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Spin-Orbit Splitting and Tensor Force. II

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General formulae of the second order perturbation energies due to the tensor force are given in the case of the closed shell+one nuclei, and useful formulae for calculating the two-body matrix elements are also derived. Using these formulae, the D-state doublet splitting in O^{17} is estimated and it is found that about a half of the observed value is explained in terms of the second order effect of the tensor force as in the case of He^5 and N^{15}.

§ 1. Introduction

In the preceding paper [I],[1] one of the authors (T.T.) has estimated the second order effect of the tensor force on the spin-orbit splitting of He^5 and $N,^{15}$ using the meson theoretic potential and a phenomenological Serber one with a strong tensor part.

In this paper general formulae for the second order perturbation energy due to the tensor force are given in the case of the closed shell+one nuclei, and also useful formulae for calculating the two-body matrix elements are derived. These formulae are adopted for estimating the D-state doublet splitting of O^{17} due to the tensor force. It is found that the strong two-body tensor force can qualitatively explain the origin of the one-body spin-orbit force in the nuclear shell model, but quantitatively this force gives about a half of the observed value of the doublet splitting. It is mainly because the deformation of the closed shell core induced by the mutual tensor interaction among the core-nucleons is affected by the presence of the outmost nucleon so as to satisfy the Pauli principle. This situation is the same as in [I].

In § 2, the general formulae giving the second order effect of the tensor force are derived in the nuclei of the zeroth order configuration, closed shell (in LS-coupling sense) +one nucleon. In § 3, assuming the average field to be a harmonic oscillator well as in [I], the two-body wave functions are transformed into the wave functions in the system of the relative and the centre of mass coordinate.

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And formulae for the transformation coefficients are obtained when the state of the centre of mass coordinate is $(1s)$, $(1p)$, $(1d)$ and $(2s)$. Also, the recursion formula for the transformation coefficients is derived. The second order effect of the tensor force on the doublet splitting in O^{17} is calculated numerically in § 4, and the discussions are given in §5.

§ 2. The second order effect of the tensor force

Throughout the present paper, the nuclei of the zeroth order configuration, closed shell+one nucleon, are treated. In these nuclei the first order perturbation energies due to the tensor force vanish. Therefore, we have computed the second order perturbation effect of the tensor force.

The zeroth order shell model wave functions for these nuclei are given in the LS-coupling scheme as follows,

$$\psi_0=|(n_1 l_1)^{8l_1+4}(000)(n_2 l_2)^{8l_2+4}(000)\cdots(n_k l_k)^{8l_k+4}(000)nl,\ T=\tfrac{1}{2}(T_z)S=\tfrac{1}{2}L=l\,;\ JM\rangle \qquad (2\cdot1)$$

where (000) means all of the resultant isotopic spin T, ordinary spin S and angular momentum L are zero, and $n_1 l_1$, $\cdots$, $n_k l_k$ are principal and azimuthal quantum numbers of the closed shells respectively and nl those of the outmost nucleon. The excited configurations ψ_n, which can mix with the shell model wave function (2·1) in the first order perturbation, must not have more than two single particle orbitals different from the configuration (2·1). Furthermore, only the excited states which have the total spin $S=3/2$ or $5/2$ can have the non-vanishing matrix elements of the tensor interaction with the shell model wave function (2·1).

Now, the second order perturbation energy due to the tensor force is expressed as follows,

$$\varDelta E=\sum_n{}' \frac{|\langle\psi_n|V_T|\psi_0\rangle|^2}{E_0-E_n}, \qquad (2\cdot2)$$

where E_0 and E_n are the energies of the zeroth order state and of the excited state n, respectively. The tensor potential, $V_T\equiv\sum_{i>j}v_T(\boldsymbol{r}_{ij})$, is written as

$$v_T(\boldsymbol{r}_{ij})=[a+b(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)]S_{ij}V(r_{ij}) \qquad (2\cdot3)$$

where a and b are constants, $\boldsymbol{\tau}_i$ the isotopic spin operator, $V(r_{ij})$ the radial part of the potential and $S_{ij}=3(\boldsymbol{\sigma}_i\cdot\boldsymbol{r}_{ij})(\boldsymbol{\sigma}_j\cdot\boldsymbol{r}_{ij})/r_{ij}^2-(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)$, with the Pauli spin operator $\boldsymbol{\sigma}_i$. In the following computation, it may be convenient to rewrite $S_{ij}V(r_{ij})$ as

$$S_{ij}V(r)=\boldsymbol{S}^{(2)}\cdot\boldsymbol{L}^{(2)}, \qquad (2\cdot4)$$

where $S_m^{(2)}$ is $[\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_j]_m^{(2)}$ and $L_m^{(2)}$ is proportional to the product of $V(r)$ and the spherical harmonics $Y_{2m}(\theta, \varphi)$, in which (r, θ, φ) are the relative coordinates between the particles i and j. Through the usual method, the matrix element of the tensor potential becomes

$$\langle TS'L';J|V_T|TSL;J\rangle=(-)^{S'-L+J}W(S'L'SL;J2)\langle S'L'\|\boldsymbol{S}^{(2)}\cdot\boldsymbol{L}^{(2)}\|SL\rangle\,\Phi_\tau(T), \tag{2·5}$$

where T, S and L are total isotopic spin, total ordinary spin and total angular momentum of the initial state, T, S' and L' those of the final state, J total spin and $\Phi_\tau(T)$ is the expectation value of $a+b(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$ in the state of the isotopic spin T. $W(S'L'SL;J2)$ and $\langle S'L'\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|SL\rangle$ are the Racah coefficient and the reduced matrix element, respectively.

In the following part of this section, the excited configurations of various types will be presented and the contributions to the second order perturbation energy will be calculated.

(I) The first kind of the configurations are

$$\psi_{Ia,n}=|[(n_1l_1)^{8l_1+4}(000),\cdots(n_il_i)^{8l_i+2}(T_11L_1)\cdots,\ (n_kl_k)^{8l_k+4}(000)\,(n_i'l_i'n_i''l_i'')\,(T_11L_2)]\,(022)nl,\ T=\tfrac{1}{2}(T_z)SL;\ JM\rangle \tag{2·6}$$

and

$$\psi_{Ib,n}=|[(n_1l_1)^{8l_1+4}(000),\cdots\{(n_il_i)^{8l_i+3}(\tfrac{1}{2}\tfrac{1}{2}l_i)\,(n_jl_j)^{8l_j+3}(\tfrac{1}{2}\tfrac{1}{2}l_j)\}\,(T_11L_1),\cdots(n_kl_k)^{8l_k+4}\,(000)\,(n_i'l_i'n_j'l_j')\,(T_11L_2)](022)nl,\ T=\tfrac{1}{2}(T_z)SL;\ JM\rangle \tag{2·7}$$

which are graphically shown in Figs. 1 and 2. In these configurations two nucleons are excited from the core into other unoccupied orbits than nl-orbit, i.e., $n_i'l_i'n_i''l_i''$ and $n_j'l_j'$ cannot coincide with nl. Both the total ordinary spin and total angular momentum of the excited core must be 2, because the tensor force is a scalar product of two second rank tensors $\boldsymbol{S}^{(2)}$ and $\boldsymbol{L}^{(2)}$ as can be seen from Eq. (2·4). These configurations interact with the shell model wave function (2·1) and give the same correction as that of the self energy of the closed shell nuclei in the second order perturbation. And these corrections cannot give the energy difference between two states $J=l+1/2$ and $J=l-1/2$. By the method of the tensor calculus, the contributions $\varDelta E$ due to these configuration mixings may be easily estimated and result into

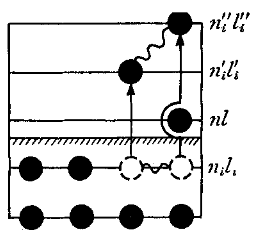

Fig. 1 Configuration Ia.

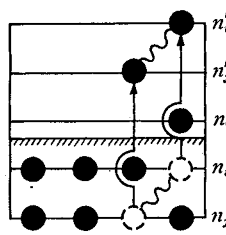

Fig. 2 Configuration Ib.

$$\varDelta E_{Ia}=-\frac{1}{5}\sum_{\substack{n_i'l_i',n_i''l_i''\\ n_il_i}}\frac{1}{\varDelta E_{on}}\sum_{T_1,L_1,L_2}(2T_1+1)\times|\langle n_i'l_i'n_i''l_i''1L_2\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|(n_il_i)^21L_1\rangle|^2\,\Phi_\tau(T_1) \tag{2·8}$$

and

$$\Delta E_{II_b}=-\frac{1}{5}\sum_{\substack{n_i l_i, n_j l_j \\ n_i' l_i', n_j' l_j'}}\frac{1}{\Delta E_{on}}\sum_{T_1 L_1 L_2}(2T_1+1)$$
$$\times|\langle n_i' l_i' n_j' l_j' 1L_2\|S^{(2)}L^{(2)}\|n_i l_i n_j l_j 1L_1\rangle|^2\Phi_\tau^2(T_1) \qquad (2\cdot 9)$$

where ΔE_{on} is the zeroth order energy difference between the states (2·1) and (2·6) or (2·7), and all two-body wave functions $|n_\alpha l_\alpha n_\beta l_\beta TSL\rangle$ should be antisymmetrized.

Mixing percentage of these configurations is rather large as discussed in § 5, but it should be noted that these configuration mixings do not influence the expectation value of any single particle operator except for a scalar one. The deformation of the closed shell core due to these configuration mixings is caused by the mutual interaction between core-nucleons and then it will hereafter be briefly called "the self-deformation of the closed shell core".

