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Spin chains with boundary inhomogeneities

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Abstract: We investigate the effect of introducing a boundary inhomogeneity in the transfer matrix of an integrable open quantum spin chain. We find that it is possible to construct a local Hamiltonian, and to have quantum group symmetry. The boundary inhomogeneity has a profound effect on the Bethe ansatz solution.

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Contents

1 Introduction

Integrable open spin chains with boundaries have been widely studied in a variety of contexts, see e.g. [1–8] and references therein. Sklyanin [3] provided a general recipe for constructing such models, based on solutions of the bulk [9] and boundary [10, 11] Yang-Baxter equations (YBEs), to which we refer here as R-matrices and K-matrices, respectively.

It was recently noticed, in the context of the $A_1^{(1)}$ R-matrix (corresponding to a spin chain of XXZ-type) and a trivial K-matrix, that Sklyanin's construction can be further generalized by introducing [12] a boundary inhomogeneity in the transfer matrix, as in (2.15) below.[1] The corresponding Hamiltonian is generally not expected to be local; however, by virtue of also having suitable staggered bulk inhomogeneities, the resulting Hamiltonian is in fact local. This model has some further remarkable features, including quantum group (QG) symmetry [13], a novel Bethe ansatz solution, and a continuum limit described by a non-compact CFT [14], see also [15–21] for the corresponding closed chain.

The goal of the present paper is to explore such models with boundary inhomogeneities more broadly, particularly by considering higher-rank R-matrices, as well as non-trivial K-matrices. For concreteness, we focus here on the infinite family of $A_{2n}^{(2)}$ R-matrices; however, we expect that similar results hold for other trigonometric R-matrices [22–24] with crossing symmetry (including $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $D_{n+1}^{(2)}$). By introducing suitable staggered bulk inhomogeneities, we find that the key features of locality of the Hamiltonian and QG symmetry appearing at rank one can still be maintained, and again find novel Bethe ansatz solutions.

The outline of this paper is as follows. In section 2 we define the model by constructing its transfer matrix, and we see explicitly that its Hamiltonian is local. We briefly

[1] This was actually a side result of [12], which was primarily devoted to solving $D_2^{(2)}$ models.

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discuss the model's QG symmetry in section 3, and we present its Bethe ansatz solution in section 4. We conclude in section 5 with a brief summary and a list of some interesting remaining questions.

2 The model

We begin with a brief review in section 2.1 of the basic ingredients that are used in section 2.2 to construct the transfer matrix. We derive the corresponding Hamiltonian in section 2.3.

2.1 Basic ingredients

As already noted, the model is constructed from solutions $R(u)$ of the bulk YBE [9]

$$R_{12}(u-v)\, R_{13}(u)\, R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u-v)\,, \tag{2.1}$$

and solutions $K^R(u)$ of the corresponding boundary YBE [10, 11]

$$R_{12}(u-v)\, K^R_1(u)\, R_{21}(u+v)\, K^R_2(v) = K^R_2(v)\, R_{12}(u+v)\, K^R_1(u)\, R_{21}(u-v)\,, \tag{2.2}$$

where the notations follow those in [25, 26]. For concreteness, we take $R(u)$ to be the $A^{(2)}_{2n}$ R-matrices ($n = 1, 2, \ldots$) [22–24], which for $n = 1$ was obtained by Izergin and Korepin [27]; we use the specific form of these R-matrices given in appendix A of [25], with anisotropy parameter η. These R-matrices have the following additional important properties: periodicity

$$R(u + 2i\pi) = R(u)\,; \tag{2.3}$$

unitarity

$$R_{12}(u)\;R_{21}(-u) = \xi(u)\xi(-u)\,\mathbb{I}\otimes\mathbb{I}\,, \qquad \xi(u) = 2\sinh(\frac{u}{2} - 2\eta)\cosh(\frac{u}{2} - (2n+1)\eta)\,; \tag{2.4}$$

regularity

$$R(0) = \xi(0)\, \mathcal{P}\,, \tag{2.5}$$

where $\mathcal{P}$ is the permutation matrix; PT symmetry

$$R_{21}(u) \equiv \mathcal{P}_{12}\, R_{12}(u)\, \mathcal{P}_{12} = R^{t_1 t_2}_{12}(u)\,; \tag{2.6}$$

and crossing symmetry

$$R_{12}(u) = V_1\, R^{t_2}_{12}(-u-\rho)\, V_1 = V^{t_2}_2\, R^{t_1}_{12}(-u-\rho)\, V^{t_2}_2\,, \qquad \rho = -2(2n+1)\eta - i\pi\,, \tag{2.7}$$

where the matrix V is given by

$$V = \sum_{\alpha=1}^{2n+1} e^{(\bar{\alpha}-\bar{\alpha}')\eta} e_{\alpha,\alpha'}\,, \quad \alpha' = 2n+2-\alpha\,, \quad \alpha = 1, \ldots, 2n+1\,, \tag{2.8}$$

and

$$\bar{\alpha} = \begin{cases} \alpha + \frac{1}{2} & 1 \le \alpha < n+1 \\ \alpha & \alpha = n+1 \\ \alpha - \frac{1}{2} & n+1 < \alpha \le 2n+1 \end{cases} . \tag{2.9}$$

