\check{Z}}$ are the associated WVs of $¢_i$ experts and $\mathfrak{S}_j$ parameters. Then the tabular presentation of $\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}(\mathfrak{G}_i)$ through operational rules and score function is given in Table 5. and Table 6. as

$$\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}(\mathfrak{G}_i) = \left(\underline{\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}_e{}^{*}{}_{\nabla\mathfrak{S}_j}}(\mathfrak{G}_i)\right) = \left(n\mathscr{P}_i\underline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i), n\mathscr{P}_i\overline{\mathfrak{R}_{e\mathfrak{S}_j}}(\mathfrak{G}_i)\right).$$

TABLE 5. Tabular representation by using operational laws for of $\mathfrak{R}^{\star}{}_{e\nabla\mathfrak{S}_j}(\mathfrak{G}_i) = (\underline{\mathfrak{R}^{\star}{}_{e\nabla\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}^{\star}{}_{e\nabla\mathfrak{S}_j}}(\mathfrak{G}_i))$.

$\mathfrak{R}_e$	$\mathfrak{S}_1$	$\mathfrak{S}_2$
$¢_1$	{(0.4846, 0.6216, 0.3261), (0.4330, 0.4729, 0.3425)}	{(0.6115, 0.2826, 0.4831), (0.1455, 0.7648, 0.2987)}
$¢_2$	{(0.4027, 0.0965, 0.1951), (0.2162, 0.5427, 0.6280)}	{(0.3199, 0.0716, 0.5572), (0.3896, 0.5020, 0.3980)}
$¢_3$	{(0.2813, 0.3490, 0.1335), (0.1329, 0.7386, 0.5908)}	{(0.0996, 0.1033, 0.5049), (0.4443, 0.3214, 0.3214)}

TABLE 6. Tabular presentation by using score function for $\mathfrak{R}^{\star}{}_{e\nabla\mathfrak{S}_j}(\mathfrak{G}_i) = (\underline{\mathfrak{R}^{\star}{}_{e\nabla\mathfrak{S}_j}}(\mathfrak{G}_i), \overline{\mathfrak{R}^{\star}{}_{e\nabla\mathfrak{S}_j}}(\mathfrak{G}_i))$.

$\mathfrak{R}_e$	$\mathfrak{S}_1$	$\mathfrak{S}_2$
$¢_1$	{(0.4027, 0.0965, 0.1951), (0.2162, 0.5427, 0.6280)}	{(0.0996, 0.1033, 0.5049), (0.4443, 0.3214, 0.3214)}
$¢_2$	{(0.4846, 0.6216, 0.3261), (0.4330, 0.4729, 0.3425)}	{(0.3199, 0.0716, 0.5572), (0.3896, 0.5020, 0.3980)}
$¢_3$	{(0.2813, 0.3490, 0.1335), (0.1329, 0.7386, 0.5908)}	{(0.6115, 0.2826, 0.4831), (0.1455, 0.7648, 0.2987)}

Based on the above information, we have:

$$SFS_{ft}RHA\left(\mathfrak{R}_{e\mathfrak{S}_1}(\mathfrak{G}_1),\ldots,\mathfrak{R}_{e\mathfrak{S}_n}(\mathfrak{G}_m)\right) = \{(0.3979, 0.1631, 0.3484), (0.3343, 0.5213, 0.4048)\}.$$

Moreover, similarity to the $SFS_{ft}ROWA$ operator, the $SFS_{ft}RHA$ operator has some important properties, such as idempotency, boundedness, monotonicity, shift invariance and homogeneity.

Remark 4:

a) If we ignore the AG, then the developed $SFS_{ft}RHA$ operator will degenerate into Pythagorean fuzzy soft rough hybrid average ($P_yFS_{ft}RHA$) operator.
b) If only one soft parameter $\mathfrak{S}_1$ i.e. ($n = 1$) is used, then the developed $SFS_{ft}RHA$ will degenerate into a spherical fuzzy rough hybrid average (SFRHA) operator.

V. A MULTI-CRITERIA DECISION-MAKING METHOD BASED ON SPHERICAL FUZZY SOFT ROUGH AVERAGE AGGREGATION OPERATORS

A. AN ALGORITHM FOR PROPOSED WORK

In this section, we will study a new MCDM method by using $SFS_{ft}RWA$, $SFS_{ft}ROWA$ and $SFS_{ft}RHA$ aggregation operators to solve MCDM problems under the environment of $SFS_{ft}R$ information.

