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SPIN GAUGE THEORY OF THE FIRST GENERATION

II. - BASIC THEORY OF STRONG, WEAK, AND ELECTROMAGNETIC INTERACTIONS

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A B S T R A C T

The (16x16) Clifford algebra used in the first paper (I) of this series is extended to a (32x32) algebra. The eight-dimensional vector bases corresponding to leptons and dions become ten-dimensional bases. Four basis vectors define the space-time metric ; three vectors define the metric of the space invariant under the SU(1,1) group introduced in I, and the remaining three vectors the space in which the SU(2) weak interactions are generated by rotations. A simple helicity-symmetric model of unified interactions is detailed in Section 2. The more complex theory incorporating the GSW weak interaction matrix elements is given in Section 3. The Clifford algebra structure ensures that SU(2) matrix elements are the same for leptons and dions. The neutral potentials generated by the SU(1,1) and SU(2) transformations mix with a third neutral potential to define the electromagnetic, Z and neutral gryphon potentials. In the final section, we discuss the models and outline a programme of further work.

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1. THE EXTENDED CLIFFORD ALGEBRA

In the first paper of this series, denoted by I, we put forward a spin gauge theory of electrons and neutrinos and "up and down dions", which interact through the electromagnetic field, a "strong" neutral vector gryphon field and charged pseudovector gryphon fields carrying charges $\pm\frac{1}{3}$ e. The theory was based on a (16 × 16) Clifford algebra; the 16-component lepton (or electron-antineutrino) and dion fields are, respectively, of the form

$$\Psi = \begin{bmatrix} \ell_L \\ \ell_R \end{bmatrix} , \quad X = \begin{bmatrix} \delta_L \\ \delta_R \end{bmatrix}, \tag{1.1}$$

where

$$\ell_{L,R} = \begin{bmatrix} \epsilon_{L,R} \\ \nu'_{L,R} \end{bmatrix} \equiv \frac{1}{2} \begin{bmatrix} (I \pm i\gamma_5)\ \epsilon \\ (I \pm i\gamma_5)\ \nu' \end{bmatrix}, \tag{1.2a}$$

$$\delta_{L,R} = \begin{bmatrix} d_{L,R} \\ u'_{L,R} \end{bmatrix} \equiv \frac{1}{2} \begin{bmatrix} (I \pm i\gamma_5)\, d \\ (I \pm i\gamma_5)\, u' \end{bmatrix}, \tag{1.2b}$$

and ε, ν', d, u$'$ are the four-component spinors corresponding to the massless Dirac electron, its antineutrino, the down dion and the up antidion. The linking of (ε,ν') and (d,u$'$) provide unusual charged gryphon interactions which can, for example, convert a dion into an antidion. In the present paper, we show how to extend the spin gauge theory and the Clifford algebra in order to include the Glashow-Salam-Weinberg (GSW) weak interaction matrix elements; the spin gauge theory allows us to include gauge invariant lepton mass terms, but we do not study the problem of boson masses in this paper.

In order to include the weak interactions, it is necessary for ε and ν, for instance, to occur in the same field. The 16-component fields (1.1) have to be extended to 32-component fields which contain the charge conjugate spinors ε', ν, d$'$, u; 4-component charge conjugation is defined by (I, 5.1) and (I, 5.2b).

The 16-component charge conjugates Ψ' and X' of Ψ and X are given by (I, 5.3) and (I, 5.4); we define 32-component fields ψ and χ by

$$\psi = n\begin{bmatrix} \Psi \\ \tau_1 \Psi' \end{bmatrix}, \quad \chi = n\begin{bmatrix} X \\ \tau_1 X' \end{bmatrix}, \tag{1.3}$$

where $n = 1/\sqrt{2}$; thus

$$\psi = n\begin{bmatrix} \varepsilon_L & \nu'_L & \varepsilon_R & \nu'_R & \nu_L & \varepsilon'_L & \nu_R & \varepsilon'_R \end{bmatrix}^T, \tag{1.4a}$$

$$\chi = n\begin{bmatrix} d_L & u'_L & d_R & u'_R & u_L & d'_L & u_R & d'_R \end{bmatrix}^T. \tag{1.4b}$$

The last sixteen components of ψ, χ may be regarded as different charge states of the respective first sixteen components.

Just as we introduced sets of Pauli matrices $\{\rho_s\}$ and $\{\tau_r\}$ in I, we introduce a third set $\{\lambda_t;\ t = 1, 2, 3\}$ together with the (2×2) unit matrix λ_4, acting on states of the form (1.3). This allows us to define the elements

$$\lambda_t \rho_s \tau_r (I, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{5\mu}, \gamma_5) \qquad (\mu, \nu, r, s, t = 1, 2, 3, 4) \tag{1.5}$$

of a (32×32) Clifford algebra, which contains (I, 2.1), multiplied by λ_4, as a sub-algebra. This algebra has a 10- or 11-dimensional vector basis. The underlying space is 10-dimensional compared with the 8-dimensional one used in I. We shall see that, within this higher dimensional space, there is a 3-dimensional subspace whose bivectors are the generators of the SU(2) transformation of the GSW theory.

When we take matrix elements of operators between states of the form (1.4) and their bar conjugates, we shall obtain sums and differences of pairs of elements of the forms, for example,

$$(\bar\varepsilon_L \not{M} \varepsilon_R), \ (\bar\varepsilon'_R \not{M} \varepsilon'_L),$$

$$(\bar d_R i\gamma_5 \not{M} d_L), \ (\bar d'_L i\gamma_5 \not{M} d'_R),$$

and

$$(\bar{u}'\, i\gamma_5 \not{B}_-\, d)\ ,\ (\bar{d}'\, i\gamma_5 \not{B}_-\, u).$$

Apart from a possible minus sign, the members of each pair are equal. So the pairs either reinforce each other or cancel, depending on the combination of λ,ρ and τ matrices in the 32-component element; if the wrong choice of λ-matrix is made, the total matrix element will be zero. In choosing the vector basis of the (32 × 32) algebra it is important to ensure that the interactions generated, with one exception, by bivectors of the algebra are not self cancelling. The basic formulae determining the relative signs of the pairs of matrix elements are (I, 2.10), but we shall require analogous identities which distinguish helicities. Again using ε,ν to denote *any* two 4-component spinors (including antiparticle spinors), these identities are

$$\bar{\varepsilon}'_L \nu'_L = -\bar{\nu}_L \varepsilon_L\ ,\quad \bar{\varepsilon}'_R \nu'_R = -\bar{\nu}_R \varepsilon_R, \tag{1.6a}$$

$$\bar{\varepsilon}'_L \not{M} \nu'_R = \bar{\nu}_R \not{M} \varepsilon_L,\quad \bar{\varepsilon}'_R \not{M} \nu'_L = \bar{\nu}_L \not{M} \varepsilon_R, \tag{1.6b}$$

$$\bar{\varepsilon}'_L i\gamma_5 \not{M} \nu'_R = -\bar{\nu}_R i\gamma_5 \not{M} \varepsilon_L,\quad \bar{\varepsilon}'_R i\gamma_5 \not{M} \nu'_L = -\bar{\nu}_L i\gamma_5 \not{M} \varepsilon_R, \tag{1.6c}$$

$$\bar{\varepsilon}' \gamma^\mu (\partial_\mu \nu') = (\partial_\mu \nu^\dagger)\, \gamma\, \gamma^\mu \varepsilon, \tag{1.6d}$$

where M_μ is any 4-vector. It is important to remember that we use the suffixes "L" and "R" to denote factors $I+i\gamma_5$ and $I-i\gamma_5$, multiplying both spinors and their conjugates. So, for example,

$$\varepsilon'_R = \gamma_5\, C^{-1} \varepsilon^*_L$$

and

$$\bar{\varepsilon}'_L = (\varepsilon'_R)^\dagger\, \gamma .$$

The scalar of the (32 × 32) Clifford algebra is $\lambda_4\rho_4\tau_4 I$, and we have chosen the 11-dimensional vector basis to be

$$\Gamma_r = \lambda_4 \rho_1 \tau_3 \gamma_\mu \qquad (r = \mu = 1,2,3,4) \tag{1.7a}$$

$$\Gamma_r = i\, \lambda_k \rho_2 \tau_4 I \qquad (r = k+4 = 5,6) \tag{1.7b}$$

$$\Gamma_7 = \lambda_4 \rho_1 \tau_3 \gamma_5 \tag{1.7c}$$

$$\Gamma_r = i\, \lambda_3 \rho_4 \tau_i I \qquad (r = i+7 = 8,9) \tag{1.7d}$$

$$\Gamma_{10} = \lambda_3 \rho_2 \tau_3 I \tag{1.7e}$$

$$\Gamma_{11} = \lambda_4 \rho_3 \tau_3 I\,. \tag{1.7f}$$

This choice of basis leads us to select

$$\Gamma \equiv \lambda_4 \rho_1 \tau_3 \gamma \tag{1.8}$$

as the (32 × 32) conjugation matrix. The basis vectors (1.7) satisfy

$$\{\Gamma_r, \Gamma_s\} = 2 g_{rs} I_{32}\,,$$

where I_{32} is the unit (32 × 32) matrix, and

$$g_{rs} = 0 \qquad (r \neq s)\,, \tag{1.9a}$$

$$\left.\begin{aligned} g_{44} &= g_{10\,10} = g_{11\,11} = 1\,, \\ g_{rr} &= -1 \quad (r \neq 4, 10, 11) \end{aligned}\right\}. \tag{1.9b}$$

The basis vectors (1.7a) are just λ_4 times the basis vectors ($\underline{\mathrm{I}}$, 2.2a). In Table 1, the other seven basis vectors are compared with the remaining five of the (16 × 16) algebra used in $\underline{\mathrm{I}}$. Apart from the λ_3 or λ_4 factors mentioned above, and with i factors omitted from the (16 × 16) vectors Γ_8 and Γ_9, the last five basis vectors are the same in the two algebras. We note the timelike metric of Γ_{10} and Γ_{11} in Table 1, the reason for which was indicated in §§ 2,3 of $\underline{\mathrm{I}}$. The additional vectors Γ_5, Γ_6 in the (32 × 32) algebra contain λ_1, λ_2; these off-diagonal matrices turn out to be associated with the charged currents of the GSW theory. It is important to note that a vector basis *must* contain the off-diagonal τ_1, τ_2 as well as λ_1, λ_2;

so in this "unified theory", it is as natural to include charged gryphonic interactions as it is to include charged weak interactions.

