

SPIN-BOSON MODEL THROUGH A POISSON-DRIVEN STOCHASTIC PROCESS

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Abstract

We give a functional integral representation of the semigroup generated by the spin-boson Hamiltonian by making use of a Poisson point process and a Euclidean field. We present a method of constructing Gibbs path measures indexed by the full real line which can be applied also to more general stochastic processes with jump discontinuities. Using these tools we then show existence and uniqueness of the ground state of the spin-boson, and analyze ground state properties. In particular, we prove super-exponential decay of the number of bosons, Gaussian decay of the field operators, derive expressions for the positive integer, fractional and exponential moments of the field operator, and discuss the field fluctuations in the ground state.

Key-words: Poisson process, càdlàg paths, Gibbs measure, spin-boson operator, ground state

1 Spin-boson model

1.1 Introduction

Gibbs measures constructed on the space of continuous paths of a random process proved to play an important role in studying ground state properties of Hamiltonians in quantum field theory ([LHB11, Chapter 6] and [LM01, LMS02, L02, BHLMS02, BH09, GL09, GHL12]). Such random processes are obtained by conditioning Brownian motion with respect to an external and a pair interaction potential. In this setting Gibbs measures are obtained as weak limits of sequences of Gibbs measures indexed by the bounded intervals of the real line by using pre-compactness or tightness arguments.

In the present paper we extend this strategy to construct Gibbs measures on paths of a random process with jump discontinuities (càdlàg paths) associated with the Hamilton operator of the spin-boson model:

$$H = -\varepsilon\sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes H_f + \alpha\sigma_z \otimes \phi_b(\hat{h}) \quad (1.1)$$

with a view of studying spectral properties of this Hamiltonian. Here $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ are parameters, σ_x, σ_z are Pauli matrices, H_f is the free field Hamiltonian, $\phi_b(\hat{h})$ is the field operator in Fock space \mathcal{F} , and \hat{h} is a form factor (see the details below). One of the merits of this approach is that it allows to carry through this analysis in a non-perturbative way. While in [Spo89, SSW90] the spectral properties of the spin-boson model are discussed through a measure on the space of paths with jump discontinuities, no attention was paid to constructing Gibbs measures.

As it will be seen below, in the case of the spin-boson model Gibbs measures involve densities dependent on a pair interaction potential alone, and no external potential contribution. We stress that, in contrast, in the case of Brownian motion under zero external potential and non-zero pair interaction potential even the very existence of Gibbs measures is poorly understood. A rigorous study of Gibbs measures with an external potential but without pair interaction on càdlàg paths was begun in [KL12a], considering them with respect to fractional $P(\phi)_1$ -processes, i.e., stable processes under an external potential.

For Hamiltonians with spin or a non-local kinetic term Feynman-Kac-type formulae can be derived by using Lévy processes [Hir12, HL08, LM12, HIL12a, HIL12b, KL12b]. There are few rigorous results on quantum models with spin using functional integration methods. In [HL08] we derived such a formula for the heat semigroup generated by a quantum field operator with spin by making use of a Euclidean quantum field and a

Poisson process. In [HIL12b] we developed similar methods for relativistic Schrödinger operators, allowing to obtain results on the decay of ground states.

In this paper we derive a Feynman-Kac-type formula for the semigroup generated by the spin-boson Hamiltonian. The spin-boson model is a much studied variant of the Caldeira-Leggett model describing a two-state quantum system linearly coupled to a scalar quantum field [LCDGZ87]. Work on the spectral properties of the spin-boson and related models includes [SD85, Spo89, SSW90, HS95, Ger96, BS98, AH97, Hir99, Ger00, Hir01, Hir02]. In particular, in [SD85, Spo89, SSW90, HS95] stochastic methods were used. Existence or absence of a spin-boson ground state was investigated in [AH97, Ger00]. Below we obtain existence and uniqueness of the ground state by a different approach. In [HH12] we apply the method developed in this paper to the so-called Rabi model, which can be regarded as a single-mode spin-boson model. We also refer to the recent papers [Abd12, HH10].

In order to study the spin-boson in a stochastic representation we describe the spin states by the set $\mathbb{Z}_2 = \{-1, +1\}$ and derive a Poisson-driven random process with càdlàg path space $\mathcal{X} = D(\mathbb{R}, \mathbb{Z}_2)$, indexed by the real line and taking values in \mathbb{Z}_2 . This will describe the spin-process. The spin-boson Hamiltonian can be defined as a self-adjoint operator on a Hilbert space $L^2(\mathbb{Z}_2 \times Q)$ instead of $\mathbb{C}^2 \otimes \mathcal{F}$.

On path space \mathcal{X} we are then able to construct a Gibbs measure μ_g associated with the unique ground state φ_g of the spin-boson Hamiltonian. Using this probability measure we represent ground state expectations for interesting choices of operators \mathcal{O} in the form

$$(\varphi_g, \mathcal{O}\varphi_g) = \int_{\mathcal{X}} f_{\mathcal{O}} d\mu_g, \quad (1.2)$$

where $f_{\mathcal{O}}$ is a function on path space \mathcal{X} uniquely associated with \mathcal{O} . We will consider the field operator $\phi(f)$ with test function f , and the second quantization $d\Gamma(\rho(-i\nabla))$ of the multiplication operator by a function ρ , and derive path integral representations of expressions of the type

$$\mathcal{O} = \xi(\sigma)F(\phi(f)) \quad \text{and} \quad \mathcal{O} = \xi(\sigma)e^{-\beta d\Gamma(\rho(-i\nabla))} \quad (1.3)$$

with suitable $\xi : \mathbb{Z}_2 \rightarrow \mathbb{C}$, $F : \mathbb{R} \rightarrow \mathbb{C}$ and $\rho : \mathbb{R}^d \rightarrow [0, \infty)$. These cases include

$$\mathcal{O} = N^m, \quad \mathcal{O} = e^{\beta N}, \quad \mathcal{O} = (-1)^N \quad \text{and} \quad \sigma(-1)^N$$

for all $m \in \mathbb{N}$ and $\beta \in \mathbb{C}$ (in particular, $\beta > 0$), where $N = d\Gamma(\mathbb{1})$ is the boson number operator, as well as

$$\mathcal{O} = e^{(\beta/2)\phi(f)^2}, \quad \mathcal{O} = e^{i\beta\phi(f)}, \quad \mathcal{O} = \phi(f)^n, \quad \mathcal{O} = |\phi(f)|^s \quad \text{and} \quad \mathcal{O} = \sigma\phi(f).$$

Here $\beta \in \mathbb{R}$, $n \in \mathbb{N}$ and $0 < s < 2$.

Specifically, we obtain explicit formulae for the positive integer and fractional order moments and exponential moments of the field operator, and show that the field fluctuations increase on switching on the coupling between the spin and the boson field. Moreover, we show that

$$\varphi_g \in D(e^{(\beta/2)\phi(f)^2})$$

for $-\infty < \beta < 1/\|f\|^2$ (i.e., Gaussian decay of the field operators) with

$$\lim_{\beta \uparrow 1/\|f\|^2} \|e^{(\beta/2)\phi(f)^2} \varphi_g\| = \infty.$$

As a consequence, we obtain another representation of the ground state. Recall that when $\varepsilon = 0$, the spin boson Hamiltonian can be diagonalized so that each matrix element is a van Hove Hamiltonian (see also below). Then it is trivial to see that

$$(\varphi_g, F(\phi(f))\varphi_g) = (\varphi_{\text{vH}}, F(\phi_b(\hat{f}))\varphi_{\text{vH}}),$$

where φ_{vH} is the ground state of the van Hove Hamiltonian in \mathcal{F} . Here we show a similar representation for the case of $\varepsilon \neq 0$, i.e., we derive

$$(\varphi_g, F(\phi(f))\varphi_g) = \int_{\mathcal{X}} (\varphi_{\text{vH}}(\chi), F(\phi_b(\hat{f}))\varphi_{\text{vH}}(\chi)) d\mu_g,$$

where $\varphi_{\text{vH}}(\chi)$ is the ground state of a random van Hove model and χ is a function of the random path. This suggests implicitly that $\varphi_g = \int_{\mathcal{X}}^{\oplus} \varphi_{\text{vH}}(\chi) d\mu_g$.

Next we discuss ground state properties of second quantized operators. In particular, we prove that

$$\varphi_g \in D(e^{\beta N}), \quad \beta > 0$$

(i.e., super-exponential decay of the number of bosons). Also, we obtain explicit formulae for the moments of the boson number operator in terms of sums involving coefficients given by the Stirling numbers of the second kind. Furthermore, although we show that $(\varphi_g, \sigma(-1)^N \varphi_g) = -1$, we obtain a positive lower bound on the ground state functional $(\varphi_g, (-1)^N \varphi_g)$. Finally, we obtain the inequality

$$(\varphi_g, N \varphi_g) \leq \frac{\alpha}{2} (\varphi_g, \phi(\omega(\mathbf{D})^{-1} h) \varphi_g) \leq \frac{\alpha^2}{2} \|\hat{h}/\omega\|^2, \quad \mathbf{D} = -i\nabla,$$

relating the mean of the field operator with the expected boson number in the ground state.

These applications to ground state properties are derived from the main results of this paper, which can be summarized as follows:

- (1) existence and uniqueness of the ground state φ_g of the spin-boson Hamiltonian H is shown in Theorem 2.3
- (2) a probability measure μ_g on càdlàg path space \mathcal{X} associated with φ_g is constructed in Theorems 3.8 and 3.9
- (3) it is shown in Theorem 3.12 that μ_g is a Gibbs measure for a pair interaction potential
- (4) path integral representations of $(\varphi_g, \mathcal{O}\varphi_g)$ in terms of μ_g are given in Section 4, in particular,
 - (i) it is shown that

$$\varphi_g \in D(e^{aN}) \cap D(e^{b\phi(f)^2}), \quad (a, b) \in \mathbb{R} \times (-\infty, 1/\|f\|^2)$$

in Theorem 4.9, Corollary 4.10 and Corollary 4.18

- (ii) it is shown that $(\varphi_g, \mathcal{O}\varphi_g)$ can be represented by the ground state $\varphi_{\text{vH}}(\chi)$ of a van Hove Hamiltonian as

$$(\varphi_g, \mathcal{O}\varphi_g) = \int_{\mathcal{X}} (\varphi_{\text{vH}}(\chi), \mathcal{O}\varphi_{\text{vH}}(\chi)) d\mu_g$$

in Theorem 4.14.

We note that path integral representations were already used to a great extent for the Nelson model describing the interaction of a charged particle with a scalar quantum field, see [BHLMS02] and [LHB11, Chapter 6]. We also note that although the result $(\varphi_g, e^{\beta N} \varphi_g) < \infty$ has been established in [Gro73] by using operator theory, our construction here is completely different and rather general. Also, our methods can be applied to further models involving càdlàg paths, for instance, the Nelson model with a relativistic kinetic term $\sqrt{-\Delta + m^2} + V$, which will be done elsewhere.

The paper is organized as follows. The remainder of Section 1 is devoted to constructing the Feynman-Kac formula of the spin-boson heat semigroup. In Section 2 we show that the spin-boson Hamiltonian has a unique ground state φ_g if an infrared regularity condition is satisfied. In Section 3 we define a Gibbs measure on \mathcal{X} for bounded time intervals associated with the density obtained from the Feynman-Kac representation, and show its local weak convergence to a path measure in the infinite time limit. We view Theorems 3.8 and 3.9 below to be pivotal results in this paper. In Section 4 we derive the expressions of the ground state expectations mentioned above.

1.2 Definition

We begin by defining the spin-boson Hamiltonian as a self-adjoint operator on a Hilbert space. Let $\mathcal{F} = \bigoplus_{n=0}^{\infty} (\otimes_{\text{sym}}^n L^2(\mathbb{R}^d))$ be the boson Fock space over $L^2(\mathbb{R}^d)$, where the subscript means symmetrized tensor product. We denote the boson annihilation and creation operators by $a(f)$ and $a^\dagger(f)$, $f, g \in L^2(\mathbb{R}^d)$, respectively, satisfying the canonical commutation relations

$$[a(f), a^\dagger(g)] = (\bar{f}, g), \quad [a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]. \quad (1.4)$$

We use the informal expression $a^\sharp(f) = \int a^\sharp(k)f(k)dk$ for notational convenience. Consider the Hilbert space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{F}. \quad (1.5)$$

Denote by $d\Gamma(T)$ be the second quantization of a self-adjoint operator T in $L^2(\mathbb{R}^d)$. The operator on Fock space defined by

$$H_f = d\Gamma(\omega) \quad (1.6)$$

is the free boson Hamiltonian with dispersion relation $\omega(k) = |k|$. The operator

$$\phi_b(\hat{h}) = \frac{1}{\sqrt{2}} \int \left(a^\dagger(k)\hat{h}(-k) + a(k)\hat{h}(k) \right) dk, \quad (1.7)$$

acting on Fock space is the scalar field operator, where $h \in L^2(\mathbb{R}^d)$ is a suitable form factor and \hat{h} is the Fourier transform of h . Denote by σ_x, σ_y and σ_z the 2×2 Pauli matrices given by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (1.8)$$

With these components, the spin-boson Hamiltonian is defined by the linear operator

$$H_{\text{SB}} = \varepsilon \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes H_f + \alpha \sigma_x \otimes \phi_b(\hat{h}) \quad (1.9)$$

on \mathcal{H} , where $\alpha \in \mathbb{R}$ is a coupling constant and $\varepsilon \geq 0$ a parameter.

1.3 A Feynman-Kac-type formula

In this section we give a functional integral representation of $e^{-tH_{\text{SB}}}$ by making use of a Poisson point process and an infinite dimensional Ornstein-Uhlenbeck process. First we transform H_{SB} in a convenient form to study its spectrum in terms of path measures.

Recall that the rotation group in \mathbb{R}^3 has an adjoint representation on $SU(2)$. Let $n \in \mathbb{R}^3$ be a unit vector and $\theta \in [0, 2\pi)$. Thus $e^{(i/2)\theta n \cdot \sigma}$ satisfies that $e^{(i/2)\theta n \cdot \sigma} \sigma_\mu e^{-(i/2)\theta n \cdot \sigma} = (R\sigma)_\mu$, where R denotes the 3×3 matrix representing the rotation around n with angle θ , and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. In particular, for $n = (0, 1, 0)$ and $\theta = \pi/2$, we have

$$e^{(i/2)\theta n \cdot \sigma} \sigma_x e^{-(i/2)\theta n \cdot \sigma} = \sigma_z, \quad (1.10)$$

$$e^{(i/2)\theta n \cdot \sigma} \sigma_z e^{-(i/2)\theta n \cdot \sigma} = -\sigma_x. \quad (1.11)$$

Let

$$U = \exp\left(i\frac{\pi}{4}\sigma_y\right) \otimes \mathbb{1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \otimes \mathbb{1} \quad (1.12)$$

be a unitary operator on \mathcal{H} . By (1.10) and (1.11) H_{SB} transforms as

$$H = UH_{\text{SB}}U^* = -\varepsilon\sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes H_f + \alpha\sigma_z \otimes \phi_b(\hat{h}). \quad (1.13)$$

Then H is realized as

$$H = \begin{bmatrix} H_f + \alpha\phi_b(\hat{h}) & -\varepsilon \\ -\varepsilon & H_f - \alpha\phi_b(\hat{h}) \end{bmatrix}.$$

In particular, $\varepsilon = 0$ makes H diagonal. If $\hat{h}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$ and h is real-valued, then $\phi_b(\hat{h})$ is symmetric and infinitesimally small with respect to H_f , hence by the Kato-Rellich theorem it follows that H is a self-adjoint operator on $D(H_f)$ and bounded from below.