Let $\mathfrak{F} = \{\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots, \mathfrak{F}_\mathfrak{r}\}$ be the set of "$\mathfrak{r}$" alternative and $\mathfrak{S} = \{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \ldots, \mathfrak{S}_n\}$ be the corresponding set of "n" parameters. Let $\text{¢} = \{\text{¢}_1, \text{¢}_2, \text{¢}_3, \ldots, \text{¢}_m\}$ be the family of "m" senior firewall software experts who provide their expertise for each alternative $\mathfrak{F}_l\ (l = 1, 2, 3, \ldots, r)$. Let $\dot{w} = \{\dot{w}_1, \dot{w}_2, \ldots, \dot{w}_n\}$ denote the WVs of "¢_i" experts and $\tilde{\hbar} = \left\{\tilde{\hbar}_1, \tilde{\hbar}_2, \ldots, \tilde{\hbar}_n\right\}$ represent the WVs of parameters "$\mathfrak{S}_\jmath$" with a condition that $\dot{w}_\mathrm{i}, \tilde{\hbar}_\jmath \in [0, 1]$ and $\sum_{\mathrm{i}=1}^{m}\dot{w}_\mathrm{i} = 1$, $\sum_{\jmath=1}^{n}\tilde{\hbar}_\jmath = 1$. Suppose the assessment data given by intellectual is in the form of $SFS_{ft}RNs$. The overall information is given in $SFS_{ft}R$ matrix $\mathcal{M} = \left[\mathfrak{R}e_{\mathfrak{S}_\jmath}(\mathfrak{G}_\mathrm{i})\right]_{n\times m}$. The main steps based on the developed operators for the MCDM method is gives as follows.

Step 1: All intellectual provide their assessment in the shape of $SFS_{ft}RNs$ for each alternative $\mathfrak{F}_l$ corresponding to their respective parameters $\mathfrak{S}_\jmath$. Then, arrange the overall assessment information in $SFS_{ft}R$ decision matrix $\mathcal{M} = \left[\mathfrak{R}e_{\mathfrak{S}_\jmath}(\mathfrak{G}_\mathrm{i})\right]_{n\times m}$.

Step 2: Normalize the $SFS_{ft}R$ decision matrix given in step 1 according to the following formula:

$$\mathfrak{R}e_{\mathfrak{S}_\jmath}(\mathfrak{G}_\mathrm{i}) = \begin{cases} \left(\mathfrak{R}e_{\mathfrak{S}_\jmath}(\mathfrak{G}_\mathrm{i})\right)^c & \text{for cost type parameter} \\ \mathfrak{R}e_{\mathfrak{S}_\jmath}(\mathfrak{G}_\mathrm{i}) & \text{for a benefit type parameter} \end{cases}$$

where

$$\left(\mathfrak{R}e_{\mathfrak{S}_\jmath}(\mathfrak{G}_\mathrm{i})\right)^c = \left(\underline{\mathfrak{R}e_{\mathfrak{S}_1}}(\mathfrak{G}_1), \overline{\mathfrak{R}e_{\mathfrak{S}_1}}(\mathfrak{G}_1)\right) = \left(\left(\underline{\mathfrak{h}_{11}}, \underline{\mathfrak{g}_{11}}, \underline{f_{11}}\right), \left(\overline{\mathfrak{h}_{11}}, \overline{\mathfrak{g}_{11}}, \overline{f_{11}}\right)\right)$$

denote the complement of

$$\mathfrak{R}e_{\mathfrak{S}_\jmath}(\mathfrak{G}_\mathrm{i}) = \left(\underline{\mathfrak{R}e_{\mathfrak{S}_1}}(\mathfrak{G}_1), \overline{\mathfrak{R}e_{\mathfrak{S}_1}}(\mathfrak{G}_1)\right) = \begin{pmatrix} \left(\underline{f_{11}}, \underline{\mathfrak{g}_{11}}, \underline{\mathfrak{h}_{11}}\right), \\ \left(\overline{f_{11}}, \overline{\mathfrak{g}_{11}}, \overline{\mathfrak{h}_{11}}\right) \end{pmatrix}.$$

Step 3: Apply the established operators of each decision matrix $\mathcal{M} = \left[\mathfrak{R}e_{\mathfrak{S}_\jmath}(\mathfrak{G}_\mathrm{i})\right]_{n\times m}$ for each alternative $\mathfrak{F}_l\ (l = 1, 2, 3, \ldots, \mathfrak{r})$ corresponding to their respective parameters $\mathfrak{S}_\jmath$ and calculate the aggregated result $\mathfrak{Q}_l = \left(\left(\underline{f}, \underline{\mathfrak{g}}, \underline{\mathfrak{h}}\right), \left(\overline{f}, \overline{\mathfrak{g}}, \overline{\mathfrak{h}}\right)\right)$.

Step 4: Utilize Definition 10 to calculate the score values for each $\mathfrak{Q}_l$.

Step 5: Rank the results and choose the best result.

B. APPLICATION OF THE PROPOSED METHOD

In this section, an explanatory example is given to view the strength of the presented work.

A firewall is a web safety system to handle incoming or outgoing network traffic by using a preset security rule. The selection of the best firewall software is a key point for a company to protect their data and secure their information. So the proposed MCDM can handle successfully the evaluation process and the ability to detect and choose the best firewall software among the given alternative.