We take the right K-matrices to be the diagonal matrices [28–30]

$$K^R(u) = \text{diag}\left(\underbrace{e^{-u}, \dots, e^{-u}}_{p}, \underbrace{\frac{\gamma e^u + 1}{\gamma + e^u}, \dots, \frac{\gamma e^u + 1}{\gamma + e^u}}_{2n+1-2p}, \underbrace{e^u, \dots, e^u}_{p} \right), \tag{2.10}$$

$$\gamma = \gamma_0 \, e^{(4p+2)\eta + \frac{1}{2}\rho}, \qquad p = 0, 1, \dots, n, \qquad \gamma_0 = \pm 1, \tag{2.11}$$

which for $n = 1$ was obtained in [31]. We emphasize that these K-matrices depend on two boundary parameters p and γ_0, which can take the set of discrete values noted in (2.11). Moreover, for the left K-matrices, we take [25, 31]

$$K^L(u) = K^R(-u-\rho)\, M, \qquad M = V^t\, V, \tag{2.12}$$

which corresponds to imposing the "same" boundary conditions on the two ends.

For later reference, we note here the useful identity [32]

$$\text{tr}_0\, K_0^L(u)\, R_{01}(2u)\, \mathcal{P}_{01} = f(u)\, V_1\, K_1^R(u)\, V_1, \tag{2.13}$$

with

$$\begin{aligned} f(u) = -4 \sinh\left(\frac{u}{2} - \frac{1}{2}(2n-1)\,\eta - \gamma_0 \frac{i\pi}{4}\right) \sinh\left(\frac{u}{2} - \frac{1}{2}(2n+3)\,\eta + \gamma_0 \frac{i\pi}{4}\right) \\ \times \sinh\left(u - (4n+2)\,\eta\right) \frac{\sinh\left(\frac{u}{2} + \frac{1}{2}(2n-4p-1)\,\eta - \gamma_0 \frac{i\pi}{4}\right)}{\sinh\left(\frac{u}{2} - \frac{1}{2}(6n-4p+1)\eta - \gamma_0 \frac{i\pi}{4}\right)}. \end{aligned} \tag{2.14}$$

2.2 Transfer matrix

We consider the following open-chain transfer matrix for a spin chain of length N [12]

$$t(u; \{\theta_l\}, u_0) = \text{tr}_0 \left\{ \bar{K}_0^L(u)\, T_0(u; \{\theta_l\})\, \bar{K}_0^R(u)\, \widehat{T}_0(u + u_0; \{\theta_l\}) \right\}, \tag{2.15}$$

whose key difference with respect to the transfer matrix in [3] is the shift by u_0 in the argument of $\widehat{T}$, which can be regarded as a boundary inhomogeneity. We shall see that this seemingly minor change in the transfer matrix in fact has a profound impact on the model. The monodromy matrices are given as usual by

$$\begin{aligned} T_0(u; \{\theta_l\}) &= R_{0N}(u - \theta_N) \dots R_{01}(u - \theta_1), \\ \widehat{T}_0(u; \{\theta_l\}) &= R_{10}(u + \theta_1) \dots R_{N0}(u + \theta_N), \end{aligned} \tag{2.16}$$

where $\{\theta_l\}$ are bulk inhomogeneities. The right K-matrix $\bar{K}_0^R(u)$ in (2.15) satisfies a generalized boundary YBE

$$R_{12}(u-v)\, \bar{K}_1^R(u)\, R_{21}(u+v+u_0)\, \bar{K}_2^R(v) = \bar{K}_2^R(v)\, R_{12}(u+v+u_0)\, \bar{K}_1^R(u)\, R_{21}(u-v), \tag{2.17}$$

which, compared with (2.2), has a shift by u_0 in the R-matrix whose argument has the sum of rapidities. The generalized boundary YBE (2.17) can be mapped to the standard one (2.2) by performing the shifts $u \mapsto u - u_0/2$ and $v \mapsto v - u_0/2$, and identifying $\bar{K}^R(u - u_0/2) = K^R(u)$. Hence, we set

$$\bar{K}^R(u) = K^R\left(u + \frac{u_0}{2}\right) , \tag{2.18}$$

with $K^R(u)$ given by (2.10). Setting [12]

$$\bar{K}^L(u) = \bar{K}^R(-u - \rho - u_0)\, M\,, \tag{2.19}$$

the transfer matrix (2.15) can be shown to satisfy the commutativity property

$$[t(u; \{\theta_l\}, u_0)\,, t(v; \{\theta_l\}, u_0)] = 0\,, \tag{2.20}$$

which is the hallmark of quantum integrability.

In terms of the $\bar{K}$-matrices, the identity (2.13) reads

$$\mathrm{tr}_0\, \bar{K}_0^L(u)\, R_{01}(2u + u_0)\, \mathcal{P}_{01} = f\left(u + \frac{u_0}{2}\right)\, V_1\, \bar{K}_1^R(u)\, V_1\,, \tag{2.21}$$

where $f(u)$ is given by (2.14).

