Spherical harmonic representation of the magnetic field in the presence of a current density

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Summary. Spherical harmonic expansions are derived to represent arbitrary current densities and magnetic fields in a spherical region. The expansions can be viewed as a generalization of the spherical-harmonic, scalar-potential theory to include regions where spatially distributed current densities exist. The new expansions are found in terms of the eigenfunctions of the curl curl operator. It is found that four basic field forms may be present; one of these is the well-known scalar potential expansion originated by Gauss. Electric and magnetic field boundary conditions appropriate to a conducting spherical shell including the case of anisotropic conductivity are presented. They are discussed in terms of their ability to couple the four field forms. It is pointed out that the four field forms can be expressed in five other orthogonal curvilinear coordinate systems.

1 Introduction

Mathematical analysis of coexisting magnetic fields and current densities is of interest to investigators in several technical disciplines. Some of these are: the electromagnetism of the Earth's interior (Rikitake 1966), electromagnetism as used in geophysical exploration (Wait 1979), studies of the nuclear electromagnetic pulse (Vance 1978), studies of astrophysical field and studies of the magnetosphere (Stern 1979; Potemra 1979, 1980; Boström 1964; Sugiura 1975; Kisabeth & Rostoker 1977; Kisabeth 1979).

In a source-free (current density equals zero) region, a static magnetic field can be expressed as a gradient of a scalar potential. The scalar potential (solution of Laplace's equation) can be expressed as a sum of orthogonal harmonic functions in many curvilinear orthogonal coordinate systems; these are developed in various electromagnetic treatises. Not as well known is the fact that orthogonal vector functions exist in which magnetic fields in the presence of current densities can be expanded.

Such functions are developed in this paper in terms of the eigenfunctions of the curl curl operator. These eigenfunctions can be expressed in curvilinear vector components in a total of six orthogonal curvilinear coordinate systems (Morse & Feshbach 1953, section 13.1).

The six appropriate coordinate systems are rectangular, circular cylinder, elliptic cylinder, parabolic cylinder, spherical and conical coordinate systems (see Morse & Feshbach 1953, pp. 655–659, for coordinate system definitions).

The vector functions are developed in this paper in spherical coordinates; the vector spherical harmonic as defined by Jackson (1975), is used in the derivation. Inclusion of the vector functions in the representation of a magnetic field can be regarded as a generalization of the scalar potential theory to include regions where a current density exists.

Scalar spherical harmonic analysis has been used extensively to model the geomagnetic field; this method was originated by Gauss (his field representation is defined as field FORM I in this paper). It is currently used to analyse the main field, the secular variation, and the diurnal variations S_q and L (Chapman & Bartels 1962; Cain *et al.* 1967; Fougere 1969). The generalization involves deriving three additional spherical harmonic expansions. The first expansion extends the magnetic field representation to include fields in regions where $\mathbf{j} \neq \mathbf{0}$, but $\nabla \cdot \mathbf{j} = 0$ and $\nabla \times \mathbf{j} = \mathbf{0}$ (defined as FORM II). The second and third expansions extend the representation to regions where $\nabla \cdot \mathbf{j} = 0$, but $\nabla \times \mathbf{j} \neq \mathbf{0}$; one such expansion is for transverse magnetic (TM) fields (FORM IV), the other is for transverse current density (TJ) fields (FORM III, analogous to transverse electric fields).

The three new expansions are reduced to their real-variable form suitable for numerical fitting procedures. The form is similar to that of the scalar potential expansions extensively used in geomagnetic field analysis. The author anticipates using the new expansions in an effort to model the field-aligned current above the ionosphere in the Earth's polar regions. Primarily because of this possible application, a section is included discussing the mathematical treatment of an anisotropic conducting spherical shell. Such a shell could be used to represent the ionosphere. Another application could be found in studying the shielding of a sphere constructed of an anistropic composite material in an ionized-gas or other conducting environment. The effect of the shell is to couple fields of the various FORMs. The permeability μ is assumed to be that of free space throughout the paper.

2 The magnetic field in the absence of currents

For the sake of completeness, the expansion for the magnetic field in a source-free ($\mathbf{j} = \mathbf{0}$) region is given. Such a magnetic field can be expressed as the gradient of a scalar potential V, where the scalar potential satisfies Laplace's equation ($\nabla^2 V = 0$). If the source currents are contained in a sphere interior to the domain of the field, the potential is given (in complex form) by

$$V^i = \sum_{n,m} a_{nm}^i \, r^{-(n+1)} \, Y_{nm} (\theta, \phi) \tag{1}$$

where

$$Y_{nm} (\theta, \phi) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} \, P_n^m (\cos \theta) \exp (im\phi)$$

is the spherical harmonic function (Jackson 1975, p. 99), (r, θ, ϕ) are spherical coordinates, a_{nm}^i are complex constants, and superscript i means sources interior to the domain of the field. V^i is a singular at $r = 0$ and finite as $r \to \infty$.

If the sources are exterior to the domain of the field, the potential is

$$V^e = \sum_{n,m} a_{nm}^e \, r^n \, Y_{nm} (\theta, \phi) \tag{2}$$

where superscript e means exterior sources. V^e is analytic at $r = 0$, but diverges as $r \to \infty$. The sum $(V^i + V^e)$ is the general solution of Laplace's equation in spherical polar coordinates. Equation (1) is often written in real form (e.g. for geomagnetic analysis) as

$$V^i = a \sum_{n=1}^{N} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} (g_n^m \cos m\phi + h_n^m \sin m\phi) \, P_n^m (\cos \theta) \tag{3}$$

where

a = a constant radius,

$P_n^m (\cos \theta)$ = associated Legendre's function,

g_n^m and h_n^m are constants.

The field is given by

$$\mathbf{H} = -\nabla V \tag{4}$$

for either $V = V^e$ or $V = V^i$.

If the currents yielding the potential of equation (3) are in a spherical surface of radius R, the potential inside the sphere due to the same current distribution is given by

$$V^e = -a \sum_{n=1}^{N} \frac{n+1}{n} \left(\frac{a}{r}\right)^{2n+1} \left(\frac{r}{a}\right)^{n} \sum_{m=0}^{n} (g_n^m \cos m\phi + h_n^m \sin m\phi) \, P_n^m (\cos \theta) \tag{5}$$

so that the radial component of the magnetic field is continuous across the surface, i.e.

$$\left.\frac{\partial V^e}{\partial r}\right|_{r=R} = \left.\frac{\partial V^i}{\partial r}\right|_{r=R}. \tag{6}$$

In practice, a spherical shell with thickness much less than R is often approximated by such a spherical surface.