Example 5: Suppose an organization X in America wants to select the best firewall software for the security and safety of their information. Let there are initially three firewall software that is to be considered and set of these firewall software is given as $\mathfrak{F} = \{\mathfrak{F}_1 = \textit{Check point next generation firewall}, \mathfrak{F}_2 = \textit{Glass wire firewall}, \mathfrak{F}_3 = \textit{Sophos XG firewall}\}$ based on the following parameters $\{\mathfrak{S}_1 = \textit{Wireless network protection}, \mathfrak{S}_2 = \textit{Internet and network access}, \mathfrak{S}_3 = \textit{protection against malware}, \mathfrak{S}_4 = \textit{Blokage against unauthorižed access}\}$. The family consisting of four highly qualified and professional software experts are invited by the organization for the selection of best firewall software. Let $\dot{w} = \{0.28, 0.24, 0.23, 0.25\}$ denote the WVs of $\text{¢}_\mathrm{i}\ (\mathrm{i} = 1, 2, 3, 4)$ experts and $\tilde{\hbar} = \{0.18, 0.29, 0.32, 0.21\}$ represent the WVs of parameters $\mathfrak{S}_\jmath$. The professional experts provide their assessment data for the alternatives $\mathfrak{F}_l$ corresponding to parameters $\mathfrak{S}_\jmath$ in the form of $SFS_{ft}RNs$. Now we apply the devised method to select the suitable alternative $\mathfrak{F}_l$.

1) BY USING $SFS_{ft}RWA$ OPERATOR

Step 1: All the experts provide their assessment in the form of $SFS_{ft}RNs$ for each alternative $\mathfrak{F}_l$ corresponding to their respective parameters $\mathfrak{S}_\jmath$. Then arrange the overall assessment information in $SFS_{ft}R$ decision matrix $\mathcal{M} = \left[\mathfrak{R}e_{\mathfrak{S}_\jmath}(\mathfrak{G}_\mathrm{i})\right]_{n\times m}$ given in Table 7-9.

Step 2: Normalize the $SFS_{ft}R$ decision matrix if necessary.

Step 3: Apply the devised $SFS_{ft}RWA$ aggregation operators of each decision matrix $\mathcal{M} = \left[\mathfrak{R}e_{\mathfrak{S}_\jmath}(\mathfrak{G}_\mathrm{i})\right]_{n\times m}$ for each alternative $\mathfrak{F}_l\ (l = 1, 2, 3, \ldots, \mathfrak{r})$ corresponding to their respective parameters $\mathfrak{S}_\jmath$ to get the aggregated result $\mathfrak{Q}_l = \left(\left(\underline{f}, \underline{\mathfrak{g}}, \underline{\mathfrak{h}}\right), \left(\overline{f}, \overline{\mathfrak{g}}, \overline{\mathfrak{h}}\right)\right)$.

$$\mathfrak{Q}_1 = \begin{pmatrix} (0.4112, 0.5641, 0.3238), \\ (0.4250, 0.3827, 0.4479) \end{pmatrix},$$
$$\mathfrak{Q}_2 = \begin{pmatrix} (0.4947, 0.3188, 0.3978), \\ (0.5410, 0.4655, 0.4591) \end{pmatrix},$$
$$\mathfrak{Q}_3 = \begin{pmatrix} (0.3935, 0.3599, 0.4388), \\ (0.4937, 0.3869, 0.3646) \end{pmatrix}$$

Step 4: Use Definition 10 to calculate the score values for each $\mathfrak{Q}_l$.

$$Sc(\mathfrak{Q}_1) = 0.4477, \quad Sc(\mathfrak{Q}_2) = 0.5337,$$

TABLE 7. ***SFS*$_{ft}$ matrix for alternative $\mathfrak{F}_1$.**

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
¢$_1$	{(0.3, 0.8, 0.5), (0.8, 0.4, 0.3)}	{(0.3, 0.9, 0.1), (0.2, 0.6, 0.7)}	{(0.6, 0.7, 0.3), (0.3, 0.3, 0.2)}	{(0.4, 0.8, 0.4), (0.1, 0.5, 0.6)}
¢$_2$	{(0.5, 0.4, 0.3), (0.5, 0.7, 0.3)}	{(0.2, 0.8, 0.5), (0.4, 0.4, 0.8)}	{(0.5, 0.4, 0.2), (0.4, 0.4, 0.3)}	{(0.4, 0.8, 0.4), (0.5, 0.4, 0.2)}
¢$_3$	{(0.2, 0.3, 0.4), (0.2, 0.4, 0.5)}	{(0.3, 0.8, 0.4), (0.6, 0.1, 0.6)}	{(0.5, 0.5, 0.2), (0.2, 0.2, 0.7)}	{(0.2, 0.6, 0.6), (0.6, 0.5, 0.3)}
¢$_4$	{(0.2, 0.5, 0.4), (0.4, 0.7, 0.5)}	{(0.3, 0.4, 0.3), (0.2, 0.5, 0.8)}	{(0.6, 0.3, 0.2), (0.4, 0.3, 0.5)}	{(0.4, 0.5, 0.4), (0.2, 0.6, 0.4)}

