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SPIN-ONE PARTICLE IN AN EXTERNAL ELECTROMAGNETIC FIELD*

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ABSTRACT

In this paper the electrodynamics of a spin-one particle-antiparticle is developed using a $(1, 0) \oplus (0, 1)$ six-component wave function. Anomalous magnetic dipole and electric quadrupole effects are included. The wave equation is manifestly covariant and has no auxiliary conditions. The invariant integral for the system is derived and the nonrelativistic limit is discussed.

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I. INTRODUCTION

Recently Joos,[1] Weinberg,[2] and Weaver, Hammer and Good[3] have developed new descriptions of a free particle with spin s = 0, 1/2, 1, ... These descriptions are of interest because they are closely analogous to the Dirac theory for a spin-1/2 particle and they permit many of the well-known discussions for the spin-1/2 theory to be extended to apply uniformly to particles of arbitrary spin. Joos[1] and Weinberg[2] gave their description in a manifestly covariant form. Covariantly defined matrices as developed by Barut, Muzinich, and Williams[4] appear as the generalization of the Dirac γ_μ matrices. Weaver, Hammer, and Good[3] gave their description in Hamiltonian form and found an algorithm for generalizing the Dirac Hamiltonian $\underset{\sim}{\alpha} \cdot \underset{\sim}{p} + \beta m$ to any spin. The wave functions in these two approaches are identical for odd-half-integral spin and are equivalent, in the sense of being related by an operator that has an inverse, for integral spin. In any case the wave function forms the basis for the $(s, o) \oplus (o, s)$ representation of the Lorentz group. Also the wave function corresponds to the momentum-space wave function used by Pursey[5] in his treatment of free particles with spin.

In later works most of the properties of the free-particle theory have been worked out. Sankaranarayanan and Good[6] studied the spin-1 case in detail and Shay, Song, and Good[7] the spin-3/2 case. Sankaranarayanan and Good gave general discussions of the polarization operators[6] and the position operators.[8] The density matrices for describing orientational properties were set up by Sankaranarayanan[9] and by Shay, Song, and Good.[7] Mathews[10] and Williams, Draayer, and Weber[11] obtained definite formulas for the Hamiltonian for any spin.

The descriptions have been applied so far only to free particles and a question is how to include effects of an external electromagnetic field. In view of the success in treating all these properties of the free particle uniformly for all spins, one might hope that electromagnetic interactions could also be introduced for any spin. The problem becomes more and more complicated as the spin increases, since a particle of spin s can have anomalous electric and magnetic multipole moments up to the 2^{2s} order.

The purpose of this paper is to give the theory of a spin-1 particle, described by a $(1, 0) \oplus (0, 1)$ wave function, interacting with an external electromagnetic field, and having arbitrary magnetic dipole and electric quadrupole moments. This new formulation turns out to be worthwhile because it permits a complete treatment of the system (some aspects involving the anomalous quadrupole moment were not covered before). The results apply exclusively to spin one and have not so far suggested a generalization to higher spins.

The spin-1 particle in an external field was originally studied by Proca[12] and Kemmer[13] using a 10-component wave function. Corben and Schwinger[14] showed how to include an anomalous magnetic dipole term in Proca's theory and Young and Bludman[15] took account of an anomalous electric quadrupole. Specializing to time-independent electric fields and space-time independent magnetic fields in the anomalous quadrupole terms, they obtained a Hamiltonian of the Sakata-Taketani[16] type which included the effects of the anomalous moments. This Hamiltonian formulation involves a 6-component wave function which has complicated Lorentz transformation properties.

The wave equation given here is manifestly covariant and requires no auxiliary conditions on the wave function. The equation has the usual symmetries with respect to space reflection, time reflection, and charge conjugation. It leads to the definition of a Lorentz-invariant inner product that includes a contribution from the anomalous quadrupole term. It was found that there are two possible choices for the anomalous quadrupole term in this wave equation, each having the correct transformation properties and giving the same type of contribution in the nonrelativistic limit to order m^{-2}.

For any spin of particle, the values of the normal electric and magnetic moments depend on the wave equation used to describe the particle. Here the normal magnetic moment g-factor is 1/2 and the normal electric quadrupole moment is $-\hbar^2/2m^2c^2$. The values of the moments were found by making a Foldy-Wouthuysen type of transformation, leading to a nonrelativistic Hamiltonian correct to order m^{-2}.

II. THE WAVE EQUATION

The equation is

$$\Big[\pi_\alpha \pi_\beta \gamma_{\alpha\beta} + \pi_\alpha \pi_\alpha + 2m^2 + (e\lambda/12)\ \gamma_{5,\alpha\beta}\, F_{\alpha\beta}$$

$$+ (eq/6m^2)\ \gamma_{6,\alpha\beta,\mu\nu}\ (\partial F_{\alpha\beta}/\partial x_\mu)\ \pi_\nu\Big]\,\psi = 0 \ , \qquad (1)$$

where π_α is $-i(\partial/\partial x_\alpha) - eA_\alpha$ and $F_{\alpha\beta}$ is the field tensor,

$$F_{\alpha\beta} = (\partial A_\beta/\partial x_\alpha) - (\partial A_\alpha/\partial x_\beta),$$

$$F_{ij} = \epsilon_{ijk}\, B_k,\ F_{i4} = -F_{4i} = -iE_i,\ F_{44} = 0 \ .$$

The Latin indices run from 1 to 3, Greek from 1 to 4 with $x_4 = it$. Factors of c and $\hbar$ are omitted. The constants λ and q are real and adjust the sizes of the intrinsic moments, as discussed below. The $\gamma_{\alpha\beta}$ are six-by-six matrices defined in terms of three-by-three Hermitian spin-1 matrices s by

$$\gamma_{ij} = \begin{pmatrix} 0 & \delta_{ij} - s_i s_j - s_j s_i \\ \delta_{ij} - s_i s_j - s_j s_i & 0 \end{pmatrix},\ \gamma_{i4} = \gamma_{4i} = \begin{pmatrix} 0 & is_i \\ -is_i & 0 \end{pmatrix},\ \gamma_{44} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (2)$$

