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Spherically symmetric brane spacetime with bulk $f(\mathcal{R})$ gravity

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Abstract Introducing $f(\mathcal{R})$ term in the five-dimensional bulk action we derive effective Einstein's equation on the brane using Gauss–Codazzi equation. This effective equation is then solved for different conditions on dark radiation and dark pressure to obtain various spherically symmetric solutions. Some of these static spherically symmetric solutions correspond to black hole solutions, with parameters induced from the bulk. Specially, the dark pressure and dark radiation terms (electric part of Weyl curvature) affect the brane spherically symmetric solutions significantly. We have solved for one parameter group of conformal motions where the dark radiation and dark pressure terms are exactly obtained exploiting the corresponding Lie symmetry. Various thermodynamic features of these spherically symmetric space-times are studied, showing existence of second order phase transition. This phenomenon has its origin in the higher curvature term with $f(\mathcal{R})$ gravity in the bulk.

1 Introduction

Our four dimensional world might be embedded in a five dimensional space-time was proposed in [1,2] in order to explain the observed hierarchy between Electroweak and Planck scale. Such extra dimensional models also have their origin in some suitable compactifications of ten dimensional $E_8 \times E_8$ heterotic string theory [3].

This scenario has attracted considerable attraction due to its elegant nature and simplicity. In this brane world scenario the standard model fields are confined on a 3-brane, while gravity can propagate both in the brane and the bulk. A single 3-brane, which is embedded in a five dimensional bulk has the five dimensional line element, $ds^2 = e^{-A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2$. The warp factor $e^{-A(y)}$ can be tuned properly to induce Einstein gravity on the brane as a leading order term. We could have also considered a two brane system, which comes with an additional field known as radion, representing separation between the branes, with interesting features [4,5]. However we will restrict ourselves only to the single brane system for the rest of the discussion.

However due to the presence of extra dimensions, we should expect deviation from Einstein theory, which play a significant role at high energies [6,7]. Gravity sector also gets modified at electroweak scale ∼1 TeV, changing the cosmological implications, which have been extensively studied in Ref. [8–12]. The effect of extra dimension on formation of black hole has been studied in References [13–15]. Also these models have very interesting properties from the point of view of particle phenomenology [16–20].

In General Relativity the exterior space-time of a spherically symmetric black hole or a compact object is standard Schwarzschild geometry. However due to the presence of an extra dimension in the brane world scenario the Schwarzschild solution gets modified non-trivially. This originates due to high energy corrections, Weyl stress on gravitons propagating in the bulk. One such solution was obtained in References [21], in the form of Reissner–Nördstrom solution. The interior solution can be matched to a brane world star having constant energy density [22–24]. A non singular solution for black holes in these models can be obtained by relaxing the condition of zero scalar curvature while retaining null energy condition [25,26]. Also the Gauss–Codazzi equations can be solved in Randall–Sundrum type II model to get exterior solution for spherically symmetric star [27]. The various classes of vacuum solutions has been obtained in Reference [28] by solving the vacuum field equations on the brane obtained from Gauss Codazzi equation. The results of various such calculations suggest that brane world black hole horizons has the peculiar structure of a "pancake".

In recent years, there has been a new concept in General Relativity suggesting modifications of Einstein–Hilbert action in order to explain the late time cosmic accelera-

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tion to inflation. This is achieved by introducing higher curvature terms in the action, and a very promising candidate among such modifications is $f(\mathcal{R})$ gravity theories (for recent reviews see [29–31]). The main difficulties with these modifications are, they become infected with ghost modes. However $f(\mathcal{R})$ theory on a constant curvature hyper surface is shown to be ghost free [32–35]. The modification due to introduction of $f(\mathcal{R})$ term in the Lagrangian can address variety of problems e.g. four cosmological phases [36,37], late time cosmic acceleration [38,39], initial power law inflation [40], rotation curves of spiral galaxies [41,42], detection of gravitational waves [43–45] and many others. This theory has also the potential to pass through all known tests of general relativity.

Motivated by such striking properties of $f(\mathcal{R})$ gravity it is also introduced in brane world models, where the five dimensional action is modified by introduction of $f(\mathcal{R})$ term in the bulk, with $\mathcal{R}$ being the Ricci scalar of the five dimensional theory. In particular for bulk geometry with high curvature $\sim$ Planck scale, such higher order corrections to gravity are expected to become extremely relevant. Effective gravitational equations on the brane have been obtained in References [46–49] while perturbations on the scalar and tensor modes on the brane has been studied in Reference [50]. Cosmology on these brane world models Reference [51] along with brane world sum rules have also been discussed in these $f(\mathcal{R})$ gravity models [52]. The nature of warped geometric models in this $f(\mathcal{R})$ gravity theory with constant bulk curvature has been obtained in Reference [53] and the graviton KK mode masses in these models have been examined in the light of recent ATLAS data in LHC.

Ever since the pioneering works of Regge and Wheeler [54–56], the stability of a four dimensional black hole under linear perturbation has been investigated extensively. The importance of linear stability of a black hole can be understood as follows: the black hole solutions should describe the final state of gravitational collapse and thus they should be stable against small fluctuations. Also technically, this implies that at the order of linear perturbation, Einstein equation reduces to a simple set of wave equations. For the static situation, these equations resemble Schrödinger equation with time dependent Hamiltonian. Thus the stability analysis becomes equivalent to a simple, quantum mechanical problem. We also mention that there are solutions which describe naked singularity, and stability of a naked singularity is an important issue from the viewpoint of cosmic censorship conjecture. In this work we will use the wave equations to study the stability [57–60].

An important aspect of black hole physics, pioneered by Bekenstein, shows a remarkable similarity between black hole and a thermodynamic system. The similarity arises from the fact that just like a thermodynamic system one can attribute temperature to a black hole (known as Hawking temperature) which is proportional to the surface gravity and also an entropy proportional to the horizon area [61–66]. Any arbitrary black hole can be characterized by three parameters, its mass, charge and angular momentum. The thermodynamic stability of such a system can be determined by the sign of heat capacity just like any normal thermodynamic system. For a black hole the criteria $c_v < 0$ makes the system thermodynamically unstable. However if the specific heat changes sign as well as diverges in its parameter space, then it indicates a second order phase transition [67,68]. Phase transitions in various black hole solutions have been studied extensively in Einstein gravity as well as in alternative gravity theories [69–75].

The purpose of this work is to consider various spherically symmetric vacuum space-times on the brane obtained from $f(\mathcal{R})$ action on the bulk. In order to achieve this we consider the decomposition of electric part of the Weyl tensor into dark radiation and dark pressure terms. It turns out that these determine the space-time geometry we are considering. Moreover some simple integrability conditions lead to different classes of vacuum solutions. These issues are addressed in Sects. 2 and 3. Then we have discussed stability of black holes and naked singularities in these spacetime in Sect. 4.

Next we consider vacuum space-time related to Lie groups of transformation. As a simple situation we consider spherically symmetric and static solutions with the metric tensor admitting one parameter group of conformal motion. With proper integrability condition an exact solution corresponding to a brane with one parameter group of motions can be obtained (see Sect. 5).

Finally we consider the thermodynamics of these black hole solutions. As these solutions are induced on the brane due to bulk action, the thermodynamic properties are related to the dark pressure and radiation terms coming from the electric part of Weyl tensor and thus the thermodynamic properties of the brane black holes are directly related to those of bulk space-time (see Sect. 6). We finally conclude with a discussion on our results.