TABLE 8. ***SFS*$_{ft}$ matrix for alternative $\mathfrak{F}_2$.**

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
¢$_1$	{(0.2, 0.3, 0.5), (0.1, 0.6, 0.3)}	{(0.3, 0.3, 0.6), (0.7, 0.5, 0.3)}	{(0.5, 0.4, 0.6), (0.2, 0.5, 0.8)}	{(0.5, 0.8, 0.2), (0.4, 0.6, 0.3)}
¢$_2$	{(0.6, 0.3, 0.4), (0.7, 0.2, 0.3)}	{(0.5, 0.4, 0.5), (0.8, 0.3, 0.4)}	{(0.3, 0.1, 0.9), (0.5, 0.3, 0.7)}	{(0.6, 0.1, 0.6), (0.8, 0.4, 0.4)}
¢$_3$	{(0.5, 0.3, 0.2), (0.2, 0.4, 0.5)}	{(0.3, 0.5, 0.4), (0.2, 0.5, 0.4)}	{(0.5, 0.4, 0.2), (0.2, 0.5, 0.4)}	{(0.3, 0.4, 0.3), (0.8, 0.5, 0.2)}
¢$_4$	{(0.5, 0.4, 0.4), (0.5, 0.5, 0.3)}	{(0.1, 0.5, 0.7), (0.6, 0.5, 0.4)}	{(0.7, 0.4, 0.1), (0.1, 0.4, 0.8)}	{(0.7, 0.2, 0.4), (0.2, 0.4, 0.7)}

TABLE 9. ***SFS*$_{ft}$ matrix for the alternative $\mathfrak{F}_3$.**

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
¢$_1$	{(0.3, 0.1, 0.5), (0.2, 0.4, 0.3)}	{(0.3, 0.6, 0.5), (0.3, 0.4, 0.2)}	{(0.3, 0.7, 0.5), (0.4, 0.4, 0.6)}	{(0.3, 0.1, 0.4), (0.3, 0.3, 0.7)}
¢$_2$	{(0.3, 0.6, 0.4), (0.5, 0.2, 0.6)}	{(0.4, 0.6, 0.5), (0.6, 0.4, 0.3)}	{(0.2, 0.2, 0.5), (0.6, 0.3, 0.3)}	{(0.3, 0.5, 0.5), (0.4, 0.4, 0.3)}
¢$_3$	{(0.1, 0.3, 0.4), (0.4, 0.5, 0.6)}	{(0.1, 0.8, 0.4), (0.2, 0.4, 0.5)}	{(0.2, 0.4, 0.5), (0.3, 0.4, 0.3)}	{(0.3, 0.4, 0.4), (0.6, 0.4, 0.3)}
¢$_4$	{(0.6, 0.4, 0.3), (0.8, 0.4, 0.3)}	{(0.6, 0.4, 0.2), (0.4, 0.5, 0.3)}	{(0.6, 0.3, 0.5), (0.2, 0.5, 0.4)}	{(0.3, 0.2, 0.5), (0.7, 0.4, 0.3)}

$$Sc(\mathfrak{Q}_3) = 0.5188$$

Step 5: Rank the results for each alternative $\mathfrak{Q}_l\ (l = 1, 2, 3, \ldots, \mathfrak{r})$ i.e.,

$$Sc(\mathfrak{Q}_2) > Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_1)$$

Hence $\mathfrak{F}_2$ is the best firewall software.

2) By USING $SFS_{ft}ROWA$ OPERATOR

Step 1: Same as above

Step 2: Same as above

Step 3: Apply the devised $SFS_{ft}ROWA$ aggregation operators of each decision matrix $\mathcal{M} = \left[\mathfrak{R}_{e\mathfrak{S}_{\jmath}}(\mathfrak{G}_{i})\right]_{n\times m}$ for each alternative $\mathfrak{F}_l\ (l = 1, 2, 3, \ldots, \mathfrak{r})$ corresponding to their respective parameters $\mathfrak{S}_{\jmath}$ to get the aggregated result $\mathfrak{Q}_l = \left(\left(\underline{f}, \underline{\mathfrak{g}}, \underline{\mathfrak{h}}\right), \left(\overline{f}, \overline{\mathfrak{g}}, \overline{\mathfrak{h}}\right)\right)$.

$$\mathfrak{Q}_1 = \begin{pmatrix}(0.4164, 0.5621, 0.2999), \\ (0.3825, 0.3917, 0.4762)\end{pmatrix},$$

$$\mathfrak{Q}_2 = \begin{pmatrix}(0.5022, 0.3169, 0.4031), \\ (0.5275, 0.4322, 0.4600)\end{pmatrix},$$

$$\mathfrak{Q}_3 = \begin{pmatrix}(0.3925, 0.3638, 0.4359), \\ (0.4333, 0.39087, 0.3678)\end{pmatrix}$$

Step 4: Use Definition 10 to calculate the score values for each $\mathfrak{Q}_l$.

$$Sc(\mathfrak{Q}_1) = 0.4317, \quad Sc(\mathfrak{Q}_2) = 0.5412,$$
$$Sc(\mathfrak{Q}_3) = 0.4977$$

Step 5: Rank the results for each alternative $\mathfrak{Q}_l\ (l = 1, 2, 3, \ldots, \mathfrak{r})$ i.e,

$$Sc(\mathfrak{Q}_2) > Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_1)$$

Hence $\mathfrak{F}_2$ is the best firewall software.