The other matrices are defined in terms of the $\gamma_{\alpha\beta}$ by

$$\gamma_{5,\alpha\beta} = i \left[\gamma_{\alpha\delta}, \gamma_{\beta\delta}\right]_- , \tag{3}$$

$$\begin{aligned}\gamma_{6,\alpha\beta,\mu\nu} = &\left[\gamma_{\alpha\mu}, \gamma_{\beta\nu}\right]_+ + 2\,\delta_{\alpha\mu}\,\delta_{\beta\nu} \\ &- \left[\gamma_{\alpha\nu}, \gamma_{\beta\mu}\right]_+ - 2\,\delta_{\alpha\nu}\,\delta_{\beta\mu} .\end{aligned} \tag{4}$$

The operators π_α are understood to act on everything to their right, including the wave function. The gradient operators inside brackets, such as in the factor $(\partial F_{\alpha\beta}/\partial x_\mu)$, act on the fields $F_{\alpha\beta}$ only and not on the wave function. The $\gamma_{\alpha\beta}$ satisfy $\gamma_{\alpha\beta} = \gamma_{\beta\alpha}$ and $\gamma_{\alpha\alpha} = 0$ so there are 9 of them independent; the $\gamma_{5,\alpha\beta}$ satisfy $\gamma_{5,\alpha\beta} = -\gamma_{5,\beta\alpha}$ and there are 6 of them independent; the symmetry properties of $\gamma_{6,\alpha\beta,\mu\nu}$ are

$$\gamma_{6,\alpha\beta,\mu\nu} = -\gamma_{6,\beta\alpha,\mu\nu} , \tag{5a}$$

$$\gamma_{6,\alpha\beta,\mu\nu} = \gamma_{6,\mu\nu,\alpha\beta} , \tag{5b}$$

$$\gamma_{6,\alpha\beta,\mu\nu} + \gamma_{6,\alpha\mu,\nu\beta} + \gamma_{6,\alpha\nu,\beta\mu} = 0 , \tag{5c}$$

and there are therefore 10 of them independent. One defines γ_5 by

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and then there are 36 independent Hermitian matrices 1, γ_5, $\gamma_{\alpha\beta}$, $i\gamma_5\gamma_{\alpha\beta}$, $\gamma_{5,\alpha\beta}$, $\gamma_{6,\alpha\beta,\mu\nu}$ which form a complete set of 6×6 matrices. Some other properties are given in Ref. 6.

III. LORENTZ TRANSFORMATIONS AND CHARGE CONJUGATION

The Lorentz transformation properties of the wave function are assigned to be the same as that of the free particle wave function. However this assignment does not settle the question because in the free-particle discussion a different wave function is used in the Hamiltonian formulation than in the manifestly covariant formulation. The relation between the two functions was given in Eq. (62) of Ref. 6. For the transformations continuous with the identity the two functions behave the same and the notation of Refs. 3 and 6 is used. For the reflections and charge conjugation there is a difference. As shown later, for zero fields Eq. (1) specializes to Weinberg's formulation and so his assignments for the discontinuous transformation properties are the appropriate ones to use.

For Lorentz transformations continuous with the identity

$$x'_\alpha = a_{\alpha\beta}\, x_\beta$$

the wave function transformation rule is

$$\psi'(x') = \Lambda \psi(x) \tag{6}$$

where x' denotes $\underset{\sim}{x}'$, t' and the matrix Λ satisfies

$$\Lambda^{-1} \gamma_{\alpha\beta} \Lambda = a_{\alpha\mu}\, a_{\beta\nu}\, \gamma_{\mu\nu} \quad , \tag{7a}$$

$$\Lambda^{+} \gamma_{44} = \gamma_{44}\, \Lambda^{-1} \quad , \tag{7b}$$

$$C\Lambda = \Lambda^{*} C \quad , \tag{7c}$$

$$\Lambda^{-1} \gamma_5 \Lambda = \gamma_5 \quad . \tag{8}$$

Here C is the charge-conjugation matrix defined by

$$C = \begin{pmatrix} 0 & C_s \\ C_s & 0 \end{pmatrix} \tag{9}$$

where C_s is a unitary matrix such that

$$C_s \underset{\sim}{s} = -\underset{\sim}{s}^* C_s \quad . \tag{10}$$

The C matrix has the property

$$C\gamma_{ij}\, C^{-1} = \gamma^*_{ij}\, ,\ C\gamma_{4i}\, C^{-1} = -\gamma^*_{4i}\, ,\ C\gamma_{44}\, C^{-1} = \gamma^*_{44}\ . \tag{11}$$

In consequence of Eqs. (6) and (7a) every Greek subscript is a vector index in the same sense as in Dirac's theory and Eq. (1) is evidently covariant.

For the space reflection

$$x'_i = -x_i,\ t' = t$$

Equations (6) and (7) apply again with Λ chosen to be γ_{44}. Since π_α and $F_{\alpha\beta}$ are regular under space reflection the covariance is again evident. For this transformation, instead of Eq. (8), the equation

$$\Lambda^{-1}\, \gamma_5\, \Lambda = -\gamma_5 \tag{12}$$

applies. By including factors of γ_5 the parity noninvariant interactions can be formed. For example $(e\lambda'/12)\ \gamma_5\ \gamma_{5,\alpha\beta}\ F_{\alpha\beta}$ is an electric dipole interaction term.

For the time reflection

$$x'_i = x_i,\ t' = -t$$

the wave function transformation rule is

$$\psi'(x') = \Lambda\left[C\psi(x)\right]^* \tag{13}$$

where again Λ is γ_{44} and satisfies Eqs. (7). By explicit calculation, using Λ_α and $F_{\alpha\beta}$ to be pseudo, one verifies that Eq. (1) is covariant.