2 Static, spherically symmetric field equations on the brane

To obtain the vacuum solution we start from the bulk action with $f(\mathcal{R})$ term as,

$$S = \int d^5x\sqrt{-g}[f(\mathcal{R}) + \mathcal{L}_m] \tag{1}$$

where $\mathcal{L}_m$ is the matter Lagrangian, g_{AB} is the bulk metric and $\mathcal{R}$ is the bulk Ricci scalar. The bulk indices A, B runs through $0 \ldots 4$ i.e. over all the space-time dimensions. The variation of the action S with respect to bulk metric g_{AB} leads to,

$$f'(\mathcal{R})\mathcal{R}_{AB} - \frac{1}{2}g_{AB}f(\mathcal{R}) + g_{AB}\Box f'(\mathcal{R}) - \nabla_A\nabla_B f'(\mathcal{R}) = \kappa_5^2 T_{AB} \quad (2)$$

Here the negative vacuum energy density Λ on the bulk and the brane energy-momentum tensor are the sources of the gravitational field. Eq. (2) can be put into the form,

$$G_{AB} \equiv \mathcal{R}_{AB} - \frac{1}{2}\mathcal{R}g_{AB} = T^{\rm tot}_{AB}$$
$$T^{\rm tot}_{AB} = \frac{1}{f'(\mathcal{R})} \times \left[\kappa_5^2 T_{AB} - \left(\frac{1}{2}\mathcal{R}f'(\mathcal{R}) - \frac{1}{2}f(\mathcal{R}) + \Box f'(\mathcal{R})\right)g_{AB} + \nabla_A\nabla_B f'(\mathcal{R})\right]$$
$$T_{AB} = -\Lambda g_{AB} + \delta(y)(-\lambda_T h_{\mu\nu} + \tau_{\mu\nu})\delta^\mu_A\delta^\nu_B \quad (3)$$

where $\tau_{\mu\nu}$ is the brane energy-momentum tensor and $\lambda_{\rm T}$ is the corresponding brane tension. Also the quantity $h_{\mu\nu}$ is the induced metric on $y =$ constant hypersurfaces.

The effective four-dimensional gravitational equations on the brane are,

$$G_{\mu\nu} = -\Lambda_4 h_{\mu\nu} + 8\pi G_N \tau_{\mu\nu} + \kappa_5^2\pi_{\mu\nu} + Q_{\mu\nu} - E_{\mu\nu} \quad (4)$$

where,

$$\Lambda_4 = \frac{1}{2}\kappa_5^2\left(\frac{\Lambda}{f'(\mathcal{R})} + \frac{1}{6}\kappa_5^2\lambda^2\right) \quad (5)$$
$$G_N = \frac{\kappa_4^4}{8\pi} \quad (6)$$
$$\pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}\tau^\alpha_\nu + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}h_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}h_{\mu\nu}\tau^2 \quad (7)$$
$$Q_{\mu\nu} = \left[g(\mathcal{R})h_{\mu\nu} + \frac{2}{3}\frac{\nabla_A\nabla_B f'(\mathcal{R})}{f'(\mathcal{R})} \times \left(h^A_\mu h^B_\nu + n^A n^B h_{\mu\nu}\right)\right]_{y=0} \quad (8)$$

with,

$$g(\mathcal{R}) \equiv \frac{1}{4}\frac{f(\mathcal{R})}{f'(\mathcal{R})} - \frac{1}{4}\mathcal{R} - \frac{2}{3}\frac{\Box f'(\mathcal{R})}{f'(\mathcal{R})} \quad (9)$$

Note that for $f(\mathcal{R}) = \mathcal{R}$, we retrieve the usual Gauss–Codazzi equation for a pure Einstein gravity in the bulk. We now proceed to simplify the expression for $Q_{\mu\nu}$. The normal to $y =$ constant hypersurface being $n_A = \partial_A y$, we have $n_\mu = 0$. In addition if we assume that $\partial_\mu\mathcal{R} = 0$ then using the relations: $\nabla_A\nabla_B f'(\mathcal{R}) = f''(\mathcal{R})\nabla_A\nabla_B\mathcal{R} + f'''(\mathcal{R})\nabla_A\mathcal{R}\nabla_B\mathcal{R}$ and $\nabla_A\mathcal{R}\nabla_B\mathcal{R}h^A_\mu h^B_\nu = \nabla_\mu\mathcal{R}\nabla_\nu\mathcal{R} - \nabla_\mu\mathcal{R}\nabla_B\mathcal{R}n^B n_\nu - \nabla_A\mathcal{R}\nabla_\nu\mathcal{R}n^A n_\mu - \nabla_A\mathcal{R}\nabla_B\mathcal{R}n^A n^B n_\mu n_\nu$ along with a similar expression for $\nabla_A\nabla_B\mathcal{R}h^A_\mu h^B_\nu$ Eq. (8) reduces to,

$$Q_{\mu\nu} = \left(g(\mathcal{R}) + \frac{2}{3}\frac{\nabla_A\nabla_B f'(\mathcal{R})}{f'(\mathcal{R})}n^A n^B\right)_{y=0} h_{\mu\nu} \equiv F(\mathcal{R})h_{\mu\nu} \quad (10)$$

Now the scalar curvature for the bulk must be a well behaved quantity, and we can expand it in a Taylor series around $y = 0$ hypersurface, as, $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 y + \mathcal{R}_2 y^2/2 + \mathcal{O}(y^3)$. Since bulk curvature depends only on the extra dimension y, all the coefficients are constants. Thus all the derivatives calculated at $y = 0$ yield a constant contribution which does not depend on any of the brane coordinates.

The electric part of the Weyl tensor $E_{\mu\nu}$ has its origin in the nonlocal effect from free bulk gravitational field. This is the projection of bulk Weyl tensor such that, $E_{AB} = C_{ABCD}n^C n^D$ along with $E_{AB} = E_{\mu\nu}\delta^\mu_A\delta^\nu_B$ on the brane $(y \to 0)$. From the Gauss–Codazzi equation we also have conservation of energy momentum tensor as, $D_\mu T^{\mu\nu} = 0$, where D_μ is the brane covariant derivative. This also imposes restrictions on projected Weyl tensor from Bianchi identities. Following Reference [6] the projected Weyl tensor can be expanded as,

$$E_{\mu\nu} = -k^4\left[U(r)\left(u_\mu u_\nu + \frac{1}{3}\xi_{\mu\nu}\right) + P_{\mu\nu} + 2Q_{(\mu}u_{\nu)}\right] \quad (11)$$

with $k = k_5/\sqrt{8\pi G_N}$ and $\xi_{\mu\nu} = h_{\mu\nu} + u_\mu u_\nu$. This decomposition is with respect to the four velocity field u_μ. The respective terms in the above expression are, the "Dark Radiation" term, $U = -\frac{1}{k^4}E_{\mu\nu}u^\mu u^\nu$, which is a scalar, $Q_\mu = \frac{1}{k^4}\xi^\alpha_\mu E_{\alpha\beta}$ is a spatial vector and $P_{\mu\nu} = -\frac{1}{k^4}\left[\xi^\alpha_{(\mu}\xi^\beta_{\nu)} - \frac{1}{3}h_{\mu\nu}h^{\alpha\beta}\right]E_{\alpha\beta}$ is a spatial, trace free, symmetric tensor. For static solutions, $Q_\mu = 0$, while the constraint becomes dependent on dark radiation $U(r)$, vector $A_\mu = A(r)r_\mu$ and a tensor $P_{\mu\nu} = P(r)\left(r_\mu r_\nu - \frac{1}{3}\xi_{\mu\nu}\right)$. Here r_μ is unit radial vector.

In order to obtain solution in a source free region on the brane, brane energy momentum tensor appearing on the right hand side of effective Einstein's equation is taken to be zero. Thus we readily obtain $\tau_{\mu\nu} = 0 = \pi_{\mu\nu}$. Also from the previous discussion it is evident that $\mathcal{R}$ is dependent only on y and on the brane (at $y = 0$) all its derivatives with respect to coordinates become constants. Then the Einstein equation becomes,

$$G_{\mu\nu} = -\Lambda_4 h_{\mu\nu} + F(\mathcal{R})h_{\mu\nu} - E_{\mu\nu} \quad (12)$$