The surface current associated with the fields of equations (3) and (5) is given by (assuming SI units)

$$\mathbf{j_s} = \hat{\mathbf{r}} \times (\mathbf{H}^e - \mathbf{H}^i) \tag{7}$$

where $\mathbf{H}^e$ and $\mathbf{H}^i$ are just exterior to and interior to the spherical surface, respectively, and $\hat{\mathbf{r}}$ is the unit vector $\mathbf{r}/|\mathbf{r}|$. Substituting into (7) using (3) and (5) yields the components

$$j_{s\theta} = \sum_{n=1}^{N} \frac{2n+1}{n} \left(\frac{a}{R}\right)^{n+2} \sum_{m=1}^{n} m \, (h_n^m \cos m\phi - g_n^m \sin m\phi) \frac{P_n^m (\cos \theta)}{\sin \theta} \tag{8}$$

and

$$j_{s\phi} = -\sum_{n=1}^{N} \frac{2n+1}{n} \left(\frac{a}{R}\right)^{n+2} \sum_{m=0}^{n} (g_n^m \cos m\phi + h_n^m \sin m\phi) \frac{dP_n^m (\cos \theta)}{d\theta}. \tag{9}$$

The surface current given by (8) and (9) simply circulates on the spherical surface — no current flow into or out of the surface is implied by this current; mathematically, this is expressed by the equation

$$\nabla_s \cdot \mathbf{j_s} = 0 \tag{10}$$

where

$$\nabla_s \cdot \mathbf{j}_s = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (j_{s\theta} \sin \theta) + \frac{\partial j_{s\phi}}{\partial \phi} \right].$$

The magnetic field implied by the form of the potentials given by (3) and (5) and the surface current given by the form of (8) and (9) will be referred to as fields and surface currents of FORM I. For the fields, FORMS I^e or I^i will designate exterior or interior type, respectively. Obviously, the Earth's main field is of FORM I^i. Near the Earth's surface, $\mathbf{j} = \mathbf{0}$, and all geomagnetic field components can be represented as a sum of fields of FORMS I^e and I^i if proper time variations are added to the representations. In the following section, expansions will be derived to represent a magnetic field in the presence of a current density. These can be employed in and above the ionosphere, in the interior of the Earth, or in other current-carrying media.

3 Magnetic fields in the presence of currents

In the previous section, magnetic fields that obey $\nabla \times \mathbf{H} = \mathbf{0}$ were discussed and designated as FORM I. In this section, expansions are derived to represent fields such that

$$\nabla \times \mathbf{H} = \mathbf{j}, \quad \mathbf{j} \neq \mathbf{0}. \tag{11}$$

If $\nabla \times \mathbf{j} = \mathbf{0}$, $\mathbf{j}$ is irrotational and the resulting current density-field structure will be designated FORM II. For this case, $\mathbf{H}$ is found to be transverse; i.e. $H_r = 0$. If $\nabla \times \mathbf{j} \neq \mathbf{0}$, two current density-field structures exist. One of these has a transverse current density (TJ) (analogous to transverse electric modes) and will be designated FORM III. The other has a transverse magnetic field (TM) and will be designated FORM IV. FORM I is a degenerate case of FORM III and FORM II is a degenerate case of FORM IV. (This is discussed below equation 31.)

The magnetic field satisfying (11) for a given current density $\mathbf{j}$ is not unique unless sufficient boundary conditions are applied to $\mathbf{H}$. That is, if $\mathbf{H}_p$ is a particular solution of (11), then $(\mathbf{H}_p + \mathbf{H}^I)$ is also a solution (where $\mathbf{H}^I$ is any magnetic field of FORM I). If $\mathbf{j}$ is specified and a solution for $\mathbf{H}$ is desired, one must find any particular solution $\mathbf{H}_p$ that satisfies (11). Then $\mathbf{H}^I$ must be chosen such that $(\mathbf{H}_p + \mathbf{H}^I)$ satisfies the desired boundary conditions. The types of boundary conditions that may be applied are, therefore, identical to those that may be applied to FORM I fields. These are discussed in many standard texts and will not be discussed here. In the remainder of this section, particular solutions will be found that satisfy (11). In Section 4, it will be shown that these solutions are complete for the representation of non-divergent current densities.

3.1 MODES WITH IRROTATIONAL CURRENT DENSITY (FORM II)

The equations satisfied by $\mathbf{j}$ that define FORM II fields, $\nabla \times \mathbf{j} = \mathbf{0}$ and $\nabla \cdot \mathbf{j} = \mathbf{0}$ are identical to the equations satisfied by $\mathbf{H}^I$. Therefore, the general form of the current density $\mathbf{j}$ for FORM II is identical to that of $\mathbf{H}$ for FORM I. That is, it is the gradient of a scalar that satisfies Laplace's equation; in spherical coordinates the scalar can be written as the internal type (equations 1 or 3) or as the external type (equations 2 or 5) or as a linear combination of both. To show that a magnetic field $\mathbf{H}$ is a particular solution of equation (11) for a general FORM II current density, one need merely show that its curl yields the general form of $\mathbf{j}$. The FORM II magnetic fields given below are superscripted i or e to indicate that their curl yields a current density that is the gradient of the form of V^i or V^e, respectively. These

forms for $\mathbf{H}^i$ and $\mathbf{H}^e$ were derived by taking limits as described below (equation 31). However, because they are non-unique, particular solutions, there derivation is of little importance.

The FORM II magnetic fields are given in complex form by

$$\mathbf{H}^i = \sum_{n,m} a_{nm}^i r^{-(n+1)} \mathbf{X}_{nm}(\theta, \phi) \tag{12}$$

and

$$\mathbf{H}^e = \sum_{n,m} a_{nm}^e r^n \mathbf{X}_{nm}(\theta, \phi) \tag{13}$$

where $\mathbf{X}_{nm}(\theta, \phi)$ is the vector spherical harmonic defined by Jackson (1975, section 16.2), as

$$\mathbf{X}_{nm}(\theta, \phi) \equiv \frac{1}{\sqrt{n(n+1)}} \mathbf{L} Y_{nm}(\theta, \phi) \tag{14}$$

$$\mathbf{L} \equiv \frac{1}{i}(\mathbf{r} \times \nabla), \quad i = \sqrt{-1}. \tag{15}$$

In real-variable component form, (12) can be written

$$H_r^i = 0 \tag{16a}$$

$$H_\theta^i = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^{n+1} \sum_{m=1}^{n} m(\alpha_n^m \sin m\phi - \beta_n^m \cos m\phi) \frac{P_n^m(\cos \theta)}{\sin \theta} \tag{16b}$$

$$H_\phi^i = \sum_{n=1}^{N} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} (\alpha_n^m \cos m\phi + \beta_n^m \sin m\phi) \frac{dP_n^m(\cos \theta)}{d\theta}, \tag{16c}$$