The charge conjugate wave function is defined by

$$\psi^c = (C\psi)^* \quad . \tag{14}$$

It satisfies an equation the same as Eq. (1) but with all terms proportional to e changed in sign. It follows from the fact that Eq. (7c) applies to all transformation matrices Λ that ψ^c has the same Lorentz transformation properties as ψ. The charge conjugation has period two,

$$(\psi^c)^c = \psi$$

as follows from the fact that $C_s^* C_s$ is unity.[18]

IV. INVARIANT INTEGRAL

Let the adjoint wave function be defined by

$$\overline{\psi} = \psi^\dagger \gamma_{44} \quad . \tag{15}$$

The equation it satisfies is

$$\tilde{\pi}_\alpha \tilde{\pi}_\beta \overline{\psi} \gamma_{\alpha\beta} + \tilde{\pi}_\alpha \tilde{\pi}_\alpha \overline{\psi} + 2m^2 \overline{\psi} + (e\lambda/12) F_{\alpha\beta} \overline{\psi} \gamma_{5,\alpha\beta}$$

$$- (eq/6m^2)(\partial F_{\alpha\beta}/\partial x_\mu) \tilde{\pi}_\nu \overline{\psi} \gamma_{6,\alpha\beta,\mu\nu} = 0 \quad , \tag{16}$$

where $\tilde{\pi}_\alpha$ is $-i(\partial/\partial x_\alpha) + eA_\alpha$. It follows from Eq. (7b) that the adjoint function transforms according to

$$\overline{\psi}'(x') = \overline{\psi}(x) \Lambda^{-1} \tag{17}$$

for isochronous Lorentz transformations and according to

$$\overline{\psi}'(x') = \left[\overline{\psi}(x) C^{-1}\right]^* \Lambda^{-1} \tag{18}$$

for time reflections.

If $\overline{\psi}^{(\ell)}$ is a solution of Eq. (16) and $\psi^{(n)}$ of Eq. (1), then the current

$$\begin{aligned} J_\alpha^{(\ell,n)} = {} & \left(\tilde{\pi}_\beta \overline{\psi}^{(\ell)}\right) \gamma_{\alpha\beta}\, \psi^{(n)} - \overline{\psi}^{(\ell)}\, \gamma_{\alpha\beta}\pi_\beta \psi^{(n)} \\ & + \left(\tilde{\pi}_\alpha \overline{\psi}^{(\ell)}\right) \psi^{(n)} - \overline{\psi}^{(\ell)}\, \pi_\alpha\, \psi^{(n)} \\ & - (eq/6m^2)\, \overline{\psi}^{(\ell)}\, (\partial F_{\mu\nu}/\partial x_\beta)\, \gamma_{6,\mu\nu,\beta\alpha}\, \psi^{(n)} \end{aligned} \tag{19}$$

is conserved,

$$\partial J_\alpha^{(\ell,n)}/\partial x_\alpha = 0 \quad .$$

Here the brackets in a factor like $\left(\tilde{\pi}_\beta \overline{\psi}^{(\ell)}\right)$ indicate that the $\tilde{\pi}_\beta$ acts only on the $\overline{\psi}^{(\ell)}$. Evidently $J_\alpha^{(\ell,n)}$ is a Lorentz four-vector so the integral of $J_4^{(\ell,n)}$ over space is a time-independent Lorentz scalar. The invariant integral is therefore defined by

$$\begin{aligned} \left(\psi^{(\ell)}, \psi^{(n)}\right) = {} & i(4m)^{-1} \int d^3x \Big[\left(\tilde{\pi}_\beta \overline{\psi}^{(\ell)}\right) \gamma_{4\beta}\, \psi^{(n)} \\ & - \overline{\psi}^{(\ell)}\, \gamma_{4\beta}\pi_\beta \psi^{(n)} + \left(\tilde{\pi}_4 \overline{\psi}^{(\ell)}\right)\psi^{(n)} - \overline{\psi}^{(\ell)}\pi_4\, \psi^{(n)} \\ & - (eq/6m^2)\, \overline{\psi}^{(\ell)}\, (\partial F_{\mu\nu}/\partial x_\beta)\, \gamma_{6,\mu\nu,\beta 4}\, \psi^{(n)}\Big] . \end{aligned} \tag{20}$$

An alternative form is

$$\begin{aligned} \left(\psi^{(\ell)}, \psi^{(n)}\right) = {} & i(4m)^{-1} \int d^3x \Big[-2\,\psi^{(\ell)\dagger}\, \gamma_{44}\, \gamma_{4i}\pi_i\, \psi^{(n)} \\ & + \left(\pi_4\, \psi^{(\ell)}\right)^\dagger (1+\gamma_{44})\, \psi^{(n)} - \psi^{(\ell)\dagger}\, (1+\gamma_{44})\pi_4\, \psi^{(n)} \\ & - (eq/6m^2)\, \overline{\psi}^{(\ell)}\, (\partial F_{\mu\nu}/\partial x_\beta)\, \gamma_{6,\mu\nu,\beta 4}\, \psi^{(n)}\Big] . \end{aligned} \tag{21}$$

The factor is chosen to give the right nonrelativistic limit, as discussed below. As in the Dirac theory, the anomalous magnetic moment term does not influence the invariant integral formula. The integral is not positive definite.

For the time-rate-of-change of matrix elements of any operator T, one finds that

$$d\left(\psi^{(\ell)},\ T\,\psi^{(n)}\right)/dt = i\,(4m)^{-1}\int d^3x\ \overline{\psi}^{(\ell)}\ [W, T]_-\,\psi^{(n)} \tag{22}$$

where W is the operator inside the square brackets on the left in Eq. (1). This applies in general, even when the fields and T are time dependent. Equation (22) can be easily derived by operating on the equation

$$W\,(T\,\psi^{(n)}) = [W, T]_-\ \psi^{(n)}$$

from the left with $\overline{\psi}^{(\ell)}$, on Eq. (16) from the right with $(-T\psi)$ and adding; the terms on the left can be rearranged into i times the divergence of a current J_α built from $\overline{\psi}^{(\ell)}$ and $T\,\psi^{(n)}$. From Eq. (22) it is seen that matrix elements of a symmetry operation of the system are time independent. The point is that if T is a symmetry operation and $\psi^{(n)}$ satisfies the equations of motion then $T\,\psi^{(n)}$ also is a solution. Then $W\,\psi^{(n)}$ and $WT\,\psi^{(n)}$ are both zero and the right-hand-side of Eq. (22) is zero.