Now we choose an ansatz for spherically symmetric solution in the form,

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2 \tag{13}$$

For this choice the effective Einstein's equation and energy-momentum conservation equation on the brane become,

$$-e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = (\Lambda_4 - F(\mathcal{R})) + \frac{3}{4\pi G\lambda_T}U \tag{14}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = F(\mathcal{R}) - \Lambda_4 + \frac{1}{4\pi G\lambda_T}(U + 2P) \tag{15}$$

$$e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) = 2(F(\mathcal{R}) - \Lambda_4) + \frac{1}{2\pi G\lambda_T}(U - P) \tag{16}$$

$$\nu' = -\frac{U' + 2P'}{2U + P} - \frac{6P}{r(2U + P)} \tag{17}$$

where we have denoted $a' \equiv da/dr$. Now Eq. (14) can be solved for $e^{-\lambda}$ to yield,

$$e^{-\lambda} = 1 - \frac{\Lambda_4 - F(\mathcal{R})}{3}r^2 - \frac{Q(r)}{r} - \frac{C_1}{r} \tag{18}$$

where C_1 is an arbitrary constant of integration. The quantity $Q(r)$ is defined as,

$$Q(r) = \frac{48\pi G}{k_4^4\lambda_b}\int r^2 U(r)dr \tag{19}$$

We can interpret the term Q as equivalent to gravitational mass originating from dark radiation and henceforth will be referred as dark mass. In the limit $f(\mathcal{R}) \to \mathcal{R}$, $\Lambda_4 \to 0$ as well as $U \to 0$ we retrieve the standard Schwarzschild solution. This helps us to identify the arbitrary constant as $C_1 = 2GM$, M being the constant mass of the gravitating body. Also we can obtain the differential equations that are satisfied by dark radiation $U(r)$ and dark pressure $P(r)$ in static spherically symmetric space-time. Eliminating ν' from Eqs. (17) and (15) and using $e^{-\lambda}$ from Eq. (18) we obtain:

$$\frac{dU}{dr} = -2\frac{dP}{dr} - 6\frac{P}{r} - \frac{(2U + P)[2GM + Q + \{\alpha(U+2P)+2\chi/3\}r^3]}{r^2\left(1 - \frac{2GM}{r} - \frac{Q(r)}{r} - \frac{\Lambda_4 - F(\mathcal{R})}{3}r^2\right)} \tag{20}$$

$$\frac{dQ}{dr} = 3\alpha r^2 U \tag{21}$$

where we introduce two extra parameters, $\alpha = (1/4\pi G\lambda_T)$ and $\chi = F(\mathcal{R}) - \Lambda_4$. Now we define the following quantities in order to transform the above differential equation into a more convenient form which will be used extensively later,

$$q = \frac{2GM + Q}{r};\ \mu = 3\alpha r^2 U;\ p = 3\alpha r^2 P;$$
$$\theta = \ln r;\ 2\chi r^2 = \ell \tag{22}$$

In terms of these variables the differential equations satisfied by the dark radiation and dark pressure are,

$$\frac{dq}{d\theta} = \mu - q \tag{23}$$

$$\frac{d\mu}{d\theta} = -(2\mu + p)\frac{q + \frac{1}{3}(\mu + 2p) + \frac{\ell}{3}}{1 - q + \frac{\ell}{6}} - 2\frac{dp}{d\theta} + 2\mu - 2p \tag{24}$$

Thus the Eqs. (14)–(17) are the effective field equations, on the brane, while the Eqs. (23)–(24) represent equations for the source terms in the bulk i.e. dark pressure and dark radiation.

3 Various classes of solutions on the brane

Equations (20) and (21) can not be solved for dark radiation U and dark pressure P simultaneously unless we have a relation connecting them. We therefore choose some possible relations between the dark radiation U and dark pressure P which essentially define different equations of state. For different such choices we get different solutions. In this section we impose certain conditions on dark radiation U and dark pressure P, to obtain the corresponding solution. It turns out that the solutions are very distinct for different choices.

3.1 Case-I: $U = 0$

This condition comes with vanishing dark radiation, which imply readily $Q = 0$. In this scenario, one of the metric elements can be given by,

$$e^{-\lambda} = 1 + \frac{F(\mathcal{R}) - \Lambda_4}{3}r^2 - \frac{2GM}{r} \tag{25}$$

The differential equation satisfied by the dark pressure $P(r)$ is given by,

$$\frac{dP}{dr} + 3\frac{P}{r} + \frac{P(GM + \alpha r^3 P + (F(\mathcal{R}) - \Lambda_4)/3r^3)}{r^2\left(1 - \frac{2GM}{r} + \frac{F(\mathcal{R}) - \Lambda_4}{3}r^2\right)} = 0 \tag{26}$$

while the differential equation satisfied by ν is given by,

$$\nu' = \frac{2(GM + \alpha r^3 P + (F(\mathcal{R}) - \Lambda_4)/3r^3)}{r^2\left(1 - \frac{2GM}{r} + \frac{F(\mathcal{R})-\Lambda_4}{3}r^2\right)} \tag{27}$$

Solution for these two differential equations give the pressure and metric for this case. Note that in this situation the metric element e^{ν} is solely determined from the pressure, which can be seen directly from Eqs. (27) and (26) as, $\nu' = -2P'/P - 6/r$. This equation can be integrated to yield, $\exp(\nu) = C_2/r^6 P^2$, where C_2 is an arbitrary constant of integration. Thus once pressure equation is solved, the metric element is also known.

In order to obtain the pressure two quantities r_1 and d would be important with the following expressions:

$$r_1 = \frac{3^{-2/3}(F(\mathcal{R}) - \Lambda_4) + \left(-GM\,(F(\mathcal{R}) - \Lambda_4)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + \left[-1 + 9\,(F(\mathcal{R}) - \Lambda_4)\,G^2M^2\right]}\right)^{2/3}}{3^{-1/3}\,(F(\mathcal{R}) - \Lambda_4)\left(-GM\,(F(\mathcal{R}) - \Lambda_4)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + \left[-1 + 9\,(F(\mathcal{R}) - \Lambda_4)\,G^2M^2\right]}\right)^{1/3}} \tag{28}$$

$$\begin{aligned} d = &\frac{1}{(F(\mathcal{R}) - \Lambda_4)^2} \\ &\times \Bigg[-3^{5/6}\sqrt{\frac{(F(\mathcal{R}) - \Lambda_4)^3}{27}\left[-1 + 9\,(F(\mathcal{R}) - \Lambda_4)\,G^2M^2\right]} \\ &\times \Bigg(-GM\,(F(\mathcal{R}) - \Lambda_4)^2 + \sqrt{3} \\ &\times \sqrt{\frac{(F(\mathcal{R}) - \Lambda_4)^3}{27} + \left[-1 + 9\,(F(\mathcal{R}) - \Lambda_4)\,G^2M^2\right]}\Bigg)^{1/3} \\ &+ \frac{(F(\mathcal{R}) - \Lambda_4)}{3}\Bigg(-3GM\,(F(\mathcal{R}) - \Lambda_4)^2 \\ &+ \sqrt{(F(\mathcal{R}) - \Lambda_4)^3\left[-1 + 9G^2M^2\,(F(\mathcal{R}) - \Lambda_4)\right]}\Bigg)^{2/3} \\ &+ \frac{(F(\mathcal{R}) - \Lambda_4)^2}{3}\Bigg(1 + 3GM\Bigg\{-3GM\,(F(\mathcal{R}) - \Lambda_4)^2 \\ &+ \sqrt{(F(\mathcal{R}) - \Lambda_4)^3\left[-1 + 9G^2M^2\,(F(\mathcal{R}) - \Lambda_4)\right]}\Bigg\}\Bigg)^{1/3}\Bigg] \end{aligned}$$

With these variables the solution for the pressure is obtained as:

$$P(r) = h(r)\Bigg[\int \frac{\alpha r^3}{r^2(1 - 2GM/r + (F(\mathcal{R} - \Lambda_4)r^2/3))} \times h(r) + C_1\Bigg]^{-1} \tag{29}$$

$$\begin{aligned} h(r) = &\frac{1}{r^3}\left(\frac{1}{r} - \frac{1}{r_1}\right)^{-\frac{3GMA}{(F(\mathcal{R})-\Lambda_4)r_1 d}} \\ &\exp\Bigg[-\frac{3GM\,(d - r_1^2)}{(F(\mathcal{R}) - \Lambda_4)\,r_1^2 d\,(1 + d/2r_1^2)\sqrt{4d - r_1^2}} \\ &\arctan\left(\frac{r_1 + 2d/r_1}{\sqrt{4d - r_1^2}}\right)\Bigg] \\ &\times\left(1 + \frac{r_1}{r} + \frac{d}{r^2}\right)^{-\frac{3GM}{2(F(\mathcal{R})-\Lambda_4)r_1 d(1+d/2r_1^2)}} \end{aligned} \tag{30}$$

From the above expression it is evident that at $r = r_1$ the metric element e^{ν} vanishes. Thus the space-time has an event horizon located at $r = r_1$ with its characteristic thermodynamic features.