3) BY USING $SFS_{ft}RHA$ OPERATOR

Step 1: Same as above

Step 2: Same as above

Step 3: Apply the devised $SFS_{ft}RHA$ aggregation operators of each decision matrix $\mathcal{M} = \left[\mathfrak{R}_{e}{}^{*}{}_{\nabla\mathfrak{S}_{\jmath}}(\mathfrak{G}_{i})\right]_{n\times m}$ for each alternative $\mathfrak{F}_l\ (l = 1, 2, 3, \ldots, \mathfrak{r})$ corresponding to their respective parameters $\mathfrak{S}_{\jmath}$ to get the aggregated result $\mathfrak{Q}_l = \left(\left(\underline{f^*_{\nabla}}, \underline{\mathfrak{g}^*_{\nabla}}, \underline{\mathfrak{h}^*_{\nabla}}\right), \left(\overline{f^*_{\nabla}}, \overline{\mathfrak{g}^*_{\nabla}}, \overline{\mathfrak{h}^*_{\nabla}}\right)\right)$ with $\wp = (0.28, 0.30, 0.17, 0.25)^{\check{Z}}, \not{b} = (0.32, 0.26, 0.18, 0.24,)^{\check{Z}}$ be the WVs of ¢$_i$ experts and $\mathfrak{S}_{\jmath}$ parameters. The results are given by

$$\mathfrak{Q}_1 = \begin{pmatrix}(0.3770, 0.5379, 0.3577), \\ (0.4701, 0.4143, 0.4418)\end{pmatrix},$$

$$\mathfrak{Q}_2 = \begin{pmatrix}(0.4479, 0.3871, 0.4758), \\ (0.5767, 0.4181, 0.4219)\end{pmatrix},$$

$$\mathfrak{Q}_3 = \begin{pmatrix}(0.3632, 0.4459, 0.4764), \\ (0.4855, 0.3926, 0.3686)\end{pmatrix}$$

Step 4: Calculate the score values for each $\mathfrak{Q}_l$.

$$Sc(\mathfrak{Q}_1) = 0.4454, \ Sc(\mathfrak{Q}_2) = 0.5202, \ Sc(\mathfrak{Q}_3) = 0.4711$$

Step 5: Rank the results for each alternative $\mathfrak{Q}_l\ (l = 1, 2, 3, \ldots, \mathfrak{r})$ i.e.

$$Sc(\mathfrak{Q}_2) > Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_1)$$

Hence $\mathfrak{F}_2$ is the best firewall software.

VI. COMPARATIVE ANALYSIS

In this section, we have to present the cooperative analysis of the exposed work by comparing the presented work with some other existing literature to show the usefulness and superiority of the proposed work. We compare our work with picture fuzzy soft rough weighted average $(PFS_{ft}RWA)$, $PFS_{ft}ROWA$, $PFS_{ft}RHA$, Wang *et al.* [41] method, Hussain *et al.* [40] method, and Zhang *et al.* [39] method. The overall discussion is given below.

Example 6: In our daily life every person travels from one place to another place. Suppose a person X wants to travel through the best airline company to his desired destination. He has a set of alternatives of four best airlines given as $\varrho = \{\mathfrak{F}_1 = Air\ Canada, \mathfrak{F}_2 = United\ Airline, \mathfrak{F}_3 = Emirates, \mathfrak{F}_4 = American\ Airline\}$ corresponding to parameter set $\mathfrak{S} = \{\mathfrak{S}_1 = Customer\ servise, \mathfrak{S}_2 = Great\ price\ with\ great\ deal, \mathfrak{S}_3 = Inflight\ Meal, \mathfrak{S}_4 = Entertainment\}$. Let $w = \{0.26, 0.21, 0.29, 0.24\}$ denote the WVs of $¢_i$ $(i = 1, 2, 3, 4)$ experts and $\hbar = \{0.30, 0.23, 021, 0.26\}$ represent the WVs of parameters $\mathfrak{S}_j$. Suppose these experts provide their assessment data for the alternatives $\mathfrak{F}_l$ $(l = 1, 2, 3, 4)$ corresponding to parameters $\mathfrak{S}_j$ $(j = 1, 2, 3, 4)$ in the form of $PFS_{ft}RNs$ as given in Table 10. The overall results are also given in Table 11.