V. NONRELATIVISTIC LIMIT

As was first emphasized by Foldy and Wouthuysen in the spin one-half case, the nonrelativistic approximation corresponds to an expansion on m^{-1}. For a Dirac particle they developed the series by making unitary transformations of the Hamiltonian which removed odd parts of the Hamiltonian to higher and higher orders of m^{-1}. Their process cannot be applied directly here because there isn't a Hamiltonian to begin with. In the Dirac case it is appropriate to consider unitary transformations because the invariant integral is $\int d^3x\ \psi^{(\ell)\dagger}\,\psi^{(n)}$ and the unitary transformations preserve this form into the nonrelativistic limit. With the different invariant integral that applies here, Eq. (21), it is not appropriate to keep wave function transformations unitary. The basic idea used in finding the limit here is to take out the rest-energy part of the wave function, make a non-unitary transformation that removes odd parts of the wave equation in a certain order, and then expand in powers of m^{-1}.

The first step is to convert the wave equation, Eq. (1), into a nonrelativistic type of notation. The appropriate matrices are

$$\beta = \gamma_{44} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\alpha_i = i\,\gamma_{44}\,\gamma_{i4} = -\gamma_5\, s_i = \begin{pmatrix} s_i & 0 \\ 0 & -s_i \end{pmatrix}.$$

In terms of them the $\gamma_{5,\alpha\beta}$ matrices are

$$\gamma_{5,ij} = -6\epsilon_{ijk}\, s_k \quad ,$$

$$\gamma_{5,i4} = -6\alpha_i \quad .$$

The $\gamma_{6,\alpha\beta,\mu\nu}$ type of matrix is easily converted into terms of $\underset{\sim}{\alpha}$ and $\underset{\sim}{s}$ by using the result

$$\gamma_{6,\alpha\beta,\mu\nu} = -(1/12)\left[\gamma_{5,\alpha\beta},\ \gamma_{5,\mu\nu}\right]_+ + 4(\delta_{\alpha\mu}\,\delta_{\beta\nu} - \delta_{\alpha\nu}\,\delta_{\beta\mu}) - 4\epsilon_{\alpha\beta\mu\nu}\,\gamma_5\ . \tag{23}$$

It is assumed that the external fields $F_{\alpha\beta}$ satisfy the homogeneous Maxwell equations

$$\epsilon_{\alpha\beta\mu\nu}\ \partial F_{\alpha\beta}/\partial x_\mu = 0$$

so the third term in Eq. (23) doesn't contribute in the wave equation. The second term in Eq. (23) leads to terms proportional to $(\partial F_{\alpha\nu}/\partial x_\nu)$ in the wave equation;

this type of term is retained so that all the results below apply even in the case when the wave function overlaps the sources of the external fields. With this notation change and after multiplication by m^{-2} the wave equation reads

$$(\pi_\alpha/m)(\pi_\alpha/m)(1+\beta) - 2\beta(\underset{\sim}{s}\cdot\underset{\sim}{\pi}/m)^2 - 2i\beta(\underset{\sim}{\alpha}\circ\underset{\sim}{\pi}/m)(\pi_4/m)$$
$$- (e/m^2)(\beta+\lambda)\,\underset{\sim}{s}\cdot(\underset{\sim}{B}+i\gamma_5\underset{\sim}{E}) + 2$$
$$+ (eq/m^4)(s_p s_k + s_k s_p - 4/3\,\delta_{pk})\Big\{\epsilon_{ik\ell}\left[\partial(B_p+i\gamma_5 E_p)/\partial x_i\right]\pi_\ell$$
$$+ \gamma_5\left[\partial(B_p+i\gamma_5E_p)/\partial x_k\right]\pi_4 - \gamma_5\left[\partial(B_p+i\gamma_5E_p)/\partial x_4\right]\pi_k\Big\}\psi = 0 \quad . \tag{24}$$

The second step is to make the substitution

$$\psi = \exp\left[(\underset{\sim}{\alpha}\circ\underset{\sim}{\pi}/m) - imt\right]\psi_1 \quad ,$$

multiply through by

$$\exp\left[-(\underset{\sim}{\alpha}\circ\underset{\sim}{\pi}/m) + imt\right] \quad ,$$

and expand for small m^{-1}. The effect of the time-factors is just that π_4 becomes $im + \pi_4$. The expansion needs to be carried out to order m^{-3} to get the quadrupole contribution. The calculation leads to

$$\Big[(1-\beta)+2i(\pi_4/m)(1+\beta)-2i(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)(\pi_4/m)+2i(\pi_4/m)(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)(1-\beta)+(\pi_4/m)^2(1+\beta)$$
$$+(\pi^2/m^2)(1+\beta)-(e/m^2)(\beta+\lambda)\,\underset{\sim}{s}\cdot(\underset{\sim}{B}+i\gamma_5\underset{\sim}{E})+i(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)^2(\pi_4/m)(1-\beta)-2i(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)(\pi_4/m)(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)$$
$$+i(\pi_4/m)(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)^2(1+\beta)-(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)(\pi_4/m)^2(1+\beta)+(\pi_4/m)^2(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)(1-\beta)+(4/3)(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)^3\beta$$
$$-(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)(\pi^2/m^2)(1+\beta)+(\pi^2/m^2)(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)(1-\beta)+(e/m^2)(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)(\beta+\lambda)\,\underset{\sim}{s}\cdot(\underset{\sim}{B}+i\gamma_5\underset{\sim}{E})$$
$$-(e/m^2)(\beta+\lambda)\,\underset{\sim}{s}\cdot(\underset{\sim}{B}+i\gamma_5\underset{\sim}{E})(\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}/m)+i(eq/m^3)(s_ps_k+s_ks_p-4/3\,\delta_{pk})\,\gamma_5\,\partial(B_p+i\gamma_5E_p)/\partial x_k\Big]\psi_1 = 0 \, . \tag{25}$$

The odd terms only begin in the m^{-2} order.