(II) The second kind of the configurations are

$$\psi_{II,n}=|(n_1 l_1)^{8l_1+4}(000),\cdots(n_i l_i)^{8l_i+3}(\tfrac{1}{2}\tfrac{1}{2}l_i),\cdots(n_k l_k)^{8l_k+4}(000),\ (n_i' l_i' n' l')(T'1L'),$$
$$T=\tfrac{1}{2}(T_z)S=\tfrac{3}{2}L;\ JM\rangle \qquad (2\cdot 10)$$

in which one core nucleon is excited by the interaction with the outmost nucleon (see Fig. 3). The resultant spin of $(n_i' l_i' n' l')$ is restricted to one, because the tensor force has non-vanishing two-body matrix element only in the spin triplet state. The second order perturbation energies due to the configuration mixings of this type result in

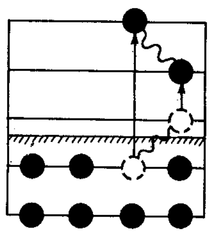

Fig. 3 Configuration II.

$$\Delta E_{J,II}=-\sum\frac{1}{\Delta E_{on}}\Big[\frac{1}{20(2l+1)}\sum_{T'}\sum_{L',L''}(2T'+1)$$
$$\times|\langle n_i' l_i' n' l' 1L'\|S^{(2)}L^{(2)}\|n_i l_i n l 1L''\rangle|^2\Phi_\tau^2(T')$$
$$-(-)^{1/2+l-J}W(\tfrac{1}{2}\tfrac{1}{2}ll;1J)\cdot\frac{3}{4\sqrt{5}}\sum_{T',L',L'',\widetilde{L}''}(-)^{L''+\widetilde{L}''}$$
$$\times(2T'+1)\sqrt{(2L''+1)(2\widetilde{L}''+1)}\,W(l_i L'' l1;l\widetilde{L}'')\,W(L'L''21;2\widetilde{L}'')$$
$$\times\langle n_i' l_i' n' l' 1L'\|S^{(2)}L^{(2)}\|n_i l_i n l 1L''\rangle$$
$$\times\langle n_i' l_i' n' l' 1L'\|S^{(2)}L^{(2)}\|n_i l_i n l 1\widetilde{L}''\rangle\cdot\Phi_\tau^2(T')\Big]. \qquad (2\cdot 11)$$

In the brace of Eq. (2·11), the first term gives the common energy shift for both states of the $J=l+1/2$ and $J=l-1/2$. The second term gives the energy difference between these states. It is very interesting to note that this term has a factor $(-)^{1/2+l-J}W(\frac{1}{2}\frac{1}{2}ll;1J)$, which appears in the expectation value of $\langle\frac{1}{2}l;JM|\boldsymbol{s}\cdot\boldsymbol{l}|\frac{1}{2}l;JM\rangle$. $n_i'l_i'$ and $n'l'$ can be any orbit as far as the Pauli principle is not violated.

Only the second order effect of this type has been considered by Kisslinger[2] and Jancovici,[3]* whose calculations have shown this contribution to be small or of wrong sign. Brueckner et al.[4] have estimated more accurately the one-body spin-orbit force induced by the tensor force between the outside nucleon and the closed shell core, and have got the negative result. The present calculation and the preceding one[1] have also led to the same conclusion. Therefore, it may be said that the configuration mixings of this type cannot explain the observed doublet splitting. The mixed configurations considered here will be called "the induced deformation of the closed shell core", because the mixings of this type are induced by the mutual interaction between the outside nucleon and the closed shell core.

(III) In the type I, it is not taken into consideration that the core-nucleons excited from the closed shells by the mutual interaction jump into the outmost orbit nl.

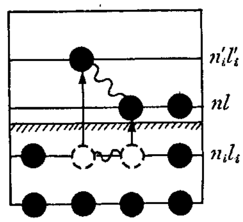

Fig. 4 Configuration IIIa.

If the Pauli principle does not work, also in these cases the contributions are same for both spin states $J=l\pm\frac{1}{2}$. However, it has been shown in [I] that this exclusion effect is important for the doublet splitting. Therefore, these cases will be considered in this paragraph. There are several configurations in which the nucleons jump from the closed shells into the most outside orbit nl. The first of them is

$$\psi_{IIIa,n}=|(n_1l_1)^{8l_1+4}(000),\cdots(n_il_i)^{8l_i+2}(T_1S_1L_1)\cdots(n_\kappa l_k)^{8l_k+4}(000)$$
$$\{n_i'l_i'\cdot(nl)^2(T_2S_2L_2)\}\,(T_3S_3L_3),\ T=\tfrac{1}{2}(T_z)SL\,;\,JM\rangle, \qquad (2\cdot12)$$

which corresponds to Fig. 4. The energy shift caused by these configuration interactions becomes

$$\Delta E_{J,IIIa}=-\sum\frac{1}{\Delta E_{on}}\Bigg[\left\{\frac{1}{5}-\frac{1}{20(2l+1)}\right\}\sum_{T_1L_1L_4}(2T_1+1)$$
$$\times|\langle n_i'l_i'nl1L_4\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|(n_il_i)^21L_1\rangle|^2\,\Phi_\tau^2(T_1)$$
$$+(-)^{1/2+l-J}W(\tfrac{1}{2}\tfrac{1}{2}ll\,;1J)\cdot\frac{3}{4\sqrt{5}}\sum_{T_1,L_1,L_4,\widetilde{L}_4}(-)^{L_4+\widetilde{L}_4}(2T_1+1)$$
$$\times\sqrt{(2L_4+1)(2\widetilde{L}_4+1)}\,W(l_i'L_4l1\,;l\widetilde{L}_4)\,W(L_1L_421\,;2\widetilde{L}_4)$$
$$\times\langle(n_il_i)^21L_1\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|n_i'l_i'nl1L_4\rangle$$
$$\times\langle(n_il_i)^21L_1\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|n_i'l_i'nl1\widetilde{L}_4\rangle\,\Phi_\tau^2(T_1)\Bigg]. \qquad (2\cdot13)$$

The second is

* Recently, Takagi et al.[12] and Jancovici[13] have calculated the other effect which is discussed in the next paragraph, using the Fermi gas model.

$$\psi_{IIIb,n}=|(n_1l_1)^{8l_1+4}(000),\cdots\{(n_il_i)^{8l_i+3}(\tfrac{1}{2}\tfrac{1}{2}l_i)\,(n_jl_j)^{8l_j+3}(\tfrac{1}{2}\tfrac{1}{2}l_j)\}\,(T_1S_1L_1),\cdots$$
$$(n_kl_k)^{8l_k+4}(000),\ \{n_i'l_i'\cdot(nl)^2(T_2S_2L_2)\}\,(T_3S_3L_3),\ T=\tfrac{1}{2}(T_z)SL\,;\,JM\rangle, \tag{2·14}$$

which is shown graphically in Fig. 5. The contribution coming from this type is given by the equation

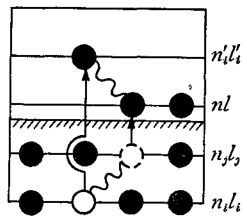

Fig. 5 Configuration IIIb.

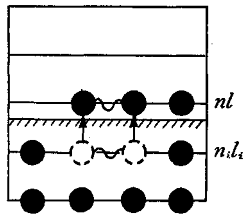

Fig. 6 Configuration IIIc.

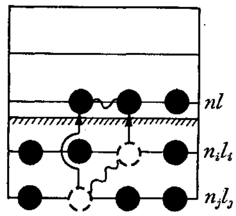

Fig. 7 Configuration IIId.

$$\begin{aligned}
\Delta E_{J,IIIb}=&-\sum\frac{1}{\Delta E_{on}}\Bigg[\left\{\frac{1}{5}-\frac{1}{20(2l+1)}\right\}\sum_{T_1,L_1,L_4}(2T_1+1)\\
&\times|\langle n_il_in_jl_j1L_1\|S^{(2)}L^{(2)}\|n_i'l_i'nl1L_4\rangle|^2\,\Phi_T^{\,2}(T_1)\\
&+(-)^{1/2+l-J}W(\tfrac{1}{2}\tfrac{1}{2}ll;1J)\cdot\frac{3}{4\sqrt{5}}\sum_{T_1,L_1,L_4,\widetilde{L}_4}(-)^{L_4+\widetilde{L}_4}(2T_1+1)\\
&\times\sqrt{(2L_4+1)(2\widetilde{L}_4+1)}\,W(l_i'L_4l1\,;\,l\widetilde{L}_4)\\
&\times W(L_1L_421\,;\,2\widetilde{L}_4)\cdot\langle n_il_in_jl_j1L_1\|S^{(2)}L^{(2)}\|n_i'l_i'nl1L_4\rangle\\
&\times\langle n_il_in_jl_j1L_1\|S^{(2)}L^{(2)}\|n_i'l_i'nl1\widetilde{L}_4\rangle\,\Phi_T^{\,2}(T_1)\Bigg].
\end{aligned}\tag{2·15}$$

In the brace of Eqs. (2·13) and (2·15), the first term gives no splitting between $J=l+\frac{1}{2}$ and $J=l-\frac{1}{2}$ states, and the correction $1/20(2l+1)$ in this term and the second term are due to the Pauli principle. It should be noted that the sign of the second term is opposite to that in Eq. (2·11). Now, the two particles excited from the closed shells may also jump into the same orbit nl. There are two possibilities whether both of the two particles come from a certain orbit or from two different orbits. Then, the third configuration is

$$\psi_{IIIc,n}=|(n_1l_1)^{8l_1+4}(000),\cdots(n_il_i)^{8l_i+2}(T_1S_1L_1),\cdots(n_kl_k)^{8l_k+4}(000),(nl)^3(T_2S_2L_2),$$
$$T=\tfrac{1}{2}(T_z)SL\,;\,JM\rangle. \tag{2·16}$$