Now that the vector basis and hence the conjugation matrix of the (32 × 32) Clifford algebra have been chosen, we may define the bar conjugate of the 32-component fields given by (1.3) as

$$\bar{\psi} = \psi^{\dagger}\lambda_4\rho_1\tau_3\gamma\,, \quad \bar{\chi} = \chi^{\dagger}\lambda_4\rho_1\tau_3\gamma.$$

The bar conjugation $\bar{\Psi}$ of the 16-component field is given by (I, 2.7) and (I, 2.8), and the bar conjugate of Ψ' is, as in (I, 5.5),

$$\bar{\Psi}' \equiv \Psi'^{\dagger}\rho_1\tau_3\gamma = -(\Psi^{T}\rho_1\tau_3\gamma^{*})\rho_2\tau_4 C.$$

So $\bar{\psi}$ becomes

$$\bar{\psi} = n\begin{bmatrix}\bar{\Psi} & (\tau_1\bar{\Psi}')\end{bmatrix}$$

$$= n\begin{bmatrix}\bar{\varepsilon}_L & -\bar{\nu}'_L & \bar{\varepsilon}_R & -\bar{\nu}'_R & \bar{\nu}_L & -\bar{\varepsilon}'_L & \bar{\nu}_R & -\bar{\varepsilon}'_R\end{bmatrix}, \quad (1.10a)$$

and similarly

$$\bar{\chi} = n\begin{bmatrix}\bar{d}_L & -\bar{u}'_L & \bar{d}_R & -\bar{u}'_R & \bar{u}_L & -\bar{d}'_L & u_R & -\bar{d}'_R\end{bmatrix}. \quad (1.10b)$$

Following the pattern laid down in I, the 10-dimensional basis $\Gamma_1, \Gamma_2, \ldots, \Gamma_{10}$ is associated with the theory of leptons; the bivector formed from Γ_{10} and Γ_{11}, namely $\lambda_3\rho_1\tau_4 I$ (a generalization of $\rho_1\tau_4 I$ of I) is used to transform from the lepton representation to the dion representations of the algebra. In the lepton representation, the six vectors $\Gamma_5, \Gamma_6, \ldots, \Gamma_{10}$ divide into two groups of three: (Γ_8, Γ_9, Γ_{10}) define the subspace whose bivectors give rise to the SU(1,1) algebra, and are the generalizations of (Γ_6, Γ_7, Γ_8) of I. The vectors (Γ_5, Γ_6, Γ_7) define a second 3-dimensional subspace; the bivectors in this space form the SU(2) algebra of the GSW theory. So the "strong" SU(1,1) algebra and the weak SU(2) algebra apppear in an almost symmetrical way in our theory; the main difference is that SU(2) governs only one helicity. In order to emphasize this near-symmetry, we shall define in § 2 a helicity-symmetric model. As we noted in I, the bivectors generated by Γ_1,

Γ_2, Γ_3, Γ_4 form the SL(2,C) algebra; we can contemplate the possibility of gauging this group also, so that the 10-dimensional space breaks up into three spaces whose bivectors generate the SL(2,C), SU(2) and SU(1,1) groups.

In Table 2, we list the bivectors used to generate the lepton gauge transformations, indicating in brackets the indices of the vectors forming the bivector; we add to the list the pseudoscalar of the space spanned by Γ_5, ... , Γ_{10}, which is also used as a generator. Under the heading "dion", we list the matrices formed by replacing Γ_{10} by Γ_{11} wherever it occurs; these matrices are used to generate dion gauge transformations. We note that the SU(2) generators, which do not involve Γ_{10} or Γ_{11}, are the same for leptons and dions. The "strong space" bivectors are generalizations of the bivectors used in I.

In the GSW theory, the weak interactions distinguish between helicity states; we therefore need to define projection operators which eliminate states of a given handedness. The non-zero interaction matrix elements are all of the form (1.6b) or (1.6c), the particle spinors being any of ε, ν, d, u. If we wish to eliminate (right-handed) matrix elements of the form $\bar{\nu}_L M \varepsilon_R$ and $\bar{\nu}_L\, i\gamma_5 M\, \varepsilon_R$, while preserving (left-handed) elements of the form $\bar{\nu}_R\, M\, \varepsilon_L$ and $\bar{\nu}_R\, i\gamma_5\, M\, \varepsilon_L$, then (1.6b) and (1.6c) tell us that we must preserve the 4-component spinors ε_L, ν_L, ε'_R, ν'_R in (1.4a), and likewise d_L, u_L, d'_R, u'_R in (1.4b). If

$$I_{32} \equiv \lambda_4\, \rho_4\, \tau_4\, I \tag{1.11}$$

is the (32 × 32) unit matrix, and we define

$$h_+ \equiv \tfrac{1}{2}\left(I_{32} + i\,\lambda_4\, \rho_4\, \tau_3\, \gamma_5\right) \tag{1.12a}$$

$$h_- \equiv \tfrac{1}{2}\left(I_{32} - i\,\lambda_4\, \rho_4\, \tau_3\, \gamma_5\right), \tag{1.12b}$$

then

$$\psi_+ \equiv h_+\psi = n\left[\varepsilon_L \;\; 0 \;\; 0 \;\; \nu'_R \;\; \nu_L \;\; 0 \;\; 0 \;\; \varepsilon'_R\right]^T \tag{1.13a}$$

$$\psi_- \equiv h_-\psi = n\left[0 \;\; \nu'_L \;\; \varepsilon_R \;\; 0 \;\; 0 \;\; \varepsilon'_L \;\; \nu_R \;\; 0\right]^T, \tag{1.13b}$$

projecting out the correct 4-component spinors; χ_+, χ_- are defined similarly from (1.4b). From (1.10a) and (1.12), we likewise define

$$\bar{\psi}_+ \equiv \bar{\psi}\, h_+ = n\left[\bar{\varepsilon}_L \;\; 0 \;\; 0 \; -\bar{\nu}_R' \;\; \bar{\nu}_L \;\; 0 \;\; 0 \; -\bar{\varepsilon}_R'\right]^T \tag{1.14a}$$

$$\bar{\psi}_- \equiv \bar{\psi}\, h_- = n\left[0 \; -\bar{\nu}_L' \;\; \bar{\varepsilon}_R \;\; 0 \;\; 0 \; -\bar{\varepsilon}_L' \;\; \bar{\nu}_R \;\; 0\right]^T, \tag{1.14b}$$

with similar definitions for $\bar{\chi}_+$, $\bar{\chi}_-$. We note that matrix elements of the forms (1.6b), (1.6c) will arise from pairs of states of the form $(\bar{\psi}_-, \psi_+)$ and $(\bar{\psi}_+, \psi_-)$. The τ_3 factors in (1.12) are significant, since they do not commute with the bi-vectors $\pm i\lambda_4\rho_2\tau_i I, \pm i\lambda_3\rho_3\tau_i I$ $(i = 1, 2)$ of Table 2. As we shall see in § 3, this fact requires us to study in more detail the gauge theory for the helicity-dependent GSW interactions, compared with the helicity-symmetric theory of § 2.

One important advantage of spin gauge theories is that we can include in the Lagrangian gauge invariant mass terms. As in (I, 2.13a, b, d), the gauge transformations on the 32-component spinors and on the basis vectors will be of the form

$$\psi \to R\psi\,, \quad \bar{\psi} \to \bar{\psi} R^{-1} \tag{1.15a}$$

and

$$T_r \to R\, T_r\, R^{-1}\,, \tag{1.15b}$$

so that the conjugation matrix transforms by

$$\lambda_4 \rho_1 \tau_3 \gamma \to (R^\dagger)^{-1} \lambda_4 \rho_1 \tau_3 \gamma R^{-1}. \tag{1.15c}$$

Then, if E is an element of the Clifford algebra (apart from factors ±1 or ±i, a product of basis vectors),

$$\bar{\psi}\, E\, \psi \tag{1.16}$$

is gauge invariant. This fact will allow us to include gauge invariant fermion mass terms in the Langrangian; the specific matrices E used to generate mass terms for the leptons and dions will be considered in Section 3.