As a third step one makes a similarity transformation so that the same equation holds but with β, γ_5, $\underset{\sim}{\alpha}$, ψ_1 replaced by β', γ_5', $\underset{\sim}{\alpha}'$, ψ_1' where

$$\psi_1' = 2^{-1/2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\psi_1 \quad ,$$

$$\beta' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\gamma_5' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\underset{\sim}{\alpha}' = \begin{pmatrix} 0 & -\underset{\sim}{s} \\ -\underset{\sim}{s} & 0 \end{pmatrix} .$$

The point of this is that, if ψ_1' is considered as two three-component functions,

$$\psi_1' = \begin{pmatrix} \psi_S \\ \psi_L \end{pmatrix} ,$$

then the upper part of Eq. (25) is

$$2\psi_S + 0(m^{-2})\ \psi_S + 0(m^{-2})\ \psi_L = 0 \quad ,$$

where $0(m^{-2})$ denotes terms of order m^{-2}. The small components are thus of order m^{-2} compared to the large. There is considerable simplification in the lower half of Eq. (25), leading to the result

$$\begin{aligned}\Big[&4i(\pi_4/m) + 2(\pi_4/m)^2 + 2(\pi^2/m^2) - (e/m^2)(1+\lambda)\,\underset{\sim}{s}\cdot\underset{\sim}{B} \\ &- 2i(\underset{\sim}{s}\cdot\underset{\sim}{\pi}/m)(\pi_4/m)(\underset{\sim}{s}\cdot\underset{\sim}{\pi}/m) + 2i(\pi_4/m)(\underset{\sim}{s}\cdot\underset{\sim}{\pi}/m)^2 \\ &+ i(e/m^2)(\underset{\sim}{s}\cdot\underset{\sim}{\pi}/m)(1-\lambda)\,\underset{\sim}{s}\cdot\underset{\sim}{E} + i(e/m^2)(1+\lambda)\,\underset{\sim}{s}\cdot\underset{\sim}{E}(\underset{\sim}{s}\cdot\underset{\sim}{\pi}/m) \\ &- (eq/m^3)(s_p s_k + s_k s_p - 4/3\delta_{pk})(\partial E_p/\partial x_k)\Big]\,\psi_L = 0 \quad . \end{aligned} \tag{26}$$

This can be rearranged into the form

$$(-i\pi_4 - (2m)^{-1}\,\pi_4^2)\;\psi_L = (H-e\phi)\,\psi_L \quad , \tag{27}$$

where H is given by

$$\begin{aligned} H = {} & e\phi + (\pi^2/2m) - (e/4m)(1+\lambda)\;\underset{\sim}{s}\cdot\underset{\sim}{B} \\ & + (e/8m^2)(1-\lambda-2q)(s_i s_j + s_j s_i - 4/3\,\delta_{ij})(\partial E_j/\partial x_i) \\ & - (e/8m^2)(1-\lambda)\,\underset{\sim}{s}\cdot(\underset{\sim}{\pi}\times\underset{\sim}{E} - \underset{\sim}{E}\times\underset{\sim}{\pi}) + (1/6)(e/m^2)(1-\lambda)(\nabla\cdot\underset{\sim}{E}) \quad , \end{aligned} \tag{28}$$

and where ϕ is $-iA_4$.

As a fourth and final step in finding the nonrelativistic limit, the equation is reorganized into Hamiltonian form. If the function Ψ is defined by

$$\Psi = \left[1 + (2m)^{-1}\,(H - e\phi)\right]\psi_L \tag{29}$$

then, to order m^{-2}, the π_4^2 term cancels out, Eq. (27) becomes

$$i\partial\Psi/\partial t = H\Psi \quad , \tag{30}$$

and Ψ is identified as the nonrelativistic wave function.

It is clear that, from what has been said so far, this identification is not unique. For example, ψ_L might be taken as the nonrelativistic wave function and the π_4^2 term manipulated into a contribution to the Hamiltonian in the m^{-2} order. However the identification above is supported by the limiting value of the invariant integral. Starting from Eq. (21) and disregarding terms of order m^{-3}, one finds

$$\begin{aligned} \left(\psi^{(\ell)}\psi^{(n)}\right) &= i(4m)^{-1}\int d^3x\left[2i\psi^{(\ell)\dagger}\,\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi}\psi^{(n)} + \left(\pi_4\psi^{(\ell)}\right)^\dagger(1+\beta)\psi^{(n)} - \psi^{(\ell)\dagger}(1+\beta)\,\pi_4\psi^{(n)}\right] \\ &= \int d^3x\,\psi_L^{(\ell)\dagger}\,\psi_L^{(n)} + i(2m)^{-1}\int d^3x\left(\pi_4\psi_L^{(\ell)}\right)^\dagger\psi_L^{(n)} - i(2m)^{-1}\int d^3x\,\psi_L^{(\ell)\dagger}\,\pi_4\,\psi_L \\ &= \int d^3x\,\Psi^{(\ell)\dagger}\,\Psi^{(n)} \quad . \end{aligned} \tag{31}$$

Therefore, except perhaps for further unitary transformations, Ψ is the correct nonrelativistic wave function.

VI DISCUSSION

The magnetic dipole term in H can be written as $-g(e/2m)\, \underset{\sim}{s} \cdot \underset{\sim}{B}$ where the g-factor is $(1+\lambda)/2$. Thus for a particle described by Eq. (1) the normal g-factor is 1/2, the same as for a Dirac particle. The conventional form for a spin s electric quadrupole interaction term is

$$H_{eq} = -\frac{Qe}{4s(2s-1)}\left(s_i s_j + s_j s_i - 2/3\,\delta_{ij}\, s^2\right)\frac{\partial E_i}{\partial x_j} \quad ,$$

where Q is the quadrupole moment. By comparison with Eq. (28) one sees that the quadrupole moment of this particle is

$$Q = (-1 + \lambda + 2q)/(2m^2) \quad ,$$

the normal moment being then $-1/(2m^2)$.

An alternative way to include the anomalous quadrupole contribution is to use the term $(qe/m^2)(\partial F_{\alpha\beta}/\partial x_\nu)\, \pi_\beta\, \delta_{\alpha\nu}$ in place of the γ_6 term in Eq. (1). The invariant integral can still be defined all right and the same nonrelativistic limit applies except for a different factor in the $\underset{\sim}{\nabla} \cdot \underset{\sim}{E}$ term. However the type of quadrupole term used in Eq. (1) has a universal application because the $\gamma_{6,\alpha\beta,\mu\nu}$ matrices, Lorentz type $(2,0) \oplus (0,2)$, exist for all spins greater than one-half whereas matrices like $\gamma_{\alpha\nu}$, Lorentz type (1, 1), exist only for spin one.