To construct the functional integral representation of the semigroup e^{-tH} , it is useful to introduce a spin variable $\sigma \in \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{-1, +1\}$ is the additive group of order 2. For $\Psi = \begin{bmatrix} \Psi(+) \\ \Psi(-) \end{bmatrix} \in \mathcal{H}$, we have

$$H\Psi = \begin{bmatrix} (H_f + \alpha\phi_b(\hat{h}))\Psi(+) - \varepsilon\Psi(-) \\ (H_f - \alpha\phi_b(\hat{h}))\Psi(-) - \varepsilon\Psi(+) \end{bmatrix}. \quad (1.14)$$

Thus we can transform H on \mathcal{H} to the operator \tilde{H} on

$$L^2(\mathbb{Z}_2; \mathcal{F}) = \left\{ f : \mathbb{Z}_2 \rightarrow \mathcal{F} \left| \|f\|_{L^2(\mathbb{Z}_2; \mathcal{F})} = \sum_{\sigma \in \mathbb{Z}_2} \|f(\sigma)\|_{\mathcal{F}}^2 < \infty \right. \right\} \quad (1.15)$$

by

$$(\tilde{H}\Psi)(\sigma) = \left(H_f + \alpha\sigma\phi_b(\hat{h}) \right) \Psi(\sigma) + \varepsilon\Psi(-\sigma), \quad \sigma \in \mathbb{Z}_2. \quad (1.16)$$

In what follows, we identify the Hilbert space \mathcal{H} with $L^2(\mathbb{Z}_2; \mathcal{F})$ through

$$\mathcal{H} \ni \begin{bmatrix} \Psi(+) \\ \Psi(-) \end{bmatrix} \mapsto \Psi(\sigma) = \begin{cases} \Psi(+), & \sigma = +1, \\ \Psi(-), & \sigma = -1 \end{cases} \in L^2(\mathbb{Z}_2; \mathcal{F}),$$

and instead of H we consider \tilde{H} , and use the notation H for \tilde{H} .

Let (Ω, Σ, P) be a probability space, and $(N_t)_{t \in \mathbb{R}}$ be a two-sided Poisson process with unit intensity on this space. We denote by $D = \{t \in \mathbb{R} \mid N_{t+} \neq N_{t-}\}$ the set of jump points, and define the integral with respect to this Poisson process by

$$\int_{(s,t]} f(r, N_r) dN_r = \sum_{\substack{r \in D \\ r \in (s,t]}} f(r, N_r)$$

for any predictable function f (we refer to Appendix of [HL08] for details). In particular, we have for any continuous function g ,

$$\int_{(s,t]} g(r, N_{r-}) dN_r = \sum_{\substack{r \in D \\ s < r \leq t}} g(r, N_{r-}). \quad (1.17)$$

We write $\int_s^{t+} \cdots dN_r$ for $\int_{(s,t]} \cdots dN_r$. Note that $\int_s^{t+} g(r, N_{r-}) dN_r$ is right-continuous in t and the integrand $g(r, N_{r-})$ is left-continuous in r and thus a predictable process. Define the random process

$$\sigma_t = \sigma(-1)^{N_t}, \quad \sigma \in \mathbb{Z}_2. \quad (1.18)$$

This process describes the spin. Since our Poisson process is indexed by the real line, we summarize its properties below.

Proposition 1.1 *The stochastic process $(N_t)_{t \in \mathbb{R}}$ has the following properties:*

1. Independence: *The random variables N_t and N_s are independent for all $s \leq 0 \leq t$, $s \neq t$.*
2. Markov property: *The stochastic processes $(N_t)_{t \geq 0}$ and $(N_t)_{t \leq 0}$ are Markov processes with respect to the natural filtrations $\mathcal{N}_t^+ = \sigma(N_s, 0 \leq s \leq t)$ and $\mathcal{N}_t^- = \sigma(N_s, t \leq s \leq 0)$, respectively, i.e.,*

$$\mathbb{E}_P [N_{t+s} | \mathcal{N}_s^+] = \mathbb{E}_P^{N_s} [N_t], \quad \mathbb{E}_P [N_{-t-s} | \mathcal{N}_{-s}^-] = \mathbb{E}_P^{N_{-s}} [N_{-t}].$$

3. Reflection symmetry: *The random variables N_t and N_{-t} are identically distributed for all $t \in \mathbb{R}$, i.e., $\mathbb{E}_P[f(N_{-t})] = \mathbb{E}_P[f(N_t)] = \sum_{n=0}^{\infty} f(n) \frac{|t|^n}{n!} e^{-|t|}$.*

4. Shift invariance: *The stochastic process $\sigma_t = \sigma(-1)^{Nt}$, $t \in \mathbb{R}$, is shift invariant, i.e.,*

$$\sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \left[\prod_{j=0}^n f_j(\sigma_{t_j}) \right] = \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \left[\prod_{j=0}^n f_j(\sigma_{s+t_j}) \right], \quad s \in \mathbb{R}.$$

Proof: The proof is a minor modification of [LHB11, Theorem 3.106] and it is omitted. \square

In the Schrödinger representation the boson Fock space \mathcal{F} can be realized as an L^2 -space over a probability space (Q, μ) , and the field operator $\phi_b(\hat{f})$ with real-valued function $f \in L^2(\mathbb{R}^d)$ as a multiplication operator, which we will denote by $\phi(f)$. The identity function $\mathbb{1}$ on Q corresponds to the Fock vacuum Ω_b in \mathcal{F} .

Next we introduce the random process describing the free boson field H_f . Let (Q_E, μ_E) be a probability space associated with the Euclidean quantum field (for details see [LHB11, Section 5]). The Hilbert spaces $L^2(Q_E)$ and $L^2(Q)$ are related through the family of isometries $\{j_s\}_{s \in \mathbb{R}}$ from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{d+1})$ defined by

$$\widehat{j_s f}(k, k_0) = \frac{e^{-itk_0}}{\sqrt{\pi}} \sqrt{\frac{\omega(k)}{|k_0|^2 + \omega(k)^2}} \hat{f}(k),$$

where \hat{f} denotes the Fourier transform of f . Let $\Phi_E(j_s f)$ be a Gaussian random variable on (Q_E, μ_E) indexed by $j_s f \in L^2(\mathbb{R}^{d+1})$ with mean zero and covariance

$$\mathbb{E}_{\mu_E}[\Phi_E(j_s f)\Phi_E(j_t g)] = \frac{1}{2} \int_{\mathbb{R}^d} e^{-|s-t|\omega(k)} \overline{\hat{f}(k)} \hat{g}(k) dk.$$

Also, let $\{J_s\}_{s \in \mathbb{R}}$ be the family of isometries from $L^2(Q)$ to $L^2(Q_E)$ defined by

$$J_s : \phi(f_1) \cdots \phi(f_n) := : \Phi_E(j_s f_1) \cdots \Phi_E(j_s f_n) :,$$

where $:X:$ denotes Wick product of X . Then we derive that

$$(J_s \Phi, J_t \Psi)_{L^2(Q_E)} = (\Phi, e^{-|t-s|H_f} \Psi)_{L^2(Q)}. \quad (1.19)$$

In [HL08] by making use of the process $(\sigma_t)_{t \in \mathbb{R}}$ a functional integral representation of the Pauli-Fierz model with spin 1/2 in non-relativistic quantum electrodynamics was obtained. By a suitable modification we can also construct the functional integral representation of e^{-tH} . In fact, the construction in the spin-boson case becomes simpler than in the case of the Pauli-Fierz model, see also [HL08, Remark 6.3] and [LHB11, Section 7.9]. We have the following Feynman-Kac-type formula for the spin-boson Hamiltonian.

We identify \mathcal{H} as

$$\mathcal{H} \cong L^2(\mathbb{Z}_2; L^2(Q)) \cong L^2(\mathbb{Z}_2 \times Q). \quad (1.20)$$

Proposition 1.2 *Let $\Phi, \Psi \in \mathcal{H}$ and $h \in L^2(\mathbb{R}^d)$ be real-valued. Then*

$$(\varepsilon \neq 0) \quad (\Phi, e^{-tH}\Psi)_{\mathcal{H}} = e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \mathbb{E}_{\mu_E} \left[\overline{J_0 \Phi(\sigma_0)} e^{-\alpha \Phi_E(\int_0^t \sigma_s j_s h ds)} \varepsilon^{N_t} J_t \Psi(\sigma_t) \right] \quad (1.21)$$

$$(\varepsilon = 0) \quad (\Phi, e^{-tH}\Psi)_{\mathcal{H}} = e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mu_E} \left[\overline{J_0 \Phi(\sigma)} e^{-\alpha \Phi_E(\int_0^t j_s h ds)} J_t \Psi(\sigma) \right]. \quad (1.22)$$

Proof: Let $\varepsilon \neq 0$. We see by (1.16) that

$$H\Psi(\sigma) = (H_t + \alpha\phi(h))\Psi(\sigma) - e^{\log \varepsilon} \Psi(-\sigma), \quad \sigma \in \mathbb{Z}_2.$$

Then by [HL08, Theorem 4.11] we have

$$(\Phi, e^{-tH}\Psi)_{\mathcal{H}} = e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \mathbb{E}_{\mu_E} \left[\overline{J_0 \Phi(\sigma_0)} e^{-\alpha \int_0^t \sigma_s \Phi_E(j_s f) ds + \int_0^{t+} \log \varepsilon dN_s} J_t \Psi(\sigma_t) \right].$$

Since $\int_0^{t+} \log \varepsilon dN_s = N_t \log \varepsilon$, (1.21) follows. Next consider the case $\varepsilon = 0$. As $\varepsilon \rightarrow 0$ on both sides of (1.21) the integrands over $\{N_t \geq 1\}$ vanish while those on $\{N_t = 0\}$ are non-vanishing. Moreover, note that $N_s = 0$, $s \leq t$, on $\{N_t = 0\}$. Hence (1.22) is obtained by taking the limit

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} (\Phi, e^{-tH}\Psi)_{\mathcal{H}} &= \lim_{\varepsilon \rightarrow 0} e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \mathbb{E}_{\mu_E} \left[\overline{J_0 \Phi(\sigma_0)} e^{-\alpha \Phi_E(\int_0^t \sigma_s j_s h ds)} \varepsilon^{N_t} J_t \Psi(\sigma_t) \right] \\ &= e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mu_E} \left[\overline{J_0 \Phi(\sigma)} e^{-\alpha \Phi_E(\int_0^t j_s h ds)} J_t \Psi(\sigma) \right]. \end{aligned}$$

□

Remark 1.3 By the Feynman-Kac formula (Proposition 1.2) we see that

$$e^{-tH}\Phi(\sigma) = e^t \mathbb{E}_P \left[J_0^* e^{\Phi_E(-\alpha \int_0^t \sigma_r j_r h dr)} J_t \Phi(\sigma_t) \right] \quad (1.23)$$

for every $\sigma \in \mathbb{Z}_2$.

Denote $\mathbb{1}_{\mathcal{H}} = \mathbb{1}_{L^2(\mathbb{Z}_2)} \otimes \mathbb{1}_{L^2(Q)}$. Using the above proposition we can compute the vacuum expectation of the semigroup e^{-tH} , which plays an important role in this paper.

Corollary 1.4 *Let $h \in L^2(\mathbb{R}^d)$ be a real-valued function. Then for every $t > 0$ it follows that*

$$(\mathbb{1}_{\mathcal{H}}, e^{-tH} \mathbb{1}_{\mathcal{H}}) = e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \left[\varepsilon^{N_t} e^{\frac{\alpha^2}{2} \int_0^t dr \int_0^t W(N_r - N_s, r-s) ds} \right], \quad (1.24)$$

where the pair interaction potential W is given by

$$W(x, s) = \frac{(-1)^x}{2} \int_{\mathbb{R}^d} e^{-|s|\omega(k)} |\hat{h}(k)|^2 dk. \quad (1.25)$$

Proof: By Proposition 1.2 we have

$$(\mathbb{1}_{\mathcal{H}}, e^{-tH} \mathbb{1}_{\mathcal{H}}) = e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \left[\varepsilon^{N_t} e^{\frac{\alpha^2}{2} \int_0^t dr \int_0^t W(N_r + N_s, r-s) ds} \right].$$

Since $W(N_r + N_s, r-s) = W(N_r - N_s, r-s)$, the corollary follows. \square

Note that equality (1.25) gives the interaction potential

$$W(N_r - N_s, r-s) = \frac{1}{2} \sigma_r \sigma_s \int_{\mathbb{R}^d} e^{-|r-s|\omega(k)} |\hat{h}(k)|^2 dk \quad (1.26)$$

of an infinite range Ising-model on the real line instead of a lattice.

1.4 Parity symmetry

It is a known fact that H_{SB} has a parity symmetry. Let

$$P = \sigma_z \otimes (-1)^N, \quad (1.27)$$

where $N = d\Gamma(\mathbb{1})$ denotes the number operator in \mathcal{F} . From $\text{Spec}(\sigma_z) = \{-1, 1\}$ and $\text{Spec}(N) = \{0, 1, 2, \dots\}$ it follows that $\text{Spec}(P) = \{-1, 1\}$. We identify \mathcal{H} with $\mathcal{F}_\uparrow \oplus \mathcal{F}_\downarrow$, where \mathcal{F}_\uparrow and \mathcal{F}_\downarrow are identical copies of \mathcal{F} . Then each Pauli matrix $\sigma_X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acts as

$$\sigma_X \begin{bmatrix} \Psi(+), \\ \Psi(-) \end{bmatrix} = \begin{bmatrix} a\Psi(+), + b\Psi(-) \\ c\Psi(+), + d\Psi(-) \end{bmatrix}$$

for $\begin{bmatrix} \Psi(+), \\ \Psi(-) \end{bmatrix} \in \mathcal{F}_\uparrow \oplus \mathcal{F}_\downarrow$. Furthermore, $\mathcal{F} = \bigoplus_n \mathcal{F}_n$ can be decomposed as $\mathcal{F}_e \oplus \mathcal{F}_o$, where \mathcal{F}_e and \mathcal{F}_o denote respectively the subspaces of \mathcal{F} consisting of even and odd numbers of bosons, i.e., $\mathcal{F}_e = \bigoplus_{m=0}^\infty \mathcal{F}_{2m}$ and $\mathcal{F}_o = \bigoplus_{m=0}^\infty \mathcal{F}_{2m+1}$. The projections from \mathcal{F} to \mathcal{F}_e and \mathcal{F}_o are denoted by P_e and P_o , respectively. Let $\mathcal{H}_+ = P_e \mathcal{F}_\uparrow \oplus P_o \mathcal{F}_\downarrow$ and $\mathcal{H}_- = P_o \mathcal{F}_\uparrow \oplus P_e \mathcal{F}_\downarrow$ be subspaces of $\mathcal{F}_\uparrow \oplus \mathcal{F}_\downarrow$.

Lemma 1.5 *The following properties hold:*

(1) *The Hilbert space \mathcal{H} can be identified with $\mathcal{H}_+ \oplus \mathcal{H}_-$ by the correspondence*

$$\mathcal{F}_\uparrow \oplus \mathcal{F}_\downarrow \ni \begin{bmatrix} \Psi(+) \\ \Psi(-) \end{bmatrix} \mapsto \begin{bmatrix} \Psi_e(+) \\ \Psi_o(-) \end{bmatrix} \oplus \begin{bmatrix} \Psi_o(+) \\ \Psi_e(-) \end{bmatrix} \in \mathcal{H}_+ \oplus \mathcal{H}_-,$$

where $\Psi_e(\pm) = P_e \Psi(\pm)$ and $\Psi_o(\pm) = P_o \Psi(\pm)$.

(2) *It follows that $[H_{\text{SB}}, P] = 0$.*

(3) *\mathcal{H}_\pm is the eigenspace associated with eigenvalue ± 1 of P .*

(4) *H_{SB} can be decomposed as $H_{\text{SB}} = H_{\text{SB}}|_{\mathcal{H}_+} \oplus H_{\text{SB}}|_{\mathcal{H}_-}$.*

Proof: (1) and (2) are straightforward. Let $\Psi = \begin{bmatrix} \Psi_e(+) \\ \Psi_o(-) \end{bmatrix} \in \mathcal{H}_+$. Then $P\Psi = \Psi$ follows by a direct calculation. Thus Ψ is eigenvector of P with eigenvalue $+1$. Similarly it follows that \mathcal{H}_- is the eigenspace associated with eigenvalue -1 of P , and (3) also follows. (4) is obtained by a combination of (1), (2) and (3). \square

Finally, for later use we show some related spin-flip properties.

Lemma 1.6 *We have the following properties:*

(1) *$(\Psi, \sigma_x \Phi) = 0$ and $(U^* \Psi, \sigma_z U^* \Phi) = 0$ for any $\Psi, \Phi \in \mathcal{H}_\pm$.*

(2) *$(\Psi, \phi(f)\Phi) = 0$ and $(U^* \Psi, \phi(f)U^* \Phi) = 0$ for any $\Psi, \Phi \in \mathcal{H}_\pm$.*

Proof: (1) It is straightforward to show that σ_x is a spin-flip transform, i.e.,

$$\sigma_x \mathcal{H}_\pm \subset \mathcal{H}_\mp,$$

which gives the first statement. The second statement follows by observing that $U\sigma_z U^* = \sigma_x$. To obtain (2) it is again straightforward to show that

$$\phi(f)(\mathcal{H}_\pm \cap D(\phi(f))) \subset \mathcal{H}_\mp.$$

The second part follows by $U\phi(f)U^* = \phi(f)$. \square

2 Ground state of the spin-boson

2.1 Positivity improving semigroup

In the remainder of this paper we assume that $h \in L^2(\mathbb{R}^d)$ is real-valued.