3.2 Case-II: $P = 0$

In this situation Eqs. (23) and (24) reduces to the following form,

$$\frac{dq}{d\theta} = \mu - q \tag{31}$$

$$\frac{d\mu}{d\theta} = 2\mu\left[\frac{6 - \ell - 2\mu - 12q}{6 + \ell - 6q}\right] \tag{32}$$

These two equations can be combined to yield a single differential equation such that,

$$(6 + \ell - 6q)\frac{d^2q}{d\theta^2} + (26q - 6 - \ell)\frac{dq}{d\theta} + 4\left(\frac{dq}{d\theta}\right)^2 + 2q(14q - 6 - \ell) = 0 \tag{33}$$

The transformations $dq/d\theta = 1/v$ and $v = w(6 - 6q + \ell)^{-2/3}$ lead to the following differential equation,

$$\frac{dw}{dq} - (26q - 6 - \ell)(6 - 6q + \ell)^{-5/3}w^2 - 2q(14q - 6 - \ell)(6 - 6q + \ell)^{-7/3}w^3 = 0 \tag{34}$$

The above differential equation has a particular solution, $w = -\frac{1}{q}(6 - 6q + \ell)^{2/3}$. However for a wider class of solutions we define a new variable $\eta = (6 - 6q + \ell)^{-1/3}$. This leads to the differential equation,

$$\frac{\mathrm{d}w}{\mathrm{d}\eta} - \frac{10\eta^3 + 10/13\ell\eta^3 - 13/16}{\eta^2} + \frac{[\eta^3(1+\ell/6) - 1][7/3 - \eta^3(10 + 4\ell/3)]}{\eta^3} w^3 = 0 \quad (35)$$

It is hard to find an exact solution of this differential equation. Therefore we resort to approximated methods. For that purpose we choose the differential equation (33) and making Laplace transform of this equation we get,

$$\mathcal{L}\left([3 + \chi \mathrm{e}^{2\theta}]\frac{\mathrm{d}^2 q}{\mathrm{d}\theta^2} - [3 + \chi \mathrm{e}^{2\theta}]\frac{\mathrm{d}q}{\mathrm{d}\theta} - 4q[3 + \chi \mathrm{e}^{2\theta}]\right) = \mathcal{L}\left(3q\frac{\mathrm{d}^2 q}{\mathrm{d}\theta^2} - 13q\frac{\mathrm{d}q}{\mathrm{d}\theta} + 4\left(\frac{\mathrm{d}q}{\mathrm{d}\theta}\right)^2 - 14q^2\right) \quad (36)$$

Then using the convolution theorem in the form,

$$\mathcal{L}^{-1}(\tilde{f}(s)\tilde{g}(s)) = \int_a^b f(t-u)g(u)\mathrm{d}u \quad (37)$$

we readily obtain the following integral solution,

$$q(\theta) = q_0(\theta) + \int_{\theta_0}^{\theta} f(\theta - y) \times \left[3q\frac{\mathrm{d}^2 q}{\mathrm{d}\theta^2} - 13q\frac{\mathrm{d}q}{\mathrm{d}\theta} + 4\left(\frac{\mathrm{d}q}{\mathrm{d}\theta}\right)^2 - 14q^2\right]\mathrm{d}x \quad (38)$$

where we have the following functions,

$$f(x-y) = \frac{1}{9}\left(\mathrm{e}^{2(x-y)} - \mathrm{e}^{-(x-y)}\right) \quad (39)$$

$$q_0(\theta) = A_1\mathrm{e}^{-\theta} + A_2\mathrm{e}^{2\theta} \quad (40)$$

$$A_1 = [(3q_0 - \mu_0) + ((3 + 2\chi)q_0 - \mu_0)]\mathrm{e}^{\theta_0}/3 \quad (41)$$

$$A_2 = \mu(\theta_0)\mathrm{e}^{-2\theta_0}/3 \quad (42)$$

Having obtained an integral solution we now move forward to determine the metric. However the solution is usually obtained by successive approximation methods, which invokes iterations. At zeroth order we get the solution by using only the linear part of the differential equation (33) and will be denoted by q_0. Then we can write our full solution as a limiting process, such that $q(\theta) = \lim_{m\to\infty} q_m(\theta)$. In this situation for $m \in N$, we have the iterative solution at mth order connected to $(m-1)$th order by the following integral equation,

$$q_m(\theta) = \int_{\theta_0}^{\theta} F(\theta - y)\left[3q_{m-1}\frac{\mathrm{d}^2 q_{m-1}}{\mathrm{d}\theta^2} - 13q_{m-1}\frac{\mathrm{d}q_{m-1}}{\mathrm{d}\theta} + 4\left(\frac{\mathrm{d}q_{m-1}}{\mathrm{d}\theta}\right)^2 - 14q_{m-1}^2\right]\mathrm{d}y + q_{m-1}(\theta) \quad (43)$$

Then following Ref. [28] the zeroth order static and spherically symmetric solution to the field equations turn out to be,

$$\mathrm{e}^{\nu} = C_0\sqrt{\frac{\alpha}{A_2}} \quad (44)$$

$$\mathrm{e}^{-\lambda} = 1 - \frac{A_1}{r} - A_2 r^2 \quad (45)$$

$$U = \frac{A_2}{\alpha} \quad (46)$$

where we have C_0 as an arbitrary integration constant. After using one more iteration i.e. up to first order approximation the metric components are obtained as,

$$\mathrm{e}^{\nu} = C_0\sqrt{\frac{\alpha r_0}{2}}\sqrt{\frac{r}{A_2(r_0 - r)[A_1 + A_2 r r_0^2 + A_2 r_0 r^2]}} \quad (47)$$

$$\mathrm{e}^{-\lambda} = 1 + \frac{A_2 r_0^2[(4A_2 r_0^2/5) + A_1]}{r} - 3A_1A_2 r - 2A_2(2A_2 r_0^2 - A_1/r_0)r^2 + 6A_2^2 r^4/5 \quad (48)$$

Note that the dependence on $f(\mathcal{R})$ gravity appears through the A_1 factor. However the dependence is quiet complicated and affects both the metric elements.

3.3 Case-III: $2U + P = 0$

For this choice Eq. (20) yields,

$$P(r) = \frac{P_0}{r^4} \quad (49)$$

$$U(r) = -\frac{P_0}{2r^4} \quad (50)$$

where P_0 is an arbitrary integration constant. Also the dark mass can be calculated from Eq. (21) as,

$$Q(r) = Q_0 + \frac{3\alpha P_0}{2r} \quad (51)$$

where again Q_0 is an integration constant. For this particular choice we have from Eqs. (14) and (15) $\nu' = -\lambda'$. Hence the metric elements are given by,

$$\mathrm{e}^{\nu} = \mathrm{e}^{-\lambda} = 1 - \frac{2GM + Q_0}{r} - \frac{3\alpha P_0}{2r^2} + \frac{F(\mathcal{R}) - \Lambda_4}{3}r^2 \quad (52)$$

This solution has several interesting features which we discuss now. Firstly this solution is asymptotically dS (AdS) or flat depending on the sign of $(F(\mathcal{R}) - \Lambda_4)$ being negative (positive) or zero. Then there is an analogous charge term which is the coefficient of $1/r^2$ term and is given by $-3\alpha P_0/2$. Finally we have a mass term given by,

$2GM + Q_0$. Thus we note that the charge term is coming solely from the dark pressure term and thus has its origin in the bulk geometry. Similar argument hold true for the mass term also. However the effect of $f(\mathcal{R})$ gravity on the bulk actually induces a dS (AdS) nature to the vacuum solutions.

3.4 Case-IV: $U + 2P = 0$

Here we consider a different condition on the dark radiation and dark pressure terms. In this case Eq. (20) leads to the expression for the dark mass Q as,

$$Q = \frac{2r}{3} - 2GM \tag{53}$$

along with the the solution for dark radiation term and dark pressure term as,

$$U(r) = -2P(r) = \frac{2}{9\alpha r^2} \tag{54}$$

The metric elements in this case can be evaluated as,

$$e^{-\lambda} = \frac{1}{3} + \frac{F(\mathcal{R}) - \Lambda_4}{3} r^2 \tag{55}$$

$$e^{\nu} = C_0 r^2 \tag{56}$$

Note that this solution actually represents a naked singularity since the event horizon is determined by the equation, $e^{\nu} = 0$. Thus though the $f(\mathcal{R})$ model modifies the e^{λ} term however it yields a naked singularity solution. Moreover $e^{-\lambda} = 0$ determines the null surface, however in this situation the null surface exists only if $\Lambda_4 > F(\mathcal{R})$ and is located at, $r_h = \sqrt{\Lambda_4 - F(\mathcal{R})}$. Hence by imposing appropriate conditions we obtain either black hole solution with event horizon or solution with naked singularity.