2. HELICITY SYMMETRIC UNIFIED THEORY

In order to set up, within the (32 × 32) matrix Clifford algebra, a spin gauge theory unifying the weak, electromagnetic and strong interactions, we shall first reconstruct the electrostrong theory of I by using generators in the higher dimensional algebra. This will involve using the same ρ and τ matrices in the generators as in those of the gauge transformation (I, 2.14), but they must now be combined with the appropriate λ matrix so that the resultant matrix element in the interaction Lagrangian is non-zero on decomposition. The λ matrix will necessarily be diagonal. We decide whether λ_3 or λ_4 is the correct choice by explicit consideration of matrix elements of various operators between states of the form (1.10) and (1.4). For example,

$$\begin{aligned}
&\bar{\psi}\lambda_3\rho_1\tau_4\not{W}\psi \\
&= \frac{1}{2}\begin{bmatrix}\bar{\varepsilon}_L & -\bar{\nu}'_L & \bar{\varepsilon}_R & -\bar{\nu}'_R\end{bmatrix}\begin{bmatrix} & & \not{W} & \\ & & & \not{W} \\ \not{W} & & & \\ & \not{W} & & \end{bmatrix}\begin{bmatrix}\varepsilon_L \\ \nu'_L \\ \varepsilon_R \\ \nu'_R\end{bmatrix} \\
&\quad - \frac{1}{2}\begin{bmatrix}\bar{\nu}_L & -\bar{\varepsilon}'_L & \bar{\nu}_R & -\bar{\varepsilon}'_R\end{bmatrix}\begin{bmatrix} & & \not{W} & \\ & & & \not{W} \\ \not{W} & & & \\ & \not{W} & & \end{bmatrix}\begin{bmatrix}\nu_L \\ \varepsilon'_L \\ \nu_R \\ e'_R\end{bmatrix} \\
&= \frac{1}{2}\left[\bar{\varepsilon}\not{W}\varepsilon - \bar{\nu}'\not{W}\nu' - \bar{\nu}\not{W}\nu + \bar{\varepsilon}'\not{W}\varepsilon'\right] = \bar{\varepsilon}\not{W}\varepsilon - \bar{\nu}\not{W}\nu. \qquad (2.1a)
\end{aligned}$$

However, it is easy to see that replacing τ_4 by τ_3 in (2.1a) leads to

$$\begin{aligned}
\bar{\psi}\lambda_3\rho_1\tau_3\not{W}\psi &= \frac{1}{2}\left[\bar{\varepsilon}\not{W}\varepsilon + \bar{\nu}'\not{W}\nu' - \bar{\nu}\not{W}\nu - \bar{\varepsilon}'\not{W}\varepsilon'\right] \\
&= 0\,, \qquad (2.1b)
\end{aligned}$$

and hence, by using (I, 2.20b, c),

$$\begin{aligned}
\bar{\psi}\lambda_4\rho_1\tau_3\not{W}\psi &= -\bar{\psi}\lambda_4\rho_2\tau_3\gamma_5\not{W}\psi \\
&= \bar{\varepsilon}\not{W}\varepsilon + \bar{\nu}\not{W}\nu. \qquad (2.1c)
\end{aligned}$$

Furthermore, from (I, 2.10c), (I, 2.21) and (I, 2.37a), we see that

$$\begin{aligned} &\bar{\chi}\,\lambda_3\,\rho_2\,\varepsilon_{3ij}\,\tau_i\,\not{B}_j\,\chi \\ &= \frac{\sqrt{2}}{2}\left(\bar{d}\,i\gamma_5\not{B}_+u' + \bar{u}'\,i\gamma_5\not{B}_-d - \bar{u}\,i\gamma_5\not{B}_+d' - \bar{d}'\,i\gamma_5\not{B}_-u\right) \\ &= \sqrt{2}\left(\bar{d}\,i\gamma_5\not{B}_+u' + \bar{u}'\,i\gamma_5\not{B}_-d\right), \end{aligned} \tag{2.2a}$$

and, for any suffix a,

$$\bar{\psi}\,\lambda_3\,\rho_3\,\tau_a\not{A}\,\psi = \bar{\psi}\,\lambda_4\,\rho_3\,\tau_a\not{A}\,\psi = 0\,. \tag{2.2b}$$

Bearing in mind that the dion representation of the algebra is a transformation of the lepton representation by the operator

$$T_\eta = \exp\left(-i\lambda_3\,\rho_1\,\tau_4\,I\,\eta\right), \tag{2.3}$$

we choose the charged gryphon interactions of the leptons to be

$$-\frac{1}{2}\,g\,\bar{\psi}\,\lambda_4\,\rho_3\,\varepsilon_{3ij}\,\tau_i\not{B}_j\,\psi\,. \tag{2.4}$$

These terms are zero on decomposition, but on applying T_η, they have the same form as (2.2a), as we require for the interaction of the dions with the charged gryphonic fields.

By combining together the results of (2.1) and (2.4), we find the generalization to 32 components of the lepton interaction terms in (I, 2.19). They are

$$\begin{aligned} &\frac{1}{2}\,e\,\bar{\psi}\,\lambda_3\,\rho_1\,\tau_4\not{A}\,\psi + \frac{1}{2}\,g'\,\bar{\psi}\,\lambda_4\,\rho_2\,\tau_3\,\gamma_5\not{W}\,\psi \\ &+ \frac{1}{2}\,g\,\bar{\psi}\left(-\lambda_4\,\rho_3\,\varepsilon_{3ij}\,\tau_i\not{B}_j + \lambda_4\,\rho_1\,\tau_3\not{B}_3\right)\psi\,. \end{aligned} \tag{2.5}$$

The matrices $\lambda_4\rho_1\tau_4\gamma_\mu$ are used to define the Lagrangian for the free ψ field, which is

$$L_0^{\ell} = \frac{1}{2}\left[\overline{\psi}\, i \lambda_4 \rho_1 \tau_4 (\not\partial \psi) - (\partial_\mu \psi)^{+} \Gamma\, i \lambda_4 \rho_1 \tau_4 \gamma^\mu \psi\right]$$

$$= \frac{1}{4} i \left[\overline{\varepsilon}_L \;\; -\overline{\nu}_L' \;\; \overline{\varepsilon}_R \;\; -\overline{\nu}_R'\right] \begin{bmatrix} & & \gamma^\mu & \\ & & & \gamma^\mu \\ \gamma^\mu & & & \\ & \gamma^\mu & & \end{bmatrix} \begin{bmatrix} \partial_\mu \varepsilon_L \\ \partial_\mu \nu_L' \\ \partial_\mu \varepsilon_R \\ \partial_\mu \nu_R' \end{bmatrix} + \text{h.c.}$$

$$+ \frac{1}{4} i \left[\overline{\nu}_L \;\; -\overline{\varepsilon}_L' \;\; \overline{\nu}_R \;\; -\overline{\varepsilon}_R'\right] \begin{bmatrix} & & \gamma^\mu & \\ & & & \gamma^\mu \\ \gamma^\mu & & & \\ & \gamma^\mu & & \end{bmatrix} \begin{bmatrix} \partial_\mu \nu_L \\ \partial_\mu \varepsilon_L' \\ \partial_\mu \nu_R \\ \partial_\mu \varepsilon_R' \end{bmatrix} + \text{h.c.}$$

$$= \frac{1}{4}\left[\overline{\varepsilon}\, i \gamma^\mu (\partial_\mu \varepsilon) - \overline{\nu}'\, i \gamma^\mu (\partial_\mu \nu') + \overline{\nu}\, i \gamma^\mu (\partial_\mu \nu) - \overline{\varepsilon}'\, i \gamma^\mu (\partial_\mu \varepsilon')\right] + \text{h.c.}$$

$$= \frac{1}{2}\left[\overline{\varepsilon}\, i \gamma^\mu (\partial_\mu \varepsilon) - (\partial_\mu \varepsilon^{\dagger}) \gamma\, i \gamma^\mu \varepsilon + \overline{\nu}\, i \gamma^\mu (\partial_\mu \nu) - (\partial_\mu \nu^{\dagger}) \gamma_i \gamma^\mu \nu\right]. \tag{2.6}$$

The interaction terms (2.5) are a direct consequence of the requirement that the free Lagrangian (2.6) be invariant under the transformations (1.15) where

$$R = \exp\left\{-\frac{1}{2} i \left[e \lambda_3 \rho_4 \tau_4 I\, \Theta(x) - i g' \lambda_4 \rho_3 \tau_3 \gamma_5 \Lambda(x) + g\left(i \lambda_4 \rho_2 \varepsilon_{3ij} \tau_i \Lambda_j(x) + \lambda_4 \rho_4 \tau_3 \Lambda_3(x)\right)\right]\right\}, \tag{2.7}$$

which is the 32 component analogue of (I, 2.14).

The mixing of the W^μ and B_3^μ potentials according to (I, 2.23) and the conditions (I, 2.25) ensure that the terms (2.5) produce the correct electromagnetic interactions and zero strong interactions for the lepton system. Furthermore, if we transform the representation of the algebra using T_η, defined by (2.3), and if we put $2\eta = \theta_0$, the interaction terms (2.5) become

$$\frac{1}{2} e\, \overline{\chi} \lambda_3 \rho_1 \tau_4 \not{A} \chi + \frac{1}{2} g'\, \overline{\chi} (\lambda_4 \rho_2)' \tau_3 \gamma_5 \not{W} \chi + \frac{1}{2} g\, \overline{\chi}\left\{-(\lambda_4 \rho_3)' \varepsilon_{3ij} \tau_i \not{B}_j + \lambda_4 \rho_1 \tau_3 \not{B}_3\right\} \chi, \tag{2.8}$$

where

$$(\lambda_4\rho_2)' = \lambda_4\rho_2 \cos\theta_0 + \lambda_3\rho_3 \sin\theta_0 \tag{2.9a}$$

$$(\lambda_4\rho_3)' = \lambda_4\rho_3 \cos\theta_0 - \lambda_3\rho_2 \sin\theta_0 . \tag{2.9b}$$

By choosing $\theta_0 = \pi/2$, the terms (2.8) decompose to the dion interaction terms evaluated in $\underline{I}$.