Let $E = \inf \text{Spec}(H)$. We estimate the dimension of $\text{Ker}(H - E)$ for $\varepsilon \neq 0$.

Corollary 2.1 *Assume that $\varepsilon \neq 0$. Then e^{-tH} , $t > 0$, is a positivity improving semigroup on $L^2(\mathbb{Z}_2 \times Q)$, i.e., $(\Psi, e^{-tH}\Phi) > 0$ for $\Psi, \Phi \geq 0$ such that $\Psi \not\equiv 0 \not\equiv \Phi$.*

Proof: The proof simplifies an argument in [Hir00]. It is trivial that $(\Psi, e^{-tH}\Phi) \geq 0$, thus it suffices showing that $(\Psi, e^{-tH}\Phi) \neq 0$. Suppose the contrary. Then we have

$$\sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \left[\left(J_0 \Psi(\sigma_0), e^{\Phi_{\mathbb{E}}(-\alpha \int_0^t \sigma_s j_s h ds)} \varepsilon^{N_t} J_t \Phi(\sigma_t) \right)_{L^2(Q_{\mathbb{E}})} \right] = 0.$$

Since J_t is positivity preserving and $e^{\Phi_{\mathbb{E}}(-\alpha \int_0^t \sigma_s j_s h ds)}$ is positive, for $\sigma \in \mathbb{Z}_2$ we have

$$\mathbb{E}_P \left[\left(J_0 \Psi(\sigma_0), e^{\Phi_{\mathbb{E}}(-\alpha \int_0^t \sigma_s j_s h ds)} \varepsilon^{N_t} J_t \Phi(\sigma_t) \right)_{L^2(Q_{\mathbb{E}})} \right] = 0,$$

which implies that $\text{supp} J_0 \Psi(\sigma_0) \cap \text{supp} J_t \Phi(\sigma_t) = \emptyset$ a.s. Hence $0 = (J_0 \Psi(\sigma_0), J_t \Phi(\sigma_t)) = (\Psi(\sigma_0), e^{-tH_t} \Phi(\sigma_t))$. Since e^{-tH_t} is positivity improving, $\Psi(\sigma_0) \equiv 0$ or $\Phi(\sigma_t) \equiv 0$. This contradicts that $\Psi \not\equiv 0$ and $\Phi \not\equiv 0$, and the claim follows. \square

2.2 Existence and uniqueness of ground state

2.2.1 The case of $\varepsilon = 0$

Whenever $\varepsilon = 0$ the Hamiltonian H is diagonal, i.e., we have

$$H = \begin{bmatrix} H_f + \alpha \phi(h) & 0 \\ 0 & H_f - \alpha \phi(h) \end{bmatrix}.$$

It is known that $H_f + \alpha \phi(h)$, the Hamiltonian of the van Hove model, has a unique ground state if and only if $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, see e.g. [Hir06], which implies that H has a two-fold degenerate ground state if and only if $\hat{h}/\omega \in L^2(\mathbb{R}^d)$.

2.2.2 The case of $\varepsilon \neq 0$

Next we consider the case of $\varepsilon \neq 0$. Write

$$\Phi_T = e^{-T(H-E)} \mathbb{1}, \quad T \geq 0, \tag{2.1}$$

and

$$\gamma(T) = \frac{(\mathbb{1}_{\mathcal{H}}, \Phi_T)^2}{\|\Phi_T\|^2} = \frac{(\mathbb{1}_{\mathcal{H}}, e^{TH} \mathbb{1}_{\mathcal{H}})^2}{(\mathbb{1}_{\mathcal{H}}, e^{-2TH} \mathbb{1}_{\mathcal{H}})}. \quad (2.2)$$

A known criterion of existence of a ground state is [LHB11, Proposition 6.8].

Proposition 2.2 *A ground state of H exists if and only if $\lim_{T \rightarrow \infty} \gamma(T) > 0$.*

By Corollary 1.4 we have

$$\|\Phi_T\|^2 = e^{2TE} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \left[\varepsilon^{N_T} e^{\frac{\alpha^2}{2} \int_{-T}^T dt \int_{-T}^T W(N_t - N_s, t-s) ds \right], \quad (2.3)$$

$$\begin{aligned} (\mathbb{1}_{\mathcal{H}}, \Phi_T) &= e^{TE} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \left[\varepsilon^{N_T} e^{\frac{\alpha^2}{2} \int_0^T dt \int_0^T W(N_t - N_s, t-s) ds \right] \\ &= e^{TE} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \left[\varepsilon^{N_T} e^{\frac{\alpha^2}{2} \int_{-T}^0 dt \int_{-T}^0 W(N_t - N_s, t-s) ds \right]. \end{aligned} \quad (2.4)$$

The second identity on (2.4) is derived from the reflection symmetry in Proposition 1.1.

Note that

$$\left| \int_{-T}^0 dt \int_0^T W(N_t - N_s, t-s) ds \right| \leq \frac{1}{2} \left\| \frac{\hat{h}}{\omega} \right\|^2 \quad (2.5)$$

uniformly in T and in the paths.

Theorem 2.3 *If $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, then H has a ground state and it is unique.*

Proof: We write $\int_{-T}^T dt \int_{-T}^T W ds = \int_{-T}^0 dt \int_{-T}^0 W ds + \int_0^T dt \int_0^T W ds + 2 \int_{-T}^0 dt \int_0^T W ds$, and by (2.5) obtain

$$\|\Phi_T\|^2 \leq e^{2TE} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \left[\varepsilon^{N_T} e^{\frac{\alpha^2}{2} (\int_{-T}^0 dt \int_{-T}^0 ds W(N_t - N_s, t-s) + \int_0^T dt \int_0^T ds W(N_t - N_s, t-s) + 2\|\hat{h}/\omega\|^2)} \right]. \quad (2.6)$$

By the independence of N_t and N_{-s} , and reflection symmetry of the paths we furthermore obtain that

$$\begin{aligned} \|\Phi_T\|^2 &\leq e^{2TE} \sum_{\sigma \in \mathbb{Z}_2} \left(\mathbb{E}_P \left[\varepsilon^{N_T} e^{\frac{\alpha^2}{2} \int_0^T dt \int_0^T ds W(N_t - N_s, t-s)} \right] \right)^2 e^{\frac{\alpha^2}{2} \|\hat{h}/\omega\|^2} \\ &\leq \left(e^{TE} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \left[\varepsilon^{N_T} e^{\frac{\alpha^2}{2} \int_0^T dt \int_0^T ds W(N_t - N_s, t-s)} \right] \right)^2 e^{\frac{\alpha^2}{2} \|\hat{h}/\omega\|^2} \\ &= (\mathbb{1}_{\mathcal{H}}, \Phi_T)^2 e^{\frac{\alpha^2}{2} \|\hat{h}/\omega\|^2}. \end{aligned}$$

Hence $\gamma(T) \geq e^{-\frac{\alpha^2}{2}\|\hat{h}/\omega\|^2}$ and a ground state φ_g of H exists. By Corollary 2.1 φ_g is strictly positive as a vector in $L^2(\mathbb{Z}_2 \times Q)$, in particular, it is unique. \square

We note that the condition $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ in Theorem 2.3 is not a mere technicality. This condition will play an essential role throughout below in the definition of a Gibbs measure and the analysis of the ground state properties, see also Section 4.6 below.

By Theorem 2.3 it follows that H_{SB} also has a unique ground state. As seen above, \mathcal{H} can be decomposed as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and H can be reduced by \mathcal{H}_\pm .

Corollary 2.4 *Let φ_{SB} be the ground state of H_{SB} . Then $\varphi_{\text{SB}} \in \mathcal{H}_-$.*

Proof: Let U be as in (1.12). Notice that $\varphi_{\text{SB}} = U^* \varphi_g$, and thus

$$\varphi_{\text{SB}} = s - \lim_{T \rightarrow \infty} \frac{U^* e^{-TH} \mathbb{1}_{\mathcal{H}}}{\|U^* e^{-TH} \mathbb{1}_{\mathcal{H}}\|} = s - \lim_{T \rightarrow \infty} \frac{e^{-TH_{\text{SB}}} U^* \mathbb{1}_{\mathcal{H}}}{\|e^{-TH_{\text{SB}}} U^* \mathbb{1}_{\mathcal{H}}\|}.$$

The function $\mathbb{1}_{\mathcal{H}} \in L^2(\mathbb{Z}_2) \otimes L^2(Q)$ corresponds to $\begin{bmatrix} \Omega \\ \Omega \end{bmatrix} \in \mathcal{F}_\uparrow \oplus \mathcal{F}_\downarrow$ and

$$U^* \mathbb{1}_{\mathcal{H}} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \Omega \\ \Omega \end{bmatrix} = \begin{bmatrix} 0 \\ \Omega \end{bmatrix} \in \mathcal{H}_-.$$

Hence by the parity symmetry of H_{SB} we have

$$P e^{-TH_{\text{SB}}} U^* \mathbb{1}_{\mathcal{H}} = e^{-TH_{\text{SB}}} P U^* \mathbb{1}_{\mathcal{H}} = -e^{-TH_{\text{SB}}} U^* \mathbb{1}_{\mathcal{H}}$$

and thus $e^{-TH_{\text{SB}}} U^* \mathbb{1}_{\mathcal{H}} \in \mathcal{H}_-$. This implies that $\varphi_{\text{SB}} \in \mathcal{H}_-$. \square

Remark 2.5 By Corollary 2.1 the ground state φ_g of H overlaps with the non-negative vector $\rho(\sigma, \phi) = \begin{cases} 1, & \sigma = +1 \\ 0, & \sigma = -1 \end{cases}$ in $L^2(\mathbb{Z}_2 \times Q)$. Hence $(\varphi_g, \rho)_{L^2(\mathbb{Z}_2 \times Q)} \neq 0$ and

$$\inf \text{Spec}(H) = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log(\rho, e^{-\beta H} \rho) = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log e^\beta \mathbb{E}_P \left[\varepsilon^{N_t} e^{\frac{\alpha^2}{2} \int_0^\beta dt \int_0^\beta ds W} \right]. \quad (2.7)$$

The expression at the right hand side above was also obtained in [Hir99, Abd12].

3 Path measure associated with the ground state

3.1 \mathbb{Z}_2 -valued paths

In Sections 3.1-3.2 we set $\varepsilon = 1$ for simplicity.

Let $\mathcal{X} = D(\mathbb{R}; \mathbb{Z}_2)$ be the space of càdlàg paths with values in \mathbb{Z}_2 , and \mathcal{G} the σ -field generated by cylinder sets. Thus $\sigma. : (\Omega, \Sigma, P) \rightarrow (\mathcal{X}, \mathcal{G})$ is an \mathcal{X} -valued random variable. We denote its image measure by \mathcal{W}^σ , i.e., $\mathcal{W}^\sigma(A) = \sigma.^{-1}(A)$ for $A \in \mathcal{G}$, and the coordinate process by $(X_t)_{t \in \mathbb{R}}$, i.e., $X_t(\omega) = \omega(t)$ for $\omega \in \mathcal{X}$. Hence Proposition 1.2 can be reformulated in terms of $(X_t)_{t \in \mathbb{R}}$ as

$$(\Phi, e^{-tH} \Psi)_{\mathcal{H}} = e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}^\sigma} \mathbb{E}_{\mu_{\mathbb{E}}} \left[\overline{J_0 \Phi(X_0)} e^{-\alpha \Phi_{\mathbb{E}}(\int_0^t X_s j_s h ds)} J_t \Psi(X_t) \right]. \quad (3.1)$$

Here $\mathbb{E}_{\mathcal{W}^\sigma} = \mathbb{E}_{\mathcal{W}}^\sigma$ so that $\mathbb{E}_{\mathcal{W}}^\sigma[X_0 = \sigma] = 1$. Then (1.23) can be converted to the form

$$e^{-tH} \Phi(\sigma) = e^t \mathbb{E}_{\mathcal{W}}^\sigma \left[J_0^* e^{\Phi_{\mathbb{E}}(-\alpha \int_0^t X_r j_r h dr)} J_t \Phi(\sigma_t) \right] \quad (3.2)$$

for every $\sigma \in \mathbb{Z}_2$.

Lemma 3.1 *For every $s, t \in \mathbb{R}$ it follows that*

$$(\Phi, e^{-tH} \Psi)_{\mathcal{H}} = e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}^\sigma} \mathbb{E}_{\mu_{\mathbb{E}}} \left[\overline{J_s \Phi(X_s)} e^{-\alpha \Phi_{\mathbb{E}}(\int_s^{s+t} X_r j_r h dr)} J_{s+t} \Psi(X_{s+t}) \right]. \quad (3.3)$$

Proof: By the Trotter product formula $(\Phi, e^{-tH} \Psi)_{\mathcal{H}} = \lim_{n \rightarrow \infty} (\Phi, (e^{-\frac{t}{n} H_0} e^{-\frac{t}{n} H_t})^n \Psi)$, and using the fact that $e^{-|t-s|H_t} = J_t^* J_s$ we have

$$(\Phi, e^{-tH} \Psi)_{\mathcal{H}} = e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \mathbb{E}_{\mu_{\mathbb{E}}} \left[\overline{J_s \Phi(\sigma_0)} e^{-\alpha \Phi_{\mathbb{E}}(\int_0^t \sigma_r j_{s+r} h dr)} J_{s+t} \Psi(\sigma_t) \right].$$

By the shift invariance of $(N_t)_{t \in \mathbb{R}}$ stated in Proposition 1.1 we have

$$(\Phi, e^{-tH} \Psi)_{\mathcal{H}} = e^t \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_P \mathbb{E}_{\mu_{\mathbb{E}}} \left[\overline{J_s \Phi(\sigma_s)} e^{-\alpha \Phi_{\mathbb{E}}(\int_0^t \sigma_{s+r} j_{s+r} h dr)} J_{s+t} \Psi(\sigma_{s+t}) \right].$$

Hence the lemma follows. \square

Let $(\mathbb{Z}_2, \mathcal{B})$ be a measurable space with σ -field $\mathcal{B} = \{\emptyset, \{-1\}, \{+1\}, \mathbb{Z}_2\}$. For later use we show a functional integral representation of Euclidean Green functions of the type $(\Phi, \mathbb{1}_{A_0} e^{-(t_1-t_0)H} \mathbb{1}_{A_1} e^{-(t_2-t_1)H} \dots e^{-(t_n-t_{n-1})H} \mathbb{1}_{A_n} \Psi)$, where $-\infty < t_0 \leq \dots \leq t_n < \infty$ and $A_0, \dots, A_n \in \mathcal{B}$. We see that the operator

$$Q_{[S,T]} = J_S^* e^{\Phi_{\mathbb{E}}(-\alpha \int_S^T X_s j_s h ds)} J_T : L^2(Q) \rightarrow L^2(Q) \quad (3.4)$$

is bounded. In fact, we have that

$$\|Q_{[S,T]}\|_{L^2(Q) \rightarrow L^2(Q)} \leq \|Q_{[S,T]}\|_{L^1(Q)} \leq e^{\frac{\alpha^2}{4} \|\int_S^T X_s j_s h ds\|^2}, \quad (3.5)$$

which was shown in e.g. [HL08, Corollary 4.4].