The connection between this formulation and other spin-one free-particle formulations is found by specializing Eq. (1) to the case e = 0 and rewriting it as

$$-P_\alpha P_\beta\, \gamma_{\alpha\beta}\psi = (P_\alpha P_\alpha + 2m^2)\psi \tag{32}$$

where P_α is $-i\partial/\partial x_\alpha$. Here one can operate with $-P_\alpha P_\beta \gamma_{\alpha\beta}$ and use the matrix property

$$(P_\alpha P_\beta \gamma_{\alpha\beta})^2 = (P_\alpha P_\alpha)^2 \quad . \tag{33}$$

This leads to

$$(P_\alpha P_\alpha)^2 \psi = (P_\alpha P_\alpha + 2m^2)^2 \psi$$

and so gives the Klein-Gordon equation

$$P_\alpha P_\alpha \psi = -m^2 \psi \, . \tag{34}$$

Furthermore this combines with Eq. (32) to yield Weinberg's equation[2]

$$P_\alpha P_\beta \gamma_{\alpha\beta} \psi = -m^2 \psi \quad . \tag{35}$$

Thus Eqs. (34) and (35) together are equivalent to Eq. (32). In Ref. 6, Section 6, the relations between Eqs. (34) and (35) and the other free-particle spin-one formulations were given.

Just as in the spin-1/2 case, the polarization of the spin-1 particle can in principle be followed throughout the interaction. One defines the four-vector polarization operator by

$$\begin{aligned} T_\mu &= (i/12m)\ \epsilon_{\mu\nu\rho\sigma} \gamma_{5,\nu\rho}\ \pi_\sigma \\ &= (-i/6m)\ \gamma_5\ \gamma_{5,\mu\sigma} \pi_\sigma \quad . \end{aligned} \tag{36}$$

This is a gauge-independent notion and it fits in with the scheme of free-particle polarization operators.[6] The fourth component is

$$T_4 = (i/m)\ \underset{\sim}{s} \cdot \underset{\sim}{\pi} \quad ;$$

for a particle in an electrostatic field, this is (ip/m) times the helicity operator $(\underset{\sim}{s} \cdot \underset{\sim}{p}/p)$ which is central to the discussion in the next section.

VII. APPLICATION: SEMICLASSICAL SOLUTION OF THE GENERAL ELECTROSTATIC PROBLEM

As an example of the usefulness of this theory the semiclassical approximation for the solutions of Eq. (1) will be set up in the case of a spin-1 particle in an electrostatic field. This applies for example to a deuteron moving relativistically through laboratory fields. The approximation is the generalization of the WKB method to the relativistic spin-1 case. Some of the ideas are the same as those developed by Pauli[21] in the relativistic spin-1/2 problem. The approximate solutions are expressed as linear combinations of functions that have, in first approximation, definite helicities and a set of differential equations is given for the expansion coefficients. The anomalous quadrupole moment term does not contribute to the wave function in first approximation.

The equation to be solved is

$$\Big[-\hbar^2\nabla^2(1+\beta) + 2\hbar^2(\underset{\sim}{s}\cdot\underset{\sim}{\nabla})^2\,\beta + 2\hbar\left(-\frac{\hbar}{c}\frac{\partial}{\partial t} - \frac{ie}{c}\phi\right)\underset{\sim}{\alpha}\cdot\underset{\sim}{\nabla}\,\beta + \frac{ie\hbar}{c}\,\underset{\sim}{E}\cdot\underset{\sim}{\alpha}\,\beta$$

$$+\left(-\frac{\hbar}{c}\frac{\partial}{\partial t} - \frac{ie}{c}\phi\right)^2(\beta+1) + 2m^2c^2 + \frac{e\hbar}{c}\lambda i\,\underset{\sim}{\alpha}\cdot\underset{\sim}{E}$$

$$+\frac{e\hbar^3}{m^2c^3}\,q\,\gamma_5\,\epsilon_{jk\ell}\,(s_is_\ell + s_\ell s_i)\,\frac{\partial^2\phi}{\partial x_i\partial x_j}\,\frac{\partial}{\partial x_k}$$

$$-i\frac{e\hbar^2}{m^2c^3}\,q\left(s_is_j + s_js_i - \frac{4}{3}\delta_{ij}\right)\frac{\partial^2\phi}{\partial x_i\partial x_j}\left(-\frac{\hbar}{c}\frac{\partial}{\partial t} - \frac{ie}{c}\phi\right)\Big]\psi = 0\ . \qquad (37)$$

This is just Eq. (1) specialized to the case $\underset{\sim}{A} = 0$, $A_4 = i\phi$ time-independent. The factors of $\hbar$ and c have been reinserted and it has been written in terms of the $\underset{\sim}{\alpha}$- and β- rather than the γ-matrices.

The results are conveniently expressed in terms of the solutions of the free-particle problem. In this case one considers solutions of the form

$$\psi = w \exp\left[i\hbar^{-1}(\underset{\sim}{p}\cdot\underset{\sim}{x} - Et)\right]\ . \qquad (38)$$

The equation determining w is then

$$\left[p^2(1+\beta) - 2\,(\underset{\sim}{s}\cdot\underset{\sim}{p})^2\,\beta - 2\,(E/c)\,\underset{\sim}{\alpha}\cdot\underset{\sim}{p}\,\beta - (E/c)^2\,(1+\beta) + 2m^2c^2\right] w = 0 \quad . \tag{39}$$

By looking at the problem in the rest frame, $\underset{\sim}{p} = 0$, one sees clearly that there are six solutions for each fixed $\underset{\sim}{p}$. It is easy to find them by supposing they are eigenstates of the six-by-six helicity operator, say

$$(\underset{\sim}{s}\cdot\underset{\sim}{p}/p)\,w = \sigma w \quad , \tag{40}$$

where $\sigma = 0, \pm 1$. Then Eq. (39) reduces to a two-by-two problem. There are solutions only if

$$E = \epsilon W \tag{41}$$

where $\epsilon = \pm 1$ and W is an abbreviation for $c(p^2 + m^2c^2)^{1/2}$, the positive root. The particle/antiparticle solutions are identified as $\epsilon = \pm 1$. The final formulas for the solutions $w_{\epsilon,\sigma}(\underset{\sim}{p})$ of the free-particle problem are

$$w_{\epsilon,0}(\underset{\sim}{p}) = \frac{1}{2^{\frac{1}{2}}}\begin{pmatrix} u_0 \\ u_0 \end{pmatrix}, \qquad w_{\epsilon,\pm 1}(\underset{\sim}{p}) = \frac{1}{2^{\frac{1}{2}}mc^2}\begin{pmatrix} (W \pm \epsilon cp)u_{\pm 1} \\ (W \mp \epsilon cp)u_{\pm 1} \end{pmatrix}, \tag{42}$$

