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Spherical Monogenic Functions and Analytic Functionals on the Unit Sphere

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ABSTRACT. The aim of this paper is to introduce the analogues of the restrictions of $z\to z^k$, $k\in \boldsymbol{Z}$, $z\in \boldsymbol{C}$, to the unit circle, in the case of the unit sphere $\partial B(0, 1)$ in $\boldsymbol{R}^{m+1}$ and to apply them to boundary value problems of left monogenic functions in $\boldsymbol{R}^{m+1}\backslash\partial B(0, 1)$.

Introduction

In [2], a theory of monogenic functions has been developed which generalizes in a natural way the theory of holomorphic functions of one complex variable to the $(m+1)$-dimensional Euclidean space ($m\geqq 1$). In this context the Cauchy kernel is determined by the function

$$\frac{1}{\omega_{m+1}}\frac{\bar{u}-\bar{x}}{|u-x|^{m+1}}\,,\quad u,\, x\in \boldsymbol{R}^{m+1}\,.$$

The aim of the first section is to give an expansion for the Cauchy kernel into a series of homogeneous monogenic polynomials in x. To obtain these polynomials we use the development of $1/|u-x|^{m-1}$ into Gegenbauer polynomials; this will also enable us to give estimates for the kernel expansion. Other expansions have been introduced in [2] and [12] and it is shown that all of them are equal.

In the second section we derive the Taylor and Laurent expansions for monogenic functions in open balls and in annular domains of $\boldsymbol{R}^{m+1}$. Furthermore, when f admits a Laurent expansion in the annular domain $\mathring{B}(0, R_1)\backslash\bar{B}(0, R_2)$, $0<R_1<R_2$, say

$$f(x)=\sum_{k=0}^{\infty} P_k(x)+\sum_{k=0}^{\infty} Q_k(x)\,,$$

an estimate for the terms $P_k(x)$ and $Q_k(x)$ is given.

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In the third section we introduce a transform which corresponds to the transformation $f(z)\to\bar{f}(1/\bar{z})(1/z)$ in the case of holomorphic functions. The method to obtain it is not analogous to the case of $m=1$ and holds only for $m\geqq 2$. Using this transform we are able to define an inner product on $\partial B(0, 1)$ (§ 4) and this for analytic functions.

In the last section we study analytic functionals, distributions, test-functions and L_2-functions on $\partial B(0, 1)$, making use of a refined version of the Laplace-Beltrami operator on $\partial B(0, 1)$, which corresponds to the operator $i(d/d\theta)$ in the complex case. The inner product will then be extended in a suitable way in order to express certain dualities on the unit sphere.

Although the case of distributions on $\partial B(0, 1)$ may also be treated by decomposing spherical harmonics into spherical monogenic functions and by using the results for spherical harmonics as has been done by Seeley in [13] and Morimoto in [9], we develop an independent theory.

In a forthcoming paper [16] we shall study wider classes of analytic functionals on $\partial B(0, 1)$ which were also investigated by Hashizume-Kowata-Minemura-Okamoto in [5], by Helgason in [7] and by Morimoto in [8], [9] and [10].

PRELIMINARIES. Let $m\in N$, $m\geqq 1$, and let $\mathscr{A}$ be the Clifford algebra constructed over a real quadratic n-dimensional vector space, $n\geqq m$. Then $\mathscr{A}$ is a 2^n-dimensional real vector space with basis $\{e_A: A\subseteq \{1, \cdots, n\}\}$, where $e_\varnothing=e_0=1$, $e_A=e_{\alpha_1}\cdots e_{\alpha_l}$, $A=\{\alpha_1, \cdots, \alpha_l\}$, $\alpha_1<\alpha_2<\cdots<\alpha_l$; $e_{\alpha_j}e_{\alpha_k}=-e_{\alpha_k}e_{\alpha_j}$ when $j\neq k$ and $e_{\alpha_j}^2=-1$, $j=1, \cdots, n$. By ordering the subsets A of $N=\{1, \cdots, n\}$ in a certain way, each $a=\sum_A a_A e_A$ may thus be identified with the element $(a_A)_{A\in\mathscr{P}N}\in \boldsymbol{R}^{2^n}$. Moreover, the e_0-component a_0 of $a\in\mathscr{A}$ will be called real. For $a=\sum_A a_A e_A\in\mathscr{A}$, we put $\bar{a}=\sum_A a_A\bar{e}_A$ where for $A\neq\varnothing$, $\bar{e}_A=\bar{e}_{\alpha_l}\cdots\bar{e}_{\alpha_1}$, with $\bar{e}_i=-e_i$, $i=1, \cdots, n$, while for $A=\varnothing$, $\bar{e}_0=e_0$.

If $x=(x_0, x_1, \cdots, x_m)\in\boldsymbol{R}^{m+1}$ is identified with $x=x_0+x_1e_1+\cdots+x_me_m$, then $\boldsymbol{R}^{m+1}$ may be considered as a subspace of $\mathscr{A}$. Note that in this case $\bar{x}=x_0-x_1e_1-\cdots-x_me_m$ corresponds to the point $\bar{x}=(x_0, -x_1, \cdots, -x_m)\in \boldsymbol{R}^{m+1}$.

The norm in $\boldsymbol{R}^{2^n}$ is denoted by $|x|$ while the norm in $\mathscr{A}$ is defined by $|a|_0^2=2^n\operatorname{Re}(\bar{a}a)=2^n|a|^2$. Furthermore the points of the unit ball $\partial B(0, 1)$ in $\boldsymbol{R}^{m+1}$ are denoted by ω.

The classical Cauchy-Riemann operator is generalized to $D=\sum_{i=0}^m e_i(\partial/\partial x_i)$ and $\bar{D}=\sum_{i=0}^m \bar{e}_i(\partial/\partial x_i)$ denotes the conjugate of D. Remark that $D\bar{D}=\bar{D}D=\Delta_{m+1}$, the $(m+1)$-dimensional Laplacian.

Let $\Omega\subset\boldsymbol{R}^{m+1}$ be open; then an $\mathscr{A}$-valued function f is called left (resp. right) monogenic in Ω if and only if $f \in C_1(\Omega; \mathscr{A})$ and $Df=0$ (resp. $fD=0$) in Ω. The space of left (resp. right) monogenic functions in Ω is denoted by $M_1(\Omega; \mathscr{A})$ (resp. $M_1^{(r)}(\Omega; \mathscr{A})$).

§ 1. The Kernel function.

In this section we give an expansion for the Cauchy kernel function $(1/\omega_{m+1})((\bar{y}-\bar{x})/|y-x|^{m+1})$ for $|y|>|x|$. It is well known that the function $1/|x-y|^{m-1}$, $x, y \in \boldsymbol{R}^{m+1}$, may be developed, for $|y|>|x|$, into the series

$$\frac{1}{|x-y|^{m-1}}=\sum_{k=0}^{\infty} C_{k,m+1} \frac{1}{|y|^{m+k-1}} I_{k,m+1,y}(x) ,$$

where

$$C_{k,m+1}=\binom{k+m-2}{k}$$

and

$$I_{k,m+1,y}(x)=|x|^k L_{k,m+1}\left(\frac{\langle x, y\rangle}{|x||y|}\right) .$$

(See [11].)
As the above series converges, together with all its derivatives, absolutely and uniformly in $|y|>|x|$, we obtain that

$$(1) \qquad \frac{\bar{y}-\bar{x}}{|y-x|^{m+1}}=\sum_{k=0}^{\infty} \frac{C_{k+1,m+1}}{|y|^{m+k}} K_{k,m+1,y}(x)$$

where

$$K_{k,m+1,y}(x)=\frac{1}{m-1} \bar{D}_x I_{k+1,m+1,y}(x) .$$

Observe that the functions $K_{k,m+1,y}(x)$ are left and right monogenic homogeneous polynomials in $x \in \boldsymbol{R}^{m+1}$ and that from the definition and [11] it may easily be proved that for some constant $C_m>0$

$$(2) \qquad |K_{k,m+1,y}(x)|_0 \leqq C_m(1+k^2)|x|^k .$$

On the other hand, it follows from [11] that for $|y|>|x|$

$$(3) \qquad \frac{\bar{y}-\bar{x}}{|y-x|^{m+1}}=\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle x, \nabla_y\rangle^k \frac{\bar{y}}{|y|^{m+1}} .$$

Hence, identifying the homogeneous polynomials of order k in (1) and (3), we get

$$\frac{(-1)^k}{k!} \langle x, \nabla_y\rangle^k \frac{\bar{y}}{|y|^{m+1}}=C_{k+1,m+1} \frac{1}{|y|^{m+k}} K_{k,m+1,y}(x) ,$$

which implies that the function $(1/|y|^{m+k})K_{k,m+1,y}(x)$ is both left and right monogenic for $y \in R^{m+1}\backslash\{0\}$.

§ 2. A Taylor and a Laurent expansion.

Let $R'>0$ and let f be left monogenic in $\mathring{B}(0, R')$. Then using Cauchy's formula (see [2]), for any $0<R<R'$ and $x \in \mathring{B}(0, R)$,

$$f(x)=\frac{1}{\omega_{m+1}}\int_{\partial B(0,R)}\frac{\bar{u}-\bar{x}}{|u-x|^{m+1}}d\sigma_u f(u) .$$

Hence by the kernel expansion (1)

$$f(x)=\sum_{k=0}^{\infty}\frac{C_{k+1,m+1}}{\omega_{m+1}R^{m+k}}\int_{\partial B(0,R)}K_{k,m+1,u}(x)d\sigma_u f(u) . \tag{4}$$

Note that this series converges absolutely and uniformly in $\mathring{B}(0, R)$. The expression (4) can also be brought into the form

$$f(x)=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{\omega_{m+1}}\int_{\partial B(0,R)}(-1)^k\langle x, \nabla_u\rangle^k\left(\frac{\bar{u}}{|u|^{m+1}}\right)d\sigma_u f(u) ,$$

which is a natural generalization of the expansion

$$f(z)=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{2\pi i}\int_{\partial B(0,R)}\frac{1}{u}\left(z\frac{d}{du}\right)^k f(u)du$$

in the case of holomorphic functions.

On the other hand, in view of [2], we have the expansion

$$f(x)=\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)}V_{l_1\cdots l_k}(x)a_{l_1\cdots l_k} , \tag{5}$$

where for each $k \in N$ and $(l_1, \cdots, l_k) \in \{1, \cdots, m\}^k$,

$$a_{l_1\cdots l_k}=\left[\frac{\partial}{\partial u_{l_1}}\cdots\frac{\partial}{\partial u_{l_k}}f(u)\right]_{u=0} ,$$

$$V_{l_1\cdots l_k}(x)=\frac{1}{k!}\sum_{\pi(l_1,\cdots,l_k)}z_{l_1}\cdots z_{l_k}$$

with $z_l=x_l e_0-x_0 e_l$, $l=1, \cdots, m$, and where the series converges as a multiple power series in $\mathring{B}(0, \sqrt{2}^{-1}R)$. Hence, identifying the homogeneous parts in (4) and (5), we obtain that for each $k \in N$,

$$\sum_{(l_1,\cdots,l_k)}V_{l_1\cdots l_k}(x)a_{l_1\cdots l_k}=\frac{C_{k+1,m+1}}{\omega_{m+1}R^{m+k}}\int_{\partial B(0,R)}K_{k,m+1,u}(x)d\sigma_u f(u) . \tag{6}$$

Using the estimate (2) it is easy to see that

$$(7)\qquad \Big|\sum_{(l_1,\cdots,l_k)} V_{l_1\cdots l_k}(x)a_{l_1\cdots l_k}\Big|_0 \leqq CC_{k+1,m+1}(1+k^2)\Big(\frac{|x|}{R}\Big)^k \sup_{\partial B(0,R)} |f(u)|_0 ,$$

where the constant $C>0$ depends only on the dimension m.