From Table 2, we see that in (2.7) the matrices

$$\{T_a ; a=1,2,3\} = \left\{-\tfrac{1}{2}\lambda_4\rho_2\tau_2 I, \tfrac{1}{2}\lambda_4\rho_2\tau_1 I, -\tfrac{1}{2}i\lambda_4\rho_4\tau_3 I\right\} \tag{2.10}$$

are bivectors of the algebra generating the group SU(1,1), which preserves the metric on the 3-dimensional subspace spanned by $(\Gamma_8, \Gamma_9, \Gamma_{10})$, with signature $(-1, -1, +1)$. The generator of the W^μ potential is the pseudoscalar in the space spanned by $\Gamma_5, \ldots, \Gamma_{10}$ and the generator of the A^μ potential is a bivector representing rotations in the plane spanned by (Γ_5, Γ_6).

Rather than consider the matrix $\lambda_3\rho_4\tau_4 I$ in (2.7) as the generator of a U(1) transformation, let us consider it as one member of a set generating rotations in the 3-dimensional subspace spanned by $(\Gamma_5, \Gamma_6, \Gamma_7)$ with signature $(-1, -1, -1)$. Then it will generate the neutral field partner of a pair of charged fields generated by $i(\lambda_1 \pm i\lambda_2)\rho_3\tau_3\gamma_5$. The set of bivectors

$$\{U_a ; a=1,2,3\} = \left\{\tfrac{1}{2}\lambda_1\rho_3\tau_3\gamma_5, \tfrac{1}{2}\lambda_2\rho_3\tau_3\gamma_5, -\tfrac{1}{2}i\lambda_3\rho_4\tau_4 I\right\} \tag{2.11}$$

form a set of infinitesimal generators of the group SU(2).

We consider the interaction terms which result from requiring that the Lagrangian (2.6) be invariant under transformations (1.15), where R is now given, not by (2.7), but by

$$R = \exp\left\{-\tfrac{1}{2}i\left[\alpha\left(i\lambda_i\rho_3\tau_3\gamma_5\,\Theta_i(x) + \lambda_3\rho_4\tau_4 I\,\Theta_3(x)\right)\right.\right.$$
$$\left.\left.-ig'\lambda_4\rho_3\tau_3\gamma_5\,\Lambda(x) + g\left(i\lambda_4\rho_2\varepsilon_{3ij}\tau_i\Lambda_j(x) + \lambda_4\rho_4\tau_3\Lambda_3(x)\right)\right]\right\} \tag{2.12a}$$

$$\equiv \exp\left(\alpha U_a \Theta_a\right)\exp\left(-ig'\lambda_4\rho_4\tau_3\gamma_5\Lambda + gT_a\Lambda_a\right), \tag{2.12b}$$

with summation over the suffix values i, j = 1, 2 and a = 1, 2, 3. We note that the pseudoscalar $\lambda_4\rho_3\tau_3\gamma_5$ is the only element of the Clifford algebra which commutes with all the generators of the SU(1,1) and SU(2) groups, and that these two sets of generators commute, belonging to mutually exclusive subspaces. This is why R can be factorized into exponentials containing SU(2), SU(1,1) and pseudoscalar generators separately. We can ensure invariance of the Lagrangian L_0^ℓ provided that, as in I, derivatives are replaced by covariant derivatives of the form

$$D\psi = \lambda_4\rho_1\tau_4\gamma_\mu\left\{\partial^\mu + \left[\alpha U_a W_a^\mu - \tfrac{1}{2}g'\lambda_4\rho_3\tau_3\gamma_5 W^\mu\right.\right.$$
$$\left.\left. + gT_a B_a^\mu\right]\right\}\psi. \tag{2.13}$$

The finite gauge transformations on the potentials are given by

$$W^\mu \to W^\mu - \partial^\mu\Lambda, \tag{2.14a}$$

$$T_a B_a^\mu \to RT_a B_a^\mu R^{-1} + g^{-1}\exp\left(gT_b\Lambda_b\right)\partial^\mu\left[\exp\left(-gT_c\Lambda_c\right)\right]$$
$$= \exp\left(gT_b\Lambda_b\right)\left(T_a B_a^\mu + g^{-1}\partial^\mu\right)\exp\left(-gT_c\Lambda_c\right), \tag{2.14b}$$

$$U_a W_a^\mu \to R\,U_a W_a^\mu R^{-1} + \alpha^{-1}\exp\left(\alpha U_b\Theta_b\right)\partial^\mu\left[\exp\left(-\alpha U_c\Theta_c\right)\right]$$
$$= \exp\left(\alpha U_b\Theta_b\right)\left(U_a W_a^\mu + \alpha^{-1}\partial^\mu\right)\exp\left(-\alpha U_c\Theta_c\right), \tag{2.14c}$$

which are derived using the commutation relations

$$\left[T_a, U_b\right] = 0 \tag{2.15a}$$

$$\left[T_a, \lambda_4 \rho_3 \tau_3 \gamma_5\right] = \left[U_a, \lambda_4 \rho_3 \tau_3 \gamma_5\right] = 0, \tag{2.15b}$$

for a, b = 1, 2, 3.

The insertion of covariant derivatives in the Lagrangian (2.6) generates the invariant Lagrangian for the ψ field, given by

$$\begin{aligned} L_1^{\ell} = L_0^{\ell} &+ \tfrac{1}{2} g' \bar{\psi} \lambda_4 \rho_2 \tau_3 \gamma_5 \not{W} \psi \\ &+ \tfrac{1}{2} g \bar{\psi} \left(-\lambda_4 \rho_3 \varepsilon_{3ij} \tau_i \not{B}_j + \lambda_4 \rho_1 \tau_3 \not{B}_3\right) \psi \\ &+ \tfrac{1}{2} \alpha \bar{\psi} \left(-\lambda_i \rho_2 \tau_3 \gamma_5 \not{W}_i + \lambda_3 \rho_1 \tau_4 \not{W}_3\right) \psi . \end{aligned} \tag{2.16}$$

Now in (2.16) the matrix elements containing W_i^{μ} decompose to give

$$\begin{aligned} &-\bar{\psi} \lambda_i \rho_2 \tau_3 \gamma_5 \not{W}_i \psi \\ &= \frac{1}{2} \left[\bar{\varepsilon}_L \; -\bar{\nu}_L' \; \bar{\varepsilon}_R \; -\bar{\nu}_R'\right] i\gamma_5 \begin{bmatrix} & & \not{W}_1 - i\not{W}_2 & \\ & & & -\not{W}_1 + i\not{W}_2 \\ -\not{W}_1 + i\not{W}_2 & & & \\ & \not{W}_1 - i\not{W}_2 & & \end{bmatrix} \begin{bmatrix} \nu_L \\ \varepsilon_L' \\ \nu_R \\ \varepsilon_R' \end{bmatrix} \\ &+ \frac{1}{2} \left[\bar{\nu}_L \; -\bar{\varepsilon}_L' \; \bar{\nu}_R \; -\bar{\varepsilon}_R'\right] i\gamma_5 \begin{bmatrix} & & \not{W}_1 + i\not{W}_2 & \\ & & & -\not{W}_1 - i\not{W}_2 \\ -\not{W}_1 - i\not{W}_2 & & & \\ & \not{W}_1 + i\not{W}_2 & & \end{bmatrix} \begin{bmatrix} \varepsilon_L \\ \nu_L' \\ \varepsilon_R \\ \nu_R' \end{bmatrix} \\ &= \frac{\sqrt{2}}{2} \left(\bar{\varepsilon} \not{W}_- \nu + \bar{\nu}' \not{W}_- \varepsilon' + \bar{\nu} \not{W}_+ \varepsilon + \bar{\varepsilon}' \not{W}_+ \nu'\right) \\ &= \sqrt{2} \left(\bar{\varepsilon} \not{W}_- \nu + \bar{\nu} \not{W}_+ \varepsilon\right), \end{aligned} \tag{2.17}$$

where $W_{\pm}^{\mu} = 1/\sqrt{2}\ (W_1^{\mu} \pm i W_2^{\mu})$.

By using (2.17) together with (2.1) and (2.4), we may write the Lagrangian (2.16) in terms of 4-component spinors as

$$L_1^{\ell} = L_0^{\ell} + \frac{1}{2}\bar{\varepsilon}\left(\alpha \not{W}_3 + g \not{B}_3 - g' \not{W}\right)\varepsilon + \frac{1}{2}\bar{\nu}\left(-\alpha \not{W}_3 + g \not{B}_3 - g' \not{W}\right)\nu + \frac{1}{2}\sqrt{2}\alpha\left(\bar{\varepsilon}\not{W}_-\nu + \bar{\nu}\not{W}_+\varepsilon\right). \tag{2.18}$$

As in I, we allow the potentials B_3^{μ} and W^{μ} to mix, producing G_0^{μ} and, instead of A^{μ} in I, an "intermediate" potential W_4^{μ}:

$$W^{\mu} = \cosh\xi \; W_4^{\mu} + \sinh\xi \; G_0^{\mu} \tag{2.19a}$$

$$B_3^{\mu} = \sinh\xi \; W_4^{\mu} + \cosh\xi \; G_0^{\mu}. \tag{2.19b}$$

Then, if we modify the conditions (I, 2.25) so that they are now written as

$$g \sinh\xi - g' \cosh\xi = \alpha' \tag{2.20a}$$

$$g \cosh\xi - g' \sinh\xi = 0, \tag{2.20b}$$

the interaction terms in (2.18) are

$$\frac{1}{2}\bar{\varepsilon}\left(\alpha \not{W}_3 + \alpha' \not{W}_4\right)\varepsilon + \frac{1}{2}\bar{\nu}\left(-\alpha \not{W}_3 + \alpha' \not{W}_4\right)\nu + \frac{1}{2}\sqrt{2}\alpha\left(\bar{\varepsilon}\not{W}_-\nu + \bar{\nu}\not{W}_+\varepsilon\right). \tag{2.21}$$

The form of interaction terms (2.21) is reminiscent of the GSW interaction terms of the electron and neutrino. In the standard model [Abers and Lee, 1973], W_3^{μ}, W_4^{μ}, α and α' are denoted by A_3^{μ}, B^{μ}, g and g′ respectively. The obvious difference between (2.21) and the corresponding terms in the GSW model is the symmetry between helicity states. The interaction terms (2.21) are the result of "filling in" all the missing helicity states, in particular the right-handed neutrino states.