Fig. 6. shows this configuration schematically. In this case the calculation of the second order perturbation is rather complicated but the result is

$$\Delta E_{J,IIIc}=-\sum\frac{1}{\Delta E_{on}}\Big[\Big\{\frac{1}{5}-\frac{1}{10(2l+1)}\Big\}\sum_{T_1,L_1,L_2}(2T_1+1)$$
$$\times|\langle(nl)^2 1L_2\|S^{(2)}L^{(2)}\|(n_i l_i)^2 1L_1\rangle|^2\Phi_\tau^2(T_1)$$
$$+(-)^{1/2+l-J}W(\tfrac{1}{2}\tfrac{1}{2}ll;1J)\cdot\frac{3}{2\sqrt{5}}\sum(2T_1+1)(2L_2+1)$$
$$\times W(lL_2l1;lL_2)W(L_1L_2 21;2L_2)$$
$$\times|\langle(nl)^2 1L_2\|S^{(2)}L^{(2)}\|(n_i l_i)^2 1L_1\rangle|^2\Phi_\tau^2(T_1)\Big].\tag{2·17}$$

Some details of this calculation are given in Appendix I. The last configuration (Fig. 7) which can be mixed with (2·1) and which can give the second order contribution to the doublet splitting is

$$\psi_{IIId,n}=|(n_1l_1)^{8l_1+4}(000),\cdots\{(n_il_i)^{8l_i+3}(\tfrac{1}{2}\tfrac{1}{2}l_i)(n_jl_j)^{8l_j+3}(\tfrac{1}{2}\tfrac{1}{2}l_j)\}$$
$$(T_1S_1L_1),\cdots(n_kl_k)^{8l_k+4}(000),\ (nl)^3(T_2S_2L_2),\ T=\tfrac{1}{2}(T_z)SL;JM\rangle.\tag{2·18}$$

By the same procedure as in Eq. (2·17), the second order correction due to these configurations may be calculated and becomes as follows,

$$\Delta E_{J,IIId}=-\sum\frac{1}{\Delta E_{on}}\Big[\Big\{\frac{1}{5}-\frac{1}{10(2l+1)}\Big\}\sum_{T_1,L_1,L_3}(2T_1+1)$$
$$\times|\langle(nl)^2 1L_3\|S^{(2)}L^{(2)}\|n_il_in_jl_j1L_1\rangle|^2\Phi_\tau^2(T_1)$$
$$+(-)^{1/2+l-J}W(\tfrac{1}{2}\tfrac{1}{2}ll;1J)\cdot\frac{3}{2\sqrt{5}}\sum_{T_1,L_1L_3}(2T_1+1)(2L_3+1)$$
$$\times W(lL_3l1;lL_3)W(L_1L_3 21;2L_3)$$
$$\times|\langle(nl)^2 1L_3\|S^{(2)}L^{(2)}\|n_il_in_jl_j1L_1\rangle|^2\Phi_\tau^2(T_1)\Big].\tag{2·19}$$

It is very interesting that, in Eqs. (2·17) and (2·19), the fractional parentage coefficients such as $\langle l^3TSL\{|l^2(T'S'L')l\rangle$ do not appear although they are inevitably used in the course of calculation.

In this section no special assumption about the average field has not been made, so that these formulae for the second order perturbation energy can be applied to any unperturbed system of independent particles. It is only necessary to estimate two-body matrix elements, for example, $\langle n_il_in_jl_j1L\|S^{(2)}L^{(2)}\|n_i'l_i'n_j'l_j'1L'\rangle$. However, these matrix elements are not easily calculated except by using the harmonic oscillator wave functions. If the harmonic oscillator wave functions are used and the wave functions of the two particle system are transformed into those of the relative and centre of mass coordinate system, the summations over the degenerate intermediate states of a same excitation energy can be carried out as in [I]. However, the matrix elements for the states which are excluded by the Pauli principle should be subtracted. This can be done by using the transformatoin coefficients between the wave functions in the two coordinate systems mentioned above. Then, the general formulae for the transformation coefficients will be investigated in the next section.

§ 3. The transformation coefficients between the wave functions in the two-particle coordinate system and the relative and centre of mass coordinate system

Two-body matrix elements can be easily evaluated, if the shell model wave function of the two-particle system can be expressed in terms of the wave functions of the relative and centre of mass coordinate system. If the single particle wave function is a plane wave, the wave function can be transformed very easily into the new coordinate system. And also, when the average field is taken to be a harmonic oscillator well, this transformation coefficient may be calculated by an elementary method, although it is not so easy.[5)]* In this section, the recurrence formula for the transformation coefficients between the wave functions in the two different coordinate systems is derived, and this formula is used to obtain the coefficients in the simple cases. At first, the spatial wave function of the two particles, (n_1l_1) and (n_2l_2), is expanded into the wave functions of the relative coordinate $(\boldsymbol{r}=\boldsymbol{r}_2-\boldsymbol{r}_1)$ and the centre of mass coordinate $(\boldsymbol{R}=\frac{1}{2}(\boldsymbol{r}_1+\boldsymbol{r}_2))$, and vice versa, that is,

$$|n_1l_1n_2l_2\,;LM\rangle=\sum_{\tilde{N}\tilde{L},\tilde{n}\tilde{l}}|\tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;LM\rangle\langle\tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;L|n_1l_1n_2l_2\,;L\rangle \tag{3·1a}$$

and

$$|\tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;LM\rangle=\sum_{n_1l_1,n_2l_2}|n_1l_1n_2l_2\,;LM\rangle\langle n_1l_1n_2l_2\,;L|\tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;L\rangle, \tag{3·1b}$$

where $\langle\tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;L|n_1l_1n_2l_2\,;L\rangle$ and $\langle n_1l_1n_2l_2\,;L|\tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;L\rangle$ are the transformation coefficients, $(\tilde{n},\tilde{l})$ are the quantum numbers of the relative wave function and $(\tilde{N},\tilde{L})$ are those of the centre of mass wave function. And it has to be borne in mind that the total energy in these two different systems must be equal, $2n_1+l_1+2n_2+l_2=2\tilde{n}+\tilde{l}+2\tilde{N}+\tilde{L}$.

In this equation (3·1), the radial wave functions for the $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ coordinates are given by

$$R_{n_il_i}(r_i)=N_{n_il_i}(\nu)\cdot\exp(-\nu r_i^2/2)\cdot r_i^{l}\cdot v_{n_il_i}(r_i), \tag{3·2}$$

where

$$N_{n_il_i}(\nu)=\left[\frac{2^{l_i-n_i+3}\cdot(2l_i+2n_i-1)!!\cdot\nu^{l_i+3/2}}{\pi^{1/2}\cdot(n_i-1)!\cdot\{(2l_i+1)!!\}^2}\right]^{1/2}$$

* Lawson and Mayer have recently made a similar calculation to that in this section (private communication).

and

$$v_{n_i l_i}(r_i)=\sum_{\kappa=0}^{n_i-1}(-)^{k}\cdot\frac{2^{k}\cdot(n_i-1)!\cdot(2l_i+1)!!}{k!\cdot(n_i-k-1)!\cdot(2l_i+2k+1)!!}\cdot(\nu r_i^2)^{\kappa},$$

The radial wave functions $R_{\tilde{n}\tilde{l}}(r)$ and $R_{\tilde{N}\tilde{L}}(R)$ for the relative and centre of mass motions are also written in the form of Eq. (3·2), but the ν in this equation must be replaced by $\nu/2$ and 2ν, respectively.

Now, the μ component of the operator $\boldsymbol{p}=(\nu\boldsymbol{r}-\boldsymbol{\nabla})/\sqrt{2\nu}$ brings the wave function $R_{nl}(r)\,Y_{lm}(\theta,\varphi)$ into

$$p_\mu\cdot R_{nl}(r)\,Y_{lm}(\theta,\varphi)=\sqrt{\frac{(l+1)(2l+2n+1)}{(2l+3)}}\cdot(l1\,m\mu|l+1\,m+\mu)$$
$$\times R_{n,l+1}(r)\,Y_{l+1,m+\mu}(\theta,\varphi)$$
$$+\sqrt{\frac{l\cdot 2n}{(2l-1)}}\cdot(l1\,m\mu|l-1\,m+\mu)\,R_{n+1,l-1}(r)\,Y_{l-1,m+\mu}(\theta,\varphi). \qquad (3\cdot3)$$

Therefore, by Eqs. (3·1b) and (3·3), the following equation is obtained,

$$\sum_\mu(1L_1\mu M_1|LM)\cdot\frac{2\nu R_\mu-\nabla_\mu}{2\sqrt{\nu}}\cdot|\tilde{N}\tilde{L}\tilde{n}\tilde{l};\,L_1M_1\rangle$$
$$=\sqrt{(\tilde{L}+1)(2\tilde{L}+2\tilde{N}+1)(2L_1+1)}\cdot W(1\tilde{L}L\tilde{l};\,\tilde{L}+1\,L_1)\cdot|\tilde{N}\tilde{L}+1,\,\tilde{n}\tilde{l};\,LM\rangle$$
$$+\sqrt{\tilde{L}\cdot2\tilde{N}\cdot(2L_1+1)}\cdot W(1\tilde{L}L\tilde{l};\,\tilde{L}-1\,L_1)\cdot|\tilde{N}+1\,\tilde{L}-1,\,\tilde{n}\tilde{l};\,LM\rangle$$
$$=\sqrt{\frac{2L_1+1}{2}}\sum_{n_1l_1,\,n_2l_2}\Big\{\sqrt{l_1\cdot(2l_1+2n_1-1)}\cdot W(1\,l_1-1\,L\,l_2;\,l_1L_1)$$
$$\times\langle n_1l_1-1,\,n_2l_2;\,L_1|\tilde{N}\tilde{L},\,\tilde{n}\tilde{l};\,L_1\rangle$$
$$+\sqrt{(l_1+1)(2n_1-2)}\cdot W(1\,l_1+1\,L\,l_2;\,l_1L_1)\cdot\langle n_1-1\,l_1+1,\,n_2l_2;\,L_1|\tilde{N}\tilde{L},\,\tilde{n}\tilde{l};\,L_1\rangle$$
$$+(-)^{1+L_1-L}\cdot\sqrt{l_2\cdot(2l_2+2n_2-1)}\cdot W(l_1\,l_2-1\,L1;\,L_1l_2)$$
$$\times\langle n_1l_1,\,n_2l_2-1;\,L_1|\tilde{N}\tilde{L},\,\tilde{n}\tilde{l};\,L_1\rangle$$
$$+(-)^{1+L_1-L}\cdot\sqrt{(l_2+1)(2n_2-2)}\cdot W(l_1\,l_2+1\,L1;\,L_1l_2)$$
$$\times\langle n_1l_1,\,n_2-1\,l_2+1;\,L_1|\tilde{N}\tilde{L},\,\tilde{n}\tilde{l};\,L_1\rangle\Big\}\,|n_1l_1n_2l_2;\,LM\rangle. \qquad (3\cdot4)$$