We now construct a Laurent expansion. Suppose that f is left monogenic in $\operatorname{co}\bar{B}(0, R')$, $R'>0$, and that $f\to 0$ for x tending to infinity. Then for $R>R'$ and $x\in\operatorname{co}\bar{B}(0, R)$ (See [2].)

$$f(x)=\frac{1}{\omega_{m+1}}\int_{\partial B(0,R)} \frac{\bar{x}-\bar{u}}{|x-u|^{m+1}} d\sigma_u f(u) .$$

Hence, using the kernel expansion (1), we obtain that

$$(8)\qquad f(x)=\sum_{k=0}^{\infty}\frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial B(0,R)} \frac{K_{k,m+1,x}(u)}{|x|^{m+k}} d\sigma_u f(u) .$$

Furthermore, in view of the estimate (4),

$$(9)\qquad \Big|\int_{\partial B(0,R)} \frac{K_{k,m+1,x}(u)}{|x|^{m+k}} d\sigma_u f(u)\Big|_0 \leqq C^*(1+k^2)\Big(\frac{R}{|x|}\Big)^{m+k} \sup_{\partial B(0,R)} |f(u)|_0 ,$$

C^* depending only on the dimension m.

In [3] the following version of a Laurent expansion has been proved:

$$(10)\qquad f(x)=\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)} X_{l_1,\cdots,l_k}(x)b_{l_1\cdots l_k} ,$$

where

$$b_{l_1\cdots l_k}=\frac{1}{\omega_{m+1}}\int_{\partial B(0,R)} V_{l_1\cdots l_k}(u)d\sigma_u f(u)$$

and

$$X_{l_1\cdots l_k}(x)=\Big[\frac{\partial^k}{\partial u_{l_1}\cdots\partial u_{l_k}}\frac{\bar{x}-\bar{u}}{|x-u|^{m+1}}\Big]_{u=0} .$$

We now show that the expansions (8) and (10) are equal.

Lemma 1. *For any* $k\in N$,

$$\sum_{(l_1,\cdots,l_k)} X_{l_1\cdots l_k}(x)b_{l_1\cdots l_k}=\frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial B(0,R)} \frac{K_{k,m+1,x}(u)}{|x|^{m+k}} d\sigma_u f(u) .$$

Proof. In view of (10) the right hand side is of the form

$$H(x)=\sum_{s=0}^{\infty}\sum_{(l_1,\cdots,l_s)} X_{l_1\cdots l_s}(x)b'_{l_1\cdots l_s} .$$

If $s>k$ then for any $R>0$,

$$b'_{l_1\cdots l_s}=\frac{1}{\omega_{m+1}}\int_{\partial B(0,R)} V_{l_1\cdots l_s}(u)d\sigma_u H(u)$$
$$=\frac{1}{\omega_{m+1}}\int_{\partial B(0,R)} V_{l_1\cdots l_s}\left(\frac{u}{|u|}\right)\frac{|u|^s}{|u|^{m+k}}H\left(\frac{u}{|u|}\right).$$

Hence, letting $R\to 0$, we obtain that $b'_{l_1\cdots l_s}=0$.

If $s<k$ we get the same result by letting R tend to infinity. Consequently

$$H(x)=\sum_{(l_1,\cdots,l_k)} X_{l_1\cdots l_k}(x)b'_{l_1\cdots l_k}$$

and so

$$\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)} X_{l_1\cdots l_k}(x)(b_{l_1\cdots l_k}-b'_{l_1\cdots l_k})=0\ ,$$

which yields that $b_{l_1\cdots l_k}=b'_{l_1\cdots l_k}$ for any $k\in N$ and $(l_1,\cdots,l_k)\in\{1,\cdots,m\}^k$.

Suppose now that $0<R'_2<R_2<R_1<R'_1$ and let f be left monogenic in $\mathring{B}(0,R'_1)\backslash\bar{B}(0,R'_2)$. Then by the representation theorem of Cauchy and the kernel expansion

$$f(x)=\sum_{k=0}^{\infty}\frac{C_{k+1,m+1}}{\omega_{m+1}R_1^{m+k}}\int_{\partial B(0,R_1)} K_{k,m+1,u}(x)d\sigma_u f(u)$$
$$+\sum_{k=0}\frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial B(0,R_2)}\frac{K_{k,m+1,x}(u)}{|x|^{m+k}}d\sigma_u f(u)\ ,$$

where both series converge absolutely and uniformly in any compact subset of $\mathring{B}(0,R_1)\backslash\bar{B}(0,R_2)$.

REMARK. Let $S\subset R^{m+1}$ be an open bounded neighbourhood of the origin with C_1-boundary. Then, in view of the Taylor and Laurent expansions and by using Cauchy's Theorem (see [2]), we obtain the following orthogonality relations on ∂S: for all $k, l\in N$,

$$\text{(11)}\qquad \frac{K_{k,m+1,y}(x)}{|y|^{m+k}}\delta_{k,l}=\frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial S}\frac{K_{l,m+1,u}(x)}{|u|^{m+l}}d\sigma_u\frac{K_{k,m+1,y}(u)}{|y|^{m+k}}$$
$$=\frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial S}\frac{K_{l,m+1,y}(u)}{|y|^{m+l}}d\sigma_u\frac{K_{k,m+1,u}(x)}{|u|^{m+k}}\ .$$

§ 3. Adjoint monogenic functions.

In this section we construct an extension of the mapping

$$f(z) \longrightarrow \bar{f}\left(\frac{1}{\bar{z}}\right)\frac{1}{z},$$

which is defined for holomorphic functions in an open $\Omega \subseteq \boldsymbol{C}$, to the case of left monogenic functions in $\Omega \subseteq \boldsymbol{R}^{m+1}$ ($m \geqq 2$). Let $m \in \boldsymbol{N}$, $m>1$. Then, for $(u, y) \in \boldsymbol{R}^{m+1} \times \boldsymbol{R}^{m+1}$, we first consider the function

$$\frac{1}{||u|y-\bar{u}/|u||^{m-1}} = \left[\frac{1}{|y-x|^{m-1}}\right]_{x=\bar{u}/|u|} \cdot \frac{1}{|u|^{m-1}},$$

which may be expanded into the series

$$\sum_{k=0}^{\infty} C_{k,m+1} \frac{L_{k,m+1}(\langle \bar{u}, y\rangle/(|u||y|))}{(|u||y|)^{m+k-1}},$$

which converges absolutely and uniformly in any compact subset of the region $|y||u|>1$ in $\boldsymbol{R}^{2(m+1)}$. Note that for any $k \in \boldsymbol{N}$,

$$\frac{L_{k,m+1}(\langle \bar{u}, y\rangle/(|u||y|))}{(|u||y|)^{m+k-1}}$$

is harmonic in $\boldsymbol{R}^{m+1}\backslash\{0\}$ and so in each variable u and y separately. Moreover it is equal to

$$\left[\frac{|x|^k}{|y|^{m+k-1}} L_{k,m+1}\left(\frac{\langle x, y\rangle}{|x||y|}\right)\right]_{x=\bar{u}/|u|^2} \cdot \frac{1}{|u|^{m-1}}.$$

We now put

$$s(y, u) = \frac{\bar{D}_y}{1-m} \frac{1}{||u|y-\bar{u}/|u||^{m-1}} \frac{\bar{D}_u}{1-m}.$$

Then in the region $|u||y|>1$,

$$s(y, u) = \sum_{k=0}^{\infty} C_{k,m+1} \frac{\bar{D}_y}{1-m} L_{k,m+1}\left(\frac{L_{k,m+1}(\langle \bar{u}, y\rangle/(|u||y|))}{(|u||y|)^{m+k-1}}\right)\frac{\bar{D}_u}{1-m}.$$

Furthermore, as for each $k \in \boldsymbol{N}$,

$$C_{k,m+1}\left(\frac{\bar{D}_y}{1-m}\right)\frac{|x|^k}{|y|^{m+k-1}} L_{k,m+1}\left(\frac{\langle x, y\rangle}{|x||y|}\right) = C_{k+1,m+1}\frac{1}{|y|^{m+k}} K_{k,m+1,y}(x),$$

we obtain the identity

$$C_{k,m+1}\frac{\bar{D}_y}{1-m}\left(\frac{L_{k,m+1}(\langle \bar{u}, y\rangle/(|u||y|))}{(|u||y|)^{m+k-1}}\right)\frac{\bar{D}_u}{1-m}$$
$$= C_{k+1,m+1}\frac{1}{|y|^{m+k}} K_{k,m+1,y}\left(\frac{\bar{u}}{|u|^2}\right) \cdot \frac{1}{|u|^{m-1}} \frac{\bar{D}_u}{1-m}.$$

Observe that all these functions are left monogenic in $y \in R^{m+1}\backslash\{0\}$ and right monogenic in $u \in R^{m+1}\backslash\{0\}$.

Using the function $s(y, u)$, we shall define the spherical transform of a function which is left monogenic in respectively $\mathring{B}(0, R)$ and co $\bar{B}(0, R)$, $R>0$. By this transform, a function $f \in M_1(\mathring{B}(0, R); \mathscr{A})$ (resp. $f \in M_1(\text{co}\,\bar{B}(0, R); \mathscr{A})$) is transformed into a function $s(f) \in M_1^{(r)}(\text{co}\,\bar{B}(0, 1/R); \mathscr{A})$ (resp. $M_1^{(r)}(\mathring{B}(0, 1/R; \mathscr{A}))$. We first consider $f \in M_1(\mathring{B}(0, R); \mathscr{A})$. Then obviously $\bar{f}(\bar{x}) \in M_1^{(r)}(\mathring{B}(0, R); \mathscr{A})$ and for any $x \in \mathring{B}(0, R-\varepsilon)$,

$$\begin{aligned}\bar{f}(\bar{x}) &= \frac{1}{\omega_{m+1}} \int_{\partial B(0, R-\varepsilon)} \bar{f}(\bar{y}) d\sigma_y \frac{\bar{y}-\bar{x}}{|y-x|^{m+1}} \\ &= \sum_{k=0}^{\infty} \frac{C_{k+1,m+1}}{\omega_{m+1}} \int_{\partial B(0, R-\varepsilon)} \bar{f}(\bar{y}) d\sigma_y \frac{K_{k,m+1,y}(x)}{|y|^{m+k}} .\end{aligned}$$

The spherical transform $s(f)$ of f is now defined by

$$\begin{aligned}s(f)(u) &= \sum_{k=0}^{\infty} \frac{C_{k+1,m+1}}{\omega_{m+1}} \int_{\partial B(0, R-\varepsilon)} \bar{f}(\bar{y}) d\sigma_y \left(\frac{K_{k,m+1,y}(\bar{u}/|u|^2)}{|y|^{m+k}} \frac{1}{|u|^{m-1}} \right) \frac{\bar{D}_u}{1-m} \\ &= \frac{1}{\omega_{m+1}} \int_{\partial B(0, R-\varepsilon)} \bar{f}(\bar{y}) d\sigma_y s(y, u) .\end{aligned}$$

Note that the above series is the Laurent expansion for $s(f)(u)$; it converges absolutely and uniformly in any compact subset of co $\bar{B}(0, 1/R)$ as a series of right monogenic functions. We also have that

$$s(f)(u) = \left(\bar{f}\left(\frac{u}{|u|^2} \right) \frac{1}{|u|^{m-1}} \right) \frac{\bar{D}_u}{1-m} .$$