In this context we should mention that naked singularities are just not some artifact, these can be used to probe structures as well. For example we can use naked singularity to take part in gravitational lensing and time delay, with centroid deformation of astrophysical objects [76,77].

4 Stability of the solutions

Stability of black holes under linearized perturbation is considered as an important problem in black hole physics. Here we consider gravitational perturbation in a static spherically symmetric background. Gauge invariant formalisms were developed in an arbitrary static background metric having the form $-g_{tt} = g^{rr} = f(r)$. It turns out that for certain ranges of the parameter space the Hamiltonian is positive guarantying the self-adjoint extension of it under suitable boundary condition.

The perturbation can be grouped into three types: scalar, vector and tensor perturbations. Expansion of each of these perturbations in harmonic functions leads to a set of equations expressed in terms of gauge covariant variables. Further reduction of these equations then reduces them to a set of decoupled wave equation in the form:

$$\left(\Box - \frac{1}{f(r)} V\right) \Phi = 0 \tag{57}$$

where as usual, $\Box$ represents the d'Alembertian operator with respect to the two dimensional metric. Also $\Phi = \Phi_{\rm S}$, $\Phi_{\rm V}$ and $\Phi_{\rm T}$ represent scalar, vector and tensor perturbations respectively. The potential function for each of these perturbation modes corresponds to [57]:

$$V_{\rm T} = \frac{f(r)}{r^2}\left(r\frac{{\rm d}f(r)}{{\rm d}r} + \ell(\ell+1)\right) \tag{58}$$

$$V_{\rm V} = \frac{f(r)}{r^2}\left(2f(r) - r\frac{{\rm d}f(r)}{{\rm d}r} + (\ell-1)(\ell+2)\right) \tag{59}$$

$$V_{\rm S} = \frac{f(r)U(r)}{16r^2(m+3x)^2} \tag{60}$$

where we have used the following expressions:

$$U(r) = 144x^3 + 144mx^2 + 48mx + 16m^3 \tag{61}$$

$$x \equiv 1 - f(r), \qquad m \equiv (\ell-1)(\ell+2) \tag{62}$$

It should be noted that the total number of independent components of the scalar, vector and tensor modes adds up to 2, the number of independent degrees of freedom for graviton in the brane. Since the tensor mode has no degrees of freedom we need to concentrate only on the vector and scalar modes.

Let us now consider the black hole and naked singularity solutions obtained in the previous section using effective gravitational field equation on the brane. Most of these solutions are quiet complex and we shall focus into some appropriate limiting cases.

- We start with the choice of vanishing dark radiation i.e. $U = 0$. From the previous section it is evident that in general the solution is complex and not in closed form. Thus we consider the limit $F(\mathcal{R}) \to \Lambda_4$, where from Eq. (25) it is evident that this leads to Schwarzschild form for $e^{-\lambda}$. However in this limit e^{ν} becomes $(1 - 2M/r)$ with some correction factors of $\mathcal{O}(\alpha)$. Thus in the small α limit the solution is Schwarzschild in nature. Hence all the potentials $V_{\rm T}$, $V_{\rm V}$ and $V_{\rm S}$ are positive implying existence of self adjoint operators and hence the stability. Thus for small $F(\mathcal{R}) - \Lambda_4$ the deviation from Schwarzschild solution would indeed be small resulting into stability of the solution.

- Next we discuss the case of vanishing dark pressure. In this case the solutions are not exact and even the zeroth order solution for $e^{-\lambda}$ looks like Schwarzschild de-Sitter. However the other one is merely a constant. Thus from the expressions for the potential it turns out they depend on the $e^{-\lambda}$ at the outside and thus will represent stable solution for the range of parameter space where $f(r) > 0$. From large r limit we observe that stability requires the condition $A_2 > 0$, which is acceptable since this in turn implies that dark radiation to be positive from Eq. (46). Thus positivity of the dark radiation term ensures stability of this solution at zeroth order. Since we have higher order solutions in a perturbative form, the stability of the full solution is expected to be dominated by the zeroth order term.
- The most important case in our hand is the situation where dark pressure and dark radiation satisfies the constraint relation $2U + P = 0$. In this case we can determine stability exactly. For this solution it turns out that V_T and V_S are positive for all choices of $F(\mathcal{R}) - \Lambda_4$. However though V_V is positive for $F(\mathcal{R}) > \Lambda_4$ it becomes negative for the other choice. Hence All these modes are positive ensuring stability of the solution for the parameter space: $F(\mathcal{R}) > \Lambda_4$. Otherwise, the solution is though stable under the tensor and scalar perturbations, is not so under vector perturbation.
- Another important aspect of this solution comes into picture when $P_0 = 0$. Then the solution represents a Schwarzschild (A)de-Sitter spacetime, which under proper limit leads to the Nariai spacetime. This has the peculiar property that a black hole in Nariai spacetime has increasing surface area due to quantum corrections as shown by Bousso and Hawking [78–80]. This phenomenon of antievaporation was then generalized for Nariai black holes in $f(\mathcal{R})$ gravity [81], with $f(\mathcal{R})$ gravity playing the role of anomaly induced effective action leading to anti evaporation. In our case as well with $P_0 = 0$, we have Nariai black hole as one limit and thus our solutions will also exhibit anti-evaporation. However for $P_0 \neq 0$, our solution cannot be reduced to the Nariai form and thus in general the solution presented here will not exhibit anti-evaporation.
- Finally we consider the solution which corresponds to the other constraint relation with $U + 2P = 0$. In this case we have both black hole and naked singularity depending on $\Lambda_4 > F(\mathcal{R})$ or otherwise. In this case at large r limit both the solutions can be taken as $1 + Cr^2$. It turns out that, V_T and V_V are positive for all choices between $F(\mathcal{R})$ and Λ_4, however V_V ensures stability for the black hole solution not for the naked singularity. Thus the black hole solution is stable under all these perturbation, while the global naked singularity is stable only under tensor and scalar perturbation, but not under vector perturbation.

Thus we observe that the solutions present here are mostly stable under perturbations, except in some specific cases where the vector mode of the perturbation shows instability. Also we have pointed out that our solution reduces to the Nariai form and thus exhibits anti-evaporation in $f(\mathcal{R})$ gravity, similar to previously obtained results.

5 Static spherically symmetric brane with conformal motion

We can use symmetries to explore the connection between geometry and matter through Einstein's equation. The most important of such symmetries can be realized through the use of conformal Killing vectors. The symmetry under which the space-time manifold admits conformal Killing vectors are known as, conformal motion. In this section we derive a particular metric which admits conformal motions. For the spherically symmetric and static solutions on the brane if one requires to have one-parameter group of conformal motion, the following condition results,

$$\mathcal{L}_\xi h_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} = \phi(r) g_{\mu\nu} \tag{63}$$

In the above relation ξ is the conformal Killing vector and $\phi(r)$ is the conformal factor, while the above symmetry of the metric is known as conformal motion. The above relation should hold for all the individual metric components. In this relation $h_{\mu\nu}$ is the metric determining the vacuum space-time configuration, ξ_μ is a vector field in this space-time with respect to which the Lie variation has been taken and $\phi(r)$ is an arbitrary function of the radial coordinate. Then following the procedure adopted in Reference [82] to determine interior structure of stellar objects, here also we can impose some symmetry requirement like, $\xi^\mu u_\mu = 0$. This symmetry enables one to determine all the unknowns exactly using the effective Einstein's equation. Thus using the metric ansatz given by Eq. (13), the above equation is shown to be equivalent to [82],

$$\begin{aligned} e^\nu &= A^2 r^2 \\ \phi(r) &= C e^{-\lambda/2} \\ \xi^\mu &= D\delta_0^\mu + \frac{\phi r}{2}\delta_1^\mu \end{aligned} \tag{64}$$

where A, C and D are arbitrary constants. With the above results the Einstein equations (14), (15) and (16) reduce to,

$$\frac{1}{r^2}\left[1-\frac{\phi^2(r)}{C^2}\right]-\frac{2\phi\phi'}{rC^2}=3\alpha U-[F(\mathcal{R})-\Lambda_4] \quad (65)$$

$$\frac{1}{r^2}\left(1-3\frac{\phi^2}{C^2}\right)=-\alpha\,(U+2P)-[F(\mathcal{R})-\Lambda_4] \quad (66)$$

$$\frac{1}{C^2}\frac{\phi^2}{r^2}+\frac{2}{C^2}\frac{\phi\phi'}{r}=\alpha(U-P)+(F(\mathcal{R})-\Lambda_4) \quad (67)$$