The real-variable component form of (13) is

$$H_r^e = 0 \tag{17a}$$

$$H_\theta^e = \sum_{n=1}^{N} \left(\frac{r}{a}\right)^{n} \sum_{m=1}^{n} m(\gamma_n^m \sin m\phi - \delta_n^m \cos m\phi) \frac{P_n^m(\cos \theta)}{\sin \theta} \tag{17b}$$

$$H_\phi^e = \sum_{n=1}^{N} \left(\frac{r}{a}\right)^{n} \sum_{m=0}^{n} (\gamma_n^m \cos m\phi + \delta_n^m \sin m\phi) \frac{dP_n^m(\cos \theta)}{d\theta}. \tag{17c}$$

By direct application of the curl operator to (16), the current density corresponding to the field $\mathbf{H}^i$ can be written

$$j_r^i = \frac{-1}{a} \sum_{n=1}^{N} n(n+1) \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} (\alpha_n^m \cos m\phi + \beta_n^m \sin m\phi) P_n^m(\cos \theta) \tag{18a}$$

$$j_\theta^i = \frac{1}{a} \sum_{n=1}^{N} n \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} (\alpha_n^m \cos m\phi + \beta_n^m \sin m\phi) \frac{dP_n^m(\cos \theta)}{d\theta} \tag{18b}$$

$$j_\phi^i = \frac{-1}{a} \sum_{n=1}^{N} n \left(\frac{a}{r}\right)^{n+2} \sum_{m=1}^{n} m(\alpha_n^m \sin m\phi - \beta_n^m \cos m\phi) \frac{P_n^m(\cos \theta)}{\sin \theta}. \tag{18c}$$

The curl of (17) is

$$j_r^e = \frac{-1}{a} \sum_{n=1}^{N} n(n+1) \left(\frac{r}{a}\right)^{n-1} \sum_{m=0}^{n} (\gamma_n^m \cos m\phi + \delta_n^m \sin m\phi) P_n^m (\cos \theta) \tag{19a}$$

$$j_\theta^e = \frac{-1}{a} \sum_{n=1}^{N} (n+1) \left(\frac{r}{a}\right)^{n-1} \sum_{m=0}^{n} (\gamma_n^m \cos m\phi + \delta_n^m \sin m\phi) \frac{dP_n^m (\cos \theta)}{d\theta} \tag{19b}$$

$$j_\phi^e = \frac{1}{a} \sum_{n=1}^{N} (n+1) \left(\frac{r}{a}\right)^{n-1} \sum_{m=1}^{n} m(\gamma_n^m \sin m\phi - \delta_n^m \cos m\phi) \frac{P_n^m (\cos \theta)}{\sin \theta}. \tag{19c}$$

The fact that $\mathbf{j}^e$ and $\mathbf{j}^i$ are irrotational is shown by simply observing that $\mathbf{j}^e$ is the gradient of a scalar of the form of V^e (equation 5) and $\mathbf{j}^i$ is the gradient of a scalar of the form of V^i (equation 3).

This mathematical form for $\mathbf{j}^e$ and $\mathbf{j}^i$ gives one a great deal of physical insight into the structure of FORM II currents and fields. That is, current $\mathbf{j}^i$ circulates out from a region near the origin of the coordinate system ($r = 0$) and returns to a region near the origin. It has the same mathematical form as the main geomagnetic field. The current $\mathbf{j}^e$ originates from a region far from the origin and circulates back to a distant region. It has the mathematical form of magnetic fields (in free space) interior to their source distribution.

Suppose FORM II currents are assumed to exist exterior to a sphere (of radius R) and terminate on it (e.g. the sphere may be a model of the ionosphere). No current is assumed to flow on to the spherical surface from inside the sphere. The resulting surface current distribution in the spherical surface is

$$\mathbf{j}_s = \mathbf{j}_s^e + \mathbf{j}_s^i \tag{20}$$

where

$$\mathbf{j}_s^e = \hat{\mathbf{r}} \times \mathbf{H}_t^e = (-H_\phi^e \hat{\theta} + H_\theta^e \hat{\phi})_{r=R} \tag{21}$$

$$\mathbf{j}_e^i = \hat{\mathbf{r}} \times \mathbf{H}_t^i = (-H_\phi^i \hat{\theta} + H_\theta^i \hat{\phi})_{r=R}. \tag{22}$$

$\hat{\theta}$ and $\hat{\phi}$ are unit vectors in their respective spherical-polar coordinate directions. Note that, for either surface current

$$\nabla \cdot \mathbf{j}_s = \nabla \cdot (\hat{\mathbf{r}} \times \mathbf{H}_t)$$

$$= -\hat{\mathbf{r}} \cdot \nabla \times \mathbf{H}_t$$

$$\nabla_s \cdot \mathbf{j}_s = -j_r. \tag{23}$$

That is, the surface divergence of the surface current is minus the current density flowing from the surface, a statement of continuity of current. Further discussion of conducting spherical surfaces is in a separate section.

3.2 MODES ADMITTING ROTATIONAL CURRENT DENSITIES (FORMS III AND IV)

For fields of FORMS I and II, $\nabla \times \mathbf{j} = \mathbf{0}$, hence, $\nabla \times \nabla \times \mathbf{H} = \mathbf{0}$. If

$$\nabla \times \nabla \times \mathbf{H} = \mathbf{G} \neq \mathbf{0} \tag{24}$$

then the vectors $\mathbf{H}$ and $\mathbf{G}$ can be expanded in series of the eigenvectors of the operator curl curl. Strictly speaking, the magnetic field terms of FORMS I and II are eigenvectors of this operator with eigenvalue zero. FORMS III and IV are defined as field expansions in eigen-

vectors of curl curl with eigenvalues greater than zero. As one might expect, if the domain of the variable r extends to infinity, the spectrum of interest is continuous in the eigenvalues and the expansion has the form of an integral. If the domain is finite in r, say $r_1 < r < r_2$, the spectrum of the expansion is discrete, but contains an infinite number of eigenvalues. The discrete form, suitable for numerical fitting, is developed here; sums over eigenvalues can be changed to integrals if that form is desired.

It should be noted at the onset that expansions of fields of FORMS III and IV involve three indices (rather than two): n, m and the eigenvalue index. For numerical analysis, the added index could result in a much greater requirement for computer storage and computation time (than two-index expansions), unless careful thought is given to the minimum required set of terms.