Corollary 3.2 *Let $-\infty < t_0 \leq \dots \leq t_n < \infty$ and $A_0, \dots, A_n \in \mathcal{B}$. Then*

$$\begin{aligned} & (\Phi, \mathbb{1}_{A_0} e^{-(t_1-t_0)H} \mathbb{1}_{A_1} e^{-(t_2-t_1)H} \dots e^{-(t_n-t_{n-1})H} \mathbb{1}_{A_n} \Psi) \\ &= e^{t_n-t_0} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \mathbb{E}_{\mu_E} \left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(X_{t_j}) \right) \overline{\Phi(X_{t_0})} Q_{[t_0, t_n]} \Psi(X_{t_n}) \right]. \end{aligned} \quad (3.6)$$

Proof: This is proven by using the Markov properties of both the Poisson process and the Euclidean field, and (1.23). Denote by $\mathcal{N}_s = \sigma(N_r, 0 \leq r \leq s)$ the natural filtration of the Poisson process $(N_t)_{t \in \mathbb{R}}$. The Markov property of $(N_t)_{t \in \mathbb{R}}$ and (1.23) yield that

$$\begin{aligned} & (e^{-sH} \mathbb{1}_A e^{-tH} \Phi)(\sigma) \\ &= e^{s+t} \mathbb{E}_P \left[J_0^* e^{\Phi_E(-\alpha \int_0^s \sigma_r j_r h dr)} J_s \mathbb{1}_A(\sigma_s) \mathbb{E}_P \left[J_0^* e^{-\alpha \Phi_E(\int_0^t \sigma_{r+s} j_r h dr)} J_t \Phi(\sigma_{t+s}) \middle| \mathcal{N}_s \right] \right] \\ &= e^{s+t} \mathbb{E}_P \left[J_0^* e^{\Phi_E(-\alpha \int_0^s \sigma_r j_r h dr)} J_s \mathbb{1}_A(\sigma_s) J_0^* e^{-\alpha \Phi_E(\int_0^t \sigma_{r+s} j_r h dr)} J_t \Phi(\sigma_{t+s}) \right] \\ &= e^{s+t} \mathbb{E}_P \left[J_0^* e^{-\alpha \Phi_E(\int_0^s \sigma_r j_r h dr)} J_s \mathbb{1}_A(\sigma_s) J_0^* e^{-\alpha \Phi_E(\int_0^t \sigma_{r+s} j_r h dr)} J_t \Phi(\sigma_{t+s}) \right]. \end{aligned}$$

Since $J_0 = U_{-s} J_s$, where $U_s : L^2(Q_E) \rightarrow L^2(Q_E)$ is the shift operator defined by $U_s \Phi_E(j_{t_1} f_1) \dots \Phi_E(j_{t_n} f_n) = \Phi_E(j_{t_1+s} f_1) \dots \Phi_E(j_{t_n+s} f_n)$, we obtain

$$\begin{aligned} & (e^{-sH} \mathbb{1}_A e^{-tH} \Phi)(\sigma) \\ &= e^{s+t} \mathbb{E}_P \left[J_0^* e^{-\alpha \Phi_E(\int_0^s j_r \sigma_r h dr)} J_s J_s^* U_s \mathbb{1}_A(\sigma_s) e^{-\alpha \Phi_E(\int_0^t \sigma_{r+s} j_r h dr)} J_t \Phi(\sigma_{t+s}) \right] \\ &= e^{s+t} \mathbb{E}_P \left[J_0^* e^{-\alpha \Phi_E(\int_0^s j_r \sigma_r h dr)} J_s J_s^* \mathbb{1}_A(\sigma_s) e^{-\alpha \Phi_E(\int_0^t \sigma_{r+s} j_{r+s} h dr)} J_{t+s} \Phi(\sigma_{t+s}) \right]. \end{aligned} \quad (3.7)$$

Furthermore, by the Markov property of the Euclidean field we can remove the projection $J_s J_s^*$ in (3.7) and obtain

$$(e^{-sH} \mathbb{1}_A e^{-tH} \Phi)(\sigma) = e^{s+t} \mathbb{E}_P \left[J_0^* e^{-\alpha \Phi_E(\int_0^{s+t} \sigma_r j_r h dr)} \mathbb{1}_A(\sigma_s) J_{t+s} \Phi(\sigma_{t+s}) \right].$$

Hence in terms of $(X_t)_{t \in \mathbb{R}}$ we have

$$\begin{aligned} & (\Phi, \mathbb{1}_{A_0} e^{-sH} \mathbb{1}_{A_1} e^{-tH} \mathbb{1}_{A_2} \Psi) \\ &= e^{s+t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\mathbb{1}_{A_0}(X_0) \mathbb{1}_{A_1}(X_s) \mathbb{1}_{A_2}(X_{s+t}) \left(J_0 \Phi(X_0), e^{-\alpha \Phi_E(\int_0^{s+t} X_r j_r h dr)} J_{t+s} \Psi(X_{t+s}) \right) \right]. \end{aligned}$$

Repeating this procedure, we have

$$\begin{aligned} & (\Phi, \mathbb{1}_{A_0} e^{-(t_1-t_0)H} \mathbb{1}_{A_1} e^{-(t_2-t_1)H} \dots e^{-(t_n-t_{n-1})H} \mathbb{1}_{A_n} \Psi) \\ &= e^{t_n-t_0} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \mathbb{E}_{\mu_E} \left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(X_{t_j-t_0}) \right) \overline{\Phi(X_0)} Q_{[0, t_n-t_0]} \Psi(X_{t_n-t_0}) \right]. \end{aligned} \quad (3.8)$$

By the shift invariance of σ_t (Proposition 1.1) we complete the proof. \square

Corollary 3.3 *Let $-\infty < t_0 \leq t_1 \leq \dots \leq t_n < \infty$ and $A_0, \dots, A_n \in \mathcal{B}$. Then*

$$\begin{aligned} & (\mathbb{1}_{A_0}, e^{-(t_1-t_0)H} \mathbb{1}_{A_1} e^{-(t_2-t_1)H} \dots e^{-(t_n-t_{n-1})H} \mathbb{1}_{A_n}) \\ &= e^{t_n-t_0} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[e^{\frac{\alpha^2}{2} \int_{t_0}^{t_n} dt \int_{t_0}^{t_n} ds W(X_s, X_t, t-s)} \prod_{j=0}^n \mathbb{1}_{A_j}(X_{t_j}) \right], \end{aligned} \quad (3.9)$$

where $W(x, y, t) = \frac{xy}{2} \int_{\mathbb{R}^d} e^{-|t|\omega(k)} \hat{h}(k)^2 dk$.

Proof: By Corollary 3.2 we have

$$\text{LHS (3.9)} = e^{t_n-t_0} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\mathbb{E}_{\mu_E} [Q_{[t_0, t_n]}] \prod_{j=0}^n \mathbb{1}_{A_j}(X_{t_j}) \right].$$

Hence the corollary follows. \square

3.2 Local weak convergence

In this section we make the assumption that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, so that there is a unique ground state $\varphi_g \in \mathcal{H}$. Let $\mathcal{G}_{[-T, T]} = \sigma(X_t, t \in [-T, T])$ be the family of sub- σ -fields of \mathcal{G} and

$$\mathcal{G} = \bigcup_{T \geq 0} \mathcal{G}_{[-T, T]}.$$

Let $\bar{\mathcal{G}} = \sigma(\mathcal{G})$. Define the probability measure μ_T on $(\mathcal{X}, \bar{\mathcal{G}})$ by

$$\mu_T(A) = \frac{e^{2T}}{Z_T} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\mathbb{1}_A e^{\frac{\alpha^2}{2} \int_{-T}^T dt \int_{-T}^T ds W(X_t, X_s, t-s)} \right], \quad A \in \bar{\mathcal{G}}, \quad (3.10)$$

where Z_T is the normalizing constant such that $\mu_T(\mathcal{X}) = 1$. This probability measure is a Gibbs measure for the pair interaction potential W , indexed by the bounded intervals $[-T, T]$ (see the next section for further details). In this section we show convergence of μ_T to a probability measure μ_∞ in a specific sense when $T \rightarrow \infty$.

Definition 3.4 Let μ_∞ be a probability measure on $(\mathcal{X}, \bar{\mathcal{G}})$, and $(T_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ be any unbounded increasing sequence of positive numbers. The sequence of probability measures $(\mu_{T_n})_{n \in \mathbb{N}}$ is said to converge to the probability measure μ_∞ in *local weak topology* whenever $\lim_{n \rightarrow \infty} |\mu_{T_n}(A) - \mu_\infty(A)| = 0$ for all $A \in \mathcal{G}_{[-t, t]}$ and $t \geq 0$.

By the above definition it is seen that whenever $\mu_T \rightarrow \mu_\infty$ in local weak sense, we have that

$$\lim_{T \rightarrow \infty} \mathbb{E}_{\mu_T}[f] = \mathbb{E}_{\mu_\infty}[f] \quad (3.11)$$

for any bounded $\mathcal{G}_{[-t,t]}$ -measurable function f .

Next we define the finite dimensional distributions indexed by $\Lambda = \{t_0, \dots, t_n\} \subset [-T, T]$ with $t_0 \leq \dots \leq t_n$. Let

$$\mu_T^\Lambda(A_0 \times \dots \times A_n) = \frac{e^{2T}}{Z_T} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(X_{t_j}) \right) e^{\frac{\alpha^2}{2} \int_{-T}^T dt \int_{-T}^T ds W(X_t, X_s, t-s)} \right] \quad (3.12)$$

be a probability measure on $(\mathbb{Z}_2^\Lambda, \mathcal{B}^\Lambda)$, where $\mathbb{Z}_2^\Lambda = \times_{j=1}^n \mathbb{Z}_2^{t_j}$ and $\mathcal{B}^\Lambda = \times_{j=1}^n \mathcal{B}^{t_j}$ for $\Lambda = \{t_1, \dots, t_n\}$, and $\mathbb{Z}_2^{t_j}$ and \mathcal{B}^{t_j} are copies of \mathbb{Z}_2 and \mathcal{B} , respectively. Clearly, \mathcal{G} is a finitely additive family of sets. Define an additive set function on $(\mathcal{X}, \mathcal{G})$ by

$$\mu(A) = e^{2Et} e^{2t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma [\mathbb{1}_A(\varphi_g(X_{-t}), Q_{[-t,t]} \varphi_g(X_t))], \quad A \in \mathcal{G}_{[-t,t]}. \quad (3.13)$$

Note that $\mu(\mathcal{X}) = (\varphi_g, e^{-2t(H-E)} \varphi_g) = 1$.

Lemma 3.5 *There exists a unique probability measure μ_∞ on $(\mathcal{X}, \overline{\mathcal{G}})$ such that $\mu_\infty \upharpoonright_{\mathcal{G}} = \mu$. In particular, $\mu_\infty(A) = \mu(A)$, for every $A \in \mathcal{G}_{[-t,t]}$ and $t \in \mathbb{R}$.*

Proof: Let $\cup_{j=1}^\infty A_j \in \mathcal{G}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Then there exists $t > 0$ such that $\cup_{j=1}^\infty A_j \in \mathcal{G}_{[-t,t]}$ by the definition of \mathcal{G} . Thus by the definition of μ we have

$$\mu(\cup_{j=1}^\infty A_j) = e^{2Et} e^{2t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma [\mathbb{1}_{\cup_{j=1}^\infty A_j}(\varphi_g(X_{-t}), Q_{[-t,t]} \varphi_g(X_t))] = \sum_{j=1}^\infty \mu(A_j)$$

by the Lebesgue dominated convergence theorem. Hence the set function μ on $(\mathcal{X}, \mathcal{G})$ is a completely additive measure. Then the Hopf extension theorem implies that there exists a unique probability measure μ_∞ on $(\mathcal{X}, \overline{\mathcal{G}})$ such that $\mu_\infty \upharpoonright_{\mathcal{G}} = \mu$. \square

In order to show that $\mu_T(A) \rightarrow \mu_\infty(A)$ for every $A \in \mathcal{G}_{[-t,t]}$, we define the probability measure ρ_T on $(\mathcal{X}, \mathcal{G}_{[-T,T]})$ for $A \in \mathcal{G}_{[-t,t]}$ with $t \leq T$ by

$$\rho_T(A) = e^{2Et} e^{2t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\mathbb{1}_A \left(\frac{\Phi_{T-t}(X_{-t})}{\|\Phi_T\|}, Q_{[-t,t]} \frac{\Phi_{T-t}(X_t)}{\|\Phi_T\|} \right) \right]. \quad (3.14)$$

The family of probability measures ρ_T^Λ on $(\mathbb{Z}_2^\Lambda, \mathcal{B}^\Lambda)$ indexed by $\Lambda = \{t_0, \dots, t_n\} \subset [-T, T]$ is defined by

$$\rho_T^\Lambda(A_0 \times \dots \times A_n) = e^{2Et} e^{2t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(X_{t_j}) \right) \left(\frac{\Phi_{T-t}(X_{-t})}{\|\Phi_T\|}, Q_{[-t,t]} \frac{\Phi_{T-t}(X_t)}{\|\Phi_T\|} \right) \right] \quad (3.15)$$

for arbitrary t such that $-T \leq -t \leq \dots \leq t_0 \leq \dots \leq t_n \leq t \leq T$. To show that $\mu_T = \rho_T$, we prove that their finite dimensional distributions coincide.

Lemma 3.6 *Let $\Lambda = \{t_0, t_1, \dots, t_n\}$ and $A_0 \times \dots \times A_n \in \mathcal{B}^\Lambda$. Then it follows that*

$$\mu_T^\Lambda(A_0 \times \dots \times A_n) = \rho_T^\Lambda(A_0 \times \dots \times A_n).$$

In particular, ρ_T^Λ is independent of the choice of t .

Proof: By Corollary 3.2 we see that

$$\mu_T^\Lambda(A_0 \times \dots \times A_n) = \frac{1}{\|\Phi_T\|^2} (\mathbb{1}_{\mathcal{H}}, e^{-(t_0+T)H} \mathbb{1}_{A_0} e^{-(t_1-t_0)H} \mathbb{1}_{A_1} \dots \mathbb{1}_{A_n} e^{-(T-t_n)H} \mathbb{1}_{\mathcal{H}}).$$

Hence we have by the definition of Φ_{T-t} that

$$\mu_T^\Lambda(A_0 \times \dots \times A_n) = \frac{e^{2Et}}{\|\Phi_T\|^2} (\Phi_{T-t}, e^{-(t_0+t)H} \mathbb{1}_{A_0} e^{-(t_1-t_0)H} \mathbb{1}_{A_1} \dots \mathbb{1}_{A_n} e^{-(t-t_n)H} \Phi_{T-t}).$$

By Corollary 3.2 we have furthermore

$$\begin{aligned} \mu_T^\Lambda(A_0 \times \dots \times A_n) &= e^{2Et} e^{2t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\prod_{j=0}^n \mathbb{1}_{A_j}(X_{t_j}) \left(\frac{\Phi_{T-t}}{\|\Phi_T\|}(X_{-t}), Q_{[-t,t]} \frac{\Phi_{T-t}(X_t)}{\|\Phi_T\|} \right) \right] \\ &= \rho_T^\Lambda(A_0 \times \dots \times A_n). \end{aligned}$$

Thus the lemma follows. \square

Denote $\mathbb{Z}_2^{(-\infty, \infty)} = \{\omega : (-\infty, \infty) \rightarrow \mathbb{Z}_2\}$.