where $u_\sigma(\underset{\sim}{p})$ are the solutions of the three-by-three helicity eigenvalue problem

$$(\underset{\sim}{s}\cdot\underset{\sim}{p}/p)\,u = \sigma u \quad ,$$

normalized so that

$$u^\dagger u = 1 \quad .$$

In the representation $(s_i)_{jk} = i\,\epsilon_{jik}$ explicit formulas for these functions are

$$u_0(\underset{\sim}{p}) = \frac{1}{p}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \; u_{\pm 1}(\underset{\sim}{p}) = \left(\frac{1}{2p^2(p^2 - p_3^2)}\right)^{1/2} \begin{pmatrix} \pm\, ipp_2 - p_1p_3 \\ \mp\, ipp_1 - p_2p_3 \\ p^2 - p_3^2 \end{pmatrix} \quad . \tag{43}$$

The factors in Eqs. (42) are chosen so as to give a normalization appropriate to Eq. (21). If the q-term can be disregarded and if ψ has time-dependence $\exp(-i\hbar^{-1} Et)$ then

$$(\psi, \psi) = \int d^3x \; I \quad ,$$

where

$$I = (2mc^2)^{-1} \left[E\psi^{\dagger} (1+\beta)\psi - c\psi^{\dagger} \underset{\sim}{\alpha} \cdot \underset{\sim}{p} \psi \right] \quad . \tag{44}$$

The normalization is such that $I = E/mc^2$ for the free-particle solutions.

The semiclassical approximation is obtained by substituting

$$\psi = \left[a_0 + (\hbar/i) \, a_1 + \ldots \right] \; \exp(iS/\hbar)$$

into Eq. (37) and formally obtaining a solution to first order for small $\hbar$. Applicability of this approximation is discussed below. In terms of the abbreviations

$$\underset{\sim}{p} = \underset{\sim}{\nabla} S \quad , \qquad E = -\partial S/\partial t$$

(this is a different use of the symbol $\underset{\sim}{p}$ than before) the terms in $\hbar$ give

$$\left[p^2(1+\beta) - 2(\underset{\sim}{s} \cdot \underset{\sim}{p})^2 \beta - 2c^{-1}(E-e\phi) \underset{\sim}{\alpha} \cdot \underset{\sim}{p} \beta - c^{-2}(E-e\phi)^2 (1+\beta) + 2m^2c^2 \right] a_0 = 0. \tag{45}$$

Solutions of this equation are known by comparison with the free-particle problem, Eq. (39). According to Eq. (41) it is necessary that

$$E - e\phi = \epsilon c \, (p^2 + m^2c^2)^{1/2} \quad . \tag{46}$$

In the following only the particle solutions, $\epsilon = +1$, are considered; then this is the equation for the classical Hamilton-Jacobi function S. Also only the solutions with definite energy E are found so that

$$S = \overline{S} - Et \quad , \tag{47}$$

where $\overline{S}$ is time-independent. This means that $\underset{\sim}{p}$ is $\underset{\sim}{\nabla}\overline{S}$ and is also time-independent. Suppose the classical problem is solved so that $\overline{S}$ as a function of $\underset{\sim}{x}$ and three constants, values of integrals of the motion, is known. The only problem then is to determine a_0. Equation (45) implies that

$$a_0 = \sum_{\sigma=0,\pm 1} A_\sigma \, w_{+1,\sigma}(\underset{\sim}{p}) \tag{48}$$

where A_σ are three functions of position still to be determined. The limitation on A_σ is that the approximation should be solvable to next order to get a_1. Thus Eq. (37) for the terms in $\hbar^1$ gives

$$\left[p^2(1+\beta) - 2(\underset{\sim}{s}\cdot\underset{\sim}{p})^2\beta - 2c^{-1}(E-e\phi)\,\underset{\sim}{\alpha}\cdot\underset{\sim}{p}\,\beta - c^{-2}(E-e\phi)^2(1+\beta) + 2m^2c^2\right] a_1$$

$$= \left[(\underset{\sim}{p}\cdot\underset{\sim}{\nabla} + \underset{\sim}{\nabla}\cdot\underset{\sim}{p})(1+\beta) - 2\beta(\underset{\sim}{s}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{s}\cdot\underset{\sim}{p} + \underset{\sim}{s}\cdot\underset{\sim}{p}\,\,\underset{\sim}{s}\cdot\underset{\sim}{\nabla}) - 2c^{-1}(E-e\phi)\,\underset{\sim}{\alpha}\cdot\underset{\sim}{\nabla}\,\beta\right.$$

$$\left. - ec^{-1}\,\underset{\sim}{E}\cdot\underset{\sim}{\alpha}\,\beta - ec^{-1}\lambda\,\underset{\sim}{E}\cdot\underset{\sim}{\alpha}\right] a_0 \quad . \tag{49}$$

(Even in this order, the q term does not contribute.) The matrix of coefficients of a_1 is the same as in Eq. (45) and has zero determinant. Equations (49) have a solution for a_1 only if the vector on the right is orthogonal to the solutions of the homogeneous equations formed with the Hermitian conjugate of the matrix on the left. Taking that Hermitian conjugate is the same as replacing $(E-e\phi)$ by $-(E-e\phi)$ so those solutions are just $w_{-1,\tau}(\underset{\sim}{p})$ and the condition for solvability is

$$w^\dagger_{-1,\tau}\left[(\underset{\sim}{p}\cdot\underset{\sim}{\nabla} + \underset{\sim}{\nabla}\cdot\underset{\sim}{p})(1+\beta) - 2\beta(\underset{\sim}{s}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{s}\cdot\underset{\sim}{p} + \underset{\sim}{s}\cdot\underset{\sim}{p}\,\underset{\sim}{s}\cdot\underset{\sim}{\nabla}) - 2c^{-1}(E-e\phi)\,\underset{\sim}{\alpha}\cdot\underset{\sim}{\nabla}\,\beta\right.$$

$$\left. - ec^{-1}\,\underset{\sim}{E}\cdot\underset{\sim}{\alpha}\,\beta - ec^{-1}\lambda\,\underset{\sim}{E}\cdot\underset{\sim}{\alpha}\right]\sum A_\sigma\, w_{+1,\sigma} = 0 \quad .$$