TABLE 10. $PFS_{ft}R$ **information.**

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
$\mathfrak{Q}_1$	$\left\{(0.23, 0.21, 0.15), (0.12, 0.14, 0.12)\right\}$	$\left\{(0.33, 0.16, 0.11), (0.43, 0.24, 0.21)\right\}$	$\left\{(0.13, 0.17, 0.25), (0.24, 0.14, 0.26)\right\}$	$\left\{(0.13, 0.11, 0.14), (0.13, 0.13, 0.27)\right\}$
$\mathfrak{Q}_2$	$\left\{(0.13, 0.11, 0.25), (0.15, 0.32, 0.13)\right\}$	$\left\{(0.45, 0.14, 0.14), (0.26, 0.24, 0.14)\right\}$	$\left\{(0.23, 0.21, 0.45), (0.26, 0.13, 0.23)\right\}$	$\left\{(0.32, 0.15, 0.25), (0.41, 0.14, 0.23)\right\}$
$\mathfrak{Q}_3$	$\left\{(0.31, 0.13, 0.22), (0.21, 0.12, 0.20)\right\}$	$\left\{(0.41, 0.28, 0.14), (0.12, 0.34, 0.45)\right\}$	$\left\{(0.21, 0.24, 0.25), (0.23, 0.14, 0.13)\right\}$	$\left\{(0.23, 0.14, 0.42), (0.26, 0.24, 0.13)\right\}$
$\mathfrak{Q}_4$	$\left\{(0.11, 0.12, 0.31), (0.21, 0.14, 0.23)\right\}$	$\left\{(0.16, 0.24, 0.32), (0.14, 0.25, 0.43)\right\}$	$\left\{(0.16, 0.13, 0.15), (0.12, 0.15, 0.14)\right\}$	$\left\{(0.13, 0.12, 0.15), (0.17, 0.14, 0.23)\right\}$

TABLE 11. **Results for data given in Table 10.**

Methods	Score results	Ranking results
$PFS_{ft}RWA$	$Sc(\mathfrak{Q}_1) = 0.0234$, $Sc(\mathfrak{Q}_2) = 0.1241$, $Sc(\mathfrak{Q}_3) = 0.1310$, $Sc(\mathfrak{Q}_4) = 0.0344$	$Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_2) > Sc(\mathfrak{Q}_1) > Sc(\mathfrak{Q}_4)$
$PFS_{ft}ROWA$	$Sc(\mathfrak{Q}_1) = 0.0520$, $Sc(\mathfrak{Q}_2) = 0.0672$, $Sc(\mathfrak{Q}_3) = 0.0885$, $Sc(\mathfrak{Q}_4) = 0.03327$	$Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_2) > Sc(\mathfrak{Q}_1) > Sc(\mathfrak{Q}_4)$
$PFS_{ft}RHA$	$Sc(\mathfrak{Q}_1 = 0.05946$, $Sc(\mathfrak{Q}_2) = 0.0704$, $Sc(\mathfrak{Q}_3) = 0.0933$, $Sc(\mathfrak{Q}_4) = -0.1125$	$Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_2) > Sc(\mathfrak{Q}_1) > Sc(\mathfrak{Q}_4)$
Wang et al. [41] method	Cannot be calculated	Cannot be calculated
Hussain et al. [40] method	Cannot be calculated	Cannot be calculated
Zhang et al. [39] method	Cannot be calculated	Cannot be calculated
$SFS_{ft}RWA$ (Proposed work)	$Sc(\mathfrak{Q}_1) = 0.0372$, $Sc(\mathfrak{Q}_2) = 0.1319$, $Sc(\mathfrak{Q}_3) = 0.1382$, $Sc(\mathfrak{Q}_4) = 0.02064$	$Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_2) > Sc(\mathfrak{Q}_1) > Sc(\mathfrak{Q}_4)$
$SFS_{ft}ROWA$ (Proposed work)	$Sc(\mathfrak{Q}_1) = 0.0589$, $Sc(\mathfrak{Q}_2) = 0.0736$, $Sc(\mathfrak{Q}_3) = 0.0906$, $Sc(\mathfrak{Q}_4) = 0.0361$	$Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_2) > Sc(\mathfrak{Q}_1) > Sc(\mathfrak{Q}_4)$
$SFS_{ft}RHWA$ (Proposed work)	$Sc(\mathfrak{Q}_1) = 0.0676$, $Sc(\mathfrak{Q}_2) = 0.0778$, $Sc(\mathfrak{Q}_3) = 0.0979$, $Sc(\mathfrak{Q}_4) = -0.1078$	$Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_2) > Sc(\mathfrak{Q}_1) > Sc(\mathfrak{Q}_4)$

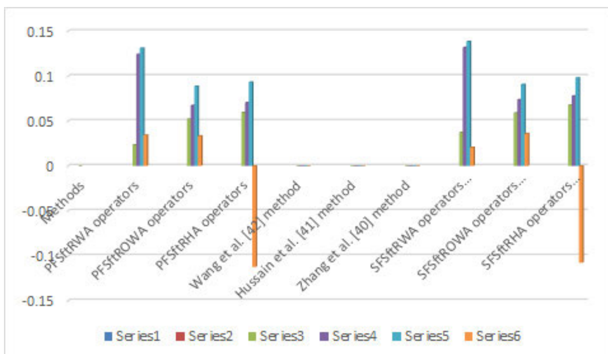

FIGURE 2. **Pictorial presentation of data given in Table 11.**

From all the above calculations, we note that Wang *et al.* [41] method, Hussain *et al.* [40] method and Zhang *et al.* [39] method consists of q-ROF soft rough numbers, PyF soft rough numbers and IF soft rough numbers respectively in which AG cannot be considered while the data given in Table 10. Consist of picture fuzzy soft rough numbers containing AG, so Wang *et al.* [41] method, Hussain *et al.* [39] method, and Zhang *et al.* [40] method cannot tackle this information. But the existing methods along with $PFS_{ft}RWA$, $PFS_{ft}ROWA$ *and* $PFS_{ft}RHA$ can consider this information, because:

If we replace power 2 by 1 in the basic definition of presented operators, then presented operators can be reduced to $PFS_{ft}RWA$, $PFS_{ft}ROWA$ *and* $PFS_{ft}RHA$. The overall results are given in Table 11.

When experts provide their information in the form of $PFS_{ft}RNs$, then this data can be handled by the presented operators because established operators are more general.

TABLE 12. $SFS_{ft}R$ **information.**

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
$\mathfrak{Q}_1$	$\{(0.5, 0.3, 0.2), (0.2, 0.4, 0.6)\}$	$\{(0.2, 0.5, 0.3), (0.4, 0.2, 0.7)\}$	$\{(0.3, 0.7, 0.2), (0.2, 0.1, 0.2)\}$	$\{(0.2, 0.4, 0.4), (0.3, 0.3, 0.3)\}$
$\mathfrak{Q}_2$	$\{(0.3, 0.3, 0.6), (0.3, 0.3, 0.4)\}$	$\{(0.5, 0.3, 0.3), (0.2, 0.2, 0.4)\}$	$\{(0.4, 0.2, 0.1), (0.3, 0.1, 0.3)\}$	$\{(0.5, 0.5, 0.3), (0.3, 0.5, 0.3)\}$
$\mathfrak{Q}_3$	$\{(0.1, 0.5, 0.2), (0.4, 0.4, 0.3)\}$	$\{(0.4, 0.8, 0.1), (0.2, 0.3, 0.6)\}$	$\{(0.2, 0.2, 0.2), (0.2, 0.4, 0.6)\}$	$\{(0.1, 0.4, 0.7), (0.5, 0.6, 0.1)\}$
$\mathfrak{Q}_4$	$\{(0.2, 0.2, 0.3), (0.3, 0.1, 0.6)\}$	$\{(0.6, 0.1, 0.1), (0.4, 0.5, 0.1)\}$	$\{(0.6, 0.3, 0.2), (0.1, 0.5, 0.5)\}$	$\{(0.4, 0.5, 0.3), (0.3, 0.1, 0.1)\}$

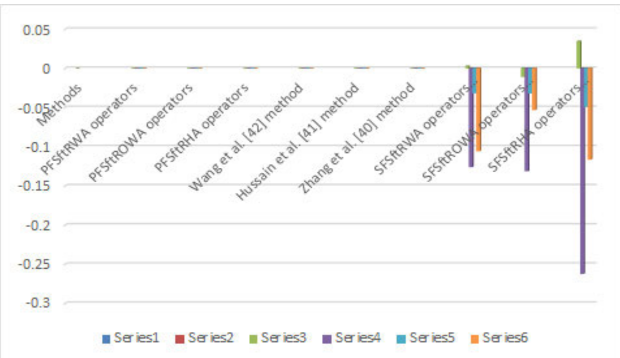

FIGURE 3. **Pictorial presentation of data given in Table 13.**

TABLE 13. **Results for data given in Table 13.**

Methods	Score results	Ranking results
$\boldsymbol{PFS_{ft}RWA}$	Cannot be calculated	Cannot be calculated
$\boldsymbol{PFS_{ft}ROWA}$	Cannot be calculated	Cannot be calculated
$\boldsymbol{PFS_{ft}RHA}$	Cannot be calculated	Cannot be calculated
Wang et al. [41] method	Cannot be calculated	Cannot be calculated
Hussain et al. [40] method	Cannot be calculated	Cannot be calculated
Zhang et al. [39] method	Cannot be calculated	Cannot be calculated
$\boldsymbol{SFS_{ft}RWA}$ (Proposed work)	$Sc(\mathfrak{Q}_1) = 0.0027$, $Sc(\mathfrak{Q}_2) = -0.1260$, $Sc(\mathfrak{Q}_3) = -0.0317$, $Sc(\mathfrak{Q}_4) = -0.1057$	$Sc(\mathfrak{Q}_1) > Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_4) > Sc(\mathfrak{Q}_2)$
$\boldsymbol{SFS_{ft}ROWA}$ (Proposed work)	$Sc(\mathfrak{Q}_1) = -0.0108$, $Sc(\mathfrak{Q}_2) = -0.1311$, $Sc(\mathfrak{Q}_3) = -0.0321$, $Sc(\mathfrak{Q}_4) = -0.0532$	$Sc(\mathfrak{Q}_1) > Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_4) > Sc(\mathfrak{Q}_2)$
$\boldsymbol{SFS_{ft}HWA}$ (Proposed work)	$Sc(\mathfrak{Q}_1) = 0.0346$, $Sc(\mathfrak{Q}_2) = -0.2624$, $Sc(\mathfrak{Q}_3) = -0.0492$, $Sc(\mathfrak{Q}_4) = -0.1165$	$Sc(\mathfrak{Q}_1) > Sc(\mathfrak{Q}_3) > Sc(\mathfrak{Q}_4) > Sc(\mathfrak{Q}_2)$