Lemma 3.7 *Let $t \leq T$ and $A \in \mathcal{G}_{[-t,t]}$. Then $\mu_T(A) = \rho_T(A)$.*

Proof: It is straightforward to see that the family of probability measures μ_T^Λ , $\Lambda \subset \mathbb{R}$, on $(\mathbb{Z}_2^\Lambda, \mathcal{B}^\Lambda)$ with $\#\Lambda < \infty$ satisfies the Kolmogorov consistency condition:

$$\mu_T^{\{t_0, \dots, t_n, s_1, \dots, s_m\}}(A_0 \times \dots \times A_n \times \prod_{j=1}^m \mathbb{Z}_2) = \mu_T^{\{t_0, \dots, t_n\}}(A_0 \times \dots \times A_n).$$

Let $\pi_\Lambda : \mathbb{Z}_2^{(-\infty, \infty)} \rightarrow \mathbb{Z}_2^\Lambda$ be the projection defined by $\pi_\Lambda(\omega) = (\omega(t_0), \dots, \omega(t_n))$ for $\omega \in \mathbb{Z}_2^{(-\infty, \infty)}$ and $\Lambda = \{t_0, \dots, t_n\}$. Then $\mathcal{A}_T = \{\pi_\Lambda^{-1}(E) \mid \Lambda \subset [-T, T], \#\Lambda < \infty, E \in \mathcal{B}^\Lambda\}$ is a finitely additive family of sets. Thus by the Kolmogorov extension theorem there exists a unique probability measure $\mu_T^{(-\infty, \infty)}$ on $(\mathbb{Z}_2^{(-\infty, \infty)}, \sigma(\mathcal{A}_T))$ such that

$$\mu_T^{(-\infty, \infty)}(\pi_\Lambda^{-1}(A_0 \times \dots \times A_n)) = \mu_T^\Lambda(A_0 \times \dots \times A_n) \quad (3.16)$$

for all $\Lambda \subset [-T, T]$ with $\#\Lambda < \infty$ and $A_j \in \mathcal{B}$. Note that $\mathbb{Z}_2^{(-\infty, \infty)} = \mathcal{X}$ and $\sigma(\mathcal{A}_T) = \mathcal{G}_{[-T, T]}$ follow. On the other hand, we have

$$\mu_T(\pi_\Lambda^{-1}(A_0 \times \dots \times A_n)) = \mu_T^\Lambda(A_0 \times \dots \times A_n),$$

and $\mu_T[\mathcal{G}_{[-T,T]}]$ is a probability measure on $(\mathcal{X}, \mathcal{G}_{[-T,T]})$. Thus the uniqueness of $\mu_T^{(-\infty, \infty)}$ satisfying (3.16) implies that $\mu_T^{(-\infty, \infty)} = \mu_T[\mathcal{G}_{[-T,T]}]$. Observing that

$$\mu_T^\Lambda(A_0 \times \cdots \times A_n) = \rho_T^\Lambda(A_0 \times \cdots \times A_n) = \rho_T(\pi_\Lambda^{-1}(A_0 \times \cdots \times A_n))$$

by Lemma 3.6, we also see that $\rho_T(A) = \mu_T^{(-\infty, \infty)}(A)$ for $A \in \mathcal{G}_{[-t,t]}$ by the uniqueness of $\mu_T^{(-\infty, \infty)}$ satisfying (3.16). Thus together with $\mu_T^{(-\infty, \infty)} = \mu_T[\mathcal{G}_{[-T,T]}]$ we conclude that $\mu_T(A) = \rho_T(A)$ for $A \in \mathcal{G}_{[-t,t]}$ and $t \leq T$. \square

Theorem 3.8 *Suppose $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then the probability measure μ_T on $(\mathcal{X}, \bar{\mathcal{G}})$ converges in local weak sense to μ_∞ as $T \rightarrow \infty$.*

Proof: By Lemma 3.7 it suffices to show that $\lim_{T \rightarrow \infty} \rho_T(A) = \mu_\infty(A)$ for every $A \in \mathcal{G}_{[-T,T]}$. Since $\Phi_{T-t}/\|\Phi_T\| \rightarrow \varphi_g$ strongly in $L^2(\mathbb{Z}_2) \otimes L^2(Q)$ as $T \rightarrow \infty$, we have $\Phi_{T-t}(\sigma)/\|\Phi_T\| \rightarrow \varphi_g(\sigma)$ for every $\sigma \in \mathbb{Z}_2$, strongly in $L^2(Q)$. Since $Q_{[-t,t]}$ is a bounded operator, it is seen that

$$\begin{aligned} \lim_{T \rightarrow \infty} \rho_T(A) &= \lim_{T \rightarrow \infty} e^{2t} e^{2Et} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\left(\frac{\Phi_{T-t}(X_{-t})}{\|\Phi_T\|}, Q_{[-t,t]} \frac{\Phi_{T-t}(X_t)}{\|\Phi_T\|} \right) \mathbb{1}_A \right] \\ &= e^{2t} e^{2Et} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[(\varphi_g(X_{-t}), Q_{[-t,t]} \varphi_g(X_t)) \mathbb{1}_A \right] = \mu_\infty(A). \end{aligned}$$

Thus the theorem follows. \square

3.3 The case of arbitrary $\varepsilon > 0$

In the case when $\varepsilon \neq 1$ a parallel discussion to the previous section can be made. Since

$$tH = \varepsilon t \left(-\sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes \frac{1}{\varepsilon} H_t + \frac{\alpha}{\varepsilon} \sigma_z \otimes \phi(\hat{h}) \right),$$

by replacing t , h and ω with εt , h/ε and ω/ε , respectively we have

$$(\Phi, e^{-tH} \Psi)_{\mathcal{H}} = e^{\varepsilon t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \mathbb{E}_{\mu_E} \left[\overline{J_0^\varepsilon \Phi(X_0)} e^{-(\alpha/\varepsilon) \Phi_E \left(\int_0^{\varepsilon t} X_s j_s^\varepsilon h ds \right)} J_t^\varepsilon \Psi(X_{\varepsilon t}) \right]. \quad (3.17)$$

Here J_t^ε and j_t^ε are defined by ω replaced by ω/ε . Thus $j_s^{\varepsilon*} j_t^\varepsilon = e^{-|t-s|\omega/\varepsilon}$ and $J_s^{\varepsilon*} J_t^\varepsilon = e^{-|t-s|H_t/\varepsilon}$. Define the probability measure μ_T^ε on $(\mathcal{X}, \bar{\mathcal{G}})$ by

$$\mu_T^\varepsilon(A) = \frac{e^{2\varepsilon T}}{Z_{\varepsilon T}} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\mathbb{1}_A e^{\frac{\alpha^2}{2} \int_{-T}^T dt \int_{-T}^T ds W(X_{\varepsilon t}, X_{\varepsilon s}, t-s)} \right], \quad A \in \bar{\mathcal{G}}. \quad (3.18)$$

Define also an additive set function on $(\mathcal{X}, \mathcal{G})$ by

$$\mu^\varepsilon(A) = e^{2E\varepsilon t} e^{2\varepsilon t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma [\mathbb{1}_A(\varphi_{\mathbf{g}}(X_{-\varepsilon t}), Q_{[-\varepsilon t, \varepsilon t]}^{(\varepsilon)} \varphi_{\mathbf{g}}(X_{\varepsilon t}), \mathcal{H}], \quad A \in \mathcal{G}_{[-\varepsilon t, \varepsilon t]},$$

where $Q_{[-\varepsilon t, \varepsilon t]}^{(\varepsilon)} = J_{-\varepsilon t}^{\varepsilon*} e^{\Phi_{\mathbb{E}}(-(\alpha/\varepsilon) \int_{-\varepsilon t}^{\varepsilon t} X_s j_s^\varepsilon h ds)} J_{\varepsilon t}^\varepsilon$. In the same way as Lemma 3.5 we see that there exists a unique probability measure μ_∞^ε on $(\mathcal{X}, \overline{\mathcal{G}})$ such that $\mu_\infty^\varepsilon[\mathcal{G}] = \mu^\varepsilon$. Furthermore, it can be derived in a similar manner to Theorem 3.8 that

$$\lim_{T \rightarrow \infty} \mu_T^\varepsilon(A) = \mu_\infty^\varepsilon(A), \quad A \in \mathcal{G}_{[-t, t]}. \quad (3.19)$$

We summarize this in the theorem below.

Theorem 3.9 *Suppose $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then the probability measure μ_T^ε on $(\mathcal{X}, \overline{\mathcal{G}})$ converges in local weak sense to μ_∞^ε as $T \rightarrow \infty$.*

We also write $\mu_{\mathbf{g}}^\varepsilon$ for μ_∞^ε for notational convenience.

3.4 Gibbs measure

In this subsection we show that $\mu_{\mathbf{g}}^\varepsilon$ is a Gibbs measure on $(\mathcal{X}, \overline{\mathcal{G}})$. First we give some definitions and basic facts on Gibbs measures needed for this proof.

Let (Ω, \mathcal{Y}, Q) be a probability space, and $(Y_t)_{t \in \mathbb{R}}$ be a Markov process with càdlàg paths on it. We write $\mathcal{Y}_T = \sigma(Y_r, r \in [-T, T])$ and $\mathcal{T}_T = \sigma(Y_r, r \in [-T, T]^c)$. Let $\mathcal{V} : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\mathcal{W} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ be Borel measurable functions, called external potential and pair potential, respectively. We call \mathcal{V} an admissible external potential whenever

$$0 < \mathbb{E}_Q[e^{-\int_I \mathcal{V}(Y_s) ds}] < \infty \quad (3.20)$$

for every bounded interval $I \subset \mathbb{R}$. Furthermore, we say that \mathcal{W} is an admissible pair interaction potential whenever

$$\int_{\mathbb{R}} \sup_{x, y \in \mathbb{R}^d} |\mathcal{W}(x, y, s)| ds < \infty. \quad (3.21)$$

For the admissible potentials \mathcal{V}, \mathcal{W} and $0 < S \leq T$ define the functionals

$$\mathcal{E}_T = \int_{-T}^T \mathcal{V}(Y_t) dt + \left(\int_{\mathbb{R}} ds \int_{-T}^T dt + \int_{-T}^T ds \int_{\mathbb{R}} dt \right) \mathcal{W}(Y_t, Y_s, |t - s|), \quad (3.22)$$

$$\mathcal{E}_{S,T} = \int_{-T}^T \mathcal{V}(Y_t) dt + \left(\int_{-S}^S ds \int_{-T}^T dt + \int_{-T}^T ds \int_{-S}^S dt \right) \mathcal{W}(Y_t, Y_s, |t - s|). \quad (3.23)$$

Also, define Q_T^Y on (Ω, \mathcal{Y}) for every $Y \in \Omega$ as the unique probability measure such that $\mathbb{E}_{Q_T^Y}[fg] = \mathbb{E}_Q[f|\mathcal{T}_T](Y)g(Y)$, for every bounded \mathcal{Y}_T -measurable function f and every bounded \mathcal{T}_T -measurable function g , i.e.,

$$Q_T^Y[A] = \mathbb{E}_Q[\mathbb{1}_A|\mathcal{T}_T](Y). \quad (3.24)$$

Definition 3.10 Suppose that \mathcal{V} and \mathcal{W} are admissible potentials.

- (1) A probability measure P_T on (Ω, \mathcal{Y}) is called a *finite volume Gibbs measure* for the interval $[-T, T]$ with respect to the reference measure Q and the potentials \mathcal{V} and \mathcal{W} whenever for all $0 < S < T$

(i) $P_T \llcorner \mathcal{Y}_T \ll Q \llcorner \mathcal{Y}_T$

- (ii) for every bounded \mathcal{Y} -measurable function f

$$\mathbb{E}_{P_T}[f|\mathcal{T}_S](Y) = \frac{\mathbb{E}_{Q_S^Y}[fe^{-\mathcal{E}_{S,T}}]}{\mathbb{E}_{Q_S^Y}[e^{-\mathcal{E}_{S,T}}]}, \quad P_T\text{-a.s.} \quad (3.25)$$

- (2) A probability measure P is called a *Gibbs measure* with respect to the reference measure Q and the potentials \mathcal{V} and \mathcal{W} whenever for all $T > 0$

(i) $P \llcorner \mathcal{Y}_T \ll Q \llcorner \mathcal{Y}_T$

- (ii) for every bounded \mathcal{Y} -measurable function f

$$\mathbb{E}_P[f|\mathcal{T}_T](Y) = \frac{\mathbb{E}_{Q_T^Y}[fe^{-\mathcal{E}_T}]}{\mathbb{E}_{Q_T^Y}[e^{-\mathcal{E}_T}]}, \quad P\text{-a.s.} \quad (3.26)$$

A sufficient condition for P_T to be a finite volume Gibbs measure and P a Gibbs measure is as follows.

Proposition 3.11 *Let \mathcal{V} and \mathcal{W} be admissible potentials.*

- (1) *For every $T > 0$*

$$dP_T = \frac{1}{Z_T} e^{-\mathcal{E}_{T,T}} dQ \quad (3.27)$$

is a finite volume Gibbs measure for $[-T, T]$, where Z_T denotes the normalizing constant.

- (2) *Suppose that there exists a probability measure P_∞ such that $P_t(A) \rightarrow P_\infty(A)$ as $t \rightarrow \infty$ for all $A \in \mathcal{Y}_T$, and $P_\infty \llcorner \mathcal{Y}_T \ll Q \llcorner \mathcal{Y}_T$ for every T . Then P_∞ is a Gibbs measure for the given potentials and reference measure.*

Proof: For (1) see Proposition 4.1, for (2) Proposition 4.2 in [LHB11]. \square

Consider on \mathbb{Z}_2 the Bernoulli measure

$$\nu(\sigma) = \frac{1}{2}(\delta_{-1}(\sigma) + \delta_{+1}(\sigma)), \quad \sigma \in \mathbb{Z}_2,$$

and define the probability measure μ_0 on $(\mathcal{X}, \overline{\mathcal{G}})$ by

$$\mu_0(A) = \mathbb{E}_\nu \mathbb{E}_{\mathcal{W}}^\sigma[\mathbb{1}_A], \quad A \in \overline{\mathcal{G}}.$$

Theorem 3.12 *Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then the probability measure μ_g is a Gibbs measure on $(\mathcal{X}, \overline{\mathcal{G}})$ with respect to reference measure μ_0 , zero external potential and pair interaction potential*

$$W(X_{\varepsilon t}, X_{\varepsilon s}, |t - s|) = \frac{1}{2} X_{\varepsilon t} X_{\varepsilon s} \int_{\mathbb{R}^d} e^{-|t-s|\omega(k)} |\hat{h}(k)|^2 dk.$$

Proof: The probability measure μ_T^ε is a finite volume Gibbs measure by part (1) of Proposition 3.11 and (3.18). By Theorem 3.9 we have that $\mu_T^\varepsilon(A) \rightarrow \mu_g(A)$ as $T \rightarrow \infty$ for every $A \in \mathcal{G}_{[-t, t]}$, $t \leq T$, and

$$\begin{aligned} \mu_g(A) &= e^{2E\varepsilon t} e^{2\varepsilon t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\left(\varphi_g(X_{-\varepsilon t}), Q_{[-\varepsilon t, \varepsilon t]}^{(\varepsilon)} \varphi_g(X_{\varepsilon t}) \right) \mathbb{1}_A \right] \\ &\leq 2e^{2E\varepsilon t} e^{2\varepsilon t} \|Q_{[-\varepsilon t, \varepsilon t]}^{(\varepsilon)}\|_{L^1(Q)} \mu_0(A) \leq 2e^{2E\varepsilon t} e^{2\varepsilon t} e^{\alpha^2 t^2 \|h\|^2} \mu_0(A). \end{aligned}$$

This bound is derived from (3.5). Hence $\mu_g \ll \mu_0$ follows for every $t > 0$. Then the theorem follows by part (2) of Proposition 3.11. \square

4 Ground state properties

4.1 Expectations of functions of the form $\xi(\sigma)F(\phi(f))$

In this section we use the Gibbs measure obtained above to derive ground state properties of the form (1.2) mentioned in Section 1.1. We start by considering ground state expectations of the form $(\varphi_g, \xi(\sigma)F(\phi(f))\varphi_g)$ with suitable functions F and ξ expressed through expectations with respect to the path measure μ_g . By the parity symmetry we know that

$$(\varphi_g, \sigma \varphi_g)_{L^2(\mathbb{Z}_2; L^2(Q))} = (\varphi_{\text{SB}}, \sigma_x \varphi_{\text{SB}})_{\mathbb{C}^2 \otimes \mathcal{F}} = 0. \quad (4.1)$$

4.1.1 Expectation of $\xi(\sigma)$

Theorem 4.1 *Let f be a $\mathcal{G}_{[-\varepsilon t, \varepsilon t]}$ -measurable function on \mathcal{X} . Then*

$$\mathbb{E}_{\mu_g}[f] = e^{2E\varepsilon t} e^{2\varepsilon t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\left(\varphi_g(X_{-\varepsilon t}), Q_{[-\varepsilon t, \varepsilon t]}^{(\varepsilon)} \varphi_g(X_{\varepsilon t}) \right) f \right]. \quad (4.2)$$

Proof: Since for $A \in \mathcal{G}_{[-\varepsilon t, \varepsilon t]}$ we have

$$\mu_g(A) = e^{2\varepsilon t} e^{2E\varepsilon t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\left(\varphi_g(X_{-\varepsilon t}), Q_{[-\varepsilon t, \varepsilon t]}^{(\varepsilon)} \varphi_g(X_{\varepsilon t}) \right) \mathbb{1}_A \right],$$

(4.2) follows. \square

An immediate consequence of Theorem 4.1 is the following.