An important observation is that the presence of a boundary inhomogeneity affects the crossing relation of the transfer matrix. Indeed, the crossing relation now becomes

$$t(-u - \rho - u_0; \{\theta_l\}, u_0) = t\left(u; \{\theta_l\}, u_0\right) , \tag{2.22}$$

i.e. there is an additional u_0-dependent shift.

For generic values of boundary and bulk inhomogeneities, the transfer matrix (2.15) does not generate a local Hamiltonian (i.e., whose range of interactions is independent of N). Following [12], we henceforth set these inhomogeneities to

$$u_0 = i\pi\,, \qquad \theta_l = \begin{cases} -i\pi & \text{for } l = \text{odd} \\ 0 & \text{for } l = \text{even} \end{cases} . \tag{2.23}$$

Note that the boundary inhomogeneity u_0 is the half-period of the R-matrix (2.3), and the bulk inhomogeneities are staggered.[2] The same transfer matrix but with no bulk or boundary inhomogeneities ($u_0 = \theta_l = 0$), to which we refer as the "homogeneous case", was investigated in [25, 26].

To summarize, we consider the transfer matrix (2.15) with inhomogeneity values given by (2.23); it depends on the discrete parameters $N \in \{1, 2, \ldots\}$, $n \in \{1, 2, \ldots\}$, $p \in \{0, 1, \ldots, n\}$ and $\gamma_0 \in \{-1, +1\}$, as well as the continuous parameters u and η.

[2]Staggered models date back at least to [33].

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2.3 Hamiltonian

We define the N-site Hamiltonian by[3]

$$\mathcal{H}^{(N)} = \frac{d}{du} \log\left(t(u)\right)\Big|_{u=0} = t^{-1}(0) \frac{d}{du} t(u)\Big|_{u=0} . \tag{2.24}$$

Note that this is the usual recipe for a closed-chain, rather than an open-chain, Hamiltonian. (For an open chain, usually $t(0) \propto \mathbb{I}$ [3], hence the definition (2.24) reduces to $t'(0)$. However, here $t(0)$ is not proportional to $\mathbb{I}$, thus these two definitions are not equivalent; and the latter definition does not yield a local Hamiltonian.) Using (2.21) and the identity

$$t(0)\, t(i\pi) = \xi^{2N}(0)\, \xi^{2N-2}(i\pi)\, f\left(\frac{i\pi}{2}\right) f\left(\frac{3i\pi}{2}\right) \mathbb{I} , \tag{2.25}$$

we obtain, after a long calculation, the following Hamiltonian for even values of $N > 2$

$$\begin{aligned}
\mathcal{H}^{(N=\text{even})} = {} & \frac{1}{\xi(0)\, \xi^4(i\pi)}\, \bar{K}_1^R(i\pi)\, R_{32}\, R_{31}\, h_{12}\, R_{13}\, R_{23}\, \bar{K}_1^R(0) \\
& + \frac{1}{\xi(0)}\, V_N\, \bar{K}_N^R(i\pi)\, V_N\, h_{N-1,N}\, V_N\, \bar{K}_N^R(0)\, V_N \\
& + \frac{1}{\xi(0)\, \xi^2(i\pi)}\, R_{N-1,N-2}\, h_{N-2,N}\, R_{N-2,N-1} \\
& + \frac{1}{\xi(0)\, \xi^2(i\pi)} \sum_{j=1,3,\ldots}^{N-3} R_{j+2,j+1}\, h_{j,j+2}\, R_{j+1,j+2} \\
& + \frac{1}{\xi(0)\, \xi^6(i\pi)} \sum_{j=2,4,\ldots}^{N-4} R_{j+3,j+2}\, R_{j+1,j}\, R_{j+3,j}\, h_{j,j+2}\, R_{j,j+3}\, R_{j,j+1}\, R_{j+2,j+3} \\
& + \frac{1}{\xi^2(i\pi)} \sum_{j=2,4,\ldots}^{N-2} \bar{h}_{j,j+1} \\
& + \frac{1}{\xi^4(i\pi)}\, \bar{K}_1^R(i\pi)\, R_{32}\, \bar{h}_{13}\, R_{23}\, \bar{K}_1^R(0) \\
& + \frac{1}{\xi^6(i\pi)} \sum_{j=2,4,\ldots}^{N-4} R_{j+3,j+2}\, R_{j+1,j}\, \bar{h}_{j,j+3}\, R_{j,j+1}\, R_{j+2,j+3} \\
& + \bar{K}_1^R(i\pi)\, \bar{K}_1^{R\prime}(0) + V_N\, \bar{K}_N^R(i\pi)\, \bar{K}_N^{R\prime}(0)\, V_N \\
& + \frac{f'(\frac{i\pi}{2})}{f(\frac{i\pi}{2})}\, \mathbb{I} ,
\end{aligned} \tag{2.26}$$

where we have introduced the following short-hand notations

$$h_{ij} = \mathcal{P}_{ij}\, R'_{ij}(0) , \qquad \bar{h}_{ij} = R_{ji}(i\pi)\, R'_{ij}(i\pi) , \qquad R_{ij} = R_{ij}(i\pi) , \tag{2.27}$$

and a prime denotes differentiation with respect to the spectral parameter u. Note that the range of interactions in this Hamiltonian does not exceed 4 sites. For the case $N = 2$,