To produce the correct "symmetric" electroweak interactions of the leptons, we must write (2.21) in terms of the physical fields A^{μ} and Z^{μ} by defining

$$W_3^\mu = \sin\theta_w \, A^\mu - \cos\theta_w \, Z^\mu \tag{2.22a}$$

$$W_4^\mu = \cos\theta_w \, A^\mu + \sin\theta_w \, Z^\mu , \tag{2.22b}$$

where θ_w is the GSW mixing angle satisfying

$$\alpha \sin\theta_w = \alpha' \cos\theta_w = e . \tag{2.23}$$

The interaction terms become

$$\begin{aligned}
& e\,\bar{\varepsilon}\,\not{A}\,\varepsilon + \tfrac{1}{2}\sqrt{2}\,\alpha\left(\bar{\varepsilon}\,\not{W}_-\,\nu + \bar{\nu}\,\not{W}_+\,\varepsilon\right) \\
& + \frac{1}{2}\frac{\alpha}{\cos\theta_w}\Big\{\bar{\varepsilon}\left(-\cos^2\theta_w + \sin^2\theta_w\right)\not{Z}\,\varepsilon \\
& \qquad + \bar{\nu}\left(\cos^2\theta_w + \sin^2\theta_w\right)\not{Z}\,\nu\Big\} \\
& = e\,\bar{\varepsilon}\,\not{A}\,\varepsilon + \tfrac{1}{2}\sqrt{2}\,\alpha\left(\bar{\varepsilon}\,\not{W}_-\,\nu + \bar{\nu}\,\not{W}_+\,\varepsilon\right) \\
& + \frac{1}{2}\frac{\alpha}{\cos\theta_w}\left(\bar{\nu}\,\not{Z}\,\nu - \bar{\varepsilon}\,\not{Z}\,\varepsilon + 2\sin^2\theta_w\,\bar{\varepsilon}\,\not{Z}\,\varepsilon\right).
\end{aligned} \tag{2.24}$$

The terms (2.24) represent the helicity symmetric electroweak interactions of the electron and its neutrino.

To obtain the corresponding terms for the dions, we consider the effect of changing the representation of the Clifford algebra on the Lagrangian (2.16). Since the bivector generators (2.11) of the W_a^μ potentials are invariant under the change of representation, we can deduce from (2.8) and (2.16) that the Lagrangian for the χ field is given by

$$\begin{aligned}
L_1^\delta = {} & L_0^\delta + \tfrac{1}{2} g' \cos\theta_0\,\bar{\chi}\,\lambda_4\,\rho_2\,\tau_3\,\gamma_5\,\not{B}\,\chi \\
& + \tfrac{1}{2} g \sin\theta_0\,\bar{\chi}\,\lambda_3\,\rho_2\,\epsilon_{3ij}\,\tau_i\,\not{B}_j\,\chi + \tfrac{1}{2} g\,\bar{\chi}\,\lambda_4\,\rho_1\,\tau_3\,\not{B}_3\,\chi \\
& + \tfrac{1}{2}\alpha\,\bar{\chi}\left(-\lambda_i\,\rho_2\,\tau_3\,\gamma_5\,\not{W}_i + \lambda_3\,\rho_1\,\tau_4\,\not{W}_3\right)\chi ,
\end{aligned} \tag{2.25}$$

where

$$L_0^{\delta} = \frac{1}{2}\left[\bar{\chi} i \lambda_4 \rho_1 \tau_4 (\delta\chi) - (\partial_\mu \chi^+) T i \lambda_4 \rho_1 \tau_4 \gamma^\mu \chi\right]. \tag{2.26}$$

The conditions (I, 2.42) and (I, 3.23) must be modified to become

$$g \sinh \xi - g' \cos \theta_0 \cosh \xi = -\frac{1}{3}\alpha' \tag{2.27a}$$

$$g \sinh \xi \cos \theta_W = -\frac{1}{3} e = -\frac{1}{3}\alpha' \cos \theta_W , \tag{2.27b}$$

which, together with (2.20), enable us to determine the four unknown parameters. The constants θ_0 and ξ have the same values as in I, namely

$$\theta_0 = \frac{1}{2}\pi , \quad \xi = \tanh^{-1}\frac{1}{2} , \tag{2.28a}$$

while g and g′ are now given by

$$g/\alpha' = -1/\sqrt{3} , \quad g'/\alpha' = -2/\sqrt{3} . \tag{2.28b}$$

By using (2.2a), (2.19), (2.22) and the solutions (2.28), the interaction terms in (2.25) become

$$\frac{1}{2}\bar{d}\left(\alpha \not{\psi}_3 - \frac{1}{3}\alpha' \not{\psi}_4\right) d + \frac{1}{2}\bar{u}\left(-\alpha\not{\psi}_3 - \frac{1}{3}\alpha'\not{\psi}_4\right) u$$
$$-\frac{1}{3}\alpha'\left(\bar{d}\not{A}_0 d + \bar{u}\not{A}_0 u\right) + \frac{1}{2}\sqrt{2}\,\alpha\left(\bar{d}\not{\psi}_- u + \bar{u}\not{\psi}_+ d\right)$$
$$+\frac{1}{2}\sqrt{2}\, g\left(\bar{d} i\gamma_5 \not{B}_+ u' + \bar{u}' i\gamma_5 \not{B}_- d\right)$$
$$= \frac{1}{3} e \bar{d}\not{A} d - \frac{2}{3} e \bar{u}\not{A} u - \frac{1}{3}\frac{e}{\cos\theta_W}\left(\bar{d}\not{A}_0 d + \bar{u}\not{A}_0 u\right)$$
$$+\frac{1}{2}\frac{\alpha}{\cos\theta_W}\Big[\bar{d}\left(-\cos^2\theta_W - \frac{1}{3}\sin^2\theta_W\right)\not{Z} d$$
$$+ \bar{u}\left(\cos^2\theta_W - \frac{1}{3}\sin^2\theta_W\right)\not{Z} u\Big] + \frac{1}{2}\sqrt{2}\,\alpha\left(\bar{d}\not{\psi}_- u + \bar{u}\not{\psi}_+ d\right)$$
$$+\frac{1}{2}\sqrt{2}\, g\left(\bar{d} i\gamma_5 \not{B}_+ u' + \bar{u}' i\gamma_5 \not{B}_- d\right)$$

$$= \frac{1}{3} e \bar{d} \not{A} d - \frac{2}{3} e \bar{u} \not{A} u - \frac{1}{3} \frac{e}{\cos\theta_W} \left(\bar{d} \not{G}_0 d + \bar{u} \not{G}_0 u \right)$$
$$+ \frac{1}{2} \frac{\alpha}{\cos\theta_W} \left(-\bar{d} \not{Z} d + \frac{2}{3} \sin^2\theta_W \bar{d} \not{Z} d + \bar{u} \not{Z} u \right.$$
$$\left. - \frac{4}{3} \sin^2\theta_W \bar{u} \not{Z} u \right) + \frac{1}{2} \sqrt{2} \alpha \left(\bar{d} \not{W}_- u + \bar{u} \not{W}_+ d \right)$$
$$+ \frac{1}{2} \sqrt{2} g \left(\bar{d} i\gamma_5 \not{B}_+ u' + \bar{u}' i\gamma_5 \not{B}_- d \right). \qquad (2.29)$$

The terms (2.29) correspond to strong and electroweak interactions of the up and down dions, with symmetry between helicity states.

In contrast to the definition of W_+ and W_-, we adopted an unusual sign convention in the definition (I, 2.38) of B_+ and B_-. The differences of sign convention show up in the last two terms of (2.29), but we can see that B_+ and W_+ both correspond to the annihilation of positive charge, while B_- and W_- correspond to the creation of positive charge. The apparently inconsistent sign conventions are thus justified.