In this derivation, the relation

$$P_{R,\mu}=\frac{2\nu R_\mu-\nabla_{R,\mu}}{2\sqrt{\nu}}=\frac{1}{\sqrt{2}}(p_{1,\mu}+p_{2,\mu}) \qquad (3\cdot5)$$

was used. By using the ortho-normality of the Racah coefficients, the following recurrence formulae are derived,

$$\langle n_1 l_1, n_2 l_2; L|\tilde{N}\tilde{L}+1, \tilde{n}\tilde{l}; L\rangle$$
$$=\frac{(2\tilde{L}+3)}{\sqrt{2(\tilde{L}+1)(2\tilde{L}+2\tilde{N}+1)}}\sum_{L_1}(2L_1+1)W(1\tilde{L}L\tilde{l};\tilde{L}+1L_1)$$
$$\times\Big\{\sqrt{l_1(2l_1+2n_1-1)}W(1l_1-1Ll_2;l_1L_1)\langle n_1 l_1-1, n_2 l_2; L_1|\tilde{N}\tilde{L}, \tilde{n}\tilde{l}; L_1\rangle$$
$$+\sqrt{(l_1+1)(2n_1-2)}W(1l_1+1Ll_2;l_1L_1)\langle n_1-1 l_1+1, n_2 l_2; L_1|\tilde{N}\tilde{L}, \tilde{n}\tilde{l}; L_1\rangle$$
$$+(-)^{1+L_1-L}\sqrt{l_2(2l_2+2n_2-1)}W(1l_2-1Ll_1;l_2L_1)$$
$$\times\langle n_1 l_1, n_2 l_2-1; L_1|\tilde{N}\tilde{L}, \tilde{n}\tilde{l}; L_1\rangle$$
$$+(-)^{1+L_1-L}\sqrt{(l_2+1)(2n_2-2)}W(1l_2+1Ll_1;l_2L_1)$$
$$\times\langle n_1 l_1, n_2-1 l_2+1; L_1|\tilde{N}\tilde{L}, \tilde{n}\tilde{l}; L_1\rangle\Big\}. \qquad (3\cdot 6)$$

Next the transformation coefficients for $\tilde{N}\tilde{L}=1\tilde{S}$ and $2\tilde{S}$ are necessary to be computed for starting the calculation with the recurrence formula. This can be easily done as follows. The directions of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are assumed to be same, and this implies that

$$\theta_1=\theta_2=\Theta=\theta \qquad (3\cdot 7)$$

and

$$\phi_1=\phi_2=\Phi=\phi \qquad (3\cdot 8)$$

in the case of $r_2 > r_1$. Then, on account of the identity,

$$\sum_{m_1,m_2}(l_1 l_2 m_1 m_2|LM)\,Y_{l_1 m_1}(\theta,\phi)\,Y_{l_2 m_2}(\theta,\phi)$$
$$=\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}}(l_1 l_2 00|L0)\,Y_{LM}(\theta,\phi), \qquad (3\cdot 9)$$

Eq. (3·1a) becomes

$$N_{n_1 l_1}(\nu)N_{n_2 l_2}(\nu)\cdot r_1^{l_1}\cdot v_{n_1 l_1}(r_1)\cdot r_2^{l_2}\cdot v_{n_2 l_2}(r_2)\cdot\sqrt{(2l_1+1)(2l_2+1)}\cdot(l_1 l_2 00|L0)$$
$$=\sum_{\tilde{N}\tilde{L},\tilde{n}\tilde{l}}N_{\tilde{N}\tilde{L}}(2\nu)N_{\tilde{n}\tilde{l}}(\nu/2)\cdot R^{\tilde{L}}\cdot v_{\tilde{N}\tilde{L}}(R)\cdot r^{\tilde{l}}\cdot v_{\tilde{n}\tilde{l}}(r)\cdot\sqrt{(2\tilde{L}+1)(2\tilde{l}+1)}$$
$$\times(\tilde{L}\tilde{l}00|L0)\cdot\langle\tilde{N}\tilde{L}, \tilde{n}\tilde{l}; L|n_1 l_1 n_2 l_2; L\rangle, \qquad (3\cdot 10)$$

where the common factors are omitted. From Eqs. (3·7) and (3·8),

$$r_1=R-r/2 \qquad (3\cdot 11)$$

and

$$r_2=R+r/2. \qquad (3\cdot 12)$$

Then, in the left-hand side of Eq. (3·10), r_1 and r_2 can be replaced by R and r. After expanding the left-hand side of Eq. (3·10) in terms of R and r, and comparing the terms of $R^0\cdot r^{2\tilde{n}+\tilde{l}-2}$ in both sides, the transformation coefficient $\langle 1\tilde{S}, \tilde{n}\tilde{l}\,;\tilde{l}|n_1l_1, n_2l_2\,;\tilde{l}\rangle$ are obtained as follows,

$$\langle 1\tilde{S}, \tilde{n}\tilde{l}\,;\tilde{l}|n_1l_1, n_2l_2\,;\tilde{l}\rangle=(-)^{n_1+n_2+l_1-\tilde{n}+1}\cdot 2^{(1/4)(l_1+l_2-3\tilde{l})-\tilde{n}+1}$$

$$\times\sqrt{\frac{(\tilde{n}-1)!\cdot(2\tilde{l}+2\tilde{n}-1)!!(2l_1+1)(2l_2+1)}{(n_1-1)!(n_2-1)!(2l_1+2n_1-1)!!(2l_2+2n_2-1)!!(2\tilde{l}+1)}}(l_1l_200|\tilde{l}0), \quad (3\cdot13)$$

where $$2\tilde{n}+\tilde{l}=2n_1+l_1+2n_2+l_2-2.$$

In the right-hand side of Eq. (3·10), there are two terms that have factor $R^2\cdot r^{2\tilde{n}+\tilde{l}-4}$, and one of them comes from $|1\tilde{S}, \tilde{n}\tilde{l}\,;\tilde{l}\rangle$ and the other from $|2\tilde{S}, \tilde{n}-1\tilde{l}\,;\tilde{l}\rangle$. By inserting Eq. (3·13) into Eq. (3·10) and comparing the terms of $R^2\cdot r^{2\tilde{n}+\tilde{l}-4}$ in both sides, the transformation coefficients

$$\langle 2\tilde{S}, \tilde{n}\tilde{l}\,;\tilde{l}|n_1l_1, n_2l_2\,;\tilde{l}\rangle=(-)^{(1/2)(\tilde{l}+l_1-l_2)}\cdot 2^{(1/4)(l_1+l_2-3\tilde{l})-\tilde{n}}$$

$$\times\sqrt{\frac{(\tilde{n}-1)!\cdot(2\tilde{l}+2\tilde{n}-1)!!\cdot(2l_1+1)(2l_2+1)}{3\cdot(n_1-1)!(n_2-1)!(2l_1+2n_1-1)!!\cdot(2l_2+2n_2-1)!!\cdot(2\tilde{l}+1)}}$$

$$\times[2(n_1-n_2)^2+2(n_1-n_2)(l_1-l_2)+(n_2+n_2-2)-\tfrac{1}{2}(l_1+l_2+\tilde{l}+1)(l_1+l_2-\tilde{l})]$$

$$\times(l_1l_200|\tilde{l}0) \quad (3\cdot14)$$

are derived. These procedures may be applied to obtain the transformation coefficients $\langle\tilde{N}\tilde{S}, \tilde{n}\tilde{l}\,;\tilde{l}|n_1l_1, n_2l_2\,;\tilde{l}\rangle$.

For $(\tilde{N}\tilde{L})=(1\tilde{P})$, Eq. (3·13) and the recurrence formula (3·6) give

$$\langle 1\tilde{P}, \tilde{n}\tilde{l}\,;\tilde{l}+1|n_1l_1, n_2l_2\,;\tilde{l}+1\rangle$$

$$=-\frac{1}{(2\tilde{l}+3)\times\sqrt{\tilde{l}+1}}\cdot[2(\tilde{l}+1)(n_1-n_2)+(l_1+l_2+\tilde{l}+2)(l_1-l_2)]$$

$$\times(l_1l_200|\tilde{l}+10)\cdot C(n_1l_1, n_2l_2\,;\tilde{n}\tilde{l}),$$

$$\langle 1\tilde{P}, \tilde{n}\tilde{l}\,;\tilde{l}\,|n_1l_1, n_2l_2\,;\tilde{l}\rangle$$

$$=\sqrt{\frac{(l_1+l_2+\tilde{l}+2)(l_1+l_2-\tilde{l}+1)(l_1-l_2+\tilde{l}+1)(l_2-l_1+\tilde{l})}{\tilde{l}\cdot(\tilde{l}+1)\cdot(2\tilde{l}+1)}}$$

$$\times(l_1+1l_200|\tilde{l}0)\cdot C(n_1l_1, n_2l_2\,;\tilde{n}\tilde{l}),$$

and

$$\langle 1\tilde{P}, \tilde{n}\tilde{l}\,;\tilde{l}-1|n_1l_1, n_2l_2\,;\tilde{l}-1\rangle$$

$$=\frac{1}{(2\tilde{l}-1)\sqrt{\tilde{l}}}\cdot[2\tilde{l}\cdot(n_1-n_2)-(l_1+l_2-\tilde{l}+1)(l_1-l_2)]$$

$$\times(l_1l_200|\tilde{l}-10)\cdot C(n_1l_1, n_2l_2\,;\tilde{n}\tilde{l}),$$

where

$$C(n_1l_1, n_2l_2; \tilde{n}\tilde{l}) = (-)^{(1/2)(\tilde{l}+l_1-l_2+1)} \cdot 2^{(1/4)(l_1+l_2+1-3\tilde{l})-\tilde{n}}$$
$$\times \sqrt{\frac{(\tilde{n}-1)! \cdot (2\tilde{l}+2\tilde{n}-1)!! \cdot (2l_1+1)(2l_2+1)}{(n_1-1)! \cdot (n_2-1)! \cdot (2l_1+2n_1-1)!! \cdot (2l_2+2n_2-1)!!}}.$$