[3] We henceforth suppress displaying the dependence of the transfer matrix on the inhomogeneities, which are given by (2.23).

we obtain

$$\mathcal{H}^{(N=2)} = \frac{1}{\xi(0)}\,\bar{K}_1^R(i\pi)\,h_{12}\,\bar{K}_1^R(0) + \frac{1}{\xi(0)}\,V_2\,\bar{K}_2^R(i\pi)\,V_2\,h_{12}\,V_2\,\bar{K}_2^R(0)\,V_2 + \bar{K}_1^R(i\pi)\,\bar{K}_1^{R'}(0) + V_2\,\bar{K}_2^R(i\pi)\,\bar{K}_2^{R'}(0)\,V_2 + \frac{f'(\frac{i\pi}{2})}{f(\frac{i\pi}{2})}\,\mathbb{I}\,. \tag{2.28}$$

A similar computation for odd values of $N>1$ gives

$$\begin{aligned}\mathcal{H}^{(N=\text{odd})} &= \frac{1}{\xi(0)\,\xi^4(i\pi)}\,\bar{K}_1^R(i\pi)\,R_{32}\,R_{31}\,h_{12}\,R_{13}\,R_{23}\,\bar{K}_1^R(0)\\ &+ \frac{1}{\xi(0)\,\xi^2(i\pi)}\,V_N\,\bar{K}_N^R(0)\,V_N\,R_{N,N-1}\,h_{N,N-1}\,R_{N-1,N}\,V_N\,\bar{K}_N^R(i\pi)\,V_N\\ &+ \frac{1}{\xi(0)\,\xi^2(i\pi)}\sum_{j=1,3,\ldots}^{N-2} R_{j+2,j+1}\,h_{j,j+2}\,R_{j+1,j+2}\\ &+ \frac{1}{\xi(0)\,\xi^6(i\pi)}\sum_{j=2,4,\ldots}^{N-3} R_{j+3,j+2}\,R_{j+1,j}\,R_{j+3,j}\,h_{j,j+2}\,R_{j,j+3}\,R_{j,j+1}\,R_{j+2,j+3}\\ &+ \frac{1}{\xi^2(i\pi)}\sum_{j=2,4,\ldots}^{N-3} \bar{h}_{j,j+1}\\ &+ \frac{1}{\xi^4(i\pi)}\,\bar{K}_1^R(i\pi)\,R_{32}\,\bar{h}_{13}\,R_{23}\,\bar{K}_1^R(0)\\ &+ \frac{1}{\xi^6(i\pi)}\sum_{j=2,4,\ldots}^{N-3} R_{j+3,j+2}\,R_{j+1,j}\,\bar{h}_{j,j+3}\,R_{j,j+1}\,R_{j+2,j+3}\\ &+ \bar{K}_1^R(i\pi)\,\bar{K}_1^{R'}(0) + V_N\,\bar{K}_N^R(i\pi)\,\bar{K}_N^{R'}(0)\,V_N\\ &+ \frac{f'(\frac{i\pi}{2})}{f(\frac{i\pi}{2})}\,\mathbb{I}\,.\end{aligned} \tag{2.29}$$

The range of interactions again does not exceed 4 sites. We conclude that the Hamiltonian is local.

3 Quantum group symmetry

The transfer matrix (2.15) with inhomogeneities (2.23) has the QG symmetry $U_q(B_{n-p})\otimes U_q(C_p)$, corresponding to removing the p^{th} node from the $A_{2n}^{(2)}$ Dynkin diagram, as follows from arguments similar to those for the homogeneous case [25].[4] The "left" algebra B_{n-p}

[4]The gauge transformations for the K-matrices are now given by

$$\tilde{\bar{K}}^R(u,p) = B\left(u+\frac{u_0}{2},p\right)\bar{K}^R(u,p)\,B\left(u+\frac{u_0}{2},p\right), \quad \tilde{\bar{K}}^L(u,p) = B\left(-u-\frac{u_0}{2},p\right)\bar{K}^L(u,p)\,B\left(-u-\frac{u_0}{2},p\right),$$

where $B(u,p)$ is given by eq. (3.3) in [25].