We now introduce the spherical transform for left monogenic functions in co $(B(0, \bar{R}))$ which tend to zero if $x \to \infty$. To this end we consider the following expansion in the region $|u||y|<1$:

$$\frac{1}{|y|u|-\bar{u}/|u||^{m-1}} = \sum_{k=0}^{\infty} C_{k,m+1}(|y||u|)^k L_{k,m+1}\left(\frac{\langle y, \bar{u} \rangle}{|y||u|} \right) .$$

It is easy to check that in $|u||y|<1$,

$$\begin{aligned}-s(y, u) &= \left(\left[\frac{\bar{x}-\bar{y}}{|x-y|^{m+1}} \right]_{x=\bar{u}/|u|^2} \cdot \frac{1}{|u|^{m-1}} \right) \frac{\bar{D}_u}{1-m} \\ &= \sum_{k=0}^{\infty} C_{k+1,m+1} \frac{\bar{D}_y}{m-1} |y|^{k+1} |u|^{k+1} L_{k+1,m+1}\left(\frac{\langle y, \bar{u} \rangle}{|y||u|} \right) \frac{\bar{D}_u}{1-m} \\ &= \sum_{k=0}^{\infty} C_{k+1,m+1} K_{k,m+1,\bar{u}/|u|^2}(y) |u|^{k+1} \frac{\bar{D}_u}{1-m} .\end{aligned}$$

Now let f be left monogenic in $\operatorname{co}\bar{B}(0, R)$ such that f tends to zero if $x \to \infty$. Then $\bar{f}(\bar{y})$ is right monogenic in $\operatorname{co}\bar{B}(0, R)$ and

$$\bar{f}(\bar{x}) = \sum_{k=0}^{\infty} \frac{C_{k+1,m+1}}{\omega_{m+1}} \int_{\partial B(0,R+\varepsilon)} \bar{f}(\bar{y}) d\sigma_y \frac{K_{k,m+1,x}(y)}{|x|^{m+k}} .$$

We now define the spherical transform $s(f)$ of f by

$$\begin{aligned} s(f)(u) &= \sum_{k=0}^{\infty} \frac{C_{k+1,m+1}}{\omega_{m+1}} \int_{\partial B(0,R+\varepsilon)} f(\bar{y}) d\sigma_y K_{k,m+1,\bar{u}/|u|^2}(y) |u|^{k+1} \frac{\bar{D}_u}{1-m} \\ &= -\frac{1}{\omega_{m+1}} \int_{\partial B(0,R+\varepsilon)} \bar{f}(\bar{y}) d\sigma_y s(y, u) \\ &= \left(\bar{f}\left(\frac{u}{|u|^2}\right) \frac{1}{|u|^{m-1}}\right) \frac{\bar{D}_u}{1-m} . \end{aligned}$$

Observe that $s(f)$ is right monogenic $\mathring{B}(0, 1/R)$. We so arrive at

DEFINITION 1. *Let f be left monogenic in $\Omega \subseteq R^{m+1}$. Then the spherical transform $s(f)$ of f is given by*

$$s(f)(u) = \left(\bar{f}\left(\frac{u}{|u|^2}\right) \frac{1}{|u|^{m-1}}\right) \frac{\bar{D}_u}{1-m} .$$

Note that $s(f)$ is right monogenic in $\{x : x/|x|^2 \in \Omega\}$.

EXAMPLES. Let P_k be a homogeneous monogenic polynomial of order k, i.e.,

$$P_k(x) = \sum_{(l_1,\cdots,l_k)} V_{l_1\cdots l_k}(x) a_{l_1\cdots l_k}$$

for some $a_{l_1\cdots l_k} \in \mathscr{A}$. Then

$$\begin{aligned} s(P_k)(u) &= \left(\bar{P}_k\left(\frac{u}{|u|^2}\right) \frac{1}{|u|^{m-1}}\right) \frac{\bar{D}_u}{1-m} \\ &= \left(\bar{P}_k(u) \frac{1}{|u|^{2k+m-1}}\right) \frac{\bar{D}_u}{1-m} \\ &= \bar{P}_k(u) \left(\frac{1}{|u|^{2k+m-1}} \frac{\bar{D}_u}{1-m}\right) \\ &= \bar{P}_k(u) \frac{\bar{u}}{|u|^{2k+m+1}} \cdot \frac{2k+m-1}{m-1} . \end{aligned}$$

Let Q_k be a left monogenic function of the form

$$Q_k(x) = \sum_{(l_1,\cdots,l_k)} X_{l_1\cdots l_k}(x) b_{l_1\cdots l_k} ;$$

then we have

$$s(Q_k)(u)=\bar{Q}_k(u)\bar{u}|u|^{2k+m-1}\frac{2k+m+1}{1-m} .$$

Observe that $s(P_k)$ may be written as $\sum_{(l_1,\cdots,l_k)} b'_{l_1\cdots l_k}X_{l_1\cdots l_k}(x)$ and that it is a homogeneous right monogenic polynomial of order k.

Before defining the adjoint of a left monogenic function we state the following

LEMMA 2. *Let $f\in M_1(\mathring{B}(0, R_1)\backslash\bar{B}(0, R_2))$ have the Laurent expansion*

$$f(x)=\sum_{k=0}^{\infty} P_k(x)+\sum_{k=0}^{\infty} Q_k(x) ,$$

with

$$P_k=\sum_{(l_1,\cdots,l_k)} V_{l_1\cdots l_k}(x)a_{l_1\cdots l_k}$$

and

$$Q_k=\sum_{(l_1,\cdots,l_k)} X_{l_1\cdots l_k}(x)b_{l_1\cdots l_k} .$$

Then the series

$$\sum_{k=0}^{\infty}\frac{m-1}{2k+m-1}s(P_k)(u)+\sum_{k=0}^{\infty}\frac{1-m}{2k+m+1}s(Q_k)(u)$$

converges to a right monogenic function in $\mathring{B}(0, 1/R_2)\backslash\bar{B}(0, 1/R_1)$.

PROOF. The lemma follows from the estimates (7) and (9) applied to the Laurent expansion of the spherical transform $s(f)$ of f.

DEFINITION 2. Let f be left monogenic in $\mathring{B}(0, R_1)\backslash\bar{B}(0, R_2)$, $0<R_2<R_1$, and let

$$f(x)=\sum_{k=0}^{\infty} P_k(x)+\sum_{k=0}^{\infty} Q_k(x)$$

be its Laurent expansion. Then the adjoint function $\tilde{f}$ of f is defined by

$$\tilde{f}(u)=\sum_{k=0}^{\infty}\frac{m-1}{2k+m-1}s(P_k)(u)+\sum_{k=0}^{\infty}\frac{1-m}{2k+m-1}s(Q_k)(u) .$$

Note that it is right monogenic in $\mathring{B}(0, 1/R_2)\backslash\bar{B}(0, 1/R_1)$.

§ 4. The inner product on $\partial B(0, 1)$.

In this section the inner product between analytic $\mathscr{A}$-valued functions on the unit sphere $\partial B(0, 1)$ of R^{m+1} will be introduced. First of all we give the definition of spherical monogenic functions.

DEFINITION 3. Let $k \in N$. Then we call

(a) P_k: $\partial B(0, 1) \to \mathscr{A}$ an inner spherical monogenic function of order k if $P_k(\omega)$ is the restriction of a left monogenic function $P_k(x)$ in R^{m+1} to $\partial B(0, 1)$ such that

$$P_k(x) = |x|^k P_k(\omega) .$$

(b) Q_k: $\partial B(0, 1) \to \mathscr{A}$ is an outer spherical monogenic function of order k if $Q_k(\omega)$ is the restriction of a left monogenic function $Q_k(x)$ in $R^{m+1} \backslash \{0\}$ to $\partial B(0, 1)$ such that

$$Q_k(x) = |x|^{-(k+m)} Q_k(\omega) .$$

We first prove the theorem of Caucky-Kowalewski for analytic functions on $\partial B(0, 1)$.

LEMMA 3. *Let f be harmonic in a ball $\mathring{B}(a, r)$; then we have $f = f_1 + f_2$, where $Df_1 = 0$ and $\bar{D}f_2 = 0$ in $\mathring{B}(a, r)$.*

PROOF. It suffices to prove the statement for harmonic functions in the unit ball. As $\bar{D}f = g$ is left monogenic in $\mathring{B}(0, 1)$, then, in view of [15], there exists h such that $(\partial/\partial x_0)h = g$ and $Dh = 0$ in $\mathring{B}(0, 1)$. Hence $\bar{D}(h/2) = g = \bar{D}f$ which implies that the problem is solved by taking $f_1 = h/2$ and $f_2 = f - h/2$.

In the following lemma, by an m-dimensional injective C_∞ surface Σ in R^{m+1}, the range of a C_∞-injection from an open subset U of R^m in R^{m+1} is meant.

LEMMA 4. *Let $\Omega \subseteq R^{m+1}$ be open and let Σ be an m-dimensional injective C_∞-surface in Ω such that $\Omega \backslash \Sigma$ is open. Furthermore let $f \in M_1(\Omega \backslash \Sigma; \mathscr{A}) \cap C_0(\Omega; \mathscr{A})$. Then $f \in M_1(\Omega; \mathscr{A})$.*

PROOF. It is easy to show that for any closed m-dimensional interval $I \subseteq \Omega$, $\int_{\partial I} d\sigma f = 0$. Hence by Morera's theorem (see [1]), f is left monogenic in Ω.

THEOREM 1 (Cauchy-Kowalewski). *Let f be an $\mathscr{A}$-valued analytic function on $\partial B(0, 1)$. Then there exist $0 < R_2 < 1 < R_1$ and a unique left monogenic function $f^*(x)$ in $\mathring{B}(0, R_1) \backslash \bar{B}(0, R_2)$ such that $f^*|_{\partial B(0,1)} = f$.*

PROOF. Choose $P \in \partial B(0, 1)$ and define a system of spherical coordinates in a neighbourhood Ω of P in R^{m+1}. Then in Ω, D may be written as

$$D=\frac{\partial}{\partial r}e_{(r)}+\frac{1}{r}\partial_\omega ,$$

where

$$e_{(r)}=\sum_{i=0}^{m}\frac{x_i}{|x|}e_i , \quad r=|x| ,$$

and

$$\partial_\omega=\sum_{i=1}^{m}\frac{1}{\sin\omega_{i-1}\cdots\sin\omega_1}e_{(\omega_i)}\frac{\partial}{\partial\omega_i} ,$$

$(\omega_1, \cdots, \omega_m)$ being the angle-coordinates and $e_{(\omega_i)}=(1/|d\omega/d\omega_i|)(d\omega/d\omega_i)$. It is easy to see that $g(\omega)=-\bar{e}_{(r)}\partial_\omega f(\omega)$, where $\bar{e}_{(r)}=\sum_{i=0}^{m}(x_i/|x|)\bar{e}_i$, is analytic in $\partial B(0, 1)\cap\Omega$.

Furthermore g does not depend on the coordinate system chosen on $\partial B(0, 1)$ so that g is defined on $\partial B(0, 1)$. By the Cauchy-Kowalewski theorem for the Laplacian Δ, there exist $0<R_2<1<R_1$ and a unique $\mathscr{A}$-valued harmonic function h in $\mathring{B}(0, R_1)\backslash\bar{B}(0, R_2)$ such that

$$h|_{\partial B(0,1)}=f \quad\text{and}\quad \frac{\partial}{\partial r}h|_{\partial B(0,1)}=-\bar{e}_r\partial_\omega f=-\bar{e}_r\frac{1}{r}\partial_\omega h|_{\partial B(0,1)} .$$

We claim that h is the desired left monogenic extension f^* of f.