Corollary 4.2 *Let $f_j : \mathbb{Z}_2 \rightarrow \mathbb{C}$, $j = 0, \dots, n$, be bounded functions. Then*

$$\mathbb{E}_{\mu_g} \left[\prod_{j=0}^n f_j(X_{\varepsilon t_j}) \right] = (\varphi_g, f_0 e^{-(t_1-t_0)(H-E)} f_1 \dots e^{-(t_n-t_{n-1})(H-E)} f_n \varphi_g). \quad (4.3)$$

In particular, we have for all bounded functions ξ , f and g that

$$\mathbb{E}_{\mu_g}[\xi(X_0)] = (\varphi_g, \xi(\sigma) \varphi_g), \quad (4.4)$$

$$\mathbb{E}_{\mu_g}[f(X_t)g(X_s)] = (f(\sigma) \varphi_g, e^{-|t-s|(H-E)} g(\sigma) \varphi_g). \quad (4.5)$$

Proof: For $A_j \in \mathcal{B}$, $j = 0, 1, \dots, n$, it follows that

$$\begin{aligned} \mathbb{E}_{\mu_g} \left[\prod_{j=0}^n \mathbb{1}_{A_j}(X_{\varepsilon t_j}) \right] &= e^{2\varepsilon t} e^{2E\varepsilon t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \left[\left(\varphi_g(X_{-\varepsilon t}), Q_{[-\varepsilon t, \varepsilon t]}^{(\varepsilon)} \varphi_g(X_{\varepsilon t}) \right) \prod_{j=0}^n \mathbb{1}_{A_j}(X_{\varepsilon t_j}) \right] \\ &= (\varphi_g, \mathbb{1}_{A_0} e^{-(t_1-t_0)(H-E)} \mathbb{1}_{A_1} \dots e^{-(t_n-t_{n-1})(H-E)} \mathbb{1}_{A_n} \varphi_g). \end{aligned}$$

Hence (4.3) is obtained. \square

4.1.2 Expectation of $\xi(\sigma)F(\phi(f))$

Lemma 4.3 *Let F be a real-valued bounded function on \mathbb{R} , $f \in L^2(\mathbb{R}^d)$ and $\xi : \mathbb{Z}_2 \rightarrow \mathbb{C}$ be a bounded function. Then*

$$\begin{aligned} &(e^{-TH} \mathbb{1}_{\mathcal{H}}, \xi(\sigma) F(\phi(f)) e^{-TH} \mathbb{1}_{\mathcal{H}}) \\ &= e^{2\varepsilon T} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^\sigma \mathbb{E}_{\mu_E} \left[\xi(X_0) e^{-\frac{\alpha}{\varepsilon} \Phi_E(\int_{-\varepsilon T}^{\varepsilon T} X_s j_s h ds)} F(\Phi_E(j_0 f)) \right]. \end{aligned}$$

Proof: By (3.2) we have

$$(e^{-TH}\xi(\sigma)F(\phi(f))e^{-TH}\mathbf{1}_{\mathcal{H}})(\sigma) = e^{2\varepsilon T}\mathbb{E}_{\mathcal{W}}^{\sigma}\left[Q_{[-\varepsilon T,0]}^{(\varepsilon)}\xi(X_0)F(\phi(f))\mathbb{E}_{\mathcal{W}}^{X_0}\left[Q_{[0,\varepsilon T]}^{(\varepsilon)}\mathbf{1}_{\mathcal{H}}(X_{\varepsilon T})\right]\right].$$

Here $Q_{[S,T]}^{(\varepsilon)} = J_{S^*}^{\varepsilon} e^{\Phi_{\mathbb{E}}(-\frac{\alpha}{\varepsilon}\int_S^T X_s j_s^{\varepsilon} h ds)} J_T^{\varepsilon}$. Then in a similar manner to the proof of Corollary 3.2 the lemma follows from the Markov property of $(N_t)_{t \in \mathbb{R}}$. \square

Theorem 4.4 *Let $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, $f \in L^2(\mathbb{R}^d)$ be real-valued, $\xi : \mathbb{Z}_2 \rightarrow \mathbb{C}$ be a bounded function, and $\beta \in \mathbb{R}$. Then*

$$(\varphi_{\mathfrak{g}}, \xi(\sigma)e^{i\beta\phi(f)}\varphi_{\mathfrak{g}}) = e^{-\frac{\beta^2}{4}\|f\|^2}\mathbb{E}_{\mu_{\mathfrak{g}}}\left[\xi(X_0)e^{i\beta K(f)}\right], \quad (4.6)$$

where $K(f)$ is a random variable on $(\mathcal{X}, \bar{\mathcal{G}})$ given by

$$K(f) = \frac{\alpha}{2} \int_{-\infty}^{\infty} (e^{-|r|\omega}\hat{h}, f)X_{\varepsilon r} dr. \quad (4.7)$$

Proof: Note that

$$(\varphi_{\mathfrak{g}}, \xi(\sigma)e^{i\beta\phi(f)}\varphi_{\mathfrak{g}}) = \lim_{T \rightarrow \infty} \left(\frac{\Phi_T}{\|\Phi_T\|}, \xi(\sigma)e^{i\beta\phi(f)} \frac{\Phi_T}{\|\Phi_T\|} \right)$$

and by Lemma 4.3 we see that

$$\left(\frac{\Phi_T}{\|\Phi_T\|}, \xi(\sigma)e^{i\beta\phi(f)} \frac{\Phi_T}{\|\Phi_T\|} \right) = \frac{1}{Z_{\varepsilon T}} e^{2\varepsilon T} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^{\sigma} \mathbb{E}_{\mu_{\mathbb{E}}} \left[\xi(X_0) e^{-\frac{\alpha}{\varepsilon}\Phi_{\mathbb{E}}(\int_{-\varepsilon T}^{\varepsilon T} X_s j_s h ds)} e^{i\beta\Phi_{\mathbb{E}}(j_0 f)} \right].$$

The expectation with respect to $\mu_{\mathbb{E}}$ can be computed explicitly and thus

$$\begin{aligned} & (\varphi_{\mathfrak{g}}, \xi(\sigma)e^{i\beta\phi(f)}\varphi_{\mathfrak{g}}) \\ &= \lim_{T \rightarrow \infty} e^{-\frac{\beta^2}{4}\|f\|^2} \frac{1}{Z_{\varepsilon T}} e^{2T} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathcal{W}}^{\sigma} \left[\xi(X_0) e^{\frac{\alpha^2}{2} \int_{-T}^T dt \int_{-T}^T W(X_{\varepsilon t}, X_{\varepsilon s}, t-s) ds} e^{i\beta \int_{-T}^T (e^{-|s|\omega}\hat{h}, f) X_{\varepsilon s} ds} \right] \\ &= \lim_{T \rightarrow \infty} e^{-\frac{\beta^2}{4}\|f\|^2} \mathbb{E}_{\mu_T^{\varepsilon}} \left[\xi(X_0) e^{i\beta \int_{-T}^T (e^{-|s|\omega}\hat{h}, f) X_{\varepsilon s} ds} \right]. \end{aligned}$$

Notice that $|\int_{-\infty}^{\infty} X_{\varepsilon s} (e^{-|s|\omega}\hat{h}, f) ds| \leq 2\|\hat{h}/\omega\|\|f\| < \infty$. By the local weak convergence of μ_T and a similar telescoping as in the proof of Theorem 4.16 below, we obtain the desired result. \square

By using Theorem 4.4 the functionals $(\varphi_{\mathfrak{g}}, \xi(\sigma)F(\phi(f))\varphi_{\mathfrak{g}})$ can be represented in terms of averages with respect to the path measure $\mu_{\mathfrak{g}}$. Consider the case when F is a polynomial or a Schwartz test function. We will show in Corollary 4.18 below that $\varphi_{\mathfrak{g}} \in D(e^{+\beta N})$ for all $\beta > 0$, thus $\varphi_{\mathfrak{g}} \in D(\phi(f)^n)$ for every $n \in \mathbb{N}$.

Corollary 4.5 *Let $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, $f \in L^2(\mathbb{R}^d)$ be real-valued, and $\xi : \mathbb{Z}_2 \rightarrow \mathbb{C}$ a bounded function. Also, let $h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$ be the Hermite polynomial of order n . Then*

$$(\varphi_g, \xi(\sigma)\phi(f)^n \varphi_g) = i^n \mathbb{E}_{\mu_g} \left[\xi(X_0) h_n \left(\frac{-iK(f)}{\|f\|2^{-1/2}} \right) \right] (\|f\|2^{-1/2})^n, \quad n \in \mathbb{N}. \quad (4.8)$$

Proof: We have

$$e^{-\beta^2 \|f\|^2/4} e^{i\beta K(f)} = \sum_{n=0}^{\infty} h_n \left(\frac{-iK(f)}{\|f\|2^{-1/2}} \right) \frac{(-\beta \|f\|2^{-1/2})^n}{n!}. \quad (4.9)$$

Hence

$$\frac{1}{i^n} \frac{d^n}{d\beta^n} e^{-\beta^2 \|f\|^2/4} e^{i\beta K(f)} \Big|_{\beta=0} = i^n h_n \left(\frac{-iK(f)}{\|f\|2^{-1/2}} \right) (\|f\|2^{-1/2})^n \quad (4.10)$$

follows. By (4.10) and the computation

$$(\varphi_g, \xi(\sigma)\phi(f)^n \varphi_g) = \frac{1}{i^n} \frac{d^n}{d\beta^n} e^{-\frac{\beta^2}{4} \|f\|^2} \mathbb{E}_{\mu_g} [\xi(X_0) e^{i\beta K(f)}] \Big|_{\beta=0},$$

we obtain (4.8). \square

In the next corollary we give the path integral representation of $(\varphi_g, \xi(\sigma)F(\phi(f))\varphi_g)$ for $F \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the space of rapidly decreasing, infinitely many times differentiable functions on \mathbb{R} .

Corollary 4.6 *Let $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, $f \in L^2(\mathbb{R}^d)$ be real-valued, $F \in \mathcal{S}(\mathbb{R})$, and $\xi : \mathbb{Z}_2 \rightarrow \mathbb{C}$ a bounded function. Then*

$$(\varphi_g, \xi(\sigma)F(\phi(f))\varphi_g) = \mathbb{E}_{\mu_g} [\xi(X_0)G(K(f))], \quad (4.11)$$

where $G = \check{F} * \check{g}$ and $g(\beta) = e^{-\beta^2 \|f\|^2/4}$.

Proof: Since $F(\phi(f)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{F}(\beta) e^{i\beta\phi(f)} d\beta$, we have

$$(\varphi_g, \xi(\sigma)F(\phi(f))\varphi_g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{F}(\beta) e^{-\frac{\beta^2}{4} \|f\|^2} \mathbb{E}_{\mu_g} [\xi(X_0) e^{i\beta K(f)}] d\beta. \quad (4.12)$$

Thus the corollary follows. \square

4.1.3 Field fluctuations in the ground state

The field fluctuations in the ground state are defined for every real-valued function $f \in L^2(\mathbb{R}^d)$ by

$$F_\alpha(f) = (\varphi_g, \phi(f)^2 \varphi_g) - (\varphi_g, \phi(f) \varphi_g)^2. \quad (4.13)$$

More generally, we also consider fluctuations of the form

$$G_\alpha(f) = (\varphi_g, (\sigma\phi(f))^2 \varphi_g) - (\varphi_g, \sigma\phi(f) \varphi_g)^2. \quad (4.14)$$

Corollary 4.7 *Let $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$ be a real-valued function. Then*

$$(1) \quad (\varphi_g, \sigma\phi(f) \varphi_g) = \mathbb{E}_{\mu_g}[X_0 K(f)],$$

$$(2) \quad (\varphi_g, (\sigma\phi(f))^2 \varphi_g) = \mathbb{E}_{\mu_g}[(X_0 K(f))^2] + \frac{1}{2} \|f\|^2.$$

In particular,

$$(3) \quad G_\alpha(f) = \mathbb{E}_{\mu_g}[(X_0 K(f))^2] - (\mathbb{E}_{\mu_g}[X_0 K(f)])^2 + \frac{1}{2} \|f\|^2,$$

$$(4) \quad (\varphi_g, \phi(f) \varphi_g) = 0 \text{ and } F_\alpha(f) = (\varphi_g, \phi(f)^2 \varphi_g) = \mathbb{E}_{\mu_g}[K(f)^2] + \frac{1}{2} \|f\|^2,$$

$$(5) \text{ whenever } f \neq 0, \text{ we furthermore have that (i) } F_\alpha(f) > 0 \text{ and } F_\alpha(f) \geq F_0(f), \text{ (ii) } G_\alpha(f) > 0 \text{ and } G_\alpha(f) \geq G_0(f).$$

Proof: Statements (1)-(3) easily follow from Corollary 4.5, which imply (4) for $\sigma = 1$. Using Schwarz inequality, we obtain (5ii), while (5i) is clear by (4). \square

Note that to prove (1)-(2) of Corollary 4.7 we can proceed, alternatively, to derive first the equality $(\varphi_g, \phi(f) \varphi_g) = \mathbb{E}_{\mu_g}[K(f)]$ by using Corollary 4.5, and from $\mathbb{E}_{\mu_g}[X_s] = (\varphi_g, \sigma \varphi_g)$ to further obtain that

$$(\varphi_g, \phi(f) \varphi_g) = -\frac{\alpha}{2} (\hat{h}/\omega, \hat{f}) (\varphi_g, \sigma \varphi_g) = 0. \quad (4.15)$$

Notice that $X_0^2 = 1$. Thus in Corollary 4.7 we have equivalently $\mathbb{E}_{\mu_g}[(X_0 K(f))^2] = \mathbb{E}_{\mu_g}[K(f)^2]$.

4.2 Gaussian decay and exponential moments of the field operator

4.2.1 Gaussian decay of the field operator

In this section we show that $(\varphi_g, e^{\beta\phi(f)^2}\varphi_g) < \infty$ for some $\beta > 0$.

Lemma 4.8 *Let $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$ be a real-valued function. Then for $\beta > 0$ we have*

$$(\varphi_g, e^{-\beta\phi(f)^2}\varphi_g) = \frac{1}{\sqrt{1 + \beta\|f\|^2}} \mathbb{E}_{\mu_g} \left[e^{-\frac{\beta K^2(f)}{1 + \beta\|f\|^2}} \right]. \quad (4.16)$$

Proof: By Theorem 4.4 we see that

$$\begin{aligned} (\varphi_g, e^{-(\beta^2/2)\phi(f)^2}\varphi_g) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\beta^2/2} (\varphi_g, e^{i\beta\phi(f)}\varphi_g) d\beta \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\beta^2/2} e^{-\beta^2\|f\|^2/4} \mathbb{E}_{\mu_g} [e^{i\beta K(f)}] d\beta \\ &= \frac{1}{\sqrt{1 + \beta^2\|f\|^2/2}} \mathbb{E}_{\mu_g} \left[e^{-\frac{\beta^2 K^2(f)/2}{1 + \beta^2\|f\|^2/2}} \right]. \end{aligned}$$

Replacing $\beta^2/2$ by β completes the proof of the lemma. \square

Theorem 4.9 *Let $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$ be a real-valued function. If $-\infty < \beta < 1/\|f\|^2$, then $\varphi_g \in D(e^{(\beta/2)\phi(f)^2})$ and*

$$\|e^{(\beta/2)\phi(f)^2}\varphi_g\|^2 = \frac{1}{\sqrt{1 - \beta\|f\|^2}} \mathbb{E}_{\mu_g} \left[e^{\frac{\beta K^2(f)}{1 - \beta\|f\|^2}} \right]. \quad (4.17)$$

Proof: The proof is a modification of [Hir04, Theorem 10.12].