3. UNIFICATION OF ELECTROSTRONG AND GSW INTERACTIONS

To truly incorporate the weak interactions of the GSW model into the electrostrong spin gauge theory of I, we must remove the symmetry between left-handed and right-handed helicity states in the model of Section 2. The $W^\mu_\pm$ and Z^μ interaction terms in (2.24) and (2.29) have a suitable form to represent the interactions of leptons and dions with the intermediate charged and neutral vector bosons of the GSW model, but with the obvious disadvantage that they are helicity symmetric. For instance, the interaction terms (2.24) should be

$$e \bar{\varepsilon} \not{A} \varepsilon + \frac{1}{2} \sqrt{2} \alpha \left(\bar{\varepsilon}_R \not{W}_- \nu_L + \bar{\nu}_R \not{W}_+ \varepsilon_L \right)$$
$$+ \frac{1}{2} \frac{\alpha}{\cos\theta_W} \left(\bar{\nu}_R \not{Z} \nu_L - \bar{\varepsilon}_R \not{Z} \varepsilon_L + 2 \sin^2\theta_W \bar{\varepsilon} \not{Z} \varepsilon \right)$$
$$= e \bar{\varepsilon} \not{A} \varepsilon + \frac{1}{4} \sqrt{2} \alpha \left[\bar{\varepsilon} (I - i\gamma_5) \not{W}_- \nu + \bar{\nu} (I - i\gamma_5) \not{W}_+ \varepsilon \right]$$
$$+ \frac{1}{4} \frac{\alpha}{\cos\theta_W} \left[\bar{\nu} (I - i\gamma_5) \not{Z} \nu - \bar{\varepsilon} (I - i\gamma_5) \not{Z} \varepsilon + 4 \sin^2\theta_W \bar{\varepsilon} \not{Z} \varepsilon \right] \qquad (3.1)$$

At first sight, it seems possible to produce the interaction terms (3.1) by incorporating helicity projection operators (1.12) into the gauge transformation (2.12b). In particular, we would envisage replacing U_a in (2.12b) by h_+U_a in order to generate, in the interaction Lagrangian, terms of the form

$$\begin{aligned}&\bar{\Psi}\, i\lambda_4 \rho_1 \tau_4 \gamma_\mu h_+ U_a W_a^\mu \psi \\ &= \bar{\Psi}\left(-\lambda_i \rho_2 \tau_3 \gamma_5 \not{W}_i + \lambda_3 \rho_1 \tau_4 \not{W}_3\right)\psi_+ ,\end{aligned}$$

where ψ_+ is defined by (1.13a). However, the combination of h_+ and U_a invalidates the commutation relations (2.15a); that is,

$$\left[T_i\, ,\, h_+ U_b\right] \neq 0 \qquad (i=1,2;\ b=1,2,3),$$

and hence the gauge transformations (2.14) would no longer apply.

To incorporate the helicity projection operators into the spin gauge transformation in a well-defined manner, we therefore consider the implications of performing a transformation of the form (1.15), where

$$R = e^M e^N \tag{3.2}$$

and such that M and N do not commute with each other. We firstly consider a transformation on the 32 component spinors and on the vector basis, of the form

$$\left.\begin{aligned}&\psi \to e^N \psi \equiv \psi^0 , \quad \bar{\Psi} \to \bar{\Psi} e^{-N} \equiv \bar{\Psi}^0 , \\ &T_r \to e^N T_r e^{-N} \equiv T_r^0 .\end{aligned}\right\} \tag{3.3}$$

In (3.3), $N \equiv gN_r(x)T_r$, the T_r being some arbitrary choice of generators from the Clifford algebra, but such that their algebra is closed. We know that the free Lagrangian (2.6) will be invariant under the transformation (3.3) provided that we replace derivatives ∂^μ in (2.6) by covariant derivatives

$$D^\mu = \partial^\mu + g T_r B_r^\mu , \tag{3.4}$$

where the gauge potentials transform according to

$$T_r B_r^\mu \to e^N \left\{ T_r B_r^\mu - g^{-1} e^{-N} (\partial^\mu e^N) \right\} e^{-N}. \tag{3.5}$$

The resultant Lagrangian density then has the form

$$\frac{1}{2} \bar{\psi} i T_\mu' D^\mu \psi + h.c. = L_0^\ell + g \bar{\psi} i \lambda_4 \rho_1 \tau_4 \gamma_\mu T_r B_r^\mu \psi. \tag{3.6}$$

Secondly, we consider the effect of applying an additional gauge transformation on the spinors and vector basis given by

$$\psi^o \to e^M \psi^o = e^M (e^N \psi), \; \bar{\psi}^o \to \bar{\psi}^o e^{-M} = (\bar{\psi} e^{-N}) e^{-M}, \tag{3.7a}$$

$$T_r^o \to e^M T_r^o e^{-M} = e^M e^N T_r e^{-N} e^{-M}, \tag{3.7b}$$

where $M \equiv \alpha M_r(x) U_r$, the algebra of U_r also being closed. We demand that the Lagrangian (3.6) be invariant under the new transformation (3.7).

Upon performing the first exponential transformation (3.3), the gauge potentials $T_r B_r^\mu$ take on the form given by (3.5), which is itself an element of the Clifford algebra. Consequently, under the additional transformation (3.7b), $T_r B_r^\mu$ are further transformed to

$$\begin{aligned} T_r B_r^\mu \to & e^M \left\{ e^N \left[T_r B_r^\mu - g^{-1} e^{-N} (\partial^\mu e^N) \right] e^{-N} \right\} e^{-M} \\ & = e^M e^N \left\{ T_r B_r^\mu - g^{-1} e^{-N} (\partial^\mu e^N) \right\} e^{-N} e^{-M}. \end{aligned} \tag{3.8}$$

Therefore, after transforming by (3.7), the Lagrangian (3.6) becomes

$$\begin{aligned} & \frac{1}{2} \bar{\psi} i T_\mu' e^{-N} e^{-M} \left[\partial^\mu + e^M e^N \left\{ g T_r B_r^\mu - e^{-N} (\partial^\mu e^N) \right\} e^{-N} e^{-M} \right] e^M e^N \psi + h.c. \\ & = \frac{1}{2} \bar{\psi} i T_\mu' \left[\partial^\mu + g T_r B_r^\mu + e^{-N} e^{-M} (\partial^\mu e^M e^N) - e^{-N} (\partial^\mu e^N) \right] \psi + h.c. \\ & = \frac{1}{2} \left[\bar{\psi} i T_\mu D^\mu \psi + \bar{\psi} e^{-N} e^{-M} (\partial^\mu e^M) e^N \psi \right] + h.c. \end{aligned} \tag{3.9}$$

The additional terms in (3.9) may be cancelled by replacing D^μ by

$$D^\mu + \alpha U_r W_r^\mu, \tag{3.10}$$

where the additional gauge potentials undergo the transformation

$$U_r W_r^\mu \to e^M e^N U_r W_r^\mu e^{-N} e^{-M} + \alpha^{-1} e^M (\partial^\mu e^{-M})$$
$$= e^M e^N \{U_r W_r^\mu - \alpha^{-1} e^{-N} e^{-M} (\partial^\mu e^M) e^N\} e^{-N} e^{-M}. \tag{3.11}$$

So the new interaction Lagrangian is

$$L_1^\ell = \frac{1}{2} \bar{\psi} i \Gamma_\mu (D^\mu + \alpha U_r W_r^\mu) \psi + h.c.$$
$$= L_0^\ell + \bar{\psi} i \lambda_4 \rho_1 \tau_4 \gamma_\mu (g T_r B_r^\mu + \alpha U_r W_r^\mu) \psi, \tag{3.12}$$

which is invariant under the combined spin gauge transformations (3.7), (3.8) and (3.11).

We now apply the techniques developed above to cope with successive non-commuting transformations to establish a spin gauge theory unifying the electromagnetic, strong and weak interactions. We take the electrostrong theory of I as the foundation for the unified theory, and build upon that the spin gauge transformations of the weak interactions. So we shall impose spin gauge invariance of the electrostrong Lagrangian density under additional helicity dependent transformations. This is equivalent to taking e^N in (3.7) as the finite gauge transformation

$$\exp\{-i g' \lambda_4 \rho_3 \tau_3 \gamma_5 \Lambda(x) + g T_a \Lambda_a(x)\} \tag{3.13a}$$

in (2.12b), generating the electrostrong interactions. To build in the weak interactions, it is necessary to apply an additional exponential transformation containing both h_+ and h_-. So we use the techniques developed above to transform firstly by (3.13a), and then by

$$\exp\{\alpha h_+ U_a \Theta_a(x) + \alpha' h_- U_3 \Theta(x)\}. \tag{3.13b}$$

We obtain as the resultant Lagrangian

$$L_1^{\ell} = L_0^{\ell} + \frac{1}{2} g' \bar{\Psi} \lambda_4 \rho_2 \tau_3 \gamma_5 \not{W} \psi + \frac{1}{2} \alpha' \bar{\Psi} \lambda_3 \rho_1 \tau_4 \not{W}_4 \psi_- $$
$$+ \frac{1}{2} g \bar{\Psi} \left(-\lambda_4 \rho_3 \varepsilon_{3ij} \tau_i \not{B}_j + \lambda_4 \rho_1 \tau_3 \not{B}_3 \right) \psi$$
$$+ \frac{1}{2} \alpha \bar{\Psi} \left(-\lambda_i \rho_2 \tau_3 \gamma_5 \not{W}_i + \lambda_3 \rho_1 \tau_4 \not{W}_3 \right) \psi_+, \tag{3.14}$$

which is invariant under the combined gauge transformations

$$\psi \to \exp\left\{\alpha h_+ U_a \Theta_a(x) + \alpha' h_- U_3 \Theta(x)\right\} \exp\left\{-ig' \lambda_4 \rho_3 \tau_3 \gamma_5 \Lambda + g T_a \Lambda_a\right\} \psi$$
$$\equiv R\psi \tag{3.15a}$$

$$\bar{\Psi} \to \bar{\Psi} \; R^{-1} \tag{3.15b}$$
$$T_r \to R \; T_r \; R^{-1} \tag{3.15c}$$

and

$$W^\mu \to W^\mu - \partial^\mu \Lambda, \tag{3.16a}$$

$$T_a B_a^\mu \to R\left\{T_a B_a^\mu - g^{-1} \exp(-g T_a \Lambda_a)\, \partial^\mu \left[\exp(g T_c \Lambda_c)\right]\right\} R^{-1}, \tag{3.16b}$$

$$h_+ U_a W_a^\mu \to R\left\{h_+ U_a W_a^\mu - \alpha^{-1} \exp(-g T_a \Lambda_a) \exp(-\alpha h_+ U_b \Theta_b) \cdot\right.$$
$$\left. \cdot\, \partial^\mu \left[\exp(\alpha h_+ U_c \Theta_c)\right] \exp(g T_d \Lambda_d)\right\} R^{-1}, \tag{3.16c}$$

$$h_- U_3 W_4^\mu \to R\left\{h_- U_3 W_4^\mu - \exp(-g T_a \Lambda_a)\, \partial^\mu (h_- U_3 \Theta) \exp(g T_b \Lambda_b)\right\} R^{-1}. \tag{3.16d}$$