Here $\underset{\sim}{\nabla}$ acts on everything to the right, including the $\underset{\sim}{x}$ dependence in $\underset{\sim}{p}$. Expressed as differential equations for A_σ this reads

$$\sum_\sigma w^\dagger_{-1,\tau}\left[2\underset{\sim}{p}(1+\beta) - 2\beta p(\tau+\sigma)\underset{\sim}{s} + 2c^{-1}W\beta\underset{\sim}{\alpha}\right] w_{+1,\sigma}\cdot\underset{\sim}{\nabla}A_\sigma$$

$$= -\sum_\sigma A_\sigma w^\dagger_{-1,\tau}\left[(\underset{\sim}{p}\cdot\underset{\sim}{\nabla}+\underset{\sim}{\nabla}\cdot\underset{\sim}{p})(1+\beta) - 2\beta(\underset{\sim}{s}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{s}\cdot\underset{\sim}{p} + \underset{\sim}{s}\cdot\underset{\sim}{p}\;\underset{\sim}{s}\cdot\underset{\sim}{\nabla})\right.$$

$$\left. + 2c^{-1}W\beta\underset{\sim}{\alpha}\cdot\underset{\sim}{\nabla} + ec^{-1}\beta\underset{\sim}{E}\cdot\underset{\sim}{\alpha} - ec^{-1}\lambda\underset{\sim}{E}\cdot\underset{\sim}{\alpha}\right] w_{+1,\sigma} \quad . \tag{50}$$

Here W still denotes $c(p^2+m^2c^2)^{1/2}$. By using the explicit formulas for $w_{\epsilon,\sigma}$, Eqs. (42), one can verify that the left-hand side simplifies to $4\underset{\sim}{p}\cdot\underset{\sim}{\nabla}A_\tau$. Let Eq. (50) be written as

$$\underset{\sim}{p}\cdot\underset{\sim}{\nabla}A_\tau = \sum_\sigma C_{\tau\sigma}(\underset{\sim}{x})\,A_\sigma \quad , \tag{51}$$

where the coefficients $C_{\tau\sigma}$ can be found given the potential $\phi(\underset{\sim}{x})$ and choice of principal function S. Since at every point $\underset{\sim}{p}$ is normal to the surface $\overline{S}$ = constant, Eqs. (51) determine A_τ everywhere if the A_τ are given on one particular surface. In this respect the semiclassical approximation is like the classical problem in which one can have various numbers of particles streaming on the various allowed trajectories. For any particular orbit $\underset{\sim}{x}(t)$, since $\underset{\sim}{p}$ is $c^{-2}(E-e\phi)\,d\underset{\sim}{x}/dt$, Eq. (51) is a set of total differential equations

$$(E-e\phi)\,dA_\tau/dt = c^2\sum_\sigma C_{\tau\sigma}A_\sigma \tag{52}$$

which determine the amplitudes A_τ if they are known at the start.

In the one-dimensional problem, when ϕ and $\overline{S}$ depend on a single coordinate, say z, one can solve for the A_τ explicitly. The principal function is

$$S = \int p\,dz - Et \tag{53}$$

where

$$p = c^{-1}\left[(E-e\phi)^2 - m^2c^4\right]^{1/2} \tag{54}$$

for particles moving in the positive z-direction. Only the z-components of the spin matrices occur in Eq. (50) so it is appropriate to use the representation in which s_z is diagonal. In place of Eqs. (43) one has

$$u_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_{+1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad u_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and their derivatives are zero.

Equation (51) uncouple and simplify to

$$p(dA_\tau/dz) = -\frac{1}{2} A_\tau (dp/dz) \quad ,$$

the λ-term dropping out. The amplitudes are then just $p^{-1/2}$ and the semi-classical solutions are

$$\psi_0 = \left(\frac{mc}{2p}\right)^{1/2} \begin{pmatrix} u_0 \\ u_0 \end{pmatrix} \exp i\hbar^{-1} \left(\int p dz - Et\right) \quad , \tag{55a}$$

$$\psi_{\pm 1} = \frac{1}{(2p\, mc^3)^{1/2}} \begin{pmatrix} (W \pm cp)\, u_{\pm 1} \\ (W \mp cp)\, u_{\pm 1} \end{pmatrix} \exp i\hbar^{-1} \left(\int p dz - Et\right) \quad . \tag{55b}$$

These are eigenstates of the helicity s_z, as are the exact solutions of the electrostatic one-dimensional problem.

Equation (44) is still appropriate for discussing the normalization, although there, $\underset{\sim}{p}$ denotes $-i\hbar\underset{\sim}{\nabla}$ whereas in Eqs. (55) p is given by Eq. (54). To first order in $\hbar$ they amount to the same thing and I is E/cp. This is a sensible result since it is inversely proportional to the classical velocity.

The semiclassical approximation is expected to apply when terms marked by higher powers of $\hbar$ are smaller than those marked by lower powers. Typically the $\hbar\underset{\sim}{\nabla}\underset{\sim}{p}$ terms are considered one higher order than p^2 in deriving Eqs. (45) and

(49). Using Eq. (46) and considering $E - e\phi$ of order mc^2, one finds that the relative size $\hbar\underset{\sim}{\nabla}\underset{\sim}{p}/p^2$ is of order $(\hbar eE_f/m^2c^3)$, where E_f is the size of the electric field. This parameter also measures the size of the $\underset{\sim}{\alpha} \cdot \underset{\sim}{E}$ terms relative to p^2, as long as λ is of order unity. Another parameter enters in when the q terms are considered. Their size relative to the $\underset{\sim}{\alpha} \cdot \underset{\sim}{E}$ terms is about $(\hbar q\underset{\sim}{\nabla}E_f/mc\,E_f)$. The approximation is expected to apply, then, when the above two parameters are small. For example, a deuteron has $\lambda = .7$, $q = 25$ and if it moves in laboratory fields, $E_f \cong 10^4$ esu, $\underset{\sim}{\nabla}E_f/E_f \cong 1\ \mathrm{cm}^{-1}$, the first parameter is about 10^{-17} and the second about 10^{-13}. The approximation would surely apply in that case.

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