Let $B = \{z \in \mathbb{C} \mid |z| < 1/\|f\|^2\}$, $\mathbb{C}_+ = \{z \mid \Re z > 0\}$ and $\mathbb{C}_- = \{z \mid \Re z < 0\}$. Consider

$$\rho(z) = \frac{1}{\sqrt{1 + z\|f\|^2}} \mathbb{E}_{\mu_g} \left[e^{-\frac{zK^2(f)}{1 + z\|f\|^2}} \right], \quad (4.18)$$

for $z > 0$. Then $\rho(z)$ can be analytically continued to $\mathbb{C}_+ \cup B$, since $|K(f)| \leq |\alpha| \|f\| \|\hat{h}/\omega\|$ uniformly in paths. We denote this extension by $\bar{\rho}(z)$. Let $w \in \mathbb{R} \cap B$ and consider the ball $B_\delta(w) = \{z \in \mathbb{C} \mid |z - w| < \delta\}$. Take any $\delta < 1/\|f\|^2$ such that for w we have $B_\delta(w) \cap \mathbb{C}_- \cap B \neq \emptyset$. We expand $\bar{\rho}(z)$ as

$$\bar{\rho}(z) = \sum_{n=0}^{\infty} (z - w)^n b_n(w), \quad z \in B_\delta(w) \cap B \quad (4.19)$$

and get in particular

$$\bar{\rho}(z) = \sum_{n=0}^{\infty} |z - w|^n |b_n(w)| < \infty \quad (4.20)$$

for $z \in B_\delta(w) \cap B$. On the other hand, $\mathbb{C}_+ \ni z \mapsto (\varphi_g, e^{-z\phi(f)^2} \varphi_g) \in \mathbb{C}$ is differentiable on \mathbb{C}_+ , since $\varphi_g \in D(\phi(f)^2)$, and is thus analytic on \mathbb{C}_+ . We have

$$(\varphi_g, e^{-z\phi(f)^2} \varphi_g) = \sum_{n=0}^{\infty} (z - w)^n \frac{1}{n!} \int_0^\infty (-\lambda)^n e^{-w\lambda} dE_\lambda, \quad z \in \mathbb{C}_+. \quad (4.21)$$

Here E_λ denotes the spectral measure of $\frac{1}{2}\phi(f)^2$ with respect to φ_g . Comparing (4.19), (4.21) and $\bar{\rho}(z) = (\varphi_g, e^{-z\phi(f)^2} \varphi_g)$ for $z \in \mathbb{C}_+$, we conclude that

$$b_n(w) = \frac{1}{n!} \int_0^\infty (-\lambda)^n e^{-w\lambda} dE_\lambda. \quad (4.22)$$

Substituting (4.22) into (4.19) we have

$$\bar{\rho}(z) = \sum_{n=0}^{\infty} (z - w)^n \frac{1}{n!} \int_0^\infty (-\lambda)^n e^{-w\lambda} dE_\lambda, \quad z \in B_\delta(w) \cap B \quad (4.23)$$

where the right hand side is absolutely convergent for every $z \in B_\delta(w) \cap B$. Thus by (4.20) for $z \in B_\delta(w) \cap B \cap \mathbb{R}$ we have

$$\int_0^M e^{-z\lambda} dE_\lambda \leq \sum_{n=0}^{\infty} \frac{|z - w|^n}{n!} \left| \int_0^M (-\lambda)^n e^{-w\lambda} dE_\lambda \right| \leq \sum_{n=0}^{\infty} \frac{|z - w|^n}{n!} \left| \int_0^\infty (-\lambda)^n e^{-w\lambda} dE_\lambda \right| < \infty,$$

which implies that $\lim_{M \rightarrow \infty} \int_0^M e^{-z\lambda} dE_\lambda < \infty$ for these z . The monotone convergence theorem then gives $\int_0^\infty e^{-z\lambda} dE_\lambda < \infty$, hence $\varphi_g \in D(e^{-(z/2)\phi(f)^2})$ and

$$\|e^{-(z/2)\phi(f)^2} \varphi_g\|^2 = \bar{\rho}(z), \quad z \in B_\delta(w) \cap B \cap \mathbb{R}. \quad (4.24)$$

Since for every $\delta < 1/\|f\|^2$ there exists $w \in \mathbb{R} \cap B$ such that $\mathbb{C}_- \cap B \cap B_\delta(w) \neq \emptyset$, the proof of the theorem is complete. \square

From Theorem 4.9 it is immediate to get the limit of $\|e^{(\beta/2)\phi(f)^2} \varphi_g\|$.

Corollary 4.10 *Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$ is a real-valued function. Then*

$$\lim_{\beta \uparrow 1/\|f\|^2} \|e^{(\beta/2)\phi(f)^2} \varphi_g\| = \infty.$$

In the previous section we investigated the moments of $\phi(f)$ of positive integer order. By using Lemma 4.8 also the moments of fractional order can be derived. Define $|\phi(f)|^s = (\phi(f)^2)^{s/2}$ for $0 \leq s \leq 2$, and let λ be the Lévy measure on $\mathbb{R} \setminus \{0\}$ such that $\int_0^\infty (1 - e^{-yu})\lambda(dy) = u^{s/2}$ for $u > 0$, i.e.,

$$\lambda(dy) = \frac{s}{2\Gamma(1-s/2)} y^{-1-\frac{s}{2}} 1_{(0,\infty)}(y) dy,$$

corresponding to the $s/2$ -subordinator. Let $\Lambda_\alpha = \|\phi(f)^{s/2} \varphi_g\|^2$.

Corollary 4.11 *Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$ is a real-valued function. Then for $0 < s < 2$,*

$$\Lambda_\alpha = \mathbb{E}_{\mu_g} \left[\int_0^\infty \left(1 - \frac{1}{\sqrt{1 + \beta \|f\|^2}} e^{-\frac{\beta K(f)^2}{1 + \beta \|f\|^2}} \right) \lambda(d\beta) \right]. \quad (4.25)$$

In particular, $\Lambda_0 \leq \Lambda_\alpha$ follows.

Proof: Notice that

$$\begin{aligned} \left(1 - \frac{1}{\sqrt{1 + \beta \|f\|^2}} e^{-\frac{\beta K(f)^2}{1 + \beta \|f\|^2}} \right) \beta^{-1-\frac{s}{2}} &\leq \left(1 - \frac{1}{\sqrt{1 + \beta \|f\|^2}} e^{-\frac{\beta \alpha^2 \|f\|^2 \|\hat{h}/\omega\|^2/4}{1 + \beta \|f\|^2}} \right) \beta^{-1-\frac{s}{2}} \\ &= \eta(\beta). \end{aligned}$$

In a neighborhood of $\beta = 0$ it holds that $\eta(\beta) = \beta^{-s/2} + o(\beta)$ locally uniformly. Then $\eta(\beta)$ is integrable in this neighborhood, and since $\eta(\beta) \leq \text{const } \beta^{1+s/2}$, $\eta \in L^1([0, \infty))$ follows. Then (4.25) is immediate from Lemma 4.8 by using the Fubini theorem, and the inequality follows from

$$\Lambda_0 = \int_0^\infty \left(1 - \frac{1}{\sqrt{1 + \beta \|f\|^2}} \right) \lambda(d\beta).$$

□

Remark 4.12 A simple computation shows that if U is a real-valued Gaussian random variable with mean m and variance σ^2 , then whenever $\theta < 1/(2\sigma^2)$, we have

$$\mathbb{E}_G[e^{\theta U^2}] = \frac{1}{\sqrt{1 - 2\theta\sigma^2}} e^{\frac{m^2\theta}{1 - 2\theta\sigma^2}},$$

where the expectation is taken with respect to this Gaussian measure. A comparison with (4.17) implies that there exists thus a real-valued Gaussian random variable U with mean $K(f)$ (or $-K(f)$) and variance $\|f\|^2/2$ such that

$$\|e^{\beta\phi(f)^2} \varphi_g\|^2 = \mathbb{E}_G[e^{\beta U^2}].$$

For some consequences see Section 4.3 below.

4.2.2 Exponential moments of the field operator

Theorem 4.9 says that $\|e^{(\beta/2)\phi(f)^2}\varphi_g\| < \infty$. Using this fact we can obtain explicit formulae of the exponential moments $(\varphi_g, e^{\beta\phi(f)}\varphi_g)$ of the field.

Corollary 4.13 *If $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$ is a real-valued function, then $\varphi_g \in D(e^{\beta\phi(f)})$ and*

$$(\varphi_g, e^{\beta\phi(f)}\varphi_g) = (\varphi_g, \cosh(\beta\phi(f))\varphi_g) = e^{\frac{\beta^2}{4}\|f\|^2} \mathbb{E}_{\mu_g} [e^{\beta K(f)}], \quad (4.26)$$

$$(\varphi_g, \sigma e^{\beta\phi(f)}\varphi_g) = (\varphi_g, \sigma \sinh(\beta\phi(f))\varphi_g) = e^{\frac{\beta^2}{4}\|f\|^2} \mathbb{E}_{\mu_g} [X_0 e^{\beta K(f)}]. \quad (4.27)$$

Proof: For simplicity we reset βf to f . By using the generating function $e^{xy - \frac{1}{2}y^2} = \sum_{n=0}^{\infty} h_n(x) \frac{y^n}{n!}$ of the Hermite polynomials, summation in (4.8) gives

$$\lim_{M \rightarrow \infty} (\varphi_g, \sum_{n=0}^M \frac{1}{n!} \phi(f)^n \varphi_g) = e^{\frac{1}{4}\|f\|^2} \mathbb{E}_{\mu_g} [e^{K(f)}]. \quad (4.28)$$

We need to check that the left hand side converges to $(\varphi_g, e^{\phi(f)}\varphi_g)$. Notice that by the spin flip property (2) in Lemma 1.6, $(\varphi_g, \phi(f)^n \varphi_g) = 0$ for odd n . Hence it suffices to show the convergence of $(\varphi_g, \sum_{n=0}^M \frac{1}{(2n)!} \phi(f)^{2n} \varphi_g)$ as $M \rightarrow \infty$. By Theorem 4.9 we have that $\|e^{\phi(f)^2/(4\|f\|^2)}\varphi_g\| < \infty$. Let E_λ be the spectral measure of $\phi(f)$ with respect to φ_g . Then

$$\sum_{n=0}^M \frac{1}{(2n)!} (\varphi_g, \phi(f)^{2n} \varphi_g) = \int_{\mathbb{R}} \sum_{n=0}^M \frac{1}{(2n)!} \lambda^{2n} e^{-\lambda^2/(4\|f\|^2)} e^{\lambda^2/(4\|f\|^2)} dE_\lambda.$$

Since $e^{\lambda^2/(4\|f\|^2)}$ is integrable by Theorem 4.9, $\sum_{n=0}^M \frac{1}{(2n)!} \lambda^{2n} e^{-\lambda^2/(4\|f\|^2)}$ is monotonously increasing to $\cosh(\lambda) e^{-\lambda^2/(4\|f\|^2)}$ as $M \uparrow \infty$, which is a bounded function, hence the monotone convergence theorem yields that $\lim_{M \rightarrow \infty} \int_{\mathbb{R}} \sum_{n=0}^M \frac{1}{n!} \lambda^n dE_\lambda = \int_{\mathbb{R}} e^\lambda dE_\lambda < \infty$, which implies $\varphi_g \in D(e^{\phi(f)})$ and (4.26). Equality (4.27) is derived in a similar way. \square

4.3 Van Hove representation

In Remark 4.12 we pointed out that the expectation of the field operator $\phi(f)$ in the ground state φ_g can be realized as an expectation of a Gaussian random variable. Here we show that this allows another representation of the ground state.

The *van Hove Hamiltonian* is defined by the self-adjoint operator

$$H_{\text{vH}}(\hat{g}) = H_{\text{f}} + \phi_{\text{b}}(\hat{g}) \quad (4.29)$$

in Fock space \mathcal{F} . Suppose that $\hat{g}/\omega \in L^2(\mathbb{R}^d)$ and define the conjugate momentum by

$$\pi_b(\hat{g}) = \frac{i}{\sqrt{2}} \int \left(a^\dagger(k) \frac{\hat{g}(k)}{\omega(k)} - a(k) \frac{\hat{g}(-k)}{\omega(k)} \right) dk.$$

Then

$$e^{i\pi_b(\hat{g})} H_{\text{vH}}(\hat{g}) e^{-i\pi_b(\hat{g})} = H_f - \frac{1}{2} \|\hat{g}/\omega\|^2 \quad (4.30)$$

and the ground state of $H_{\text{vH}}(\hat{g})$ is given by

$$\varphi_{\text{vH}}(\hat{g}) = e^{-i\pi_b(\hat{g})} \Omega_b.$$

On the other hand, clearly the spin-boson Hamiltonian H with $\varepsilon = 0$ is the direct sum of van Hove Hamiltonians since

$$H = \begin{bmatrix} H_f + \alpha\phi_b(\hat{h}) & 0 \\ 0 & H_f - \alpha\phi_b(\hat{h}) \end{bmatrix}$$

(see Section 2) and $H_f \pm \alpha\phi_b(\hat{h})$ are equivalent. Therefore the ground state of H with $\varepsilon = 0$ can be realized as $\varphi_g = \begin{bmatrix} \varphi_{\text{vH}}(\alpha\hat{h}) \\ \varphi_{\text{vH}}(-\alpha\hat{h}) \end{bmatrix}$. Thus in this case

$$(\varphi_g, e^{i\beta\phi_b(f)} \varphi_g)_{\mathcal{H}} = (\varphi_{\text{vH}}(\alpha\hat{h}), e^{i\beta\phi_b(f)} \varphi_{\text{vH}}(\alpha\hat{h}))_{\mathcal{F}}. \quad (4.31)$$

and the right hand side above equals

$$(\Omega_b, e^{i\beta(\phi_b(f) + \alpha(\hat{h}/\omega, f))} \Omega_b)_{\mathcal{F}} = e^{-\beta^2 \|f\|^2/4 + i\beta\alpha(\hat{h}/\omega, f)}. \quad (4.32)$$

When $\varepsilon \neq 0$ we can derive similar but non-trivial representations. Define the random boson field operator

$$\Psi(\hat{f}) = \phi_b(\hat{f}) + K(f) \quad (4.33)$$

on \mathcal{F} , where $K(f)$ is the random variable on \mathcal{X} defined by (4.7). Then we see that

$$(\Omega_b, \Psi(\hat{f}) \Omega_b) = K(f), \quad (4.34)$$

$$(\Omega_b, \Psi(\hat{f})^2 \Omega_b) - (\Omega_b, \Psi(\hat{f}) \Omega_b)^2 = \|\hat{f}\|^2/2, \quad (4.35)$$

$$(\Omega_b, e^{i\beta\Psi(\hat{f})} \Omega_b) = e^{-\beta^2 \|f\|^2/4 + i\beta K(f)}. \quad (4.36)$$

Let

$$\chi = \frac{\alpha}{2} \omega(k) \hat{h}(k) \int_{-\infty}^{\infty} e^{-|s|\omega(k)} X_{\varepsilon s} ds. \quad (4.37)$$

Note that $\chi \in L^2(\mathbb{R}^d)$

$$K(f) = (\chi, \hat{f}), \quad (4.38)$$

moreover, $\chi/\omega \in L^2(\mathbb{R}^d)$, whenever $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, and $\chi = \alpha\hat{h}$ for $\varepsilon = 0$. We define the *random van Hove Hamiltonian* by $H_{\text{vH}}(\chi)$.

Theorem 4.14 *If $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, then*

$$(\varphi_g, e^{i\beta\phi(f)}\varphi_g) = \mathbb{E}_{\mu_g} \left[(\Omega_b, e^{i\beta\Psi(\hat{f})}\Omega_b) \right] = \mathbb{E}_{\mu_g} \left[(\varphi_{\text{vH}}(\chi), e^{i\beta\phi_b(\hat{f})}\varphi_{\text{vH}}(\chi)) \right]. \quad (4.39)$$

Proof: The first equality can be directly derived from Theorem 4.4. The second equality follows from $e^{i\pi_b(\chi)}\Psi(\hat{f})e^{-i\pi_b(\chi)} = \phi_b(\hat{f})$. \square

Corollary 4.15 *Suppose $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $F \in \mathcal{S}(\mathbb{R})$. Then we have*

$$(\varphi_g, F(\phi(f))\varphi_g) = \mathbb{E}_{\mu_g} \left[(\Omega_b, F(\Psi(\hat{f}))\Omega_b) \right] = \mathbb{E}_{\mu_g} \left[(\varphi_{\text{vH}}(\chi), F(\phi(\hat{f}))\varphi_{\text{vH}}(\chi)) \right], \quad (4.40)$$

$$\|e^{\beta\phi(f)^2/2}\varphi_g\|^2 = \mathbb{E}_{\mu_g} \left[\|e^{\beta\Psi(\hat{f})^2/2}\Omega_b\|^2 \right] = \mathbb{E}_{\mu_g} \left[\|e^{\beta\phi_b(\hat{f})^2/2}\varphi_{\text{vH}}(\chi)\|^2 \right]. \quad (4.41)$$

Proof: This is proven from Corollary 4.6 and Theorem 4.9. \square

4.4 Expectations of second quantized operators

4.4.1 General results

In this section we consider expectations of the form $(\varphi_g, e^{-\beta d\Gamma(\rho(-i\nabla))}\varphi_g)$, where ρ is a real-valued multiplication operator given by the function ρ . An important example is $\rho = \mathbb{1}$ giving the boson number operator $N = d\Gamma(\mathbb{1})$.