Let P and Ω be as before and choose $r>0$ such that $\bar{B}(P, r)\subseteq\mathring{B}(0, R_1)\backslash\bar{B}(0, R_2)$. Then by Lemma 3, $h(x)=f_1(x)+f_2(x)$, where $Df_1(x)=0$ and $\bar{D}f_2(x)=0$ in $\bar{B}(P, r)$. Hence we have that

$$Dh(x)|_{\bar{B}(P,r)\cap\partial B(0,1)}=2\frac{\partial}{\partial x_0}f_2(x)|_{\bar{B}(P,r)\cap\partial B(0,1)}$$

and

$$Dh(x)|_{\bar{B}(P,r)\cap\partial B(0,1)}=e_{(r)}\left(\frac{\partial}{\partial r}+\frac{1}{r}\bar{e}_{(r)}\partial_\omega\right)h|_{\bar{B}(P,r)\cap\partial B(0,1)}=0 ,$$

which implies that $\gamma=2(\partial/\partial x_0)f_2(x)$ satisfies $\bar{D}\gamma=0$ and $\gamma|_{\bar{B}(P,r)\cap\partial B(0,1)}\equiv 0$. Consequently, using the analogue of Lemma 4 for $\bar{D}$, $\gamma\equiv 0$ so that $Df_2=0$ in $\bar{B}(P, r)$. Hence we proved that $Dh=0$ in $B(P, r)$ from which it follows that h is left monogenic in $\mathring{B}(0, R_1)\backslash\bar{B}(0, R_2)$. Moreover, by virtue of Lemma 4, h is unique.

The extension f^* of f obtained in Theorem 1 is called the Cauchy-Kowalewski extension.

COROLLARY. *Let f be analytic on $\partial B(0, 1)$. Then f admits a canonical expansion into spherical monogenic functions:*

$$f(\omega)=\sum_{k=0}^{\infty} P_k(\omega)+\sum_{k=0}^{\infty} Q_k(\omega) .$$

Furthermore the series $\sum_{k=0}^{\infty} P_k(x)+\sum_{k=0}^{\infty} Q_k(x)$ *converges in a neighborhood of* $\partial B(0, 1)$ *to the Cauchy-Kowalewski extension* f^* *of* f *and there exist* $1>\varepsilon>0$ *and* $C>0$ *such that*

$$\sup_{\omega\in\partial B(0,1)} |P_k(\omega)|_0 \leqq C(1-\varepsilon)^k .$$

and

$$\sup_{\omega\in\partial B(0,1)} |Q_k(\omega)|_0 \leqq C(1-\varepsilon)^k .$$

Conversely, if a sequence $(P_k(\omega), Q_k(\omega))_{k\in N}$ *satisfies the above estimates, then*

$$f(\omega)=\sum_{k=0}^{\infty} P_k(\omega)+\sum_{k=0}^{\infty} Q_k(\omega)$$

is analytic on $\partial B(0, 1)$.

PROOF. Corollary follows immediately from the previous theorem and the estimates (7) and (9).

We now come to

DEFINITION 4. The inner product of two analytic functions f and g on $\partial B(0, 1)$ is defined by

$$(f, g)=\frac{1}{\omega_{m+1}}\int_{\partial B(0,1)} \tilde{f}(\omega)d\sigma_\omega g(\omega) .$$

REMARKS. Let f and g be analytic functions on $\partial B(0, 1)$ which may be extended to left monogenic functions f^* and g^* in $\mathring{B}(0, R_1)\backslash\bar{B}(0, R_1^{-1})$ and put

$$f(\omega)=\sum_{k=0}^{\infty} P_k(\omega)+\sum_{k=0}^{\infty} Q_k(\omega)$$

and

$$g(\omega)=\sum_{k=0}^{\infty} P_k'(\omega)+\sum_{k=0}^{\infty} Q_k'(\omega) .$$

(1) If S is an open neighbourhood of the origin with C_1-boundary $\partial S\subseteqq\mathring{B}(0, R_1)\backslash\bar{B}(0, R_1^{-1})$, then

$$(f, g)=\frac{1}{\omega_{m+1}}\int_{\partial S} \tilde{f}^*(u)d\sigma_u g(u) .$$

(2)
$$\begin{aligned}\tilde{f}(\omega)&=\sum_{k=0}^{\infty}\tilde{P}_k(\omega)+\sum_{k=0}^{\infty}\bar{Q}_k(\omega)\\&=\Big(\sum_{k=0}^{\infty}\bar{P}_k(\omega)+\sum_{k=0}^{\infty}\bar{Q}_k(\omega)\Big)\bar{e}_{(r)}\\&=\bar{f}(\omega)\bar{e}_{(r)}\ .\end{aligned}$$

Hence, as $d\sigma_\omega=e_{(r)}dS_\omega$ and $\bar{e}_{(r)}e_{(r)}=1$,

$$(f,g)=\frac{1}{\omega_{m+1}}\int_{\partial B(0,1)}\bar{f}(\omega)g(\omega)dS_\omega\ .$$

Furthermore $\mathrm{Re}\,(f,f)\geqq 0$ and $\mathrm{Re}\,(f,f)=0$ implies that $f=0$.

(3) From the orthogonality relations (11) it follows easily that

$$(P_k, P_l')=0\ ,\quad k\neq l$$
$$(Q_k, Q_l')=0\ ,\quad k\neq l$$

and, as $\int_{\partial B(0,R)}\tilde{P}_k(u)d\sigma_u Q_l'(u)\to 0$ when $R\to\infty$,

$$(P_k, Q_l')=0\quad\text{for all}\quad k,l\in N\ .$$

Analogously, as $\int_{\partial B(0,R)}\tilde{Q}_k(u)d\sigma_u P_l'(u)\to 0$ when $R\to 0$,

$$(Q_k, P_l')=0\quad\text{for all}\quad k,l\in N\ .$$

Hence we obtain that

$$(f,g)=\sum_{k=0}^{\infty}(P_k, P_k')+\sum_{k=0}^{\infty}(Q_k, Q_k')\ .$$

§ 5. Boundary values on $\partial B(0, 1)$.

5.1. Analytic functionals on $\partial B(0, 1)$.

Let $M_1(\partial B(0,1);\mathscr{A})$ (resp. $M_1^{(r)}(\partial B(0,1);\mathscr{A})$) be the space of functions which are left (resp. right) monogenic in a neighbourhood of $\partial B(0, 1)$. Then in view of Lemma 2 and the uniqueness of the Cauchy-Kowalewski extension, we may define the mappings

$$f\to\tilde{f}=(\bar{f}(\omega)\bar{e}_{(r)})^{*(r)}\ ,\quad\text{when}\quad f\in M_1(\partial B(0,1);\mathscr{A})$$

and

$$f\to\tilde{f}=(\bar{e}_{(r)}\bar{f}(\omega))^{*}\ ,\quad\text{when}\quad f\in M_1^{(r)}(\partial B(0,1);\mathscr{A})\ ,$$

where * and $*^{(r)}$ denote the left and right monogenic extensions. As for any $f \in M_1(\partial B(0,1); \mathscr{A})$ (resp. $f \in M_1^{(r)}(\partial B(0,1); \mathscr{A})$), $\tilde{\tilde{f}}=f$ (resp. $f=\tilde{\tilde{f}}$), the mapping $f \to \tilde{f}$ is an algebraic isomorphism between $M_1(\partial B(0,1); \mathscr{A})$ and $M_1^{(r)}(\partial B(0,1); \mathscr{A})$ having as inverse the mapping $f \to \tilde{f}$ on $M_1^{(r)}(\partial B(0,1); \mathscr{A})$. Furthermore the above isomorphisms are also topological when we define a topology in the following way. Since for each $0<\varepsilon<1$, $M_1(\mathring{B}(1+\varepsilon)\backslash \bar{B}(1-\varepsilon); \mathscr{A})$ and $M_1^{(r)}(\mathring{B}(1+\varepsilon)\backslash \bar{B}(1-\varepsilon)$ are Fréchet $\mathscr{A}$-modules (see [2]), it is natural to introduce

DEFINITION 5. Call

$$M_1(\partial B(0,1); \mathscr{A}) = \lim_{1>\varepsilon>0} \operatorname{ind} M_1(\mathring{B}(1+\varepsilon)\backslash \bar{B}(1-\varepsilon); \mathscr{A})$$

and

$$M_1^{(r)}(\partial B(0,1); \mathscr{A}) = \lim_{1>\varepsilon>0} \operatorname{ind} M_1^{(r)}(\mathring{B}(1+\varepsilon)\backslash \bar{B}(1-\varepsilon); \mathscr{A}) .$$

By Theorem 1 we know that the set of $\mathscr{A}$-valued analytic functions on $\partial B(0,1)$ coincides with the set of restrictions of functions belonging to $M_1(\partial B(0,1); \mathscr{A})$ (resp. $M_1^{(r)}(\partial B(0,1); \mathscr{A})$. Hence it is natural to idedtify the module $M_1(\partial B(0,1); \mathscr{A})$ (resp. $M_1^{(r)}(\partial B(0,1); \mathscr{A})$) with the right (resp. left) $\mathscr{A}$-module of analytic functions on $\partial B(0,1)$. As furthermore the elements of the dual module of $M_1^{(r)}(\partial B(0,1); \mathscr{A})$ are left $\mathscr{A}$-linear functionals, this module will be called the module of left analytic functionals on $\partial B(0,1)$.

Let T be a left analytic functional on $\partial B(0,1)$ and consider the function

$$\hat{T}(x) = \frac{1}{\omega_{m+1}} \left\langle T_\omega, \frac{\bar{x}-\bar{\omega}}{|x-\omega|^{m+1}} \right\rangle ;$$

then $\hat{T}(x)$ is left monogenic in $\boldsymbol{R}^{m+1}\backslash \partial B(0,1)$ and tends to zero if $x \to \infty$. Furthermore if $\varphi \in M_1^{(r)}(\partial B(0,1); \mathscr{A})$ then for ε sufficiently small (see [4])

$$\langle T, \varphi \rangle = \int_{\partial B(0,1+\varepsilon)} \varphi(x) d\sigma_x \hat{T}(x) - \int_{\partial B(0,1-\varepsilon)} \varphi(x) d\sigma_x \hat{T}(x) .$$

Conversely, if $f \in M_1(\boldsymbol{R}^{m+1}\backslash \partial B(0,1); \mathscr{A})$ tends to zero when $x \to \infty$, T_f defined by

$$\langle T_f, \varphi \rangle = \int_{\partial B(0,1+\varepsilon)} \varphi(x) d\sigma_x f(x) - \int_{\partial B(0,1-\varepsilon)} \varphi(x) d\sigma_x f(x)$$

is a left analytic functional on $\partial B(0,1)$. We now give an expression of T into spherical monogenic functions. To this end we consider the Laurent expansion of $\hat{T}$:

$$\hat{T}(x) = -\sum_{k=0}^{\infty} P_k(x)\ , \quad \text{in} \quad \mathring{B}(0, 1)$$

and

$$\hat{T}(x) = \sum_{k=0}^{\infty} Q_k(x)\ , \quad \text{in} \quad \text{co}\,\bar{B}(0, 1)\ ,$$

which will be written as

$$\hat{T}(x) = \sum_{k=0}^{\infty} (\chi_{\text{co}\bar{B}(0,1)} Q_k(x) - \chi_{\mathring{B}(0,1)} P_k(x))\ ,$$

where χ_A is the characteristic function of $A \subseteq R^{m+1}$. It follows directly from the estimates (7) and (9) that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\sup_{\omega \in \partial B(0,1)} (|P_k(\omega)|_0, |Q_k(\omega)|_0) \leq C_\varepsilon (1+\varepsilon)^k\ .$$