Further the transformation coefficients for $(\widetilde{N}\widetilde{L}) = (1\widetilde{D})$ are derived as

$$\langle 1\widetilde{D}, \tilde{n}\tilde{l}; \tilde{l}+2|n_1l_1, n_2l_2; \tilde{l}+2\rangle = -\frac{1}{2(2\tilde{l}+5)\sqrt{(2\tilde{l}+2)(2\tilde{l}+3)(2\tilde{l}+4)}}$$
$$\times[\{2(\tilde{l}+1)(n_1-n_2)+(l_1+l_2+\tilde{l}+1)(l_1-l_2)\}$$
$$\times\{2(\tilde{l}+2)(n_1-n_2)+(l_1+l_2+\tilde{l}+3)(l_1-l_2)\}$$
$$-(\tilde{l}+2)(l_1+l_2+\tilde{l}+1)(l_1+l_2+\tilde{l}+3)-4(\tilde{l}+1)(\tilde{l}+2)(n_1+n_2-2)$$
$$-2(l_1-l_2)(n_1-n_2)(l_1+l_2-\tilde{l}-1)]\times(l_1l_200|\tilde{l}+20)\cdot K(n_1l_1, n_2l_2; \tilde{n}\tilde{l}),$$

$$\langle 1\widetilde{D}, \tilde{n}\tilde{l}; \tilde{l}+1|n_1l_1, n_2l_2; \tilde{l}+1\rangle$$
$$= \sqrt{\frac{(l_1+l_2+\tilde{l}+2)(l_1-l_2+\tilde{l}+1)(l_2-l_1+\tilde{l}+1)(l_1+l_2-\tilde{l})}{2\tilde{l}\cdot(2\tilde{l}+1)(2\tilde{l}+2)(2\tilde{l}+3)(2\tilde{l}+4)}}$$
$$\times\{2\tilde{l}\cdot(n_1-n_2)+(l_1-l_2)(l_1+l_2+\tilde{l}+1)\}\cdot(l_1l_200|\tilde{l}0)K(n_1l_1, n_2l_2; \tilde{n}\tilde{l}),$$

$$\langle 1\widetilde{D}, \tilde{n}\tilde{l}; \tilde{l}|n_1l_1, n_2l_2; \tilde{l}\rangle = \frac{1}{\sqrt{2\cdot3\cdot(2\tilde{l}-1)2\tilde{l}\cdot(2\tilde{l}+1)(2\tilde{l}+2)(2\tilde{l}+3)}}$$
$$\times[2(n_1-n_2)(l_1-l_2)\{3(l_1+l_2+1)+2\tilde{l}(\tilde{l}+1)\}+(l_1+l_2-\tilde{l})(l_1+l_2+\tilde{l}+1)$$
$$\times\{-3(l_1-l_2)^2+2\tilde{l}(\tilde{l}+1)\}+4(n_1-n_2)^2\tilde{l}(\tilde{l}+1)-4(n_1+n_2-2)\tilde{l}(\tilde{l}+1)]$$
$$\times(l_1l_200|\tilde{l}0)\cdot K(n_1l_1, n_2l_2; \tilde{n}\tilde{l}),$$

$$\langle 1\widetilde{D}, \tilde{n}\tilde{l}; \tilde{l}-1|n_1l_1, n_2l_2; \tilde{l}-1\rangle$$
$$= \sqrt{\frac{(l_1+l_2+\tilde{l}+1)(l_1-l_2+\tilde{l})(l_2-l_1+\tilde{l})(l_1+l_2-\tilde{l}+1)}{(2\tilde{l}-2)(2\tilde{l}-1)2\tilde{l}\cdot(2\tilde{l}+1)(2\tilde{l}+2)}}$$
$$\times\{2(\tilde{l}+1)(n_1-n_2)-(l_1-l_2)(l_1+l_2-\tilde{l})\}\cdot(l_1l_200|\tilde{l}0)\cdot K(n_1l_1, n_2l_2; \tilde{n}\tilde{l}),$$

and

$$\langle 1\widetilde{D}, \tilde{n}\tilde{l}; \tilde{l}-2|n_1l_1, n_2l_2; \tilde{l}-2\rangle = -\frac{1}{2\cdot(2\tilde{l}-3)\sqrt{(2\tilde{l}-2)(2\tilde{l}-1)\cdot2\tilde{l}}}$$
$$\times[\{-2\tilde{l}(n_1-n_2)+(l_1+l_2-\tilde{l})(l_1-l_2)\}$$
$$\times\{-2(\tilde{l}-1)(n_1-n_2)+(l_1+l_2-\tilde{l}+2)(l_1-l_2)\}$$
$$+(\tilde{l}-1)(l_1+l_2-\tilde{l})(l_1+l_2-\tilde{l}+2)-4\tilde{l}\cdot(\tilde{l}-1)(n_1+n_2-2)$$
$$-2(l_1-l_2)(n_1-n_2)(l_1+l_2+\tilde{l})](l_1l_200|\tilde{l}-20)\cdot K(n_1l_1, n_2l_2; \tilde{n}\tilde{l}),$$

where

$$K(n_1l_1,\, n_2l_2\,;\, \tilde{n}\tilde{l}) = (-)^{(1/2)(\tilde{l}+l_1-l_2)} \cdot 2^{(1/4)(l_1+l_2-3\tilde{l})-\tilde{n}+1}$$

$$\times \sqrt{\frac{(\tilde{n}-1)!\cdot(2\tilde{l}+2\tilde{n}-1)!!\cdot(2l_1+1)\,(2l_2+1)}{(n_1-1)!\cdot(n_2-1)!\cdot(2l_1+2n_1-1)!!\cdot(2l_2+2n_2-1)!!}}\,.$$

By these methods, other transformation coefficients may be calculated. However, in the present paper, the above coefficients are enough for estimating the second order effect of the tensor force.

The transformation coefficients used in the following section are tabulated in Appendix II.

Finally, some remarks are given on the relations between the transformation coefficients. By the transformation of the coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ into the coordinates $\boldsymbol{x}_1=\sqrt{2}\boldsymbol{R}=(\boldsymbol{r}_1+\boldsymbol{r}_2)/\sqrt{2}$ and $\boldsymbol{x}_2=\boldsymbol{r}/\sqrt{2}=(\boldsymbol{r}_2-\boldsymbol{r}_1)/\sqrt{2}$ we obtain the relation,

$$\langle \tilde{N}\tilde{L}(\boldsymbol{R})\tilde{n}\tilde{l}(\boldsymbol{r})\,;\, L|n_1l_1(\boldsymbol{r}_1)n_2l_2(\boldsymbol{r}_2)\,;\, L\rangle$$

$$=(-)^{\tilde{l}-l_2}\langle n_1l_1(\boldsymbol{X})n_2l_2(\boldsymbol{x})\,;\, L|\tilde{N}\tilde{L}(\boldsymbol{x}_1)\tilde{n}\tilde{l}(\boldsymbol{x}_2)\,;\, L\rangle$$

where $\boldsymbol{X}=(\boldsymbol{x}_1+\boldsymbol{x}_2)/2$ and $\boldsymbol{x}=\boldsymbol{x}_2-\boldsymbol{x}_1$.

The transformation coefficient $\langle \tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;\, TSL|n_1l_1n_2l_2\,;\, TSL\rangle$ of the antisymmetrized wave function $|n_1l_1n_2l_2\,;\, TSL\rangle$ can be obtained by the following relation:

$$\langle \tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;\, TSL|n_1l_1n_2l_2\,;\, TSL\rangle$$

$$=\begin{cases} \dfrac{1-(-)^{T+S+\tilde{l}}}{2}\sqrt{2}\,\langle \tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;\, L|n_1l_1n_2l_2\,;\, L\rangle & \text{for } n_1l_1 \neq n_2l_2 \\ \dfrac{1-(-)^{T+S+\tilde{l}}}{2}\langle \tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;\, L|nlnl\,;L\rangle & \text{for } n_1l_1=n_2l_2=nl. \end{cases}$$

§ 4. Numerical calculation

In this section the general formulae are applied to the case of the D-state doublet splitting in O^{17}. The numerical results have been obtained using the two kinds of the potentials as in [I]. One of them is the phenomenological tensor potential of the Serber type,[6)]

$$v_T=\tfrac{1}{4}(1-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\cdot S_{12}\cdot V_0\cdot\exp(-r^2/r_t^2), \tag{4·1}$$

where $V_0=-25.8$ Mev, $r_t=2.41\times10^{-13}$ cm and S_{12} is the tensor force operator. The others are the meson theoretic tensor potential[7)] for the triplet odd state;

$$v_T=\begin{cases} V_T^{(1\pi)}(\kappa r) & (\kappa r \geq 1.0) \\ 0 & (\kappa r < 1.0) \end{cases} \tag{4·2a}$$

and that for the triplet even state:

$$v_T = \begin{cases} V_T^{(1\pi)}(\kappa r) & (\kappa r \geq 0.7) \\ 3V_T^{(1\pi)}(0.7) & (\kappa r < 0.7) \end{cases} \tag{4·2b}$$

In the above equation, the one pion exchange potential is given by

$$V_T^{(1\pi)}(\kappa r) = \frac{g_e^2}{4\pi} \cdot \mu c^2 \cdot \frac{(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)}{3} \cdot S_{12} \cdot \left(1 + \frac{3}{\kappa r} + \frac{3}{\kappa^2 r^2}\right) \cdot \exp(-\kappa r)/\kappa r,$$

where $\boldsymbol{\tau}$ is the isotopic spin operator, $\kappa^{-1} = \hbar / c\mu$ the Compton wave length of a pion, and 0.08 is used for the coupling constant $g_e^2/4\pi$.