In a similar way to [GHPS09, Section 3.2] we obtain the expression

$$\frac{(\Phi_T, \xi(\sigma)e^{-\beta d\Gamma(\rho(-i\nabla))}\Phi_T)}{\|\Phi_T\|^2} = \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 \int_{-T}^0 dt \int_0^T W^{\rho, \beta}(X_{\varepsilon t}, X_{\varepsilon s}, t-s) ds} \right], \quad (4.42)$$

where

$$W^{\rho, \beta}(x, y, T) = \frac{xy}{2} \int_{\mathbb{R}^d} |\hat{h}(k)|^2 e^{-|T|\omega(k)} (1 - e^{-\beta\rho(k)}) dk.$$

Denote

$$W_\infty^{\rho, \beta} = \int_{-\infty}^0 dt \int_0^\infty W^{\rho, \beta}(X_{\varepsilon t}, X_{\varepsilon s}, t-s) ds. \quad (4.43)$$

Notice that $|W_\infty^{\rho, \beta}| \leq \|\hat{h}/\omega\|^2/2 < \infty$, uniformly in the paths in \mathcal{X} .

Theorem 4.16 *Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $\xi : \mathbb{Z}_2 \rightarrow \mathbb{C}$ is a bounded function. Then*

$$(\varphi_g, \xi(\sigma)e^{-\beta d\Gamma(\rho(-i\nabla))}\varphi_g) = \mathbb{E}_{\mu_g} \left[\xi(X_0) e^{-\alpha^2 W_\infty^{\rho, \beta}} \right], \quad \beta > 0. \quad (4.44)$$

Proof: This is shown by using Theorem 3.8 and telescoping. For a shorthand we write $W_T^{\rho,\beta} = \int_{-T}^0 ds \int_0^T W^{\rho,\beta}(X_{\varepsilon t}, X_{\varepsilon s}, t-s) dt$. Note that for every $\delta > 0$ there is S_δ such that $|W_T^{\rho,\beta} - W_\infty^{\rho,\beta}| \leq \delta$ for all $T > S_\delta$, uniformly in the paths, and write

$$\begin{aligned} & \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 W_T^{\rho,\beta}} \right] - \mathbb{E}_{\mu_g} \left[\xi(X_0) e^{-\alpha^2 W_\infty^{\rho,\beta}} \right] \\ &= \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 W_T^{\rho,\beta}} \right] - \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 W_\infty^{\rho,\beta}} \right] \\ & \quad + \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 W_\infty^{\rho,\beta}} \right] - \mathbb{E}_{\mu_g} \left[\xi(X_0) e^{-\alpha^2 W_\infty^{\rho,\beta}} \right]. \end{aligned}$$

We have

$$\left| \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 W_T^{\rho,\beta}} \right] - \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 W_\infty^{\rho,\beta}} \right] \right| \leq C\delta \quad (4.45)$$

with a constant C . The second term can be evaluated as

$$\begin{aligned} & \left| \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 W_\infty^{\rho,\beta}} \right] - \mathbb{E}_{\mu_g} \left[\xi(X_0) e^{-\alpha^2 W_\infty^{\rho,\beta}} \right] \right| \\ & \leq \left| \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 W_\infty^{\rho,\beta}} \right] - \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 W_{S_\delta}^{\rho,\beta}} \right] \right| \end{aligned} \quad (4.46)$$

$$+ \left| \mathbb{E}_{\mu_T^\varepsilon} \left[\xi(X_0) e^{-\alpha^2 W_{S_\delta}^{\rho,\beta}} \right] - \mathbb{E}_{\mu_g} \left[\xi(X_0) e^{-\alpha^2 W_{S_\delta}^{\rho,\beta}} \right] \right| \quad (4.47)$$

$$+ \left| \mathbb{E}_{\mu_g} \left[\xi(X_0) e^{-\alpha^2 W_{S_\delta}^{\rho,\beta}} \right] - \mathbb{E}_{\mu_g} \left[\xi(X_0) e^{-\alpha^2 W_\infty^{\rho,\beta}} \right] \right|. \quad (4.48)$$

For (4.46) and (4.48) we have again the same upper bound as in (4.45), and (4.47) goes to zero as $T \rightarrow \infty$ by Theorem 3.8. \square

4.4.2 Super-exponential decay of the boson number

In this section we discuss the expectation of $e^{-\beta N}$, which can be obtained by a minor modification of Theorem 4.16.

Corollary 4.17 *Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $\xi : \mathbb{Z}_2 \rightarrow \mathbb{C}$ is a bounded function. Then*

$$(\varphi_g, \xi(\sigma) e^{-\beta N} \varphi_g) = \mathbb{E}_{\mu_g} \left[\xi(X_0) e^{-\alpha^2 (1-e^{-\beta}) W_\infty} \right], \quad (4.49)$$

where

$$W_\infty = \int_{-\infty}^0 dt \int_0^\infty W(X_{\varepsilon t}, X_{\varepsilon s}, t-s) ds. \quad (4.50)$$

Proof: By replacing ρ by $\mathbb{1}$ in Theorem 4.16, the claim readily follows. \square

The following result says that the distribution of the number of bosons in the ground state has a super-exponentially short tail.

Corollary 4.18 *If $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, then $\varphi_g \in D(e^{\beta N})$ for all $\beta \in \mathbb{C}$ and*

$$(\varphi_g, e^{\beta N} \varphi_g) = \mathbb{E}_{\mu_g} \left[e^{-\alpha^2(1-e^\beta)W_\infty} \right] \quad (4.51)$$

follows. In particular, $\varphi_g \in D(e^{+\beta N})$ for all $\beta > 0$.

Proof: The proof is similar to that of Theorem 4.9 and [Hir04, Theorem 10.12], and is left to the reader. \square

Corollary 4.19 *Suppose $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then*

$$(\varphi_g, (-1)^N \varphi_g) = \mathbb{E}_{\mu_g} \left[e^{-2\alpha^2 W_\infty} \right], \quad (4.52)$$

$$(\varphi_g, \xi(\sigma)(-1)^N \varphi_g) = \mathbb{E}_{\mu_g} \left[\xi(X_0) e^{-2\alpha^2 W_\infty} \right]. \quad (4.53)$$

In particular, it follows that

$$(\varphi_g, (-1)^N \varphi_g) = \mathbb{E}_{\mu_g} \left[e^{-2\alpha^2 W_\infty} \right] \geq e^{-\alpha^2 \|\hat{h}/\omega\|^2} > 0, \quad (4.54)$$

$$(\varphi_g, \sigma(-1)^N \varphi_g) = \mathbb{E}_{\mu_g} \left[X_0 e^{-2\alpha^2 W_\infty} \right] = -1 < 0. \quad (4.55)$$

Proof: Equality (4.52) is derived from (4.49) with $\xi(\sigma) = 1$ and $\beta = -i\pi$, and (4.53) with $\xi(\sigma) = \sigma$. Equality (4.54) follows from the estimate of the right hand side of (4.53). Noticing that $\varphi_{\text{SB}} \in \mathcal{H}_-$, we obtain $P\varphi_{\text{SB}} = \sigma_x(-1)^N \varphi_{\text{SB}} = -\varphi_{\text{SB}}$. In particular, this gives

$$\mathbb{E}_{\mu_g} \left[X_0 e^{-2\alpha^2 W_\infty} \right] = (\varphi_g, \sigma(-1)^N \varphi_g) = (\varphi_{\text{SB}}, P\varphi_{\text{SB}}) = -1. \quad \square$$

4.4.3 Moments of the boson number operator

We can derive the expectation of N^m , $m = 1, 2, \dots$, with respect to the ground state φ_g by using Corollary 4.17.

Corollary 4.20 *Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then*

$$(\varphi_g, N^m \varphi_g) = \sum_{r=1}^m a_r(m) \alpha^{2r} \mathbb{E}_{\mu_g} [W_\infty^r], \quad m = 1, 2, 3, \dots \quad (4.56)$$

where $a_r(m) = \frac{(-1)^r}{r!} \sum_{s=1}^r (-1)^s \binom{r}{s} s^m$ are the Stirling numbers of the second kind.

Proof: It can be checked that

$$\frac{d^m}{d\beta^m} e^{-C(1-e^{-\beta})} = (-1)^m \sum_{r=1}^m a_r(m) e^{-r\beta} (-C)^r e^{-a(1-e^{-\beta})}.$$

Then the corollary follows from $(\varphi_g, N^m \varphi_g) = (-1)^m \frac{d^m}{d\beta^m} (\varphi_g, e^{-\beta N} \varphi_g) \Big|_{\beta=0}$ and Corollary 4.17. \square

4.5 A relation between the expectations of $\sigma\phi(f)$ and N

By the results obtained in the previous subsections we can derive an inequality connecting the expectations $(\varphi_g, \sigma\phi(f)\varphi_g)$ and $(\varphi_g, N\varphi_g)$.

Corollary 4.21 *Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, $f \in L^2(\mathbb{R}^d)$ is real-valued, and $\xi : \mathbb{Z}_2 \rightarrow \mathbb{C}$ a bounded function. Then*

$$(\varphi_g, \xi(\sigma)\phi(f)\varphi_g) = \alpha \int_{\mathbb{R}^d} (\xi(\sigma)\varphi_g, (H - E + \omega(k))^{-1}\sigma\varphi_g)\overline{\hat{h}(k)}\hat{f}(k)dk. \quad (4.57)$$

In particular,

$$(\varphi_g, \sigma\phi(h)\varphi_g) = \alpha \int_{\mathbb{R}^d} \|(H - E + \omega(k))^{-1/2}\sigma\varphi_g\|^2 |\hat{h}(k)|^2 dk. \quad (4.58)$$

Proof: By Theorem 4.4 we have

$$(\varphi_g, \xi(\sigma)\phi(f)\varphi_g) = \mathbb{E}_{\mu_g}[\xi(X_0)K(f)] = \frac{\alpha}{2} \int_{-\infty}^{\infty} dr(e^{-|r|\omega}\hat{h}, \hat{f})\mathbb{E}_{\mu_g}[\xi(X_0)X_{\varepsilon r}].$$

By Corollary 4.2 we also see that this furthermore is

$$\begin{aligned} &= \frac{\alpha}{2} \int_{-\infty}^{\infty} dr \int dk (\xi(\sigma)\varphi_g, e^{-|r|(H-E+\omega(k))}\sigma\varphi_g)\overline{\hat{h}(k)}\hat{f}(k) \\ &= \alpha \int_{\mathbb{R}^d} (\xi(\sigma)\varphi_g, (H - E + \omega(k))^{-1}\sigma\varphi_g)\overline{\hat{h}(k)}\hat{f}(k)dk. \end{aligned}$$

Then the corollary follows. □

A standard inequality, see e.g. [LHB11, Proposition 5.1], says that

$$(\Phi, (\sigma\phi(f))^2\Phi) \leq \frac{\|f\|^2}{2}(\Phi, (N + \mathbb{1})\Phi).$$

From Corollary 4.21 we obtain the following inequality.

Corollary 4.22 *Let $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then*

$$(\varphi_g, N\varphi_g) \leq \frac{\alpha}{2}(\varphi_g, \sigma\phi(\omega(D)^{-1}h)\varphi_g) \leq \frac{\alpha^2}{2}\|\hat{h}/\omega\|^2, \quad D = -i\nabla. \quad (4.59)$$

Proof: By (4.56) we have

$$(\varphi_g, N\varphi_g) = \frac{\alpha^2}{2} \int_{\mathbb{R}^d} |\hat{h}(k)|^2 \int_{-\infty}^0 dt \int_0^{\infty} ds e^{-|t-s|\omega(k)} \mathbb{E}_{\mu_g}[X_{\varepsilon t}X_{\varepsilon s}].$$

Since $\mathbb{E}_{\mu_{\text{gl}}}[X_{\varepsilon s}X_{\varepsilon t}] = (\sigma\varphi_{\text{g}}, e^{-|t-s|(H-E)}\sigma\varphi_{\text{g}})$, by Corollary 4.3 we see that

$$(\varphi_{\text{g}}, N\varphi_{\text{g}}) = \frac{\alpha^2}{2} \int_{\mathbb{R}^d} |\hat{h}(k)|^2 \|(H - E + \omega(k))^{-1}\sigma\varphi_{\text{g}}\|^2 dk. \quad (4.60)$$

The first inequality is derived from (4.60) and (4.58). The second inequality follows through (4.58). \square

Since $\varphi_{\text{g}} \in D(N)$, we have that $f \mapsto (\varphi_{\text{g}}, \sigma\phi(f)\varphi_{\text{g}})$ is linear and the bound $|(\varphi_{\text{g}}, \sigma\phi(f)\varphi_{\text{g}})| \leq C\|f\|$ with a constant C follows. By the Riesz representation theorem there exists $G \in L^2(\mathbb{R}^d)$ such that $(\varphi_{\text{g}}, \sigma\phi(f)\varphi_{\text{g}}) = (\hat{G}, \hat{f})_{L^2(\mathbb{R}^d)}$.

Corollary 4.23 *If $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, then*

$$\hat{G}(k) = \frac{\alpha}{2} (\sigma\varphi_{\text{g}}, (H - E + \omega(k))^{-1}\sigma\varphi_{\text{g}})\hat{h}(k). \quad (4.61)$$

Proof: This is obtained directly from Corollary 4.21. \square

4.6 Comparison with the Nelson model

From (4.61) we see that, formally, $G(\cdot) = (\varphi_{\text{g}}, \sigma\phi(\cdot)\varphi_{\text{g}})$. Recall that the Nelson model is defined by a linear coupling between a particle described by the Schrödinger operator $H_{\text{p}} = -\Delta + V$ and a boson field described by H_{f} . The coupling term is given by

$$\phi_{\text{b}}(x) = \frac{1}{\sqrt{2}} \int \left(a^\dagger(k)e^{-ikx}\hat{h}(-k) + a(k)e^{-ikx}\hat{h}(k) \right) dk \quad (4.62)$$

so that the Nelson Hamiltonian is defined by

$$H_{\text{N}} = H_{\text{p}} + H_{\text{f}} + \phi_{\text{b}}(x). \quad (4.63)$$

For this model a similar kernel to (4.61) is obtained, see [LHB11, eq. (6.5.60)]. Using this kernel we can show [LHB11, Sect. 6.5] that

$$a\|\hat{h}/\omega\|^2 - b \leq (\varphi_{\text{g}}^{\text{N}}, N\varphi_{\text{g}}^{\text{N}}) \leq c\|\hat{h}/\omega\|^2, \quad (4.64)$$

where $\varphi_{\text{g}}^{\text{N}}$ is the ground state of H_{N} , and $a, c \geq 0$ and $b \in \mathbb{R}$ are suitable constants. Thus $(\varphi_{\text{g}}^{\text{N}}, N\varphi_{\text{g}}^{\text{N}}) \rightarrow \infty$ as $\|\hat{h}/\omega\| \rightarrow \infty$, and it follows that H_{N} has no ground state whenever $\hat{h}/\omega \notin L^2(\mathbb{R}^d)$. The key point is that $(\varphi_{\text{g}}^{\text{N}}, e^{i(\cdot, x)}\varphi_{\text{g}}^{\text{N}}) \neq 0$; for some discussions see [AHH99]. While in (4.59) we show the upper bound $(\varphi_{\text{g}}, N\varphi_{\text{g}}) \leq \frac{\alpha^2}{2}\|\hat{h}/\omega\|^2$, we have no lower bound like in (4.64) due to the fact that $(\varphi_{\text{g}}, \sigma\varphi_{\text{g}}) = 0$. Although there

is no mathematical proof known, the physics literature supports the conjecture that whenever a ground state exists, the boson number expectation $(\varphi_g^N, N\varphi_g^N)$ is finite. Thus $\hat{h}/\omega \notin L^2(\mathbb{R}^d)$ would then mean absence of a ground state. For the spin boson model it can be conjectured for similar physical reasons that the same mechanism applies and $\hat{h}/\omega \notin L^2(\mathbb{R}^d)$ implies absence of a ground state. Then some interesting open questions are if it is possible to prove $(\varphi_g, N\varphi_g) \rightarrow \infty$ as $\|\hat{h}/\omega\| \rightarrow \infty$ and, secondly, if a ground state of H exists when $\hat{h}/\omega \notin L^2(\mathbb{R}^d)$. These questions will be considered elsewhere.

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