The eigenvectors and eigenvalues of the operator curl curl, expressed in spherical coordinates, are easily deduced from the work of Jackson (1975, section 16.2). Let

$$\mathbf{F}_{nm\lambda} = f_n (r\sqrt{\lambda}) \, \mathbf{X}_{nm} (\theta, \phi) \tag{25}$$

where f_n is any spherical Bessel function. Then, $\mathbf{F}_{nm\lambda}$ satisfies

$$\nabla \times \nabla \times \mathbf{F}_{nm\lambda} = \lambda \mathbf{F}_{nm\lambda}. \tag{26}$$

This is shown in the following way. Jackson has shown that $\mathbf{B} = f_n (kr)\mathbf{X}_{nm}$ (with $\mathbf{E} = i\nabla \times \mathbf{B}/k$) is a solution of Maxwell's equations in the form $\nabla \times \mathbf{E} = ik\mathbf{B}$ and $\nabla \times \mathbf{B} = -ik\mathbf{E}$ (see Jackson's equations 16.31 and 16.46). Combining the two curl equations, one obtains $\nabla \times \nabla \times \mathbf{B} = k^2 \mathbf{B}$. Setting λ equal to k^2 and $\mathbf{F}_{nm\lambda} = \mathbf{B} = f_n (r\sqrt{\lambda}) \mathbf{X}_{nm}$ yields (26).

By taking the curl of both sides of (26), it becomes obvious that the function $\nabla \times \mathbf{F}_{nm\lambda}$ satisfies the same equation, that is

$$\nabla \times \nabla \times (\nabla \times \mathbf{F}_{nm\lambda}) = \lambda \nabla \times \mathbf{F}_{nm\lambda}. \tag{27}$$

Hence, there are two linearly independent (and orthogonal) sets of eigenfunctions. The functions $\mathbf{F}_{nm\lambda}$ are transverse; a current density that can be expressed as a linear combination of the functions $\mathbf{F}_{nm\lambda}$ is transverse and therefore of FORM III or identified as TJ (transverse j),

$$\mathbf{j}^{TJ} = \sum_{n,m,i} a_{nmi}^{TJ} \, \mathbf{F}_{nm\lambda_i}. \tag{28}$$

Then, the corresponding magnetic field is

$$\mathbf{H}^{TJ} = \sum_{n,m,i} (a_{nmi}^{TJ}/\lambda_i) \, \nabla \times \mathbf{F}_{nm\lambda_i}. \tag{29}$$

Transverse magnetic (TM) or FORM IV magnetic fields are written as

$$\mathbf{H}^{TM} = \sum_{n,m,i} b_{nmi}^{TM} \, \mathbf{F}_{nm\lambda_i} \tag{30}$$

they correspond to the current density

$$\mathbf{j}^{TM} = \sum_{n,m,i} b_{nmi}^{TM} \, \nabla \times \mathbf{F}_{nm\lambda_i}. \tag{31}$$

If λ_i is allowed to approach zero for a term of (30) and (31), and b_{nmi}^{TM} is also scaled with λ_i such that a finite limit is approached, the limiting expressions are of FORM II. Similarly, (28) and (29) approach fields of FORM I. Letting the spherical Bessell function $f_n = j_n$ (the

spherical Bessel function of the first kind), the fields approach exterior solution forms; letting $f_n = n_n$ (the spherical Bessel function of the second kind), the fields approach interior solution forms.

3.2.1 FORM III – TJ modes

The transverse current density of (28), written in terms of real functions, has a form similar to the transverse magnetic vector for FORM II fields:

$$j_r^{TJ} = 0 \tag{32a}$$

$$j_\theta^{TJ} = \sum_{i=1}^{I} \sum_{n=1}^{N} j_n\left(\frac{r}{\rho_i}\right) \sum_{m=1}^{n} m\left(\zeta_{ni}^m \sin m\phi - \eta_{ni}^m \cos m\phi\right) \frac{P_n^m(\cos\theta)}{\sin\theta} \tag{32b}$$

$$j_\phi^{TJ} = \sum_{i=1}^{I} \sum_{n=1}^{N} j_n\left(\frac{r}{\rho_i}\right) \sum_{m=0}^{n} \left(\zeta_{ni}^m \cos m\phi + \eta_{ni}^m \sin m\phi\right) \frac{dP_n^m(\cos\theta)}{d\theta}. \tag{32c}$$

The spherical Bessel function of the first kind (j_n) has been written in the expansion, because it is well known that arbitrary functions can be expressed as sums (or integrals) of it (Watson 1966). For the definition of spherical Bessel functions in terms of Bessel functions, see Jackson (1975, equation 16.9) or Abromowitz & Stegun 1964, p. 437. A term of (32) has eigenvalue

$$\lambda_i = \frac{1}{\rho_i^2}. \tag{33}$$

If the maximum radius (r) of the expansion (32) is r_{max}, then

$$\rho_i = r_{max}/x_i \tag{34}$$

where the x_i are the positive roots of $j_n(x) = 0$. (This selection of roots yields a Fourier-Bessel series in r for each index pair (n, m) in the expansion. The Fourier-Bessel series will converge to any suitably well-behaved function over an interval (a, b) contained in $(0, r_{max})$; see Watson 1966, chapter XVIII.)

The real-function expansion (corresponding to 29) for the magnetic field associated with the current density of (32) is

$$H_r^{TJ} = -\frac{1}{r} \sum_{i=1}^{I} \rho_i^2 \sum_{n=1}^{N} n(n+1) j_n\left(\frac{r}{\rho_i}\right) \sum_{m=0}^{n} \left(\zeta_{ni}^m \cos m\phi + \eta_{ni}^m \sin m\phi\right) P_n^m(\cos\theta) \tag{35a}$$

$$H_\theta^{TJ} = -\sum_{i=1}^{I} \rho_i^2 \sum_{n=1}^{N} \left[\frac{1}{\rho_i} j_n'\left(\frac{r}{\rho_i}\right) + \frac{1}{r} j_n\left(\frac{r}{\rho_i}\right)\right] \sum_{m=0}^{n} \left(\zeta_{ni}^m \cos m\phi + \eta_{ni}^m \sin m\phi\right) \frac{dP_n^m(\cos\theta)}{d\theta} \tag{35b}$$

$$H_\phi^{TJ} = \sum_{i=1}^{I} \rho_i^2 \sum_{n=1}^{N} \left[\frac{1}{\rho_i} j_n'\left(\frac{r}{\rho_i}\right) + \frac{1}{r} j_n\left(\frac{r}{\rho_i}\right)\right] \sum_{m=1}^{n} m\left(\zeta_{ni}^m \sin m\phi - \eta_{ni}^m \cos m\phi\right) \frac{P_n^m(\cos\theta)}{\sin\theta} \tag{35c}$$

where $j_n'(x) \equiv dj_n(x)/dx$.