Now take $f \in M_1(\partial B(0, 1); \mathscr{A})$. Then by the Corollary to Theorem 1, f may developed into a series of spherical monogenic functions

$$f(\omega) = \sum_{k=0}^{\infty} (P'_k(\omega) + Q'_k(\omega))$$

such that for some $1 > \delta > 0$ and $C > 0$,

$$\sup_{\omega \in \partial B(0,1)} (|P'_k(\omega)|_0, |Q'_k(\omega)|_0) \leq C(1-\delta)^k\ .$$

Hence, as the sequence $(\tilde{P}'_k(\omega), \tilde{Q}'_k(\omega))_{k \in N}$ satisfies the same estimate, the series

$$\sum_{k=0}^{\infty} [(P'_k(\omega), P_k(\omega)) + (Q'_k(\omega), Q_k(\omega))]$$

converges absolutely. Furthermore, as in $M_1^{(r)}(\partial B(0, 1); \mathscr{A})$, $\tilde{f}(\omega) = \sum_{k=0}^{\infty} (P'_k(\omega) + Q'_k(\omega))$,

$$\begin{aligned}
\langle T, \tilde{f} \rangle &= \sum_{k=0}^{\infty} (\langle T, \tilde{P}'_k(\omega) \rangle + \langle T, \tilde{Q}'_k(\omega) \rangle) \\
&= \sum_{k=0}^{\infty} \Big(\int_{\partial B(0,1+\eta)} \tilde{P}'_k(x) d\sigma_x \hat{T}(x) - \int_{\partial B(0,1-\eta)} \tilde{P}'_k(x) d\sigma_x \hat{T}(x) \Big) \\
&\quad + \sum_{k=0}^{\infty} \Big(\int_{\partial B(0,1+\eta)} \tilde{Q}'_k(x) d\sigma_x \hat{T}(x) - \int_{\partial B(0,1-\eta)} \tilde{Q}'_k(x) d\sigma_x \hat{T}(x) \Big) \\
&= \omega_{m+1} \sum_{k=0}^{\infty} [(P'_k(\omega), P_k(\omega)) + (Q'_k(\omega), Q_k(\omega))]\ .
\end{aligned}$$

Hence it is natural to extend the inner product on $\partial B(0, 1)$ as follows.

Let T be a left analytic functional on $\partial B(0, 1)$ and let $f \in M_1(\partial B(0, 1); \mathscr{A})$; then we define

$$(f, T) = \frac{1}{\omega_{m+1}} \langle T, \tilde{f} \rangle \ .$$

Furthermore as any $\varphi \in M_1^{(r)}(\partial B(0, 1); \mathscr{A})$ may be written in the form $\varphi = \tilde{f}$, with $f \in M_1(\partial B(0, 1); \mathscr{A})$, we have that

$$\langle T, \varphi \rangle = \omega_{m+1}(f, T) = \omega_{m+1} \sum_{k=0}^{\infty} [(f, P_k) + (f, Q_k)] = \sum_{k=0}^{\infty} [\langle P_k, \varphi \rangle + \langle Q_k, \varphi \rangle]$$

or $T = \sum_{k=0}^{\infty} (P_k(\omega) + Q_k(\omega))$ in the weak topology on $M_1^{(r)}(\partial B(0, 1); \mathscr{A})'$. Conversely, if a sequence $(P_k, Q_k)_{k \in N}$ of spherical monogenic functions satisfies the estimate

$$\sup_{\omega \in \partial B(0,1)} (|P_k(\omega)|_0, |Q_k(\omega)|_0) \leqq C_\varepsilon (1+\varepsilon)^k \ ,$$

then obviously $T = \sum_{k=0}^{\infty} (P_k(\omega) + Q_k(\omega))$ is a left analytic functional on $\partial B(0, 1)$.

5.2. Distributions on $\partial B(0, 1)$.

We consider the left module $\mathscr{E}_{(l)}(\partial B(0, 1); \mathscr{A})$ of $\mathscr{A}$-valued C_∞-functions on $\partial B(0, 1)$ and its dual module $\mathscr{E}'_{(l)}(\partial B(0, 1); \mathscr{A})$ of left linear $\mathscr{A}$-valued distributions (called "distributions" for short) on $\partial B(0, 1)$.

DEFINITION 6. Let $f \in M_1(\boldsymbol{R}^{m+1} \backslash \partial B(0, 1))$. Then f has a distributional boundary value on $\partial B(0, 1)$ if and only if

$$\lim_{\varepsilon \to 0+} \int_{\partial B(0, 1 \pm \varepsilon)} \varphi(x) d\sigma_x f(x) \ , \quad \varphi \in \mathscr{E}_{(l)}(\partial B(0, 1); \mathscr{A}) \ ,$$

define two distributions on $\partial B(0, 1)$, denoted by $\partial^+ f$ and $\partial^- f$ respectively.

In this case the boundary value of f is denoted by BV $f = \partial^+ f - \partial^- f$.

First of all we shall introduce a refined version of the Laplace-Beltrami operator on $\partial B(0, 1)$. To this end we consider the operators $D = (\partial/\partial r) e_{(r)} + (1/r)\partial_\omega$ and $\bar{D} = (\partial/\partial r)\bar{e}_r + (1/r)\bar{\partial}_\omega$ in $\boldsymbol{R}^{m+1} \backslash \{0\}$. It is easy to see that ∂_ω is an operator in $\partial B(0, 1)$ and that for any function $\varphi \in \mathscr{E}_{(l)}(\partial B(0, 1); \mathscr{A})$ and $k \in \boldsymbol{N}$, $(\bar{e}_{(r)} \partial_\omega)^k \varphi(\omega)$ belongs to $\mathscr{E}_{(l)}(\partial B(0, 1); \mathscr{A})$.

DEFINITION 7. The operator $\Gamma = \bar{e}_{(r)} \partial_\omega$ is called the spherical Cauchy-Riemann operator. The operator $\tilde{\Gamma} = \bar{\partial}_\omega e_{(r)}$ is called the adjoint of Γ. Observe that for $m=1$, $\Gamma = i(d/d\theta)$.

THEOREM 2. *The only (real) eigenvalues of Γ on the spaces of $\mathscr{A}$-valued analytic functions in $\partial B(0, 1)$ are $-k$ and $k+m$, $k \in \boldsymbol{N}$. The* *corresponding eigenspaces are the spaces of inner and outer spherical monogenic functions of order k.*

PROOF. Let $P_k(\omega)$ be an inner spherical monogenic function having $P_k(x)$ as a left monogenic extension. Then

$$\frac{\partial}{\partial r}P_k(x)+\frac{1}{r}\Gamma P_k(x)\equiv 0 .$$

However, as $(\partial/\partial r)P_k(x)=(k/r)P_k(x)$ in $\boldsymbol{R}^{m+1}\backslash\{0\}$, P_k is an eigenvector of Γ with eigenvalue $-k$. Analogously any outer spherical monogenic function $Q_k(\omega)$ is an eigenvector of Γ with eigenvalue $k+m$. Obviously these are the only (real) eigenvalues and eigenvectors of Γ.

Let us recall that on a Riemannian manifold S with metric tensor $g^{\mu\nu}$, the Laplace-Beltrami operator, acting on C_∞-functions on S, is given by

$$\Delta_S=\sum_{\mu,\nu}\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\sqrt{|g|}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu} .$$

We now want to express the Laplace-Beltrami operator Δ_S on the unit sphere in terms of the spherical Cauchy-Riemann operator Γ and its adjoint $\tilde{\Gamma}$. Therefore we need the expression of the Laplacian in $\boldsymbol{R}^{m+1}$ in spherical coordinates: in $\boldsymbol{R}^{m+1}\backslash\{0\}$ (see [17])

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{m}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta_S .$$

Unless it is specified explicitly, operators are supposed to act on C_∞-functions.

THEOREM 3. *The Laplace-Beltrami operator equals*

$$\Delta_S=(\tilde{\Gamma}-1)\Gamma .$$

Furthermore for any inner and outer spherical monogenic function $P_k(\omega)$ *and* $Q_k(\omega)$

$$\tilde{\Gamma}P_k=e_{(r)}\widetilde{\tilde{P}_k\partial_\omega}$$

$$\tilde{\Gamma}Q_k=e_{(r)}\widetilde{\tilde{Q}_k\partial_\omega} .$$

We also have the identity

$$\Gamma+\tilde{\Gamma}=m1 .$$

PROOF. In $\boldsymbol{R}^{m+1}\backslash\{0\}$ we have that

$$\begin{aligned}\Delta &= \bar{D}D = \bar{D}e_{(r)}\bar{e}_{(r)}D \\ &= \Big(\frac{\partial}{\partial r} + \frac{1}{r}\tilde{\Gamma}\Big)\Big(\frac{\partial}{\partial r} + \frac{1}{r}\Gamma\Big) \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}(\Gamma + \tilde{\Gamma})\frac{\partial}{\partial r} + \frac{1}{r^2}\tilde{\Gamma}\Gamma + \frac{\partial}{\partial r}\Big(\frac{1}{r}\Big)\Gamma \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}(\Gamma + \tilde{\Gamma})\frac{\partial}{\partial r} + \frac{1}{r^2}(\tilde{\Gamma} - 1)\Gamma \ .\end{aligned}$$

As on the other hand in $\boldsymbol{R}^{m+1}\backslash\{0\}$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{m}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S \ ,$$

we immediately obtain the identities

$$\Gamma + \tilde{\Gamma} = m1$$

and

$$\Delta_S = (\tilde{\Gamma} - 1)\Gamma \ .$$

Let P_k be an inner spherical monogenic function of order k; then $\tilde{P}_k\partial_\omega = (\bar{P}_k\bar{e}_{(r)})\partial_\omega$ and hence $e_{(r)}\widetilde{\tilde{P}_k\partial_\omega} = e_{(r)}(\bar{e}_{(r)}\bar{\partial}_\omega(e_r P_k)) = \tilde{\Gamma}P_k$. Analogusly for any outer spherical monogenic function Q_k, $e_{(r)}\widetilde{\tilde{Q}_k\partial_\omega} = \tilde{\Gamma}Q_k$.

Observe that for any inner and outer spherical monogenic function P_k and Q_k, $\tilde{\Gamma}P_k = (k+m)P_k$ and $\tilde{\Gamma}Q_k = -kQ_k$. Note also that $P_k(\omega)$ and $Q_{k-1}(\omega)$ are eigenfunctions of the Laplace-Beltrami operator Δ_S with eigenvalue $-k(k+m-1)$. Hence $P_k(\omega)$ and $Q_{k-1}(\omega)$ are $\mathscr{A}$-valued spherical harmonics of order k.