The decomposition of matrix elements calculated in Section 2 can be applied directly to most terms in the Lagrangian (3.14). The interaction terms containing the helicity projections of the field ψ, defined by $\psi_\pm$ in (1.13), can be decomposed similarly. Using (2.1a) and (1.6b), it is easy to check that

$$\bar{\Psi} \lambda_3 \rho_1 \tau_4 \not{M} \psi_+$$
$$= \frac{1}{2} \left(\bar{\varepsilon}_R \not{M} \varepsilon_L - \bar{\nu}'_L \not{M} \nu'_R - \bar{\nu}_R \not{M} \nu_L + \bar{\varepsilon}'_L \not{M} \varepsilon'_R \right)$$
$$= \bar{\varepsilon}_R \not{M} \varepsilon_L - \bar{\nu}_R \not{M} \nu_L, \tag{3.17a}$$

and also that

$$\bar{\Psi} \lambda_3 \rho_1 \tau_4 \not{M} \psi_- = \bar{\varepsilon}_L \not{M} \varepsilon_R - \bar{\nu}_L \not{M} \nu_R. \tag{3.17b}$$

Furthermore, from (2.17) and (1.6b),

$$
\begin{aligned}
&-\bar{\Psi}\,\lambda_i\,\rho_2\,\tau_3\,\gamma_5\,\not{W}_i\,\psi_+ \\
&= \frac{\sqrt{2}}{2}\left(\bar{\varepsilon}_R \not{W}_- \nu_L + \bar{\nu}'_L \not{W}_- \varepsilon'_R + \bar{\nu}_R \not{W}_+ \varepsilon_L + \bar{\varepsilon}'_L \not{W}_+ \nu'_R\right) \\
&= \sqrt{2}\left(\bar{\varepsilon}_R \not{W}_- \nu_L + \bar{\nu}_R \not{W}_+ \varepsilon_L\right).
\end{aligned}
\tag{3.17c}
$$

Hence, by using (2.1) and (3.17), we may write (3.14) in terms of 4-component spinors as

$$
\begin{aligned}
L_1^\ell = L_0^\ell &+ \tfrac{1}{2}\bar{\varepsilon}\left(g\not{B}_3 - g'\not{W}\right)\varepsilon + \tfrac{1}{2}\bar{\nu}\left(g\not{B}_3 - g'\not{W}\right)\nu \\
&+ \tfrac{1}{2}\alpha\,\bar{\varepsilon}_R \not{W}_3 \varepsilon_L - \tfrac{1}{2}\alpha\,\bar{\nu}_R \not{W}_3 \varepsilon_L + \tfrac{1}{2}\alpha'\,\bar{\varepsilon}_L \not{W}_4 \varepsilon_R \\
&- \tfrac{1}{2}\alpha'\,\bar{\nu}_L \not{W}_4 \nu_R + \tfrac{1}{2}\sqrt{2}\,\alpha\left(\bar{\varepsilon}_R \not{W}_- \nu_L + \bar{\nu}_R \not{W}_+ \varepsilon_L\right).
\end{aligned}
\tag{3.18}
$$

The potentials B_3^μ and W^μ are mixed according to (2.19), although the potential W_4^μ is no longer "intermediate", since it is now explicitly generated by the θ transformation in (3.15). By introducing the mixing (2.19) together with the conditions (2.20) into (3.18), we obtain

$$
\begin{aligned}
L_1^\ell = L_0^\ell &+ \tfrac{1}{2}\alpha'\left(\bar{\varepsilon}\not{W}_4\varepsilon + \bar{\nu}\not{W}_4\nu\right) + \tfrac{1}{2}\alpha\,\bar{\varepsilon}_R\not{W}_3\varepsilon_L \\
&- \tfrac{1}{2}\alpha\,\bar{\nu}_R\not{W}_3\nu_L + \tfrac{1}{2}\alpha'\,\bar{\varepsilon}_L\not{W}_4\varepsilon_R - \tfrac{1}{2}\alpha'\,\bar{\nu}_L\not{W}_4\nu_R \\
&+ \tfrac{1}{2}\sqrt{2}\,\alpha\left(\bar{\varepsilon}_R\not{W}_-\nu_L + \bar{\nu}_R\not{W}_+\varepsilon_L\right).
\end{aligned}
\tag{3.19}
$$

The usual GSW mixing of the W_3^μ and W_4^μ potentials, given by (2.22) and (2.23), then ensures that the Lagrangian (3.19) reduces to the required form (3.1).

It is only the terms in (3.4) containing the W^μ and B_i^μ (i = 1, 2) potentials which are altered under the transformation to the dion representation. Hence it is a simple matter to deduce from (2.27), by analogy with (2.24) and (3.1), that the Lagrangian representing the electromagnetic, strong and weak interactions of of the dions is given by

$$
\begin{aligned}
L_1^{\delta} = {} & L_0^{\delta} + \tfrac{1}{3} e \bar{d} \not{A} d - \tfrac{2}{3} e \bar{u} \not{A} u \\
& - \tfrac{1}{3} \frac{e}{\cos\theta_W} \left(\bar{d} \not{G}_0 d + \bar{u} \not{G}_0 u \right) \\
& + \tfrac{1}{2} \frac{\alpha}{\cos\theta_W} \Big(\bar{u}_R \not{Z} u_L - \bar{d}_R \not{Z} d_L + \tfrac{2}{3} \sin^2\theta_W \bar{d} \not{Z} d \\
& - \tfrac{4}{3} \sin^2\theta_W \bar{u} \not{Z} u \Big) + \tfrac{1}{2}\sqrt{2} \left(\bar{d}_R \not{W}_- u_L + \bar{u}_R \not{W}_+ d_L \right) \\
& + \tfrac{1}{2}\sqrt{2}\, g \left(\bar{d}\, i\gamma_5 \not{B}_+ u' + \bar{u}'\, i\gamma_5 \not{B}_- d \right). \qquad (3.20)
\end{aligned}
$$

We have shown that, by transforming the 32-component fields ψ and χ, given by (1.4), using a spin gauge transformation (3.15) in the appropriate representation of the (32 × 32) Clifford algebra, it is possible to unify our electrostrong theory of I̲ with the GSW theory of the weak interactions. The transformations (3.15) are associated with the group action

$$
U(1)_R \times SU(2)_L \times SU(1,1) \times U(1) ,
$$

which will be discussed further in the concluding section.

We can also introduce gauge invariant mass terms for the leptons and dions in the Lagrangians (3.18) and (3.20). As we observed in Section 1, it is possible to include gauge invariant terms of the form (1.16) in, for example, the lepton Lagrangian (3.14). The matrices E in (1.16) which we use to give mass terms for electrons, neutrinos and down and up dions are $\lambda_4\rho_4\tau_4 I$, $i\lambda_4\rho_3\tau_4\gamma_5$, $\lambda_3\rho_4\tau_3 I$, $i\lambda_3\rho_3\tau_3\gamma_5$. The matrix elements between, for example, states (1.4a) and (1.10a) are

$$
\begin{aligned}
& \bar{\psi}\, \lambda_4 \rho_4 \tau_4 I\, \psi = \bar{\psi}\, i \lambda_4 \rho_3 \tau_4 \gamma_5\, \psi \\
& = \tfrac{1}{2} \left(\bar{\varepsilon}\varepsilon - \bar{\nu}'\nu' + \bar{\nu}\nu - \bar{\varepsilon}'\varepsilon \right) = \bar{\varepsilon}\varepsilon + \bar{\nu}\nu ,
\end{aligned}
\qquad (3.21a)
$$

$$\bar{\Psi}\,\lambda_3\rho_4\tau_3 I\,\psi = \bar{\Psi}\,i\lambda_3\rho_3\tau_3\gamma_5\,\psi$$
$$= \frac{1}{2}\left(\bar{\varepsilon}\,\varepsilon + \bar{\nu}'\nu' - \bar{\nu}\,\nu - \bar{\varepsilon}'\varepsilon'\right) = \bar{\varepsilon}\,\varepsilon - \bar{\nu}\,\nu, \qquad (3.21b)$$

using (1.6a). When we "rotate" from the lepton to the dion representation, the matrices $\lambda_4\rho_4\tau_4 I$, $\lambda_3\rho_4\tau_3 I$ remain invariant. The matrices $i\lambda_4\rho_3\tau_4\gamma_5$, $i\lambda_3\rho_3\tau_3\gamma_5$, however, transform, and introduce matrices which contain ρ_2, which then have zero matrix elements between states of the forms (1.4b) and (1.10b); so using these types of term enables us to give different masses to the leptons and the dions.