If a current density of FORM III is assumed to flow above $(r > R)$ a conducting spherical surface and no current exists inside the sphere, a magnetic field of FORM I must exist inside the sphere such that H_r is continuous across the spherical surface.

3.2.2 FORM IV – TM modes

The magnetic field $\mathbf{H}^{TM}$ has the same form as $\mathbf{j}^{TJ}$ (see 28 and 30). In terms of real functions, $\mathbf{H}^{TM}$ has the form of (32):

$$H_r^{TM} = 0 \qquad\qquad (36a)$$

$$H_\theta^{TM} = \sum_{i=1}^{I} \sum_{n=1}^{N} j_n\left(\frac{r}{\rho_i}\right) \sum_{m=1}^{n} m \left(\mu_{ni}^m \sin m\phi - \nu_{ni}^m \cos m\phi\right) \frac{P_n^m(\cos\theta)}{\sin\theta} \qquad (36b)$$

$$H_\phi^{TM} = \sum_{i=1}^{I} \sum_{n=1}^{N} j_n\left(\frac{r}{\rho_i}\right) \sum_{m=0}^{n} \left(\mu_{ni}^m \cos m\phi + \nu_{ni}^m \sin m\phi\right) \frac{dP_n^m(\cos\theta)}{d\theta}. \qquad (36c)$$

Comparing (29) and (31), the form is the same except for a factor of λ_i; thus, the curl of (36) yields the form of (35) with each term multiplied by $\lambda_i = 1/\rho_i^2$:

$$j_r^{TM} = -\frac{1}{r} \sum_{i=1}^{I} \sum_{n=1}^{N} n(n+1) j_n\left(\frac{r}{\rho_i}\right) \sum_{m=0}^{n} \left(\mu_{ni}^m \cos m\phi + \nu_{ni}^m \sin m\phi\right) P_n^m(\cos\theta) \qquad (37a)$$

$$j_\theta^{TM} = -\sum_{i=1}^{I} \sum_{n=1}^{N} \left[\frac{1}{\rho_i} j_n'\left(\frac{r}{\rho_i}\right) + \frac{1}{r} j_n\left(\frac{r}{\rho_i}\right)\right] \sum_{m=0}^{n} \left(\mu_{ni}^m \cos m\phi + \nu_{ni}^m \sin m\phi\right) \frac{dP_n^m(\cos\theta)}{d\theta}$$

$$(37b)$$

$$j_\phi^{TM} = \sum_{i=1}^{I} \sum_{n=1}^{N} \left[\frac{1}{\rho_i} j_n'\left(\frac{r}{\rho_i}\right) + \frac{1}{r} j_n\left(\frac{r}{\rho_i}\right)\right] \sum_{m=1}^{n} m \left(\mu_{ni}^m \sin m\phi - \nu_{ni}^m \cos m\phi\right) \frac{P_n^m(\cos\theta)}{\sin\theta}.$$

$$(37c)$$

4 Completeness and linear dependence

The completeness arguments presented in this section depend on the completeness of the spherical harmonic functions over a spherical surface and the completeness of the Bessel function of the first kind (and, hence, of the *spherical* Bessel function of the first kind) over finite intervals (i.e. for r in the interval (a, b), where $0 < a < b < r_{max}$). The completeness of the spherical harmonic function is discussed in many standard references. Your attention is called to Watson (1966, chapter XVIII), for a discussion of the completeness of the Bessel functions. It will be assumed that the functions to be expanded are sufficiently well behaved for the expansions to converge.

Hence, it is assumed that an arbitrary scalar function of the spherical polar coordinates (r, θ, ϕ) can be expanded in the form of equations (35a) (or 37a) where the summation limits I and N are set to infinity. It is assumed that the domain is a spherical shell such that $0 < a < r < b < r_{max}$. To perform such an expansion, one could first expand the function in spherical harmonics for every value of r, then expand the resulting coefficients (functions of r) in a Fourier-Bessel series.

4.1 COMPLETENESS OF FORMS III AND IV FOR THE CURRENT DENSITY

Suppose one is given a current density $\mathbf{j}(r, \theta, \phi)$ such that $\nabla \cdot \mathbf{j} = \mathbf{0}$. It will be demonstrated that $\mathbf{j}$ can be expanded as a sum of current densities of FORMS III and IV (equations 32 and 37). To perform such an expansion, one first expands the r-component of $\mathbf{j}$ in the form of

equation (37a). This yields the coefficients μ_{ni}^m and ν_{ni}^m and the other TM current-density components given by (37b) and (37c). Let the remaining current density be

$$j^R = j - j^{TM}. \tag{38}$$

The divergence of j^R is zero because it is assumed that $\nabla \cdot j = 0$ and $\nabla \cdot j^{TM} = 0$ because j^{TM} is the curl of a vector (31). Note that $j_r^R = 0$, hence,

$$\nabla \cdot j^R = \nabla_s \cdot j^R = 0. \tag{39}$$

Define the vector j^0 orthogonal to j^R on each spherical surface

$$j^0 = \hat{r} \times j^R = -j_\phi^R \hat{\theta} + j_\theta^R \hat{\phi}. \tag{40}$$

The curl of j^0 in the $\hat{r}$-direction is zero; by direct computation one obtains

$$(\nabla \times j^0)_r = \nabla_s \cdot j^R = 0. \tag{41}$$

Hence, j^0 is conservative on each spherical surface and can be written as the surface gradient of a scalar U on each spherical surface:

$$j^0 = \nabla_s U = \frac{1}{r} \frac{\partial U}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \hat{\phi}. \tag{42}$$

The scalar U can be expressed as a line integral of j^0, where the path of integration is confined to a spherical surface (for each r)

$$U = \int_{P_0}^{P} j^0 \cdot dl \tag{43}$$

where P_0 is a fixed spherical coordinate (θ_0, ϕ_0) and P is the point (θ, ϕ) on each sphere. The current density j^R is given in terms of U by [equate components of (40) and (42)]

$$j^R = \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \hat{\theta} - \frac{1}{r} \frac{\partial U}{\partial \theta} \hat{\phi}. \tag{44}$$

The next step is to expand U in the form of (35a). Substitution of that expansion into (44) yields expansions for j_θ^R and j_ϕ^R of the forms of (32b) and (32c); equating coefficients yields the ζ_{ni}^m and η_{ni}^m of (32). Hence, $j^R = j^{TJ}$ and j can be expressed as the sum of j^{TJ} and j^{TM} for any j with $\nabla \cdot j = 0$.