We now prove an expansion of $\mathscr{A}$-valued C_∞-functions on $\partial B(0, 1)$ in spherical monogenic functions. We first define the operator Γ on the space of left analytic functionals on $\partial B(0, 1)$. Let $T \in M_1^{(r)}(\partial B(0, 1); \mathscr{A})'$ be a left analytic functional and let $\hat{T}(x) = 1/\omega_{m+1}\langle T_\omega, (\bar{x} - \bar{\omega})/|x - \omega|^{m+1}\rangle$ admit the Laurent expansion

$$\hat{T}(x) = \sum_{k=0}^{\infty} (\chi_{\mathrm{co}\bar{B}(0,1)}Q_k(x) - \chi_{\mathring{B}(0,1)}P_k(x)) \ .$$

Then

$$g(x) = \sum_{k=0}^{\infty} ((k+m)\chi_{\mathrm{co}B(0,1)}Q_k(x) - (-k)\chi_{\mathring{B}(0,1)}P_k(x)) \ ,$$

which is completely determined by T, belongs to $M_1(\boldsymbol{R}^{m+1}\backslash\partial B(0, 1); \mathscr{A})$ and tends to zero at infinity. Hence $g(x)$ determines an element T_g of

$M_1^{(r)}(\partial B(0,1);\mathscr{A})'$ and we define $\Gamma T=T_g$. Observe that Γ is well defined and s-bounded on the space $M_1^{(r)}(\partial B(0,1);\mathscr{A})'$.

Furthermore, when T is an analytic function on $\partial B(0,1)$, then we may extend T to a left monogenic function f in a neighbourhood of $\partial B(0,1)$ admitting a Laurent expantion

$$f(x)=\sum_{k=0}^{\infty}(P_k'(x)+Q_k'(x)) .$$

Consequently

$$\bar{e}_{(r)}\partial_\omega f(x)=\sum_{k=0}^{\infty}((-k)P_k'(x)+(k+m)Q_k'(x)) ,$$

which implies that ΓT, in the sense of C_∞-functions, equals ΓT in the sense of analytic functionals.

We also have that for any left analytic functional T and any $f\in M_1(\partial B(0,1);\mathscr{A})$

$$\begin{aligned}(f,\Gamma T)&=\sum_{k=0}^{\infty}[(-k)(P_k', P_k)+(k+m)(Q_k', Q_k)]\\&=(\Gamma f, T) .\end{aligned}$$

Let now $\mathscr{E}_{(r)}(\partial B(0,1);\mathscr{A})$ be the right module of all $\mathscr{A}$-valued C_∞-functions on $\partial B(0,1)$ and let $\varphi\in\mathscr{E}_{(r)}(\partial B(0,1);\mathscr{A})$. Then by the density of $M_1(\partial B(0,1);\mathscr{A})$ in $\mathscr{E}_{(r)}(\partial B(0,1);\mathscr{A})$ it is obvious that $\Gamma\varphi$ in the sense of analytic functionals equals $\Gamma\varphi$ in the sense of C_∞-functions.

Furthermore the representing function

$$\hat{\varphi}(x)=\frac{1}{\omega_{m+1}}\int_{\partial B(0,1)}\frac{\bar{x}-\bar{\omega}}{|x-\omega|^{m+1}}d\sigma_\omega\varphi(\omega)$$

of the analytic functional determined by φ admits a Laurent expansion

$$\hat{\varphi}(x)=\sum_{k=0}^{\infty}(\chi_{\mathrm{co}\bar{B}(0,1)}Q_k(x)-\chi_{\mathring{B}(0,1)}P_k(x)) ,$$

where

$$P_k(x)=\frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial B(0,1)}K_{k,m+1,\omega}(x)d\sigma_\omega\varphi(\omega)$$

and

$$Q_k(x)=\frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial B(0,1)}\frac{K_{k,m+1,x}(\omega)}{|x|^{m+k}}d\sigma_\omega\varphi(\omega) .$$

Consequently we obtain the estimate

$$\sup_{\omega\in\partial B(0,1)}\{|P_k(\omega)|_0,\ |Q_k(\omega)|_0\}\leqq CC_{k+1,m+1}(1+k^2)\sup_{\omega\in\partial B(0,1)}|\varphi(\omega)|_0\ .$$

THEOREM 4. *Let* $\varphi\in\mathscr{E}_{(r)}(\partial B(0,1);\mathscr{A})$ *and let*

$$\hat{\varphi}(x)=\sum_{k=0}^{\infty}\chi_{\mathrm{co}\bar{B}(0,1)}Q_k(x)-\chi_{\mathring{B}(0,1)}P_k(x)\ .$$

Then for any $s\in N$ *there exists a constant* $C_s>0$ *such that*

$$\sup_{\omega\in\partial B(0,1)}\{|P_k(\omega)|_0,\ |Q_k(\omega)|_0\}\leqq C_s(1+k)^{-s}\ .$$

Conversely if a sequence $(P_k, Q_k)_{k\in N}$ *satisfies the above estimate,*

$$\varphi(\omega)=\sum_{k=0}^{\infty}[P_k(\omega)+Q_k(\omega)]\in\mathscr{E}_{(r)}(\partial B(0,1);\mathscr{A})\ ,$$

$$\hat{\varphi}(\omega)=\sum_{k=0}^{\infty}(\chi_{\mathrm{co}B(0,1)}Q_k(x)-\chi_{\mathring{B}(0,1)}P_k(x))$$

and $\lim_{\varepsilon\to 0+}\hat{\varphi}(\omega(1\pm\varepsilon))$ *exists in the* C_∞*-topology.*

PROOF. We know that $\varphi(\omega)=\sum_{k=0}^{\infty}(P_k(\omega)+Q_k(\omega))$ in the sense of analytic functionals. Hence

$$(1-\Gamma)^s\varphi(\omega)=\sum_{k=0}^{\infty}((k+1)^sP_k(\omega)+(-k-m+1)^sQ_k(\omega))\ .$$

But as $(1-\Gamma)^s\varphi(\omega)$ in the sense of analytic functionals equals $(1-\Gamma)^s\varphi$ in the sense of C_∞-functions, $(1-\Gamma)^s\varphi$ is a C_∞-function and so

$$(1+k)^s\sup_{\omega\in\partial B(0,1)}|P_k(\omega)|_0\leqq C_s'C_{k+1,m+1}(1+k^2)\ ,$$

$$(k+m-1)^s\sup_{\omega\in\partial B(0,1)}|Q_k(\omega)|_0\leqq C_s'C_{k+1,m+1}(1+k^2)\ ,$$

where $C_s'=C\sup_{\omega\in B(0,1)}|(1-\Gamma)^s\varphi(\omega)|_0$.

As furthermore $(C_{k+1,m+1}(1+k^2))_{k\in N}$ is a slowly growing sequence which does not depend on s, the stated estimate is obvious.

Conversely consider a sequence $(P_k, Q_k)_{k\in N}$ which satisfies the stated estimate, then $(P_k)_{k\in N}$ and $(Q_k)_{k\in N}$ are sequences of spherical harmonics satisfying the same estimate. Hence it follows from Seeley [13] that $\sum_{k=0}^{\infty}P_k(\omega)+Q_k(\omega)$ converges in the sense of C_∞-functions to a function φ. When we provide the module of sequences $(P_k, Q_k)_{k\in N}$ satisfying the stated estimates with the Fréchet topology determined by the seminorms

$$\sup_{k\in N}\sup_{\omega\in\partial B(0,1)}\{(1+k)^s|P_k(\omega)|_0,\ (1+k)^s|Q_k(\omega)|_0\}=p_s((P_k, Q_k))\ ,\quad s\in N\ ,$$

it is easy to see that this module is topologically isomorphic to $\mathscr{E}_{(r)}(\partial B(0,1);\mathscr{A})$.

Consider now $\hat{\varphi}(x)=\sum_{k=0}^{\infty}\chi_{\mathrm{co}\bar{B}(0,1)}Q_k(x)-\chi_{\mathring{B}(0,1)}P_k(x)$. Then $\hat{\varphi}(\omega(1+\varepsilon))$ corresponds to the sequence $(0,(1+\varepsilon)^{-k}Q_k(\omega))_{k\in N}$, which converges to $(0,Q_k(\omega))_{k\in N}$ for the p_s-seminorms. Hence $\lim_{\varepsilon\to 0+}\hat{\varphi}(\omega(1+\varepsilon))$ exists in $\mathscr{E}_{(r)}(\partial B(0,1);A)$. Analogously $\lim_{\varepsilon\to 0+}\hat{\varphi}(\omega(1-\varepsilon))$ exists in $\mathscr{E}_{(r)}(\partial B(0,1);\mathscr{A})$ and

$$\varphi(\omega)=\lim_{\varepsilon\to 0+}(\hat{\varphi}(\omega(1+\varepsilon))-\hat{\varphi}(\omega(1-\varepsilon))\ .$$

PROPOSITION 1. *For $m\geqq 2$ the operator $1-\Gamma$, acting on $M_1^{(r)}(\partial B(0,1);\mathscr{A})$, has an inverse.*

PROOF. Let $T\in M_1^{(r)}(\partial B(0,1);\mathscr{A})'$ admit the expansion $T=\sum_{k=0}^{\infty}(P_k+Q_k)$. Then $S=\sum_{k=0}^{\infty}P_k/(k+1)+Q_k/(1-(k+m))$ belongs to $M_1^{(r)}(\partial B(0,1);\mathscr{A})'$ and $(1-\Gamma)S=T$. Furthermore $(1-\Gamma)S=0$ implies that $S=0$, so that $(1-\Gamma)^{-1}$ is defined. Obviously $(1-\Gamma)^{-1}$ is s-bounded.

The indicatrix $\widehat{(1-\Gamma)^{-1}T}$ of $(1-\Gamma)^{-1}T$ may easily be found in the following way. Let $\hat{T}(x)|_{\mathring{B}(0,1)}=-\sum_{k=0}^{\infty}P_k(x)$; then

$$\begin{aligned}\frac{1}{|x|}\int_0^{|x|}\hat{T}(s\omega)ds&=-\sum_{k=0}^{\infty}P_k(\omega)\frac{1}{|x|}\int_0^{|x|}s^k ds\\&=-\sum_{k=0}^{\infty}\frac{P_k(x)}{k+1}\ .\\&=\widehat{(1-\Gamma)^{-1}T}|_{\mathring{B}(0,1)}\ .\end{aligned}$$

If $\hat{T}(x)|_{\mathrm{co}\bar{B}(0,1)}=\sum_{k=0}^{\infty}Q_k(x)$, then

$$\begin{aligned}\frac{1}{|x|}\int_{-\infty}^{|x|}\hat{T}(s\omega)ds&=\sum_{k=0}^{\infty}Q_k(\omega)\frac{1}{|x|}\int_{-\infty}^{|x|}s^{-(k+m)}ds\\&=\sum_{k=0}^{\infty}\frac{Q_k(\omega)}{1-(k+m)}\\&=\widehat{(1-\Gamma)^{-1}T}|_{\mathrm{co}\bar{B}(0,1)}\ .\end{aligned}$$

PROPOSITION 2. *Let $\varphi\in M_1^{(r)}(\partial B(0,1);\mathscr{A})'$. Then $\varphi\in\mathscr{E}_{(r)}(\partial B(0,1);\mathscr{A})$ if and only if $(1-\Gamma)^s\varphi$ is a continuous function for all $s\in N$. Furthermore a system of seminorms on $\mathscr{E}_{(r)}(\partial B(0,1);\mathscr{A})$ may be given by*

$$\{\sup_{\omega\in\partial B(0,1)}\sup_{l\leq s}|(1-\Gamma)^l\varphi(\omega)|_0\colon s\in N\}\ .$$

PROOF. When $\varphi\in\mathscr{E}_{(r)}(\partial B(0,1);\mathscr{A})$, then $(1-\Gamma)^s\varphi$ is continuous for all $s \in N$. When $(1-\Gamma)^s\varphi$ is continuous for all s, then, if $\varphi(\omega)=\sum_{k=0}^{\infty}(P_k(\omega)+Q_k(\omega))$, the sequence $(P_k(\omega), Q_k(\omega))_{k\in N}$ satisfies the estimate of Theorem 4, which implies that $\varphi \in \mathscr{E}_{(r)}(\partial B(0,1); \mathscr{A})$. Using this result one sees immediately that $\{\sup_{\omega\in\partial B(0,1)}\sup_{l\leq s}|(1-\Gamma)^l\varphi(\omega)|_0: s\in N\}$ induces a Fréchet-topology on $\mathscr{E}_{(r)}(\partial B(0,1); \mathscr{A})$ which is equivalent to the usual topology.