(with $p = 0, 1, \ldots, n-1$) has generators

$$\begin{aligned} H_j^{(l)}(p) &= e_{p+j,p+j} - e_{2n+2-p-j,2n+2-p-j}\,, \\ E_j^{+\,(l)}(p) &= e_{p+j,p+j+1} + e_{2n+1-p-j,2n+2-p-j}\,, \\ E_j^{-\,(l)}(p) &= \left(E_j^{+\,(l)}(p)\right)^t\,, \qquad j = 1, \ldots, n-p\,, \end{aligned} \tag{3.1}$$

and the "right" algebra C_p (with $p = 1, 2, \ldots, n$) has generators

$$\begin{aligned} H_j^{(r)}(p) &= -e_{p+1-j,p+1-j} + e_{2n+1-p+j,2n+1-p+j}\,, \\ E_j^{+\,(r)}(p) &= \begin{cases} e_{p-j,p+1-j} + e_{2n+1-p+j,2n+2-p+j} & \text{for } 1 \le j \le p-1 \\ \sqrt{2}e_{2n+1,1} & \text{for } j = p \end{cases}\,, \\ E_j^{-\,(r)}(p) &= \left(E_j^{+\,(r)}(p)\right)^t\,, \qquad j = 1, \ldots, p\,, \end{aligned} \tag{3.2}$$

where e_{ij} are the elementary $(2n+1) \times (2n+1)$ matrices with elements $(e_{ij})_{\alpha\beta} = \delta_{i,\alpha}\delta_{j,\beta}$. The coproducts for the "left" generators are given by

$$\begin{aligned} \Delta(H_j^{(l)}) &= H_j^{(l)} \otimes \mathbb{I} + \mathbb{I} \otimes H_j^{(l)}\,, \qquad j = 1, \ldots, n-p\,, \\ \Delta(E_j^{\pm\,(l)}) &= E_j^{\pm\,(l)} \otimes e^{(\eta+i\pi)H_j^{(l)} - \eta H_{j+1}^{(l)}} + e^{-(\eta+i\pi)H_j^{(l)} + \eta H_{j+1}^{(l)}} \otimes E_j^{\pm\,(l)}\,, \qquad j = 1, \ldots, n-p-1\,, \\ \Delta(E_{n-p}^{\pm\,(l)}) &= E_{n-p}^{\pm\,(l)} \otimes e^{(\eta+i\pi)H_{n-p}^{(l)}} + e^{-(\eta+i\pi)H_{n-p}^{(l)}} \otimes E_{n-p}^{\pm\,(l)}\,, \end{aligned} \tag{3.3}$$

and the coproducts for the "right" generators are given by

$$\begin{aligned} \Delta(H_j^{(r)}) &= H_j^{(r)} \otimes \mathbb{I} + \mathbb{I} \otimes H_j^{(r)}\,, \qquad j = 1, \ldots, p\,, \\ \Delta(E_j^{\pm\,(r)}) &= E_j^{\pm\,(r)} \otimes e^{(\eta+i\pi)H_j^{(r)} - \eta H_{j+1}^{(r)}} + e^{-(\eta+i\pi)H_j^{(r)} + \eta H_{j+1}^{(r)}} \otimes E_j^{\pm\,(r)}\,, \qquad j = 1, \ldots, p-1\,, \\ \Delta(E_p^{\pm\,(r)}) &= E_p^{\pm\,(r)} \otimes e^{2\eta H_p^{(r)}} - e^{-2\eta H_p^{(r)}} \otimes E_p^{\pm\,(r)}\,. \end{aligned} \tag{3.4}$$

These expressions for the coproducts are the same as in [25] (where many further details can also be found), except for the relative minus sign in $\Delta(E_p^{\pm\,(r)})$ (3.4).

Due to the relative minus sign in $\Delta(E_p^{\pm\,(r)})$ (3.4), this coproduct does not obey the standard co-associativity property. Indeed,

$$\begin{aligned} (\mathbb{I} \otimes \Delta)\Delta(E_p^{\pm\,(r)}) &= (\mathbb{I} \otimes \Delta)\left(E_p^{\pm\,(r)} \otimes e^{2\eta H_p^{(r)}} - e^{-2\eta H_p^{(r)}} \otimes E_p^{\pm\,(r)}\right) \\ &= E_p^{\pm\,(r)} \otimes e^{2\eta \Delta(H_p^{(r)})} - e^{-2\eta H_p^{(r)}} \otimes \Delta(E_p^{\pm\,(r)}) \\ &= \Delta(E_p^{\pm\,(r)}) \otimes e^{2\eta H_p^{(r)}} + e^{-2\eta \Delta(H_p^{(r)})} \otimes E_p^{\pm\,(r)} \\ &\ne (\Delta \otimes \mathbb{I})\Delta(E_p^{\pm\,(r)})\,, \end{aligned} \tag{3.5}$$

which suggests that there is instead an underlying quasi-Hopf algebra structure [13, 34]. We define the higher coproducts for $E_p^{\pm\,(r)}$ recursively by

$$\begin{aligned} \Delta_N(E_p^{\pm\,(r)}) &= (\mathbb{I} \otimes \Delta)\Delta_{N-1}(E_p^{\pm\,(r)}) \\ &= E_p^{\pm\,(r)} \otimes e^{2\eta \Delta_{N-1}(H_p^{(r)})} - e^{-2\eta H_p^{(r)}} \otimes \Delta_{N-1}(E_p^{\pm\,(r)})\,, \qquad N > 2\,, \end{aligned} \tag{3.6}$$

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where $\Delta_2 = \Delta$.