4.2 COMPLETENESS OF FORMS III AND IV FOR $\nabla \times j$

Suppose one is given the curl of j, say

$$\nabla \times j = G \tag{45}$$

or

$$\nabla \times \nabla \times H = G. \tag{46}$$

It will be shown that for any vector G, such that $\nabla \cdot G = 0$, a corresponding H satisfying (46) can be found in terms of a sum of FORMS III and IV (35 and 36). One can demonstrate this by repeating the same steps that were taken in the previous subsection. The curl curl operator does not change the forms of the expansions for H^{TJ} and H^{TM}, it merely intro-

duces as a multiplier the eigenvalue of each term. Hence, the form of $\nabla \times \nabla \times H^{TJ}$ is the same as that of H^{TJ} and is also the same as the form of j^{TM}. Likewise, the form of $\nabla \times \nabla \times H^{TM}$ is the same as that of H^{TM} and is the same as the form of j^{TJ}. Hence, one carries out the same procedure as described above for j, to expand G. To obtain the expansion for H, from the expansion for G, one must merely divide each term by its eigenvalue (for FORMs III and IV the eigenvalues are all greater than zero).

Note that j is not completely determined by its curl. In fact, if j satisfies (45), then $(j + j^{II})$ also satisfies (45), where j^{II} is any FORM II current density. Therefore, if one is given $\nabla \times j$ as the source function for H, an appropriate FORM II field should be included that is determined by boundary conditions on j. (Also a FORM I field should be included that is determined by boundary conditions on H as described in Section 3.)

4.3 LINEAR DEPENDENCE

It has been shown that FORMS III and IV can be used to represent current densities in a domain such that r satisfies $0 < a < r < b < r_{max}$. The current densities must have zero divergence, but otherwise have only the mathematical limitations required for the convergence of the expansions; these requirements are normally obeyed by physical quantities. Therefore, one can expand current densities that are zero or have zero curl over some interior domain, say over the domain where r satisfies $\bar{a} < r < \bar{b}$, where $0 < a < \bar{a} < \bar{b} < b < r_{max}$. If the current density is non-zero outside of this interior domain [i.e. for r in the interval $(a, \bar{a})$ and/or $(\bar{b}, b)$], the magnetic field inside the interior domain will, in general, be non-zero. It will be of FORM I if $j = 0$ in the interior domain; it will be of FORM II if $\nabla \times j = 0$ ($j \neq 0$) in the interior domain. Therefore, fields of FORMS I and II can be expanded in terms of fields of FORMS III and IV. Because of the nature of the fields (TJ or TM), FORM I fields can be expanded in terms of FORM III and FORM II fields can be expanded in terms of FORM IV. To perform such an expansion, one must simply express the r-dependence of the terms of FORMS I and II in Fourier-Bessel series over the finite domain of interest. (Note that such an expansion cannot be extended to $r \to \infty$ for FORMS I^e or IIe that are singular as $r \to \infty$; nor can they be extended to $r = 0$ for FORMS I^i or II^i that are singular at $r = 0$). The expansion of each term of FORMS I or II requires an infinite series of Bessel functions. That is, by examining the Bessel function in series form, it is apparent that the Fourier-Bessel series representations for powers of r will not truncate. Therefore, a linear dependence exists between FORMS I and III and between FORMS II and IV, but only if all the terms of the Bessel function expansion are included (i.e. $I \to \infty$ in the FORM III and FORM IV series). Another way to state this is that FORMS I and III (and FORMS II and IV) are asymptotically linearly dependent as the summation limit I approached infinity. Any single eigenvector or any finite sum of eigenvectors of FORMS III or IV are linearly independent of FORMS I and II.

If only a few eigenvalues of FORMS III and/or IV fields are to be used in a numerical fit, no attention need be paid to the asymptotic linear dependence. However, if models are attempted that include large numbers of eigenvalues, this linear dependence must be taken into account. In a least-squares fit model, the linear-equation matrix will become asymptotically singular as the number of included eigenvalues approaches infinity, if FORMS I and III (or FORMS II and IV) are included. (In practice, the matrix will become ill-conditioned when a large number of eigenvalues are used.)

5 Boundary conditions at a conducting spherical shell

The effect of a conducting spherical shell on the magnetic field is pertinent to the study of shielding, the effect of the ionosphere on the geomagnetic field, and other situations where

currents flow on a spherical shell. Therefore, the basic equations that are pertinent to such a boundary are included in this section. No provision is made for motion of the shell, but this can be added as needed. Emphasis is placed on the coupling of the field forms due to a conducting shell. Current is allowed to flow to and from the conducting shell consistent with field FORMS II, III, and IV adjacent to it. Anisotropic conductivity is discussed and a simplified model for the ionosphere is suggested.

5.1 SURFACE CURRENT RELATIONSHIPS

The surface current in the shell is related to the magnetic field by

$$\mathbf{j_s} = \hat{\mathbf{r}} \times (\mathbf{H^e} - \mathbf{H^i}) \tag{47}$$

where $\mathbf{H^e}$ is evaluated just exterior to the spherical shell $(r = R^+)$ and $\mathbf{H^i}$ is evaluated just interior to the shell $(r = R^-)$. The divergence of (47) yields

$$\nabla \cdot \mathbf{j_s} = - [\nabla \times \mathbf{H^e} - \nabla \times \mathbf{H^i}]_r$$

or

$$\nabla_s \cdot \mathbf{j_s} = - j_r^e + j_r^i. \tag{48}$$

Fields of FORMS I and III have $j_r = 0$; they do not contribute to $\nabla_s \cdot \mathbf{j_s}$ on the spherical shell, only FORMS II and IV contribute.

The r-component of the curl of (47) yields

$$(\nabla \times \mathbf{j_s})_r = \nabla_s \cdot (\mathbf{H^e} - \mathbf{H^i}). \tag{49}$$

Since,

$$\nabla \cdot \mathbf{H} = 0 \tag{50}$$

and

$$\nabla \cdot \mathbf{H} = \nabla_s \cdot \mathbf{H} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 H_r) \tag{51}$$

equation (49) can be written

$$(\nabla \times \mathbf{j_s})_r = - \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 (H_r^e - H_r^i)]. \tag{52}$$

Fields of FORMS II and IV have $H_r = 0$; they do not contribute to $(\nabla \times \mathbf{j_s})_r$ on the spherical shell, only FORMS I and III contribute.

5.2 MAGNETIC-FIELD BOUNDARY CONDITION

The magnetic field normal to the surface (the r-component) must be continuous across the boundary; that is

$$H_r^e = H_r^i. \tag{53}$$

Only magnetic fields of FORMS I and III couple magnetically (through 53). Using superscripts to indicate the form, (53) becomes

$$H_r^{Ie} + H_r^{IIIe} = H_r^{Ii} + H_r^{IIIi}. \tag{54}$$

Equation (54) has already been applied to FORM I fields to obtain equation (5).