We now consider distributions on $\partial B(0,1)$. Let $T\in \mathscr{E}'_{(l)}(\partial B(0,1); \mathscr{A})$; then

$$\hat{T}(x)=\frac{1}{\omega_{m+1}}\left\langle T_\omega, \frac{\bar{x}-\bar{\omega}}{|x-\omega|^{m+1}}\right\rangle$$

may be written in the form

$$\hat{T}(x)=\left(\sum_{k=0}^{\infty}\chi_{\mathrm{co}\bar{B}(0,1)}Q_k(x)-\chi_{\mathring{B}(0,1)}P_k(x)\right),$$

where

$$P_k(x)=\frac{C_{k+1,m+1}}{\omega_{m+1}}\langle T_\omega, K_{k,m+1,\omega}(x)\rangle$$

and

$$Q_k(x)=\frac{C_{k+1,m+1}}{\omega_{m+1}}\left\langle T_\omega, \frac{K_{k,m+1,x}(\omega)}{|x|^{m+k}}\right\rangle .$$

Note that T may act also on $\mathscr{E}_{(r)}(\partial B(0,1); \mathscr{A})$ as follows: Let $\varphi=\sum_{k=0}^{\infty}(P'_k+Q'_k)\in \mathscr{E}_{(r)}(\partial B(0,1); \mathscr{A})$; then we put

$$(\varphi, T)=\frac{1}{\omega_{m+1}}\langle T, \tilde{\varphi}\rangle ,$$

where

$$\tilde{\varphi}(\omega)=\sum_{k=0}^{\infty}(\tilde{P}'_k(\omega)+\tilde{Q}'_k(\omega))=\bar{\varphi}(\omega)\bar{e}_{(r)}$$

establishes an isomorphism between $\mathscr{E}_{(l)}(\partial B(0,1); \mathscr{A})$ and $\mathscr{E}_{(r)}(\partial B(0,1); \mathscr{A})$. As $T\in \mathscr{E}'_{(l)}(\partial B(0,1); \mathscr{A})$ may be considered as a left analytic functional, ΓT is defined and, for any function $\varphi\in M_1(\partial B(0,1); \mathscr{A})$,

$$(\varphi, \Gamma T)=(\Gamma\varphi, T) .$$

As $\varphi\in \mathscr{E}_{(r)}(\partial B(0,1); \mathscr{A})$ is the limit of a sequence $\varphi_k\in M_1(\partial B(0,1); \mathscr{A})$ in the C_∞-topology, for any $\varphi\in \mathscr{E}_{(r)}(\partial B(0,1); \mathscr{A})$

$$(\varphi, \Gamma T)=(\Gamma\varphi, T) .$$

In the following lemma we make use of the fact that a constant k_m, only depending on m, may be found such that for any continuous function φ on $\partial B(0, 1)$, $(1-\Gamma)^{-s}T$ is continuous on $\partial B(0, 1)$ whenever $s \geq k_m$.

LEMMA 6. *Let $T \in \mathscr{E}'_{(l)}(\partial B(0, 1); \mathscr{A})$. Then there exists $s \in N$ such that $(1-\Gamma)^{-s}T$ is a Radon measure on $\partial B(0, 1)$.*

PROOF. For the functional $\tau: \varphi \rightarrow (\varphi, T)$ on $\mathscr{E}_{(r)}(\partial B(0, 1); \mathscr{A})$, which is $\boldsymbol{R}$-linear, we have that $\tau(\varphi) = (1/\omega_{m+1})\langle T, \bar{\varphi}\rangle$ and that for all $c \in \mathscr{A}$, $\tau(\varphi c) = c\tau(\varphi)$. Moreover it is continuous on $\mathscr{E}_{(r)}(\partial B(0, 1); \mathscr{A})$. Hence there exists $s \in N$ such that

$$|(\varphi, T)|_0 \leq C \sup_{l \leq s} \sup_{\omega \in \partial B(0,1)} |(1-\Gamma)^l \varphi|_0 ,$$

which implies that for any $\varphi \in \mathscr{E}_{(r)}(\partial B(0, 1); \mathscr{A})$

$$|(\varphi, (1-\Gamma)^{-(s+k_m)}T)|_0 \leq C \sup_{-s \leq l \leq 0} \sup_{\omega \in \partial B(0,1)} |(1-\Gamma)^{-(l+k_m)}\varphi|_0 + C \sup_{\omega \in \partial B(0,1)} |\varphi(\omega)|_0 .$$

By a classical density argument T is a Radon measure.

We now prove the characterizing theorem for distributions on the unit sphere.

THEOREM 5. *Let $f \in M_1(\boldsymbol{R}^{m+1} \backslash \partial B(0, 1); \mathscr{A})$, $m \geq 2$. Then are equivalent:*

(1) *$f = \hat{T}$ for some $T \in \mathscr{E}'_{(l)}(\partial B(0, 1); \mathscr{A})$;*

(2) *f is of slow growth in $\boldsymbol{R}^{m+1} \backslash \partial B(0, 1)$ and it tends to zero if $x \rightarrow \infty$;*

(3) *$f = \sum_{k=0}^{\infty} (\chi_{\mathrm{co}\bar{B}(0,1)} Q_k(x) - \chi_{\mathring{B}(0,1)} P_k(x))$ and for some $s \in N$ and $C > 0$*

$$\sup_{\omega \in \partial B(0,1)} \{|P_k(\omega)|_0, |Q_k(\omega)|_0\} \leq C(1+k)^s .$$

PROOF. (1)$\Leftrightarrow$(3). Suppose that $f = \hat{T}$ for some $T \in \mathscr{E}'_{(l)}(\partial B(0, 1); \mathscr{A})$. Then for some $s' \in N$, $(1-\Gamma)^{-s'}$ is a Radon measure. The Laurent expansion of $\widehat{(1-\Gamma)^{-s'}T}(x)$ equals

$$\widehat{(1-\Gamma)^{-s'}T}(x) = \sum_{k=0}^{\infty} ((1-k-m)^{-s'} \chi_{\mathrm{co}\bar{B}(0,1)} Q_k(x) - (k+1)^{-s'} \chi_{\mathring{B}(0,1)} P_k(x)) ,$$

where $T = \sum_{k=0}^{\infty} P_k(\omega) + Q_k(\omega)$. As $(1-\Gamma)^{-s'}T$ is a Radon measure

$$|(k+1)^{-s'} P_k(\omega)|_0 \leq CC_{k+1,m+1}(1+k^2) ,$$
$$|(1-k-m)^{-s'} Q_k(\omega)|_0 \leq CC_{k+1,m+1}(1+k^2)$$

and hence (3) holds.

Conversely if f satisfies (3) and $\varphi \in \mathscr{E}_{(r)}(\partial B(0, 1); \mathscr{A})$ with $\varphi = \sum_{k=0}^{\infty} (P'_k(\omega) + Q'_k(\omega))$,

$$\langle T, \tilde{\varphi}\rangle = \omega_{m+1}(\omega, T) = \sum_{k=0}^{\infty} [(P'_k, P_k) + (Q'_k, Q_k)]$$

defines a distribution $T = \sum_{k=0}^{\infty} (P_k(\omega) + Q_k(\omega))$ in $\tilde{\mathscr{E}}_{(r)}(\partial B(0, 1); \mathscr{A})' = \mathscr{E}'_{(l)}(\partial B(0, 1); \mathscr{A})$ and $f = \hat{T}$. Note that by the same argument $\partial^+ f$ and $\partial^- f$ exist in $\mathscr{E}'_{(l)}(\partial B(0, 1); \mathscr{A})$, $\measuredangle$.

(2) ⟺ (3). Suppose that f satisfies (3). Then in $\mathring{B}(0, 1)$

$$|f(x)|_0 \leqq \sum_{k=0}^{\infty} |P_k(\omega|x|)|_0 \leqq C \sum_{k=0}^{\infty} |x|^k (1 + k^s)$$
$$\leqq C' \left(\frac{1}{1 - |x|}\right)^s .$$

Analogously in $\mathrm{co}\,\bar{B}(0, 1)$

$$|f(x)|_0 \leqq C'' \left(1 + \left(\frac{1}{|x| - 1}\right)^s\right)$$

and $f \to 0$ when $x \to \infty$.

Conversely, if f satisfies (2), then f is of the form

$$\sum_{k=0}^{\infty} (\chi_{\mathrm{co}\bar{B}(0,1)} Q_k(x) - \chi_{\mathring{B}(0,1)} P_k(x)) .$$

As for any $x \in \mathring{B}(0, 1)$ and $\omega \in \partial B(0, 1)$

$$|P_k(\omega)|_0 \leqq C C_{k+1, m+1} (1 + k^2) \frac{1}{|x|^k} \left(\frac{1}{1 - |x|}\right)^l ,$$

we obtain for $|x| = 1 - 1/(k+1)$, $k > 0$, that

$$|P_k(\omega)|_0 \leqq C C_{k+1, m+1} (1 + k^2) \left(1 + \frac{1}{k}\right)^k (k+1)^l$$
$$\leqq C e C_{k+1, m+1} (1 + k)^{l+2} .$$

Hence $(P_k(\omega))$ satisfies (2). Analogously $(Q_k(\omega))_{k \in N}$ satisfies (2).

5.3. *L_2-functions on $\partial B(0, 1)$.*

Let f and g be $\mathscr{A}$-valued L_2-functions on $\partial B(0, 1)$ ($f, g \in L_2(\partial B(0, 1); \mathscr{A})$); then it is obvious to define an inner product and a norm by

$$(f, g) = \frac{1}{\omega_{m+1}} \int_{\partial B(0,1)} \bar{f}(\omega) g(\omega) dS_\omega ,$$

$$\|f\|^2 = \mathrm{Re}\left(\frac{1}{\omega_{m+1}}\int_{\partial B(0,1)} \bar{f}(\omega)f(\omega)dS_\omega\right)$$
$$= \frac{1}{\omega_{m+1}}\int_{\partial B(0,1)} \mathrm{Re}\,(\bar{f}(\omega)f(\omega))dS_\omega\ .$$

THEOREM 6. *Let f and g belong to $L_2(\partial B(0,1); \mathscr{A})$. Then f and g may be expressed in spherical monogenic functions:*

$$f(\omega) = \sum_{k=0}^{\infty}(P_k(\omega)+Q_k(\omega))\ , \qquad g(\omega) = \sum_{k=0}^{\infty}(P'_k(\omega)+Q'_k(\omega))\ .$$

Furthermore

$$\|f\|^2 = \sum_{k=0}^{\infty}(\|P_k\|^2 + \|Q_k\|^2)$$

and

$$(f, g) = \sum_{k=0}^{\infty}[(P_k, P'_k) + (Q_k, Q'_k)]\ .$$

PROOF. We consider the operators

$$\Pi'_k\colon f(\omega) \longrightarrow \frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial B(0,1)} K_{k,m+1,\omega}(x)d\sigma_\omega f(\omega) = P_k(x)$$

and

$$\Pi'_{-(k+1)}\colon f(\omega) \longrightarrow \frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial B(0,1)} \frac{K_{k,m+1,x}(\omega)}{|x|^{m+k}}d\sigma_\omega f(\omega) = Q_k(x)$$

and there restrictions to the unit sphere

$$\Pi_k\colon f(\omega) \longrightarrow P_k(\omega)$$
$$\Pi_{-(k+1)}\colon f(\omega) \longrightarrow Q_k(\omega)\ .$$