The N-fold coproducts of the "left" and "right" generators commute with the transfer matrix (2.15), (2.23)

$$\begin{aligned}
\left[\Delta_N(H_j^{(l)})\,, t(u)\right] &= \left[\Delta_N(E_j^{\pm\,(l)})\,, t(u)\right] = 0\,, \quad j = 1,\ldots,n-p\,, \quad p = 0,\ldots,n-1\,,\\
\left[\Delta_N(H_j^{(r)})\,, t(u)\right] &= \left[\Delta_N(E_j^{\pm\,(r)})\,, t(u)\right] = 0\,, \quad j = 1,\ldots,p\,, \qquad p = 1,\ldots,n\,, \qquad (3.7)
\end{aligned}$$

for both values $\gamma_0 = \pm 1$. At least for real values of the anisotropy parameter η, the rich degeneracies in the spectrum of the transfer matrix are completely accounted for by its QG symmetry, as in the homogeneous case [25].

4 Analytical Bethe ansatz

The eigenvalues of the transfer matrix (2.15), (2.23) can be determined by analytical Bethe ansatz [35] similarly to the homogeneous case [26]; however, there are some surprises. Indeed, let $|\Lambda^{(m_1,\ldots,m_n)}\rangle$ be simultaneous eigenvectors of the transfer matrix and Cartan generators

$$\begin{aligned}
t(u)\,|\Lambda^{(m_1,\ldots,m_n)}\rangle &= \Lambda^{(m_1,\ldots,m_n)}(u)\,|\Lambda^{(m_1,\ldots,m_n)}\rangle\,,\\
\Delta_N(H_i^{(l)}(p))\,|\Lambda^{(m_1,\ldots,m_n)}\rangle &= h_i^{(l)}\,|\Lambda^{(m_1,\ldots,m_n)}\rangle\,, \qquad i = 1,\ldots,n-p\,,\\
\Delta_N(H_i^{(r)}(p))\,|\Lambda^{(m_1,\ldots,m_n)}\rangle &= h_i^{(r)}\,|\Lambda^{(m_1,\ldots,m_n)}\rangle\,, \qquad i = 1,\ldots,p\,. \qquad (4.1)
\end{aligned}$$

We propose that the eigenvalues of the transfer matrix for general values of n, p and γ_0 are given by the following TQ-equation

$$\begin{aligned}
\Lambda^{(m_1,\ldots,m_n)}(u) = \phi(u,p)\Big\{ & A(u)\, z_0(u)\, y_0(u,p)\, [-\sinh(u-4\eta)\sinh(u-2(2n+1)\eta)]^N\\
&+ \tilde{A}(u)\, \tilde{z}_0(u)\, \tilde{y}_0(u,p)\, [-\sinh(u)\sinh(u-2(2n-1)\eta)]^N\\
&+ \Big\{\sum_{l=1}^{n-1}\Big[z_l(u)\, y_l(u,p)\, B_l(u) + \tilde{z}_l(u)\, \tilde{y}_l(u,p)\, \tilde{B}_l(u)\Big]\\
&\quad + w(u)\, y_n(u,p)\, B_n(u)\Big\}\, [-\sinh(u)\sinh(u-2(2n+1)\eta)]^N\Big\}. \qquad (4.2)
\end{aligned}$$

The overall factor $\phi(u,p)$ is given by

$$\phi(u,p) = \left(\frac{\gamma e^{u+\frac{i\pi}{2}} + 1}{\gamma + e^{u+\frac{i\pi}{2}}}\right)\left(\frac{\gamma e^{-u-\rho-\frac{i\pi}{2}} + 1}{\gamma + e^{-u-\rho-\frac{i\pi}{2}}}\right)\,, \qquad (4.3)$$

where γ is defined in (2.11). The tilde denotes crossing e.g. $\tilde{A}(u) = A(-u-\rho-i\pi)$, in view of the crossing relation (2.22). The functions $A(u)$ and $B_l(u)$ are defined as

$$\begin{aligned}
A(u) &= \frac{Q^{[1]}(u+2\eta)}{Q^{[1]}(u-2\eta)}\,,\\
B_l(u) &= \frac{Q^{[l]}(u-2(l+2)\eta)}{Q^{[l]}(u-2l\eta)}\frac{Q^{[l+1]}(u-2(l-1)\eta)}{Q^{[l+1]}(u-2(l+1)\eta)}\,, \qquad l=1,\dots,n-1\,,\\
B_n(u) &= \frac{Q^{[n]}(u-2(n+2)\eta)}{Q^{[n]}(u-2n\eta)}\frac{Q^{[n]}(u-2(n-1)\eta+i\pi)}{Q^{[n]}(u-2(n+1)\eta+i\pi)}\,,
\end{aligned} \tag{4.4}$$

where the functions $Q^{[l]}(u)$ are given by

$$Q^{[l]}(u) = \prod_{j=1}^{m_l} \sinh\left(\frac{1}{2}\left(u-u_j^{[l]}\right)\right)\cosh\left(\frac{1}{2}\left(u+u_j^{[l]}\right)\right)\,, \qquad Q^{[l]}(-u) = Q^{[l]}(u+i\pi)\,,$$
$$l=1,\dots,n\,, \tag{4.5}$$

whose zeros $\{u_j^{[l]}\}$ remain to be determined. The functions $z_l(u)$ and $w(u)$ are given by