5.3 ELECTRIC-FIELD BOUNDARY CONDITION

Though analysis of electric fields, *per se,* is not a part of this paper, the role of electric fields in coupling magnetic fields across a spherical-shell boundary will be considered. The conductance of a spherical shell can be represented by an impedance function that is, in general

(1) a function of the coordinates (θ, ϕ) (i.e. it is inhomogeneous);
(2) a dyadic (i.e. it is anisotropic); and
(3) it is complex (i.e. it can be both resistive and reactive).

Property 3 is beyond the scope of this paper; only static fields and quasi-static fields (i.e. the displacement current is negligible) are discussed. It is assumed that the spherical shell is characterized by a real dyadic impedance Z that is, in general, a function of the coordinates.

The electric field tangential to the shell is given by

$$\mathbf{E} = \mathbf{Z} \cdot \mathbf{j}_s \equiv (Z_{\theta\theta} j_{s\theta} + Z_{\theta\phi} j_{s\phi})\hat{\theta} + (Z_{\phi\theta} j_{s\theta} + Z_{\phi\phi} j_{s\phi})\hat{\phi}. \tag{55}$$

Substituting $\mathbf{j}_s$ from (47), one obtains $\mathbf{E}$ in terms of the magnetic field tangential to the shell as

$$\mathbf{E} = [-Z_{\theta\theta}(H_\phi^e - H_\phi^i) + Z_{\theta\phi}(H_\theta^e - H_\theta^i)]\hat{\theta} + [-Z_{\phi\theta}(H_\phi^e - H_\phi^i) + Z_{\phi\phi}(H_\theta^e - H_\theta^i)]\hat{\phi}. \tag{56}$$

The condition that couples the various field terms through the electric field is Faraday's law applied in the radial direction:

$$(\nabla \times \mathbf{E})_r = -\dot{B}_r. \tag{57}$$

If the fields are static, $\dot{B}_r = 0$ and

$$(\nabla \times \mathbf{E})_r = 0. \tag{58}$$

Applying the curl operator to (56), one obtains

$$(\nabla \times \mathbf{E})_r = \frac{1}{r \sin \theta} \left\{ \frac{\partial}{\partial \theta} \left[-Z_{\phi\theta}(H_\phi^e - H_\phi^i) + Z_{\phi\phi}(H_\theta^e - H_\theta^i)] \sin \theta \right] \right.$$
$$\left. + \frac{\partial}{\partial \phi} [Z_{\theta\theta}(H_\phi^e - H_\phi^i) - Z_{\theta\phi}(H_\theta^e - H_\theta^i)] \right\}. \tag{59}$$

Equation (59), combined with (57) or (58), constitutes the electric-field coupling equation in general form. Since this equation has no symmetry (in general form), it is possible for it to provide coupling between any magnetic-field terms. In particular, it can couple FORMS II and/or IV to FORMS I and/or III. This was found not to be possible with only magnetic-field coupling (53 and 54).

5.4 A UNIFORM, SCALAR IMPEDANCE

Assume that $Z_{\theta\theta} = Z_{\phi\phi} = Z$ (a constant) and that $Z_{\theta\phi} = Z_{\phi\theta} = 0$. Then

$$\mathbf{E} = Z\mathbf{j}_s \tag{60}$$

and

$$(\nabla \times \mathbf{E})_r = Z(\nabla \times \mathbf{j}_s)_r. \tag{61}$$

Substitution using (52) shows that

$$(\nabla \times \mathbf{E})_r = -\frac{Z}{r^2}\frac{\partial}{\partial r}[r^2(H_r^e - H_r^i)]. \tag{62}$$

Only fields of FORMS I and III have $H_r \neq 0$ and can be coupled by this equation. If only FORM I fields are present, it can be easily shown that their Legendre-function terms of degree n decay with a time constant of $\tau = R\mu_0/Z(2n+1)$ (where R is the spherical shell radius and μ_0 is the permeability of free space), unless supported by sources in the spherical shell.

In the case of the ionosphere, such sources could be present due to motion of the ionosphere in the presence of the main geomagnetic field. However, the uniform, scalar impedance model is useless for describing the effects of field-aligned current above the ionosphere in the polar region since FORM III fields are TJ ($j_r = 0$), and the field-aligned current flows into and out of the ionosphere (Potemra 1979); an appropriate model for this purpose must relate surface current in the ionosphere to fields of FORM II and/or IV.

5.5 AN APPROXIMATE IMPEDANCE FOR THE IONOSPHERE

An approximate form for Z that is illustrative and may have value for ionospheric (or other) modelling is (shown in matrix form);

$$Z = \begin{bmatrix} Z_d & -Z_h \\ Z_h & Z_d \end{bmatrix}. \tag{63}$$

That is, $Z_{\theta\theta} = Z_{\phi\phi} = Z_d$ (diagonal terms – yield E-field parallel to $\mathbf{j}$) and $-Z_{\theta\phi} = Z_{\phi\theta} = Z_h$ (Hall impedance – yields E-field perpendicular to $\mathbf{j}$; see Potemra 1980). (Note that Z_h must be positive in the northern polar region and negative in the southern polar region.) Then,

$$\mathbf{E} = Z \cdot \mathbf{j}_s = Z_d \mathbf{j}_s - Z_h(j_{s\phi}\hat{\theta} - j_{s\theta}\hat{\phi})$$

$$\mathbf{E} = Z_d \mathbf{j}_s - Z_h(\mathbf{H}_t^e - \mathbf{H}_t^i) \tag{64}$$

where subscript t indicates tangential component. The curl of (64) yields

$$(\nabla \times \mathbf{E})_r = Z_d(\nabla \times \mathbf{j}_s)_r + (\nabla_s Z_d \times \mathbf{j}_s)_r - Z_h(j_r^e - j_r^i) + \nabla_s Z_h \cdot \mathbf{j}_s. \tag{65}$$

The field forms that contribute to the various terms of (65) are:

Term	Contributing field form
$Z_d(\nabla \times \mathbf{j}_s)_r$	FORMS I, III (see 52)
$(\nabla_s Z_d \times \mathbf{j}_s)_r$	FORMS I, II, III, IV
$-Z_h(j_r^e - j_r^i)$	FORMS II, IV
$\nabla_s Z_h \cdot \mathbf{j}_s$	FORMS I, II, III, IV.

Equation (65) illustrates that, if *both* Z_d and Z_h are finite in some region, fields of all forms can be coupled even in regions where the impedance is spatially constant; i.e. $\nabla_s Z_d = \mathbf{0}$, $\nabla_s Z_h = \mathbf{0}$ and only the first and third terms survive. (This condition may be approximated in the polar regions. Constant Z_h is unphysical globally, however, since Z_h must have approximate odd symmetry about the magnetic equator.)