The calculated splitting energies, $\Delta\Delta E$, are graphically shown in Fig. 8. as the function of the parameter $\rho\,(10^{-13}\text{cm}) \equiv (\nu/2)^{-1/2}$ which measures the extension of the harmonic oscillator wave function. Now, ρ can be estimated from the Coulomb energy difference[8] between O^{17} and F^{17}, and also from the high energy electron scattering experiment,[9] if the wave function of O^{17} is assumed to be a shell model one. From these experiments the following numerical value is obtained, $\rho = 2.37$. In the case of the Gaussian potential (4·1), the doublet splitting energy corresponding to this value of ρ is about a half of the observed value (5.08 Mev), while the splitting energy is too small in the case of the meson theoretic one. However, the results are rather sensitive to ρ, as can be seen from Fig. 8, and the splitting of the correct magnitude can be obtained, if ρ is some 20% smaller than the above value. There appears to be some reasons for using a smaller ρ value than the above value, $\rho = 2.37$. At first, the numerical calculation shows the large mixing probability of higher configurations into the zeroth order shell model configuration, and the wave functions of these higher configurations spread out more than that of the zeroth order configuration. Therefore, if the effect of the mixed higher configurations is taken into account, ρ should become smaller than the above value. Next, the effects of the strong correlation in the closed shell core are very important. The main effect of the correlation on the doublet splitting seems to come from the change of ρ, for the effect of the short range correlation function taking care of the singular repulsive core potentials is much reduced. It is because of r^2 or higher power of r in the integrand of the tensor matrix element which comes from the radial wave function, as the matrix element vanishes between S-states. Dabrowski[10] calculated the binding energy of O^{16} using the variational trial function in the form

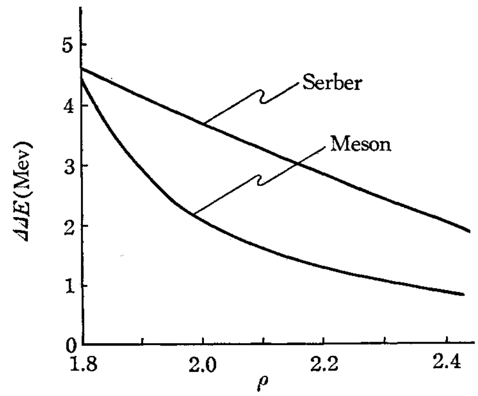

Fig. 8 Dependence of the doublet splitting energy $\Delta\Delta E$ in O^{17} on the parameter ρ.

$$\Phi(1, \cdots, A) = N \cdot \prod_{i>j}^{A} f(r_{ij}) \, \Phi_0(1, \cdots, A) \tag{4·3}$$

where N is a normalization constant and

$$f(r_{ij}) = \begin{cases} 0 & \text{for } r_{ij} < a \\ 1 - \exp[-\beta\{(r_{ij}/a)^2 - 1\}] & \text{for } r_{ij} \geq a \end{cases} \tag{4·4}$$

are the short range correlation function and

$$\Phi_0(1, \cdots, A) = (A\,!)^{-1/2} \cdot \det\{\phi_k(i)\}, \tag{4·5}$$

with single particle orbitals $\phi_k(i)$'s. By this method he determined $\rho = 1.57$ for $a = 0.2 \times 10^{-13}$ cm which is rather small. Sawicki and Folk[11] used this result in their calculation of the effect of the two-body spin-orbit force on the doublet splitting in O^{17}. If we also take this value, the calculated doublet splitting becomes larger than 5 Mev for both potentials, as can be expected from Fig. 8. Furthermore, for example, if $\rho = 2.37$ and 2.00 are used in cases of the induced deformation (II in § 2) and the self-deformation (III in § 2), respectively, we can obtain $\Delta\Delta E \sim 4$ Mev for both potentials because the positive contribution from the self-deformation becomes larger. From these considerations, it may be said that at least a considerable amount of the observed doublet splitting can be explained in terms of the tensor force.

§ 5. Discussions

Through the present calculation it has been found that qualitatively same situations as those in He^5 and N^{15} also hold in the case of O^{17}, i.e., the important effects on getting the splitting are that (1) the tensor force is strong and (2) the deformation of the closed shell core induced by the tensor interaction between the core-nucleons are restricted so as to satisfy the Pauli principle with the outside nucleon (see Table I). Therefore it may generally be concluded that at least a considerable part of the experimental spin-orbit splitting is explained in terms of the second order effect of the tensor force on account of the above mentioned effects.

Table 1. The doublet splitting energies due to the configurations (II) and (III), in case of $\rho = 2.00$. (in Mev)

Configuration	(II)	(III)
Serber	−1.6	5.2
Meson	−5.5	7.5

Some problems should be solved for obtaining more definite conclusion. Numerical calculations show that the mixing percentage of the configurations of the higher excitation energies into the zeroth order configuration is very large and becomes, for example, about 50%. However, the major part of the mixed configurations comes from the self-deformation of the closed shell core and does not contribute to the moments, i.e., magnetic moment, quadrupole moment, etc., because the total spin of the closed shell part in these mixed configurations is zero. On the other hand, the induced deformation of the closed shell core and the effect of the Pauli principle on the self-deformation induce only small mixing of the higher configurations as can be seen from Fig. 9, where the mixing percentage (P) of the configurations of the excitation energy $2N\hbar\omega$ is plotted as a function of N. From these results, if the wave function of the closed shell core could be obtained in good accuracy, it might be expected that the effects of the outside nucleon added to the closed shell core could reasonably be treated by the perturbation method. And such calculation would be useful also for settling other problems, i. e., (1) there are some ambiguities in the determination of the parameter ρ, as discussed in § 4, and (2) the effect of the higher order perturbation seems to be not so small.

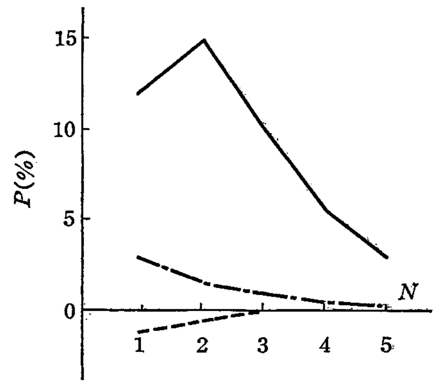

Fig. 9 Mixing percentages of various configurations.
——— Configuration I and III without the Pauli principle.
—·—·— Configuration II
- - - - - - - Effect of the Pauli principle in configuration III

In the present paper and [I], only the doublet splitting in the bound states has been calculated. Then finally, we shall briefly discuss the spin-orbit coupling in the high energy nucleon-nucleus scattering. In this case, the effect of the Pauli principle mentioned above is much reduced. The reason for this is two-fold; first because the overlap of the self-deformed closed shell core with the incident nucleon becomes much smaller, and second because the corresponding energy denominator in the second order perturbation becomes larger. On the other hand, as has been shown by several authors,[3)14)] the effect of the induced deformation can reasonably explain the spin-orbit coupling in the high-energy nucleon-nucleus scattering, although its contribution is very small or negative in the case of the bound states. Therefore, the spin-orbit couplings in these two cases seem to be caused by the different effects.

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Appendix I.

According to the usual method, the matrix element can be expressed as follows,

$$\langle (n_1 l_1)^{8l_1+4}(000),\cdots(n_i l_i)^{8l_i+2}(T_1 S_1 L_1),\cdots(n_k l_k)^{8l_k+4}(000),\ (nl)^3(T_2 S_2 L_2),$$
$$T=\tfrac{1}{2}(T_z)SL;\ JM|V_T|(n_1 l_1)^{8l_1+4}(000),\cdots(n_i l_i)^{8l_i+4}(000),\cdots(n_k l_k)^{8l_k+4}(000)\,nl,$$
$$T=\tfrac{1}{2}(T_z)S=\tfrac{1}{2}\ L=l;\ JM\rangle$$
$$=\sum_{L_3}(-)^{T_2-T_1+L+L_1-L_3-J}\cdot\sqrt{\frac{3}{2}}$$
$$\times\sqrt{(2T_2+1)(2S_2+1)(2L_2+1)(2S+1)(2L+1)}\langle l^3 T_2 S_2 L_2\{|l^2(T_1 1 L_3)l\rangle$$
$$\times W(SL\tfrac{1}{2}l;\ J2)\,W^2(11S\tfrac{1}{2};\ 2S_2)\,W(L_1 L_3 Ll;\ 2L_2)$$
$$\times\langle(nl)^2 1L_3\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|(n_i l_i)^2 1L_1\rangle\,\Phi_\tau(T_1). \qquad (\mathrm{A}\cdot 1)$$

The corresponding second order energy is

$$\Delta E_J=-\frac{1}{\Delta E_{on}}\sum\frac{3}{2}\cdot(2T_2+1)(2S_2+1)(2L_2+1)(2S+1)(2L+1)$$
$$\times W^2(SL\tfrac{1}{2}l;\ J2)\,W^2(11S\tfrac{1}{2};\ 2S_2)\,W(L_1 L_3 Ll;\ 2L_2)\,W(L_1\widetilde{L}_3 Ll;\ 2L_2)$$
$$\times\langle l^3 T_2 S_2 L_2\{|l^2(T_1 1 L_3)l\rangle\langle l^3 T_2 S_2 L_2\{|l^2(T_1 1\widetilde{L}_3)l\rangle$$
$$\times\langle(nl)^2 1L_3\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|(n_i l_i)^2 1L_1\rangle\langle(nl)^2 1\widetilde{L}_3\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|(n_i l_i)^2 1L_1\rangle\,\Phi_\tau^2(T_1). \qquad (\mathrm{A}\cdot 2)$$