$$\begin{aligned}
z_l(u) &= \frac{\cosh(u)\cosh(u-2(2n+1)\eta)\sinh(u-(2n-1)\eta)}{\cosh(u-2l\eta)\cosh(u-2(l+1)\eta)\sinh(u-(2n+1)\eta)}\,,\\
w(u) &= \frac{\cosh(u)\cosh(u-2(2n+1)\eta)}{\cosh(u-2n\eta)\cosh(u-2(n+1)\eta)}\,,
\end{aligned} \tag{4.6}$$

and the functions $y_l(u,p)$ are given by

$$y_l(u,p) = \begin{cases} F(u) & \text{for} \quad 0\le l\le p-1 \\ G(u) & \text{for} \quad p\le l\le n \end{cases}\,, \tag{4.7}$$

where

$$\begin{aligned}
G(u) &= \frac{\cosh\left(\frac{1}{2}(u-(2n-1)\eta+i\pi\varepsilon)\right)\cosh\left(\frac{1}{2}(u-(2n+3)\eta+i\pi\varepsilon)\right)}{\cosh\left(\frac{1}{2}(u-(2n-4p-1)\eta+i\pi\varepsilon)\right)\cosh\left(\frac{1}{2}(u-(2n+4p+3)\eta+i\pi\varepsilon)\right)}\,,\\
F(u) &= -\left(\frac{\sinh\left(\frac{1}{2}(u+(2n-4p-1)\eta+i\pi\varepsilon)\right)}{\cosh\left(\frac{1}{2}(u-(2n-1)\eta+i\pi\varepsilon)\right)}\right)^2 G(u)\,,
\end{aligned} \tag{4.8}$$

with

$$\varepsilon = \frac{1}{2}(1-\gamma_0) \in \{0,1\}\,. \tag{4.9}$$

The Bethe equations for the zeros $\{u_k^{[l]}\}$ of the Q-functions, which we determine by requiring that the transfer-matrix eigenvalues (4.2) have vanishing residues at the poles $u=u_k^{[l]}+2l\eta$, are given by

$$\left[\frac{\sinh\left(u_k^{[1]}+2\eta\right)}{\sinh\left(u_k^{[1]}-2\eta\right)}\right]^N \Phi_{1,p,n}(u_k^{[1]}) = \frac{Q_k^{[1]}\left(u_k^{[1]}+4\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-4\eta\right)}\frac{Q^{[2]}\left(u_k^{[1]}-2\eta\right)}{Q^{[2]}\left(u_k^{[1]}+2\eta\right)}\,, \quad k=1,\dots,m_1\,, \tag{4.10}$$

$$\Phi_{l,p,n}(u_k^{[l]}) = \frac{Q^{[l-1]}\left(u_k^{[l]}-2\eta\right)}{Q^{[l-1]}\left(u_k^{[l]}+2\eta\right)} \frac{Q_k^{[l]}\left(u_k^{[l]}+4\eta\right)}{Q_k^{[l]}\left(u_k^{[l]}-4\eta\right)} \frac{Q^{[l+1]}\left(u_k^{[l]}-2\eta\right)}{Q^{[l+1]}\left(u_k^{[l]}+2\eta\right)}, \qquad k=1,\ldots,m_l,$$
$$l=2,\ldots,n-1, \tag{4.11}$$

$$\Phi_{n,p,n}(u_k^{[n]}) = \frac{Q^{[n-1]}\left(u_k^{[n]}-2\eta\right)}{Q^{[n-1]}\left(u_k^{[n]}+2\eta\right)} \frac{Q_k^{[n]}\left(u_k^{[n]}+4\eta\right)}{Q_k^{[n]}\left(u_k^{[n]}-4\eta\right)} \frac{Q_k^{[n]}\left(u_k^{[n]}-2\eta+i\pi\right)}{Q_k^{[n]}\left(u_k^{[n]}+2\eta+i\pi\right)}, \qquad k=1,\ldots,m_n, \tag{4.12}$$

where $Q^{[l]}(u)$ is given by (4.5), and $Q_k^{[l]}(u)$ is defined by a similar product with the k^{th} term omitted

$$Q_k^{[l]}(u) = \prod_{j=1, j\neq k}^{m_l} \sinh\left(\frac{1}{2}(u-u_j^{[l]})\right)\cosh\left(\frac{1}{2}(u+u_j^{[l]})\right). \tag{4.13}$$

Finally, the important factor $\Phi_{l,p,n}(u)$ appearing in the Bethe equations is given by

$$\begin{aligned}\Phi_{l,p,n}(u) &= \frac{y_l(u+2l\eta,p)}{y_{l-1}(u+2l\eta,p)} = \begin{cases}\frac{G(u+2p\eta)}{F(u+2p\eta)} & \text{for } l=p\\ 1 & \text{for } l\neq p\end{cases}, \\ &= \left(\frac{\sinh\left(\frac{1}{2}(u-\delta_{l,p}[(2n-2p-1)\eta+i\pi\delta_{\varepsilon,0}])\right)}{\sinh\left(\frac{1}{2}(u+\delta_{l,p}[(2n-2p-1)\eta+i\pi\delta_{\varepsilon,1}])\right)}\right)^2,\end{aligned} \tag{4.14}$$

where ε is defined in (4.9). Note that $\Phi_{l,p,n}(u)$ is different from 1 only if $l=p$.