In (3.14) we introduce the following terms

$$a_1\bar{\Psi}\,\lambda_4\rho_4\tau_4 I\,\psi + a_2\bar{\Psi}\,\lambda_3\rho_4\tau_3 I\,\psi$$
$$+ a_3\bar{\Psi}\,i\lambda_4\rho_3\tau_4\gamma_5\,\psi + a_4\bar{\Psi}\,\lambda_3\rho_3\tau_3\gamma_5\,\psi$$
$$= (a_1 + a_2 + a_3 + a_4)\,\bar{\varepsilon}\,\varepsilon + (a_1 + a_3 - a_2 - a_4)\,\bar{\nu}\,\nu, \qquad (3.22)$$

where the $\{a_r;\ r = 1, 2, 3, 4\}$ are arbitrary parameters. After transforming to the dion representation, the terms (3.22) become, remembering from (2.28a) that $\theta_0 = \frac{1}{2}\pi$,

$$a_1\bar{\chi}\,\lambda_4\rho_4\tau_4 I\,\chi + a_2\bar{\chi}\,\lambda_3\rho_4\tau_3 I\,\chi$$
$$= (a_1 + a_2)\,\bar{d}\,d + (a_1 - a_2)\,\bar{u}\,u. \qquad (3.23)$$

So by choosing the parameters $\{a_r\}$ such that

$$a_1 = \frac{1}{2}(m_d + m_u)$$
$$a_2 = \frac{1}{2}(m_d - m_u)$$
$$a_3 = \frac{1}{2}(m_\varepsilon + m_\nu - m_d - m_u)$$
$$a_4 = \frac{1}{2}(m_\varepsilon - m_\nu - m_d + m_u), \qquad (3.24)$$

we generate mass terms

$$m_{\varepsilon} \bar{\varepsilon} \varepsilon + m_{\nu} \bar{\nu} \nu \tag{3.25a}$$

$$m_{d} \bar{d} d + m_{u} \bar{u} u \tag{3.25b}$$

in the respective Lagrangians (3.18) and (3.20). The choice (3.22) of mass terms is not unique, since we could have replaced ρ_3 by ρ_2, requiring a different choice of constants a_1, a_2, a_3, a_4.

As in Section 3 of I, conserved currents can be defined. In particular, we can generalize the electromagnetic current (I, 3.25a). From (I, 3.21) and (I, 2.38),

$$B^{\mu} = B_{2\nu} G_{1}^{\mu\nu} - B_{1\nu} G_{2}^{\mu\nu} \tag{3.26a}$$

$$= i \left(B_{+\nu} G_{-}^{\mu\nu} - B_{-\nu} G_{+}^{\mu\nu} \right), \tag{3.26a}$$

where

$$G_{\pm}^{\mu\nu} = \partial^{\mu} B_{\pm}^{\nu} - \partial^{\nu} B_{\pm}^{\mu} \pm i \left(B_{\pm}^{\mu} B_{3}^{\nu} - B_{\pm}^{\nu} B_{3}^{\mu} \right). \tag{3.27}$$

We define, corresponding to B^{μ},

$$W^{\mu} = W_{1\nu} W_{2}^{\mu\nu} - W_{2\nu} W_{1}^{\mu\nu}. \tag{3.28a}$$

There is sign difference between the definitions (3.26a) and (3.28a). However, if we take, analogous to (3.27),

$$W_{\pm}^{\mu\nu} = \partial^{\mu} W_{\pm}^{\nu} - \partial^{\nu} W_{\pm}^{\mu} \pm i \left(W_{\pm}^{\mu} W_{3}^{\nu} - W_{\pm}^{\nu} W_{3}^{\mu} \right), \tag{3.29}$$

then the difference in sign between the definitions of $B_{\pm}^{\mu}$ and $W_{\pm}^{\mu}$ ensures that (3.28a) gives

$$W^{\mu} = i \left(W_{+\nu} W_{-}^{\mu\nu} - W_{-\nu} W_{+}^{\mu\nu} \right). \tag{3.28b}$$

So, in terms of the potentials $B^{\mu}_{\pm}$ and $W^{\mu}_{\pm}$, the definition of W^{μ} is exactly analogous to that of B^{μ}.

In terms of the B^{μ}, W^{μ} and the fermion currents, the generalization of (I, 3.25a) can be shown to be

$$j^{\mu}_{I} = e\bar{\varepsilon}\gamma^{\mu}\varepsilon + \frac{1}{3}e\bar{d}\gamma^{\mu}d - \frac{2}{3}\bar{u}\gamma^{\mu}u + \frac{1}{3}eB^{\mu} + eW^{\mu}. \tag{3.30}$$

The difference in sign between the definitions of $B^{\mu}_{\pm}$ and $W^{\mu}_{\pm}$ ensures that the coefficients of B^{μ} and W^{μ} in (3.30) are both positive.

4. CONCLUSION

By extending the Clifford algebra of I, we have been able to unify our electrostrong theory with the standard GSW interactions, for which helicity symmetry is broken. Because of this symmetry breaking, the gauge symmetry has to be imposed by considering two separate sets of transformations. The helicity symmetric model of § 2 is simpler and more attractive than the unsymmetric theory of § 3: not only is there a single gauge transformation (2.12a); the transformation and the resulting interactions are symmetric between weak space and strong space, with invariance under the group action of $SU(2) \times SU(1,1) \times U(1)$. It is natural to ask whether the SU(2) group of this model can be broken to give the group

$$SU(2)_L \times U(1)_R \times SU(1,1) \times U(1)$$

of the final theory; this symmetry breaking would have to result in different coupling constants for the left-handed and right-handed interactions. One possible source of symmetry breaking is the introduction of higher dimensional dynamics, which we discussed in I.

In the electrostrong theory of I, the vector basis for leptons or for dions contains eight vectors: four define the space-time metric, and three define the space whose metric is invariant under the SU(1,1) group, so that one of the vectors stands alone. In the unified theories, this vector combines with the two new vectors to give the space with metric invariant under the weak SU(2) group. The symmetry between weak and strong spaces would allow us to formulate the electroweak theory in terms of a (16 × 16) algebra. The 16-component lepton field would be

$$\left[\varepsilon_L \quad \varepsilon_R \quad \nu_L \quad \nu_R\right]^T,$$

formed by omitting the anti-particle spinors from (1.4a). The vector basis would be formed from that of Table 1 by omitting vectors 8 and 9, and omitting the τ-matrices from the other basis vectors, giving (for leptons)

$$\lambda_4\rho_1\gamma_\mu \quad i\lambda_{1,2}\rho_2 I \quad \lambda_4\rho_1\gamma_5 \quad \lambda_3\rho_2 I\,.$$

The electroweak gauge transformations, equivalent in form to (3.13b), would only involve bivectors of the space of the fifth, sixth and seventh vectors; as in the electrostrong theory, one basis vector ($\lambda_3\rho_2 I$) is separated from the others. In the unified theory, there are no "extra" vectors; in this way, the electrostrong and electroweak theories complement and complete each other.

Another consequence of unification is the invariance of the $SU(2)_L$ and $U(1)_R$ matrix elements under the transformation from lepton to dion representations, which agrees with experimental facts on quarks. In our theory, this invariance is an inevitable consequence of the fact that the charged gryphonic interactions are different for leptons and dions.

Although we have now written down a complete Lagrangian for our unified theory, much remains to be done. There are a number of problems associated with the gryphonic interactions: the possibility of negative energies, the confinement problem, and an explanation of the observed strength of strong interactions. We need to study bound states of dions, which should include pions and nucleons; it may be necessary to use a symmetry breaking mechanism to explain boson masses. One arbitrariness that we introduced in I is the choice (I, 4.5) of three colour phases. We are studying the possibility of explaining these phases in terms of the representations of a group; this explanation could well be linked with colour indistinguishability and colour confinement. When these various questions have been answered, we can proceed with specific calculations. We have not yet seriously considered the problem of generation structure.

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Table 1

Comparison of basis vectors $\{\Gamma_r\}$ of the (16 × 16) algebra of $\underline{I}$ and the (32 × 32) algebra

16 × 16		32 × 32	
r	Basis vector	r	Basis vector
		5	$i\lambda_1\rho_2\tau_4 I$
		6	$i\lambda_2\rho_2\tau_4 I$
5	$\rho_1\tau_3\gamma_5$	7	$\lambda_4\rho_1\tau_3\gamma_5$
6	$i\rho_4\tau_1 I$	8	$i\lambda_3\rho_4\tau_1 I$
7	$i\rho_4\tau_2 I$	9	$i\lambda_3\rho_4\tau_2 I$
8	$\rho_2\tau_3 I$	10	$\lambda_3\rho_2\tau_3 I$
9	$\rho_3\tau_3 I$	11	$\lambda_4\rho_3\tau_3 I$

Table 2

Bivectors and pseudoscalars used in the lepton and dion representations of the (32 × 32) algebra

	Lepton		Dion	
Weak space	$\lambda_3\rho_4\tau_4 I$	(5 × 6)	$\lambda_3\rho_4\tau_4 I$	(5 × 6)
	$i\lambda_1\rho_3\tau_3\gamma_5$	(5 × 7)	$i\lambda_1\rho_3\tau_3\gamma_5$	(5 × 7)
	$i\lambda_2\rho_3\tau_3\gamma_5$	(6 × 7)	$i\lambda_2\rho_3\tau_3\gamma_5$	(6 × 7)
Strong space	$\lambda_4\rho_4\tau_3 I$	(8 × 9)	$\lambda_4\rho_4\tau_3 I$	(8 × 9)
	$i\lambda_4\rho_2\tau_2 I$	(8 × 10)	$i\lambda_3\rho_3\tau_2 I$	(8 × 11)
	$-i\lambda_4\rho_2\tau_1 I$	(9 × 10)	$-i\lambda_3\rho_3\tau_1 I$	(9 × 11)
Pseudoscalar in 6-space	$-\lambda_4\rho_3\tau_3\gamma_5$		$\lambda_3\rho_2\tau_3\gamma_5$	

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