Using the relations

$$W^2(SL\tfrac{1}{2}l;\ J2)=\sum_x(-)^{1/2+l-J-x}\cdot(2x+1)\,W(\tfrac{1}{2}l\tfrac{1}{2}l;\ Jx)\,W(L2lx;\ l2)$$
$$\times W(S2\tfrac{1}{2}x;\ \tfrac{1}{2}2),$$
$$\sum_S(2S+1)\,W^2(11S\tfrac{1}{2};\ 2S_2)\,W(S2\tfrac{1}{2}x;\ \tfrac{1}{2}2)=W(112x;\ 21)\,W(S_2 1\tfrac{1}{2}x;\ \tfrac{1}{2}1)$$

and

$$\sum_L(2L+1)\,W(L2lx;\ l2)\,W(L_1 L_3 Ll;\ 2L_2)\,W(L_1\widetilde{L}_3 Ll;\ 2L_2)=W(L_2 L_3 lx;\ l\widetilde{L}_3)$$
$$\times W(L_1 L_3 2x;\ 2\widetilde{L}_3),$$

and summing over S and L, we obtain

$$\Delta E_J=-\frac{1}{\Delta E_{on}}\cdot(-)^{1/2+l-J}\cdot\sum\frac{3}{2}\cdot(2T_2+1)(2S_2+1)(2L_2+1)\cdot(-)^x\cdot(2x+1)$$
$$\times W(\tfrac{1}{2}l\tfrac{1}{2}l;\ Jx)\,W(112x;\ 21)\,W(S_2 1\tfrac{1}{2}x;\ \tfrac{1}{2}1)\,W(L_2 L_3 lx;\ l\widetilde{L}_3)$$
$$\times W(L_1 L_3 2x;\ 2\widetilde{L}_3)\langle l^3 T_2 S_2 L_2\{|l^2(T_1 1 L_3)l\rangle\langle l^3 T_2 S_2 L_2\{|l^2(T_1 1\widetilde{L}_3)l\rangle$$
$$\times\langle(nl)^2 1L_3\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|(n_i l_i)^2 1L_1\rangle\langle(nl)^2 1\widetilde{L}_3\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|(n_i l_i)^2 1L_1\rangle\,\Phi_\tau^2(T_1)$$
$$=-\frac{1}{\Delta E_{on}}\left[\frac{1}{20(2l+1)}\sum\frac{(2T_2+1)(2S_2+1)(2L_2+1)}{(2L_3+1)}\right.$$
$$\times\langle l^3 T_2 S_2 L_2\{|l^2(T_1 1 L_3)l\rangle^2\cdot|\langle(nl)^2 1L_3\|\boldsymbol{S}^{(2)}\boldsymbol{L}^{(2)}\|(n_i l_i)^2 1L_1\rangle|^2\,\Phi_\tau^2(T_1)$$

$$-(-)^{1/2+l-J}W(\tfrac{1}{2}l\tfrac{1}{2}l\,;J1)\cdot\frac{9}{4\sqrt{5}}\cdot\sum(2T_2+1)(2S_2+1)(2L_2+1)$$
$$\times\langle l^3T_2S_2L_2\{|l^2(T_11L_3)l\rangle^2W(S_21\tfrac{1}{2}1\,;\tfrac{1}{2}1)W(L_2L_3l1\,;lL_3)W(L_1L_321\,;2L_3)$$
$$\times|\langle(nl)^21L_3\|S^{(2)}L^{(2)}\|(n_il_i)^21L_1\rangle|^2\Phi_\tau^2(T_1)]. \quad (\text{A}\cdot3)$$

Combining the relations,

$$\langle l^3(T_2S_2L_2)\{|l^2(T_11L_3)l\rangle^2=\frac{(8l+2)(2T_1+1)(2L_3+1)}{(2T_2+1)(2S_2+1)(2L_2+1)}$$
$$\times\langle l^{8l+2}(T_11L_3)\{|l^{8l+1}(T_2S_2L_2)l\rangle^2$$

and

$$\langle l^{8l+2}T_11L_3\|\sum_i s_i\cdot l_i\|l^{8l+2}T_11L_3\rangle=3\cdot(8l+2)\sum(2L_3+1)W(S_21\tfrac{1}{2}1\,;\tfrac{1}{2}1)$$
$$\times W(L_2L_3l1\,;lL_3)\langle\tfrac{1}{2}\|s\|\tfrac{1}{2}\rangle\langle l\|l\|l\rangle\langle l^{8l+2}T_11L_3\{|l^{8l+1}(T_2S_2L_2)l\rangle^2,$$

Eq. (A·3) is rewritten as

$$\Delta E_J=-\frac{1}{\Delta E_{on}}\Bigg[\frac{(8l+2)}{20(2l+1)}\cdot\sum(2T_1+1)\cdot|\langle(nl)^21L_3\|S^{(2)}L^{(2)}\|(n_il_i)^21L_1\rangle|^2\Phi_\tau^2(T_1)$$
$$-(-)^{1/2+l-J}W(\tfrac{1}{2}l\tfrac{1}{2}l\,;J1)\cdot\frac{3}{4\sqrt{5}}\cdot\sum(2T_1+1)\cdot\frac{\langle l^{8l+2}T_11L_3\|\sum_i s_il_i\|l^{8l+2}T_11L_3\rangle}{\langle\frac{1}{2}\|s\|\frac{1}{2}\rangle\cdot\langle l\|l\|l\rangle}$$
$$\times W(L_1L_321\,;2L_3)|\langle(nl)^21L_3\|S^{(2)}L^{(2)}\|(n_il_i)^21L_1\rangle|^2\Phi_\tau^2(T_1)\Bigg]. \quad (\text{A}\cdot4)$$

Since

$$\langle l^{8l+2}T_11L_3\|\sum_i s_i\cdot l_i\|l^{8l+2}T_11L_3\rangle=-\langle l^2T_11L_3\|\sum_i s_i\cdot l_i\|l^2T_11L_3\rangle$$
$$=-2(2L_3+1)W(lL_3l1\,;lL_3)\langle\tfrac{1}{2}\|s\|\tfrac{1}{2}\rangle\langle l\|l\|l\rangle,$$

Eq. (A·4) reduces to Eq. (2·17) in § 2.

Appendix II.

Table of $2^{2N}\langle\tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;L|n_1l_1n_2l_2\,;L\rangle^2$

The transformation coefficient multiplied by 2^N, $2^N\langle\tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;L|n_1l_1n_2l_2\,;L\rangle$, is minus or plus the square root of the entry in the table according to whether this entry is, or is not, preceded by an asterisk. Here, it should be noted that, for example, the square root of $(\alpha N+\beta)^2$ means not $|\alpha N+\beta|$ but $(\alpha N+\beta)$.

$\tilde{N}\tilde{L}$ $\tilde{n}\tilde{l}$; n_1l_1 n_2l_2 ; L	$2^{2N}\langle\tilde{N}\tilde{L}\tilde{n}\tilde{l}\,;L\|n_1l_1n_2l_2\,;L\rangle^2$
$1S$ Nl ; $1s$ Nl ; l	2^{2-l}
$1P$ Ns ; $1s$ Np ; 1	$2(2N+1)/3$
$1P$ Np ; $1s$ $(N+1)$ s ; 0	$2N$

$1P\ Np\ ;\ 1s\ \ Nd\ \ ;\ 2$	$2(2N+3)/5$
$1P\ Nd\ ;\ 1s\ (N+1)\ p\ ;\ 1$	$2N/3$
$1P\ Nd\ ;\ 1s\ \ Nf\ \ ;\ 3$	$3(2N+5)/7\cdot 2$
$1P\ Nf\ ;\ 1s\ (N+1)\ d\ ;\ 2$	$3N/5\cdot 2$
$1P\ Nf\ ;\ 1s\ \ Ng\ \ ;\ 4$	$(2N+7)/9$
$1P\ Ng\ ;\ 1s\ (N+1)\ f\ ;\ 3$	$N/7$
$1P\ Ng\ ;\ 1s\ \ Nh\ \ ;\ 5$	$5(2N+9)/11\cdot 8$
$1D\ Np\ ;\ 1s\ (N+1)\ p\ ;\ 1$	$2(2N+3)N/5\cdot 3$
$1D\ Np\ ;\ 1s\ \ Nf\ \ ;\ 3$	$3(2N+5)(2N+3)/7\cdot 5\cdot 2$
$1D\ Nd\ ;\ 1s\ (N+2)\ s\ ;\ 0$	$(N+1)N/3$
$1D\ Nd\ ;\ 1s\ (N+1)\ d\ ;\ 2$	$(2N+5)N/7\cdot 3$
$1D\ Nd\ ;\ 1s\ \ Ng\ \ ;\ 4$	$(2N+7)(2N+5)/7\cdot 3\cdot 2$
$1D\ Nf\ ;\ 1s\ (N+2)\ p\ ;\ 1$	$(N+1)N/5\cdot 2$
$1D\ Nf\ ;\ 1s\ (N+1)\ f\ ;\ 3$	$(2N+7)N/9\cdot 5$
$1D\ Nf\ ;\ 1s\ \ Nh\ \ ;\ 5$	$5(2N+9)(2N+7)/11\cdot 9\cdot 4$
$2S\ Np\ ;\ 1s\ (N+1)\ p\ ;\ 1$	$(2N+3)N/3\cdot 2$
$2S\ Nf\ ;\ 1s\ (N+1)\ f\ ;\ 3$	$(2N+7)N/4\cdot 3\cdot 2$
$1F\ Nd\ ;\ 1s\ (N+2)\ p\ ;\ 1$	$(2N+5)(N+1)N/7\cdot 5\cdot 2$
$1F\ Nd\ ;\ 1s\ (N+1)\ f\ ;\ 3$	$(2N+7)(2N+5)N/9\cdot 7\cdot 5$
$1F\ Nd\ ;\ 1s\ \ Nh\ \ ;\ 5$	$5(2N+9)(2N+7)(2N+5)/11\cdot 9\cdot 7\cdot 4$
$1S\ Nd\ ;\ 1p\ \ Np\ \ ;\ 2$	$*2(2N+3)/5$
$1S\ Nd\ ;\ 1p\ (N-1)\ f\ ;\ 2$	$*6(N-1)/5$
$1S\ Ng\ ;\ 1p\ (N-1)\ h\ ;\ 4$	$*5(N-1)/9\cdot 2$
$1S\ Nh\ ;\ 1p\ (N-1)\ i\ ;\ 5$	$*3(N-1)/11\cdot 2$
$1P\ Np\ ;\ 1p\ \ Np\ \ ;\ 1$	4
$1P\ Np\ ;\ 1p\ \ Np\ \ ;\ 2$	$*16(N-1)^2/5^2$
$1P\ Np\ ;\ 1p\ (N-1)\ f\ ;\ 2$	$*12(2N+3)(N-1)/5^2$
$1P\ Nd\ ;\ 1p\ (N+1)\ s\ ;\ 1$	$*2(2N+3)N/9$
$1P\ Nd\ ;\ 1p\ \ Nd\ \ ;\ 1$	$*2(2N-3)^2/9$
$1P\ Nd\ ;\ 1p\ \ Nd\ \ ;\ 2$	2