Also, note that the best alternative in all cases is the same, i.e., $\mathfrak{F}_3$ is the best alternative that shows the authenticity of the introduced work. Moreover the pictorial presentation of the data given in Table 11 and in Figure 2.

We can note that if the experts provide their assessment information in the form of $SFS_{ft}RNs$ as given in Table 12, then all the existing literature cannot handle this kind of information while initiated work can do so because existing literature can only deal with MG and NMG while the data given in Table 12 consists of MG, NMG, and AG as well. So the proposed work is more superior to all given literature. The overall results are given in Table 13. Also, note that when decision-makers provide $\left\{\begin{matrix}(0.5,\ 0.5,\ 0.3),\\ (0.3,\ 0.5,\ 0.3)\end{matrix}\right\}$ as given in Table 13, then this data is not a $PFS_{ft}R$ data. Due to this reason this data cannot be handled by $PFS_{ft}RWA$, $PFS_{ft}ROWA$ and $PFS_{ft}RHA$ aggregation operators and only the presented work can deal with this information. Moreover, note that the presented work fills up the flaws of the existing literature and provides more space to decision-makers to make their decisions in decision-making problems. Furthermore, a pictorial presentation of Table 13 results is given in Figure 3.

VII. CONCLUSION

In this paper, we have initiated a hybrid structure called spherical fuzzy soft rough set that is the combination of spherical fuzzy set, soft set, and rough set. Moreover, some new aggregation operators like spherical fuzzy soft rough weighted average, spherical fuzzy soft rough ordered weighted average, and spherical fuzzy soft rough hybrid average aggregation operators are developed and their properties are elaborated. An algorithm along with an illustrative example is given to prove the validity of the developed work. Also, a comparative study of the established work is given to show the advantages of defined operators.

Note that the presented work is also limited notion because when decision-makers present information in the form of T-spherical fuzzy soft rough set by lower and upper approximation operators like $\{(0.5,\ 0.6,\ 0.8),\ (0.6,\ 0.7,\ 0, 8)\}$, then necessary condition $0 \leq \left(\underline{f_{\jmath}}(\text{¢}_i)\right)^2 + \left(\underline{g_{\jmath}}(\text{¢}_i)\right)^2 + \left(\underline{h_{\jmath}}(\text{¢}_i)\right)^2 \leq 1$ and $0 \leq \left(\overline{f_{\jmath}}(\text{¢}_i)\right)^2 + \left(\overline{g_{\jmath}}(\text{¢}_i)\right)^2 + \left(\overline{h_{\jmath}}(\text{¢}_i)\right)^2 \leq 1$ fail to tackle such sort of information, because $(0.5)^2 + (0.6)^2 + (0.8)^2 \notin [0,\ 1]$ and $(0.6)^2 + (0.7)^2 + (0.8)^2 \notin [0,\ 1]$ while the necessary condition for T-spherical fuzzy soft rough set is more general that use the condition $0 \leq \left(\underline{f_{\jmath}}(\text{¢}_i)\right)^q +$

$\left(\underline{\mathfrak{g}_{\jmath}}(¢_{\mathrm{i}})\right)^{q}+\left(\underline{\mathfrak{h}_{\jmath}}(¢_{\mathrm{i}})\right)^{q}\leq 1$ and $0\leq\left(\overline{f_{\jmath}}(¢_{\mathrm{i}})\right)^{q}+\left(\overline{\mathfrak{g}_{\jmath}}(¢_{\mathrm{i}})\right)^{q}+\left(\overline{\mathfrak{h}_{\jmath}}(¢_{\mathrm{i}})\right)^{q}\leq 1$ for $q\geq 1$.

In the future directions, using the proposed definition and operational laws, some methods can be developed as given in [43]. Moreover, this work can be extended to T-spherical fuzzy set, and real-life problems can be resolved given in [44]–[46]. Furthermore, some new notions given in [47] can be developed based on established work. Also, some new methods like TODIM and VIKOR methods can be defined for the proposed spherical fuzzy soft rough set as given in [48], [49].

ACKNOWLEDGMENT

DATA AVAILABILITY

The data used in this manuscript is hypothetical and anyone can use it without prior permission of the authors by just citing this article.

AUTHORSHIP CONTRIBUTION

(Leina Zheng, Tahir Mahmood, Jabbar Ahmmad, Ubaid ur Rehman, and Shouzhen Zeng contributed equally in this work.)

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