From Eqs. (66) and (67) we obtain the dark radiation and dark pressure in terms of the unknown function ϕ as,

$$P(r)=-\frac{1}{3\alpha}\left[\frac{2}{C^2}\frac{\phi\phi'}{r}+\frac{1}{r^2}\left(1-2\frac{\phi^2}{C^2}\right)\right] \quad (68)$$

$$U(r)=\frac{1}{3\alpha}\left[\frac{4}{C^2}\frac{\phi\phi'}{r}-\frac{1}{r^2}\left(1-5\frac{\phi^2}{C^2}\right)-3(F(\mathcal{R})-\Lambda_4)\right] \quad (69)$$

Then from Eq. (65) and the expression for dark radiation, the differential equation satisfied by $\phi(r)$ turns out to be,

$$\frac{3}{C^2}\phi\phi'=\frac{1}{r}\left(1-3\frac{\phi^2}{C^2}\right)+4r(F(\mathcal{R})-\Lambda_4) \quad (70)$$

This can be solved with little effort to yield the general solution as,

$$\phi^2=\frac{C^2}{3}\left[1+\frac{B}{r^2}+2(F(\mathcal{R})-\Lambda_4)r^2\right] \quad (71)$$

where, B is an integration constant. Thus full solution corresponding to this one parameter symmetry group of conformal motion leads to,

$$e^{\nu}=A^2r^2 \quad (72)$$

$$e^{-\lambda}=\frac{1}{3}\left[1+\frac{B}{r^2}+2(F(\mathcal{R})-\Lambda_4)r^2\right] \quad (73)$$

$$U(r)=\frac{1}{9\alpha r^2}\left[2+\frac{B}{r^2}+9(F(\mathcal{R})-\Lambda_4)r^2\right] \quad (74)$$

$$P(r)=\frac{1}{9\alpha r^2}\left[-1+\frac{4B}{r^2}\right] \quad (75)$$

There exists another important properties of the field equations. Having obtained a single solution we can make a transformation such that, $r\rightarrow\bar{r}(r)$, $U\rightarrow\bar{U}(U)$, $P\rightarrow\bar{P}(P)$ and $Q\rightarrow\bar{Q}(Q)$ [83], called homology transformations. The homology properties of the equations determining dark radiation and dark pressure can be simplified by assuming $\gamma=P(U)/U=$ constant and $c_s=\mathrm{d}P/\mathrm{d}U=$ constant. The above transformations are being generated with the infinitesimal generator as, $\hat{L}=\zeta(r)\partial/\partial r+\psi^1(U)\partial/\partial U+\psi^2(Q)\partial/\partial Q$. Then in order to have consistent solutions we must have, $\zeta=0$, $\psi^1=U$ and $\psi^2=Q+2GM$. Thus with inclusion of $f(\mathcal{R})$ gravity the infinitesimal generator for the homologous transformation becomes restricted compared to that in Einstein gravity.

6 Some thermodynamic features

In this section we will discuss thermodynamics associated with these spherically symmetric vacuum spacetime. Our main motive is to observe if there exists any thermodynamic interpretation which is induced solely by the bulk. We focus on the line element obtained for the condition $2U+P=0$ which has the following expression,

$$\mathrm{d}s^2=-\left(1-\frac{2GM+Q_0}{r}-\frac{3\alpha P_0}{2r^2}+\frac{F(\mathcal{R})-\Lambda_4}{3}r^2\right)\mathrm{d}t^2+\left(1-\frac{2GM+Q_0}{r}-\frac{3\alpha P_0}{2r^2}+\frac{F(\mathcal{R})-\Lambda_4}{3}r^2\right)^{-1}\times\mathrm{d}r^2+r^2d\Omega^2 \quad (76)$$

The horizon is determined by setting coefficient of g_{tt} to zero, which in turn leads to the equation,

$$1-\frac{2GM+Q_0}{r}-\frac{3\alpha P_0}{2r^2}+\frac{F(\mathcal{R})-\Lambda_4}{3}r^2=0 \quad (77)$$

Then the mass term equivalent to internal energy of a thermodynamic system can be obtained in terms of the horizon radius as,

$$M(r_h)=\frac{r_h}{2}-\frac{Q_0}{2}-\frac{3\alpha P_0}{4r}+\frac{F(\mathcal{R})-\Lambda_4}{6}r^3 \quad (78)$$

The surface area of the event horizon is given by, $A=\pi r_h^2$, while the entropy for the black hole is given by, $S=k_BA/4\hbar=k_B\pi r_h^2/4\hbar r$. Choosing $\hbar=1$ and Boltzmann constant appropriately we readily obtain,

$$S=r_h^2 \quad (79)$$

Thus the mass of the black hole in terms of the entropy becomes,

$$M(S)=\frac{\sqrt{S}}{2}-\frac{Q_0}{2}-\frac{3\alpha P_0}{4\sqrt{S}}+\frac{F(\mathcal{R})-\Lambda_4}{6}S^{3/2} \quad (80)$$

This leads to the first law of black hole mechanics as,

$$\mathrm{d}M=T\mathrm{d}S+\Phi\mathrm{d}(F(\mathcal{R})-\Lambda_4) \quad (81)$$

from which the black hole temperature turns out to be:

$$T=\frac{1}{4\sqrt{S}}+\frac{3\alpha P_0}{8S^{3/2}}+\frac{F(\mathcal{R})-\Lambda_4}{4}\sqrt{S} \quad (82)$$

while the chemical potential has the following expression:

$$\Phi=\frac{S^{3/2}}{6} \quad (83)$$

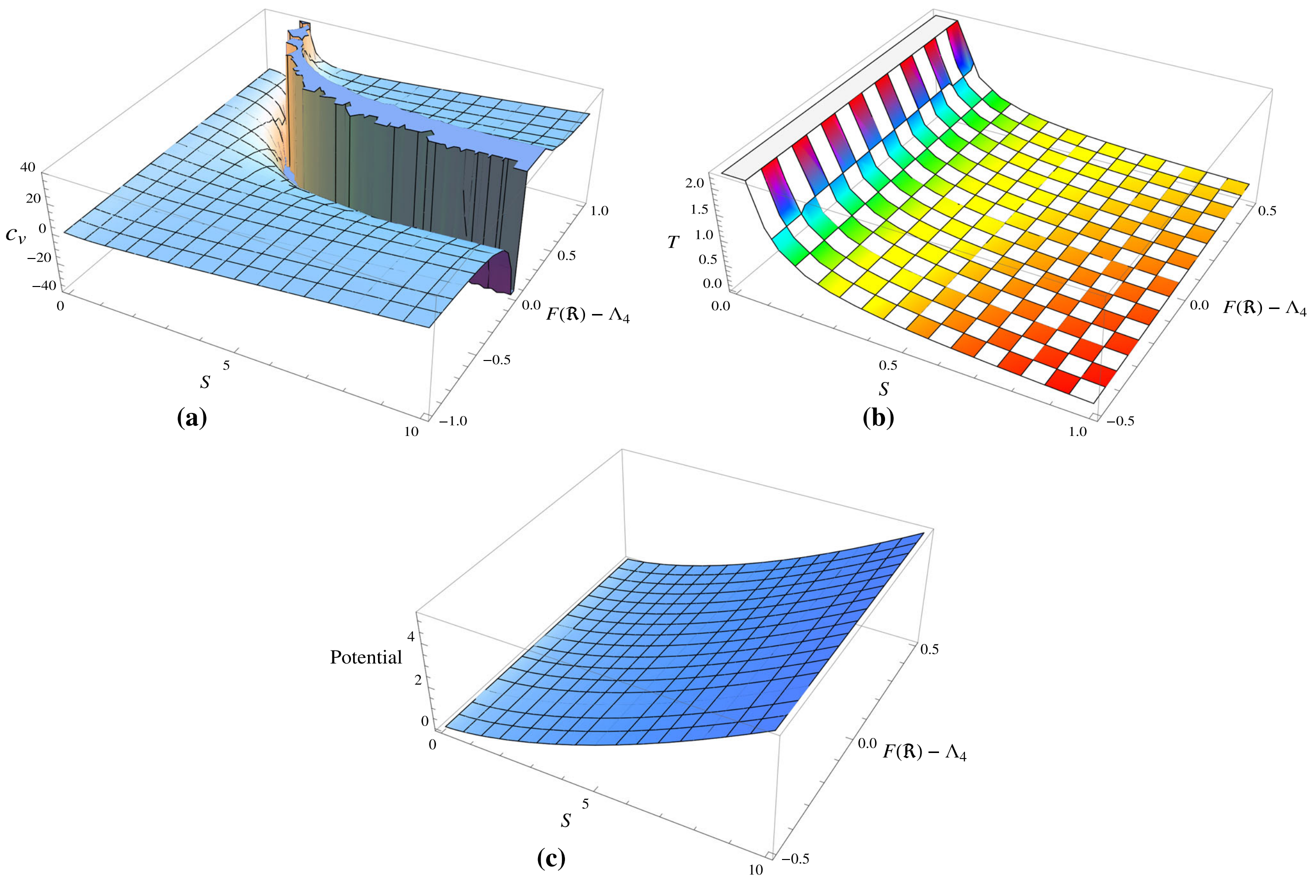

Fig. 1 The above figures show variation of three thermodynamic quantities: **a** specific heat, **b** temperature, **c** potential with entropy and $F(\mathcal{R}) - \Lambda_4$. **a** Clearly shows the existence of phase transition in this black hole spacetime through the discontinuity and divergence of the specific heat on some surface in entropy and $F(\mathcal{R}) - \Lambda_4$. While continuity of both temperature and thermodynamic potential in **b** and **c** show that this phase transition is of second order

From the expression of temperature as a function of entropy it turns out that the specific heat has the following behavior:

$$C_\mathrm{V} = T\left(\frac{\partial S}{\partial T}\right)_{F(\mathcal{R})-\Lambda_4} = \frac{\frac{1}{4\sqrt{S}} + \frac{3\alpha P_0}{8S^{3/2}} + \frac{F(\mathcal{R})-\Lambda_4}{4}\sqrt{S}}{\frac{F(\mathcal{R})-\Lambda_4}{8\sqrt{S}} - \frac{1}{8S^{3/2}} - \frac{9\alpha P_0}{16S^{5/2}}} \tag{84}$$

Figure 1 shows that while temperature T and potential ϕ are continuous with both the entropy and the quantity $F(\mathcal{R}) - \Lambda_4$, specific heat shows discontinuity indicating a second order phase transition. The surface of discontinuity in the specific heat is given by,

$$F(\mathcal{R}) - \Lambda_4 = \frac{1}{S} + \frac{9\alpha P_0}{2S} \tag{85}$$

In order to understand the physics behind these results, it is always illuminating to discuss some limiting cases. For example, if we assume pure Einstein gravity, where $F(\mathcal{R}) = 0$, then with the assumption of $\Lambda_4 \sim 0$, we arrive at the Reissner–Nordström solution. From Eq. (84) the specific heat turns out to be $C_\mathrm{v} = -S(6\alpha P_0 + 4S)/(2S + 9\alpha P_0)$. This can also be divergent, provided the entropy satisfies the criteria: $S = -(9\alpha P_0/2)$. In general P_0 is taken to be positive and thus the above relation cannot be satisfied in general. Hence the bulk term with positive dark pressure cannot lead to second order phase transition. However the other limit is interesting. For $P_0 = 0$, we get the divergence of specific heat to correspond to the condition: $S = 1/(F(\mathcal{R}) - \Lambda_4)$. Thus for $F(\mathcal{R}) > \Lambda_4$ we have second order phase transition. Our calculations therefore confirm that Schwarzschild anti-de Sitter solution shows second order phase transition.

The case for which the dark radiation vanishes, i.e. $U = 0$ also exhibits the appearance of black hole horizon. The actual calculations are quiet complex, and we have presented them in Appendix A. However here we consider some limiting cases and discuss the corresponding thermodynamic features. The first case corresponds to $F(\mathcal{R}) - \Lambda_4 = 0$. In this situation the solution for the metric elements resemble Schwarzschild solution with no associated phase transition. We cannot take $P = 0$ as in that case the metric elements would diverge. Thus another obvious choice is $M = 0$.

Then also horizon appears and the specific heat diverges for $\Lambda_4 > F(\mathcal{R})$. Thus this configuration exhibits an opposite effect in respect to $2U + P = 0$ case.

We therefore observe that in both the black hole solutions the specific heat diverge showing second order phase transition, due to the presence of $F(\mathcal{R})$ gravity in the bulk. Thus bulk $F(\mathcal{R})$ gravity plays a crucial role in determining the thermodynamic feature of the brane world black holes.

7 Discussion

In this work we have considered a bulk action with a $f(\mathcal{R})$ term, where $\mathcal{R}$ is the bulk curvature. Starting from the bulk action we have derived the full effective Einstein's equation on the brane located at $y = 0$, which under $f(\mathcal{R}) \to \mathcal{R}$ limit goes to the usual Gauss Codazzi equation in Einstein gravity. In order to get spherically symmetric solutions we have assumed that in the region of interest there is no matter field present on the brane and also the four dimensional scalar curvature is constant. Under these conditions the Einstein equation simplifies considerably, however the Weyl tensor on bulk has non trivial decomposition on the brane leading to the appearance of dark pressure and dark radiation in the effective Einstein's equation. Also the induced four dimensional cosmological constant and contribution from $f(\mathcal{R})$ term have significant effects on the solutions of the effective Einstein's equation on the brane.

Due to the presence of $f(\mathcal{R})$ gravity in the bulk, Einstein's equation on the brane picks up an extra contribution which acts as an effective cosmological constant having expression: $F(\mathcal{R}) - \Lambda_4$. Thus though the four dimensional parameter Λ_4 is not small, an effective small cosmological constant can be generated by fine tuning Λ_4 and $F(\mathcal{R})$. Hence we can argue that the observed smallness of four dimensional cosmological constant is due to a fine tuning of induced cosmological constant on the brane with the $f(\mathcal{R})$ term in the bulk.

From the effective Einstein's equation we can solve for the metric elements as well as for dark radiation and dark pressure term provided a relation between dark pressure and dark radiation term is assumed. For four such choices the equations get sufficiently simplified such that analytic solutions can be obtained. We have derived all the metric elements for these four choices. Among the four solutions two of them show the presence of event horizon and thus is important from thermodynamic point of view. On the other hand the other two solutions lead to naked singularity and thus does not have much astrophysical importance. The important features of these solutions are the asymptotic non-flatness due to presence of $f(\mathcal{R})$ term. This might be of some relevance in the context of AdS-CFT correspondence.

After obtaining various solutions leading to either a black hole or a naked singularity, we have performed a stability analysis of our solutions in some appropriate limit. It turns out that the solutions are stable under tensor and scalar perturbations, while under certain choices of parameters the vector mode leads to instability. Also some solutions can be reduced to Nariai form, where the well known anti-evaporation in $f(\mathcal{R})$ gravity takes place leading to an increase in the area of the event horizon. However we have argued that in general the solutions are stable under perturbations.

In order to get some idea about solutions representing stellar interior, a symmetry transformation, known as conformal motion is invoked. For this particular symmetry class we can solve the field equations exactly. This leads to direct evaluation of dark pressure and radiation using these symmetries. Also there exists another class of transformations known as homology transformations. For this class of solutions the homology operator has been evaluated and it turns out that $f(\mathcal{R})$ term makes the homology class restricted compared to that in Einstein gravity.

Finally we consider thermodynamical behavior of these spherically symmetric space-times. Since thermodynamics is intimately connected to existence of a horizon, we consider only the two relevant cases. Here also the $f(\mathcal{R})$ term plays a dominant role in determining the thermodynamic behavior. In both the cases, the temperature and chemical potentials are found to be continuous, while the specific heat turns out to be discontinuous along a surface indicating a second order phase transition. Such features of these spherically symmetric solutions have their origin in the $f(\mathcal{R})$ term in the bulk action and only because of the presence of higher curvature terms in the action, the black hole solutions exhibit a phase transition, which, is second order in nature.

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Appendix A: Some detailed expressions

Here we present general expressions for various thermodynamic quantities for the case $U = 0$ which have been discussed in Sect. 3.