Ignoring the $\nabla_s Z$ terms of (65), and setting j_r^i and $(\nabla \times \mathbf{E})_r$ to zero, one obtains

$$Z_d(\nabla \times \mathbf{j}_s)_r = Z_h j_r^e. \tag{66}$$

This equation describes the basic features of the coupling of field-aligned currents above the ionosphere ($\mathbf{j}^e$) to the surface currents in the ionospheric shell ($\mathbf{j}_s$) in the polar regions. Only the solenoidal part of $\mathbf{j}_s$ (FORM I and possibly some FORM III fields) is involved in the

coupling; the irrotational part of j_s is given by the same terms that describe j_r^e (FORM II and possibly some FORM IV fields).

The qualitative correctness of (66) can be verified by observing the behaviour of the current flow as illustrated by Potemra (1980). That is, in regions where j_r^e is positive as shown in his fig. 10, the curl of j_s is also positive (in the $\hat{r}$-direction) as shown in his fig. 4, and vice versa. If the field-aligned current is significant only in tubes carrying current to small regions of the ionosphere, then (to the approximation implied by 66) outside of these regions $(\nabla \times j_s)_r \simeq 0$ and j_s can be expressed as the gradient of a scalar. Because $\nabla_s \cdot j_s \simeq 0$ in these regions, the scalar is a solution of the (two-dimensional) Laplace equation. Hence, the circulating part of j_s is exactly analogous to magnetic field lines surrounding current-carrying conductors, where the conductors are the analogue of the field-aligned current tubes. Large-scale current circulation can result from small field-aligned current tubes. The stream lines of j_s will be compact between the field-aligned current tubes (in analogy to the compactness of magnetic field lines between conductors carrying currents of opposite sign), forming the auroral electrojets. The non-circulating part of j_s (the Pedersen current) is analogous to E-field lines connecting point charges.

In the case of diffuse distributions of field-aligned current, the integral form of (66) provides additional insight. Integration of (66) over an area A on the surface of the sphere and application of Stoke's theorem yields

$$\oint j_s \cdot d\mathbf{x} = (Z_h/Z_d) I_A \tag{67}$$

where

$$I_A = \int_A j_r^e d^2 x$$

and the path of integration of j_s encloses A. That is, the line integral of j_s around a closed path (on the sphere) is (Z_h/Z_d) times the field-aligned current enclosed by the path. Verification of these relationships and, hence, determination of the degree of accuracy (66) will require computer analysis.

Boundary conditions, such as that given by (58) and (59) and approximated by (66), yield only consistency relations; they do not differentiate between cause and effect. Nevertheless, this boundary condition does show that it is possible for the field-aligned currents (primarily of FORM II) to drive directly the ionospheric circulating currents (of FORM I) in the ionosphere. The boundary condition provides the coupling relationship. The ionospheric currents completing the circuit for the field-aligned currents (i.e. the Pedersen currents) are given directly by the FORM II (and possibly some FORM IV) fields, that is, by (21) and (22).

6 Generality of the field forms

It should be emphasized that any distribution of magnetic field and current density that obeys the equation

$$\nabla \times H = j \tag{68}$$

where $\nabla \cdot j = 0$ can be represented by a sum of the field FORMS I, II, III and IV. Hence, given a sufficiently accurate physical model or a sufficient amount of data concerning the magnetic field and/or currents, the magnetospheric system of currents and magnetic fields can, in principle, be represented (to the approximation that displacement current is neglected for time-varying fields). Hence, the ionosphere could be represented in detail in three dimensions.

The expansions presented in this paper were derived in terms of spherical coordinates. This choice was motivated by the near spherical shape of the ionosphere, recent interest in fields and currents near the ionosphere, and the successful use of spherical coordinates and spherical harmonics to describe the main field and other geomagnetic-field components. The primary ingredient required for defining the various field forms is the set of eigenfunctions of the operator curl curl. These vector eigenfunctions correspond to solutions of the vector Helmholtz equation discussed by Morse & Feshbach (1953). In the notation of Morse & Feshback (p. 1766), magnetic fields given by their component L are of FORM I, their component N yields FORM III, and their component M yields FORM IV. FORM II would be obtained as a degenerate case of their component M. FORM II also corresponds to their component M of the vector Laplace equation solution (pp. 1799, 1800). In describing geomagnetic fields whose behaviour departs drastically from that of low order (generalized) multipole terms, one could employ one of the other five coordinate systems appropriate for representing these FORMS. For instance, one of the cylindrical coordinate systems might be used to model the geomagnetic tail.

7 Modelling field-aligned currents

It is obvious that the transverse magnetic FORMS II and IV will be predominant in representing field-aligned currents near the ionosphere. FORM III current density has no *r*-component and therefore cannot represent currents flowing to and from the ionosphere. A possible current density with non-zero curl j that would require FORM III field terms can be visualized. It is current density due to charged particles circulating around main field lines. Such a distribution of current density may or may not be significant as a magnetic field source. Possible sources of curl j requiring FORM IV field terms are the regions of shear where the field-aligned current density changes sign (Sugiura 1975, fig. 8; or the same figure in Potemra 1979). Whether significant curl j exists in those regions depends on the details of the physics. If there exist sheets where curl j is significant, such as these thin regions, then the modeller has two choices. He can either include FORM IV fields to represent curl j on the sheets using sufficient spherical harmonic terms to resolve them (or their time average) to some accuracy, or he can divide the space into regions where curl j is very nearly zero and use separate (FORM II) expansions in those regions. It may be possible to express almost all of the field-aligned current as an expansion in FORM II field terms, at least near the ionosphere. To verify or disprove his conjecture, one can perform a field-fitting procedure including FORM II fields for the representation of field-aligned currents; then the residual field can be examined to determine its character and then decide what additional terms or refinements, if any, are required.

An expansion of detailed field information is often impractical because of the number of terms required to resolve geometrically small features with large gradients. Such small features are generally of two types: (1) time-invariant, slowly changing, or periodic features that can be represented by special physical models; the expansion can then represent the difference between the observed field values and those predicted by the model, and (2) stochastically changing features, whose behaviour is not well enough understood to create a physical model; an expansion is then used to represent the time-averaged behaviour of such a feature. The resolution required to describe the average behaviour is often much less than that required to describe one element of the average. Study of the cause, structure, etc. of the individual elements becomes a separate issue.

It appears, from a review of reduced polar-orbit satellite data (*MAGSAT*, Cain 1981) that the resolution required to expand time-averaged fields due to field-aligned current does not preclude such an expansion. The availability of sufficient data to obtain a meaningful expansion is uncertain. An effort to examine this approach is in progress.

Acknowledgments

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