Then Π_s, $s \in Z$, is bounded from L_2 to L_2, and $\Pi_s^2 = \Pi_s$. Moreover

$$(\Pi_k f, g) = \frac{1}{\omega_{m+1}}\int_{\partial B(0,1)} \overline{P_k(\omega)}g(\omega)dS_\omega$$
$$= \frac{1}{\omega_{m+1}}\int_{\partial B(0,1)}\int_{\partial B(0,1)} \frac{C_{k+1,m+1}}{\omega_{m+1}}\bar{f}(\omega')d\bar{\sigma}_{\omega'}\bar{K}_{k,m+1,\omega'}(\omega)g(\omega)dS_\omega$$
$$= \frac{1}{\omega_{m+1}}\int_{\partial B(0,1)} \bar{f}(\omega')\bar{e}_{(r)}\frac{C_{k+1,m+1}}{\omega_{m+1}}\left(\overline{\int_{\partial B(0,1)} \bar{g}(\omega)\bar{e}_{(r)}d\sigma_\omega K_{k,m+1,\omega'}(\omega))dS_{\omega'}}\right)$$
$$= \frac{1}{\omega_{m+1}}\int_{\partial B(0,1)} \bar{f}(\omega')\bar{e}_{(r)}dS_{\omega'}\overline{\overline{P'_k(\omega')}\bar{e}_{(r)}(\omega')}$$
$$= \frac{1}{\omega_{m+1}}\int_{\partial B(0,1)} \bar{f}(\omega')P'_k(\omega')dS_{\omega'} = (f, \Pi_k g)\ .$$

Analogous results hold for $\Pi_{-(k+1)}$. Furthermore $\Pi_s\Pi_{s'}=0$ whenever $s\neq s'$. Hence the operators Π_s, $s\in \boldsymbol{Z}$, form an othogonal set of projections and so, by Bessel's inequality (see [2]),

$$\sum_{k=0}^{\infty}(\|P_k\|^2+\|Q_k\|^2)\leqq\|f\|^2 .$$

Consequently $\sum_{k=0}^{n}(P_k+Q_k)$ converges in $L_2(\partial B(0, 1); \mathscr{A})$ and, as $f=\sum_{k=0}^{\infty}(P_k+Q_k)$ in the sense of distributions, the limit of $\sum_{k=0}^{\infty}(P_k+Q_k)$ in $L_2(\partial B(0, 1); \mathscr{A})$ equals f.

This implies that

$$\|f\|^2=\sum_{k=0}^{\infty}(\|P_k\|^2+\|Q_k\|^2)$$

and

$$(f, g)=\sum_{k=0}^{\infty}[(P_k, P_k')+(Q_k, Q_k')] .$$

REMARKS. (1) In the proof of the previous theorem we used the equalities

$$\frac{C_{k+1,m+1}}{\omega_{m+1}}\overline{\int_{\partial B(0,1)}\bar{g}(\omega)\bar{e}_{(r)}d\sigma_\omega K_{k,m+1,\omega'}(\omega)}=\overline{\overline{P_k'(\omega)}\bar{e}_{(r)}}$$
$$=\overline{\Pi_k g}\bar{e}_{(r)} .$$

In this expression g is considered as a distribution with the expansion $g=\sum_{k=0}^{\infty}(P_k'+Q_k')$. Hence

$$\bar{g}(\omega)\bar{e}_{(r)}=\sum_{k=0}^{\infty}(\tilde{Q}_k'+\tilde{P}_k')$$

and

$$\frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial B(0,1)}\bar{g}(\omega)\bar{e}_{(r)}d\sigma_\omega K_{k,m+1,\omega'}(\omega)=\tilde{P}_k'(\omega')=\overline{P_k'(\omega')}\bar{e}_{(r)} .$$

Furthermore $P_k'(\omega')=\Pi_k(g)(\omega')$.

(2) When we put $\tilde{f}=\bar{f}(\omega)\bar{e}_{(r)}$, then

$$(f, g)=\frac{1}{\omega_{m+1}}\int_{\partial B(0,1)}\tilde{f}(\omega)d\sigma_\omega g(\omega) ,$$

which may be considered as an extension of the inner product between analytic functions.

(3) Let $f=\sum_{k=0}^{\infty}(P_k+Q_k)$ belong to $L_2(\partial B(0, 1); \mathscr{A})$; then

$$\hat{f}(x)=\frac{1}{\omega_{m+1}}\int_{\partial B(0,1)}\frac{\bar{x}-\bar{u}}{|x-u|^{m+1}}d\sigma_u f(u)$$
$$=\sum_{k=0}^{\infty}(\chi_{\mathrm{co}\bar{B}(0,1)}Q_k(x)-\chi_{\mathring{B}(0,1)}P_k(x)) .$$

Furthermore by a direct calculation one easily sees that

$$\hat{f}(|x|\omega)\in L_2(\partial B(0,1);\mathscr{A}) ,\quad |x|\neq 1 ,$$
$$\lim_{\varepsilon\to 0+}\hat{f}(\omega(1\pm\varepsilon)) \quad \text{exists in} \quad L_2(\partial B(0,1);\mathscr{A})$$

and that the operators Π_+ and Π_- defined by

$$\lim_{\varepsilon\to 0+}\hat{f}(\omega(1+\varepsilon))=\sum_{k=0}^{\infty}Q_k(\omega)=\Big(\sum_{s=-\infty}^{-1}\Pi_s\Big)f=\Pi_+f$$
$$-\lim_{\varepsilon\to 0+}\hat{f}(\omega(1-\varepsilon))=\Big(\sum_{s=0}^{\infty}\Pi_s\Big)f=\Pi_-f$$

are projections satisfying

$$\Pi_+\Pi_-=\Pi_-\Pi_+=0 \quad \text{and} \quad 1=\Pi_++\Pi_- .$$

Hence

$$L_2(\partial B(0,1);\mathscr{A})=\Pi_+L_2(\partial B(0,1);\mathscr{A})\oplus_\perp\Pi_-L_2(\partial B(0,1);\mathscr{A}) .$$

(4) Let S_k be an $\mathscr{A}$-valued spherical harmonic function of order k. As S_k is analytic, S_k admits an expansion

$$S_k(\omega)=\sum_{l=0}^{\infty}(P_l(\omega)+Q_l(\omega)) .$$

But from the equality

$$\Delta_S=(\tilde{\Gamma}-1)\Gamma=(m-1-\Gamma)\Gamma ,$$

we obtain that

$$\Delta_S P_l=-l(l+m-1)P_l ,$$
$$\Delta_S Q_{l-1}=-l(l+m-1)Q_{l-1} .$$

Consequently as $\Delta_S S_k=-k(k+m-1)S_k$, all terms P_l and Q_{l-1} for which $l\neq k$ vanish. So we obtain that

$$S_k(\omega)=P_k(\omega)+Q_{k-1}(\omega) ,$$

where

$$P_k(\omega)=\frac{C_{k+1,m+1}}{\omega_{m+1}}\int_{\partial B(0,1)} K_{k,m+1,\omega'}(\omega)d\sigma_{\omega'}S_k(\omega') ,$$

$$Q_{k-1}(\omega)=\frac{C_{k,m+1}}{\omega_{m+1}}\int_{\partial B(0,1)} K_{k-1,m+1,\omega}(\omega')d\sigma_{\omega'}S_k(\omega') .$$

This implies that

$$\sup_{\omega\in\partial B(0,1)} |P_k(\omega)|_0 \leqq CC_{k+1,m+1}(1+k^2) \sup_{\omega\in\partial B(0,1)} |S_k(\omega)|_0 ,$$

$$\sup_{\omega\in\partial B(0,1)} |Q_{k-1}(\omega)|_0 \leqq CC_{k,m+1}(1+k^2) \sup_{\omega\in\partial B(0,1)} |S_k(\omega)|_0 .$$

Hence several spaces of analytic functions and functionals, expressed in terms of sequences of spherical harmonics provided with well defined estimates, admit a represention in terms of sequences of spherical monogenic functions.

Observe furthermore that the decomposition $S_k=P_k+Q_{k-1}$ is consistent with the dimension of the space of $\mathscr{A}$-valued spherical harmonics of order k. In fact this dimension equals $2^n N(m+1, k)$ where 2^n is the dimension of $\mathscr{A}$ and $N(m+1, k)=M(m, k)+M(m, k-1)$, $M(m, k)$ being the number of coefficients in a homogeneous polynomial of degree k in m variables (see [11]). Indeed, by the theorem of Cauchy-Kowalewski for analytic functions in $\boldsymbol{R}^m$ we obtain the dimension of the space of inner spherical monogenics of order k equals $2^n M(m, k)$, whereas the dimension of the space of outer spherical monogenics of order $(k-1)$ equals $2^n M(m, k-1)$.

References

[1] R. DELANGHE, Morera's theorem for functions with values in a Clifford algebra, Simon Stevin, **43** (1970), 129–140.
[2] R. DELANGHE and F. BRACKX, Hypercomplex function theory and Hilbert modules with reproducing kernel, Proc. London Math. Soc., **37** (1978), 545–576.
[3] R. DELANGHE and F. BRACKX, Runge's theorem in hypercomplex function theory, to appear in J. Approximation Theory.
[4] R. DELANGHE and F. BRACKX, Duality in hypercomplex function theory, to appear in J. Functional Analysis.
[5] M. HASHIZUME, A. KOWATA, K. MINEMURA and K. OKAMOTO, An integral representation of the Laplacian on the Euclidean space, Hiroshima Math. J., **2** (1972), 535–545.
[6] M. HASHIZUME, K. MINEMURA and K. OKAMOTO, Harmonic functions on Hermitian hyperbolic spaces, Hiroshima Math. J., **3** (1973), 81–108.
[7] S. HELGASON, Eigenspaces of the Laplacian; integral representations and irreducibility, J. Functional Analysis, **17** (1974), 328–353.
[8] M. MORIMOTO, A generalization of the Fourier-Borel transformation for the analytic functionals with non convex carrier, Tokyo J. Math., **2** (1979), 301–322.
[9] M. MORIMOTO, Analytic functionals on the sphere and their Fourier-Borel transformations, to appear in a volume of the Banach Center Publications.

[10] M. MORIMOTO, Analytic functionals on the Lie sphere, Tokyo J. Math., **3** (1980), 1–35.
[11] C. MÜLLER, Spherical Harmonics, Lecture Notes in Math., **17**, Springer, 1966.
[12] J. RYAN, Clifford Analysis, preprint.
[13] R. T. SEELEY, Spherical harmonics, Amer. Math. Monthly, **73** (1966), 115-121.
[14] R. T. SEELEY, Eigenfunction expansions of analytic fuctions, Proc. Amer. Math. Soc., **21** (1969), 734–738.
[15] F. SOMMEN, Distributional extension properties of hypercomplex functions, to apper in Bull. Soc. Math. Belg.
[16] F. SOMMEN, A version of the Fourier-Borel transform, to appear.
[17] M. VILENKIN, Special Functions and the Theory of Group Representations, Translation of Math. Monographs **22**, AMS, Providence, 1968.

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