The energy is given, in view of (2.24) and (4.2), by

$$\begin{aligned}E &= \frac{d}{du}\log\left(\Lambda^{(m_1,\ldots,m_n)}(u)\right)\Big|_{u=0} \\ &= -\sum_{j=1}^{m_1}\frac{\sinh(4\eta)}{\sinh(u_j^{[1]}+2\eta)\sinh(u_j^{[1]}-2\eta)} - \frac{N\sinh(2(2n+3)\eta)}{\sinh(4\eta)\sinh(2(2n+1)\eta)} + c_0,\end{aligned} \tag{4.15}$$

where

$$c_0 = \frac{d}{du}\log\left[\phi(u,p)\, z_0(u)\, y_0(u,p)\right]\Big|_{u=0}. \tag{4.16}$$

As in the homogeneous case [26], the Dynkin labels $[a_1^{(l)},\ldots,a_{n-p}^{(l)}]$ of the representations of the "left" algebra B_{n-p} (with $p=0,1,\ldots,n-1$) are given by

$$\begin{aligned}a_i^{(l)} &= m_{p+i-1} - 2m_{p+i} + m_{p+i+1}, \qquad i=1,\ldots,n-p-1. \\ a_{n-p}^{(l)} &= 2m_{n-1} - 2m_n,\end{aligned} \tag{4.17}$$

where $m_0 = N$. Similarly, the Dynkin labels $[a_1^{(r)},\ldots,a_p^{(r)}]$ of the representations of the "right" algebra C_p (with $p=1,2,\ldots,n$) are given by

$$\begin{aligned}a_i^{(r)} &= m_{i-1} - 2m_i + m_{i+1}, \qquad i=1,\ldots,p-1, \\ a_p^{(r)} &= m_{p-1} - m_p.\end{aligned} \tag{4.18}$$

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Given the cardinalities of the Bethe roots of each type (namely, $m_1, \ldots, m_n$) for an eigenvalue $\Lambda^{(m_1,\ldots,m_n)}(u)$, eqs. (4.17)–(4.18) determine the Dynkin labels of the corresponding "left" and "right" representations, from which one can deduce (e.g., using LieART [36]) their dimensions, and therefore the eigenvalue's degeneracy.

We have numerically verified the completeness of this Bethe ansatz solution for small values of n and N, namely, $n = 1, 2$ with $N = 1, 2, 3$, and $n = 3$ with $N = 1, 2$, for all $p = 0, 1, \ldots, n$ and $\gamma_0 = \pm 1$, for some generic value of the anisotropy η, along the lines in [32].

We emphasize that in the homogeneous case [26], the Q-functions are given by

$$Q^{[l]}(u) = \prod_{j=1}^{m_l} \sinh\left(\frac{1}{2}\left(u - u_j^{[l]}\right)\right) \sinh\left(\frac{1}{2}\left(u + u_j^{[l]}\right)\right) ;$$

but in the presence of the boundary inhomogeneity (2.23), the Q-functions are instead given by (4.5), with a cosh instead of sinh in the second factor. Consequently, it is not just the "left-hand-side", but also the "right-hand-side" of the Bethe equations (4.10)–(4.12) that is affected by the boundary inhomogeneity, contrary to the conventional wisdom that the boundary affects only the former. Note also that, contrary to what usually happens for open spin chains, the power appearing in the left-hand-side of the first Bethe equation (4.10) is N instead of $2N$.

5 Conclusions

We have seen that an integrable open quantum spin chain with a boundary inhomogeneity (2.15) can have a local Hamiltonian (2.26)–(2.29), as well as QG symmetry (3.7) that accounts for rich degeneracies in the spectrum. The presence of a boundary inhomogeneity affects the crossing relation (2.22), and has a profound effect on the Bethe ansatz solution, most notably on the Q-functions (4.5).

We have focused here on a boundary inhomogeneity whose value is half the period of the R-matrix, and with corresponding staggered bulk inhomogeneities (2.23). It may be interesting to consider other choices of boundary and bulk inhomogeneities, especially if they give rise to local Hamiltonians. Although we have also focused here on the infinite family of models constructed with $A_{2n}^{(2)}$ R-matrices, we expect that similar results hold for other models with crossing symmetry, such as those considered in [25, 26]. In the simpler $A_1^{(1)}$ case [12], the Hamiltonian can be formulated in a beautiful compact form [14] in terms of Temperley-Lieb (TL) operators [37]. It would be interesting if the Hamiltonians obtained here (2.26)–(2.29) could be reformulated in a similar way in terms of some sort of generalized TL operators, at least for the "extremal" cases $p = 0, n$, where the QG symmetry is $U_q(B_n)$, $U_q(C_n)$, respectively. The continuum limit of the $A_1^{(1)}$ model [12] is described [14] by a non-compact CFT; it would be very interesting if a similar phenomenon occurs for the higher-rank models introduced here.

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