Under the condition of vanishing dark radiation also we have a horizon structure to our solution. Therefore we can work out the thermodynamic features. In this case, the horizon radius turns out to be in terms of the mass M and the parameter $F(\mathcal{R}) - \Lambda_4$ with unit $G = 1$ as:

$$r_1 = \frac{3^{-2/3}(F(\mathcal{R})-\Lambda_4) + \left(-M(F(\mathcal{R})-\Lambda_4)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + [-1+9(F(\mathcal{R})-\Lambda_4)M^2]}\right)^{2/3}}{3^{-1/3}(F(\mathcal{R})-\Lambda_4)\left(-M(F(\mathcal{R})-\Lambda_4)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + [-1+9(F(\mathcal{R})-\Lambda_4)M^2]}\right)^{1/3}} \tag{A1}$$

Then by the previous conditions: $\hbar = 1$ and an appropriate choice of Boltzmann constant we get entropy to be $S = r_h^2$. From the first law of black hole mechanics as presented in Eq. (81) the temperature turns out to be,

$$\begin{aligned} T^{-1} &= \left(\frac{\partial S}{\partial M}\right)_{F(\mathcal{R})-\Lambda_4} \\ &= \frac{2r_h}{3}\frac{\left(-M(F(\mathcal{R})-\Lambda_4)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + [-1+9(F(\mathcal{R})-\Lambda_4)M^2]}\right)^{2/3} - 3^{-2/3}(F(\mathcal{R})-\Lambda_4)}{\left(-M(F(\mathcal{R})-\Lambda_4)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + [-1+9(F(\mathcal{R})-\Lambda_4)M^2]}\right)^{4/3}} \\ &\quad\times\frac{1}{3^{-1/3}(F(\mathcal{R})-\Lambda_4)}\left[-(F(\mathcal{R})-\Lambda_4)^2 + \frac{9\sqrt{3}M(F(\mathcal{R})-\Lambda_4)}{\sqrt{(F(\mathcal{R})-\Lambda_4)^3/27 + (-1+9M^2(F(\mathcal{R})-\Lambda_4))}}\right] \end{aligned} \tag{A2}$$

while the potential ϕ can be obtained by solving the equation:

$$\begin{aligned} 0 = &\left[3^{-1/3}(F(\mathcal{R})-\Lambda_4)\left(-M(F(\mathcal{R})-\Lambda_4)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + [-1+9(F(\mathcal{R})-\Lambda_4)M^2]}\right)^{1/3}\right]^{-1} \\ &\times\left[3^{1/3} + \frac{2}{3}\left(-M(F(\mathcal{R})-\Lambda_4)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + [-1+9(F(\mathcal{R})-\Lambda_4)M^2]}\right)^{-1/3}\right. \\ &\times\left.\left\{-3\phi\left(F(\mathcal{R})-\Lambda_4\right)^2 - 6M\left(F(\mathcal{R})-\Lambda_4\right) + \frac{\sqrt{3}}{2}\frac{\left(F(\mathcal{R})-\Lambda_4\right)^2/3 + 27M^2 + 54M\left(F(\mathcal{R})-\Lambda_4\right)\phi}{\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + \left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}}\right\}\right] \\ &-\left[3^{-2/3}\left(F(\mathcal{R})-\Lambda_4\right) + \left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + \left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{2/3}\right] \\ &\times\left[3^{-1/3}\left(F(\mathcal{R})-\Lambda_4\right)\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + \left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{1/3}\right]^2 \\ &\times\left[3^{2/3}\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + \left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)\right. \\ &+3^{-1/3}\left(F(\mathcal{R})-\Lambda_4\right)\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2 + \sqrt{3}\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + \left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{-2/3} \\ &\times\left.\left\{-3\phi\left(F(\mathcal{R})-\Lambda_4\right)^2 - 6M\left(F(\mathcal{R})-\Lambda_4\right) + \frac{\sqrt{3}}{2}\frac{\left(F(\mathcal{R})-\Lambda_4\right)^2/3 + 27M^2 + 54M\left(F(\mathcal{R})-\Lambda_4\right)\phi}{\sqrt{\frac{(F(\mathcal{R})-\Lambda_4)^3}{27} + \left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}}\right\}\right] \end{aligned}$$

In this case the specific heat becomes,

$$
\begin{aligned}
C_{\mathrm{v}} &= T\left(\frac{\partial S}{\partial T}\right)_{F(\mathcal{R})-\Lambda_4} = \left(\frac{\partial M}{\partial T}\right)_{F(\mathcal{R})-\Lambda_4} \\
&= \frac{2r_h}{3^{2/3}\left(F(\mathcal{R})-\Lambda_4\right)}\left[-\frac{1}{3^{2/3}\left(F(\mathcal{R})-\Lambda_4\right)}\right. \\
&+\frac{4}{3}\frac{\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{1/3}}{\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{2/3}-3^{-2/3}\left(F(\mathcal{R})-\Lambda_4\right)} \\
&-\frac{2}{3}\frac{\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)}{\left\{\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{2/3}-3^{-2/3}\left(F(\mathcal{R})-\Lambda_4\right)\right\}^2} \\
&-\frac{\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{4/3}}{\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{2/3}-3^{-2/3}\left(F(\mathcal{R})-\Lambda_4\right)} \\
&\times\left\{\frac{9\sqrt{3}\left(F(\mathcal{R})-\Lambda_4\right)}{\sqrt{\left(F(\mathcal{R})-\Lambda_4\right)^3/27+\left(-1+9M^2\left(F(\mathcal{R})-\Lambda_4\right)\right)}}-\frac{\sqrt{3}81M^2\left(F(\mathcal{R})-\Lambda_4\right)^2}{\left(F(\mathcal{R})-\Lambda_4\right)^3/27+\left(-1+9M^2\left(F(\mathcal{R})-\Lambda_4\right)\right)^{3/2}}\right\} \\
&\times\left.\left[-\left(F(\mathcal{R})-\Lambda_4\right)^2+\frac{9\sqrt{3}M\left(F(\mathcal{R})-\Lambda_4\right)}{\sqrt{\left(F(\mathcal{R})-\Lambda_4\right)^3/27+\left(-1+9M^2\left(F(\mathcal{R})-\Lambda_4\right)\right)}}\right]^{-2}\right]^{-1}
\end{aligned}
\tag{A3}
$$

It is evident from the expression of the specific heat that it diverges at the surface given by:

$$
\begin{aligned}
&\frac{4}{3}\times\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{1/3} \\
&=\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{4/3} \\
&\quad\times\left\{\frac{9\sqrt{3}\left(F(\mathcal{R})-\Lambda_4\right)}{\sqrt{\left(F(\mathcal{R})-\Lambda_4\right)^3/27+\left(-1+9M^2\left(F(\mathcal{R})-\Lambda_4\right)\right)}}-\frac{81\sqrt{3}M^2\left(F(\mathcal{R})-\Lambda_4\right)^2}{\left(F(\mathcal{R})-\Lambda_4\right)^3/27+\left(-1+9M^2\left(F(\mathcal{R})-\Lambda_4\right)\right)^{3/2}}\right\} \\
&\quad\times\left[-\left(F(\mathcal{R})-\Lambda_4\right)^2+\frac{9\sqrt{3}M\left(F(\mathcal{R})-\Lambda_4\right)}{\sqrt{\left(F(\mathcal{R})-\Lambda_4\right)^3/27+\left(-1+9M^2\left(F(\mathcal{R})-\Lambda_4\right)\right)}}\right]^{-2} \\
&\quad+\frac{\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{2/3}-3^{-2/3}\left(F(\mathcal{R})-\Lambda_4\right)}{3^{2/3}\left(F(\mathcal{R})-\Lambda_4\right)} \\
&\quad+\frac{2}{3}\frac{\left(F(\mathcal{R})-\Lambda_4\right)\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)}{\left(-M\left(F(\mathcal{R})-\Lambda_4\right)^2+\sqrt{3}\sqrt{\frac{\left(F(\mathcal{R})-\Lambda_4\right)^3}{27}+\left[-1+9\left(F(\mathcal{R})-\Lambda_4\right)M^2\right]}\right)^{2/3}-3^{-2/3}\left(F(\mathcal{R})-\Lambda_4\right)}
\end{aligned}
\tag{A4}
$$

This again shows that the black hole solution presented by the condition of vanishing dark radiation has a divergent behavior on the above surface which in turn indicates that the black hole undergoes a second order phase transition on this surface.

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