$1P\ Nd\,;\ 1p\quad Nd\quad ;\ 3$	$*(6N+1)^2/7^2\cdot 2$
$1P\ Nd\,;\ 1p\ (N-1)\ g\,;\ 3$	$*12(2N+5)(N-1)/7^2$
$1P\ Nf\,;\ 1p\quad Nf\quad ;\ 2$	$*(3N-5)^2/5^2$
$1P\ Nf\,;\ 1p\quad Nf\quad ;\ 3$	1
$1P\ Nf\,;\ 1p\quad Nf\quad ;\ 4$	$*(4N+5)^2/9^2$
$1P\ Nf\,;\ 1p\ (N-1)\ h\,;\ 4$	$*10(2N+7)(N-1)/9^2$
$1P\ Ng\,;\ 1p\ (N-1)\ i\,;\ 5$	$*15(2N+9)(N-1)/11^2\cdot 2$
$1D\ Ns\,;\ 1p\quad Np\quad ;\ 2$	$*2(2N+1)(2N-7)^2/5^2\cdot 3$
$1D\ Ns\,;\ 1p\ (N-1)\ f\,;\ 2$	$*2(2N+3)(2N+1)(N-1)/5^2$
$1D\ Np\,;\ 1p\ (N+1)\ s\,;\ 1$	$*2(2N-7)^2N/5\cdot 3^2$
$1D\ Np\,;\ 1p\ (N-1)\ g\,;\ 3$	$*12(2N+5)(2N+3)(N-1)/7^2\cdot 5$
$1D\ Nd\,;\ 1p\ (N+1)\ p\,;\ 0$	$*2N(N-1)^2/3$
$1D\ Nd\,;\ 1p\ (N+1)\ p\,;\ 1$	$2N$
$1D\ Nd\,;\ 1p\ (N+1)\ p\,;\ 2$	$*8N(N-1)^2/7\cdot 5\cdot 3$
$1D\ Nd\,;\ 1p\quad Nf\quad ;\ 2$	$*2(2N+5)(N-1)^2/7\cdot 5$
$1D\ Nd\,;\ 1p\quad Nf\quad ;\ 3$	$2(2N+5)/7$
$1D\ Nd\,;\ 1p\quad Nf\quad ;\ 4$	$*8(2N+5)(N-1)^2/9\cdot 7\cdot 3$
$1D\ Nf\,;\ 1p\ (N+2)\ s\,;\ 1$	$*(2N+5)(N+1)N/5\cdot 3\cdot 2$
$1D\ Nf\,;\ 1p\quad Ng\quad ;\ 3$	$*(2N+7)(4N-5)^2/9\cdot 7\cdot 5\cdot 2$
$1D\ Nf\,;\ 1p\quad Ng\quad ;\ 4$	$(2N+7)/3\cdot 2$
$1D\ Nf\,;\ 1p\quad Ng\quad ;\ 5$	$*(2N+7)(10N+1)^2/11^2\cdot 4\cdot 3^2$
$1D\ Nf\,;\ 1p\ (N-1)\ i\,;\ 5$	$*5(2N+9)(2N+7)(N-1)/11^2\cdot 3$
$1D\ Ng\,;\ 1p\ (N+2)\ p\,;\ 2$	$*3(2N+7)(N+1)N/7\cdot 5^2$
$1D\ Ng\,;\ 1p\ (N+1)\ f\,;\ 2$	$*(3N-7)^2N/7\cdot 5^2$
$1D\ Ng\,;\ 1p\ (N+1)\ f\,;\ 3$	$5N/7\cdot 2$
$1D\ Ng\,;\ 1p\ (N+1)\ f\,;\ 4$	$*5(4N+7)^2N/11\cdot 9\cdot 7\cdot 3\cdot 2$
$2S\ Np\,;\ 1p\quad Nd\quad ;\ 1$	$*2(2N+3)(N-2)^2/9$
$2S\ Nd\,;\ 1p\ (N+1)\ p\,;\ 2$	$*(2N+1)^2N/5\cdot 3\cdot 2$
$2S\ Nd\,;\ 1p\quad Nf\quad ;\ 2$	$*(2N+5)(N-2)^2/5\cdot 2$
$2S\ Nf\,;\ 1p\ (N+1)\ d\,;\ 3$	$*(2N+3)^2N/7\cdot 4\cdot 2$

$2S\ Nf;\ 1p\ \ Ng\ ;\ 3$	$*(2N+7)(N-2)^2/7\cdot 3$
$1F\ Nd;\ 1p\ (N+2)\ s;\ 1$	$*(2N-9)^2(N+1)N/7\cdot 5\cdot 3\cdot 2$
$1F\ Nd;\ 1p\ (N+1)\ d;\ 1$	$*2(2N+5)N^3/7\cdot 5\cdot 3$
$1F\ Nd;\ 1p\ (N+1)\ d;\ 2$	$4(2N+5)N/7\cdot 5$
$1F\ Nd;\ 1p\ (N+1)\ d;\ 3$	$*(2N+5)(2N-5)^2N/7^2\cdot 5\cdot 3$
$1F\ Nd;\ 1p\ \ Ng\ ;\ 3$	$*(2N+7)(2N+5)(4N-3)^2/7^2\cdot 5\cdot 3^2\cdot 2$
$1F\ Nd;\ 1p\ \ Ng\ ;\ 4$	$(2N+7)(2N+5)/7\cdot 3\cdot 2$
$1F\ Nd;\ 1p\ \ Ng\ ;\ 5$	$*(2N+7)(2N+5)\cdot(10N-21)^2/11^2\cdot 7\cdot 4\cdot 3^2$
$1F\ Nd;\ 1p\ (N-1)\ i;\ 5$	$*5(2N+9)(2N+7)(2N+5)(N-1)/11^2\cdot 7\cdot 3$
$2P\ Nd;\ 1p\ (N+2)\ s;\ 1$	$*(2N+1)^2(N+1)N/9\cdot 5\cdot 2$
$2P\ Nd;\ 1p\ (N+1)\ d;\ 1$	$*(2N+5)(2N-5)^2N/9\cdot 5\cdot 2$
$2P\ Nd;\ 1p\ (N+1)\ d;\ 2$	$(2N+5)\cdot N/5\cdot 2$
$2P\ Nd;\ 1p\ (N+1)\ d;\ 3$	$*(2N+5)(6N-5)^2N/7^2\cdot 5\cdot 4\cdot 2$
$2P\ Nd;\ 1p\ \ Ng\ ;\ 3$	$*3(2N+7)(2N+5)(N-2)^2/7^2\cdot 5$
$1S\ Ns;\ 1d\ (N-2)\ d;\ 0$	$16(N-1)(N-2)/3$
$1S\ Nd;\ 1d\ \ Ns\ ;\ 2$	$(2N+3)(2N+1)/5\cdot 3$
$1S\ Nd;\ 1d\ (N-1)\ d;\ 2$	$4(2N+3)(N-1)/7\cdot 3$
$1S\ Nd;\ 1d\ (N-2)\ g;\ 2$	$24(N-1)(N-2)/7\cdot 5$
$1S\ Ng;\ 1d\ \ Nd\ ;\ 4$	$(2N+7)(2N+5)/7\cdot 3\cdot 2$
$1S\ Ng;\ 1d\ (N-1)\ g;\ 4$	$10\cdot(2N+7)(N-1)/11\cdot 7\cdot 3$
$1S\ Ng;\ 1d\ (N-2)\ i;\ 4$	$5(N-1)(N-2)/11\cdot 3$
$1P\ Nd;\ 1d\ \ Np\ ;\ 1$	$2(2N+3)(2N-1)^2/9\cdot 5$
$1P\ Nd;\ 1d\ (N-1)\ f;\ 1$	$8(N-1)(N-3)^2/5\cdot 3$
$1P\ Nd;\ 1d\ \ Np\ ;\ 2$	$*2(2N+3)/5$
$1P\ Nd;\ 1d\ (N-1)\ f;\ 2$	$*16(N-1)/5$
$1P\ Nd;\ 1d\ \ Np\ ;\ 3$	$(2N+3)(6N-13)^2/7^2\cdot 5\cdot 2$
$1P\ Nd;\ 1d\ (N-1)\ f;\ 3$	$4(N-1)(2N-1)^2/7\cdot 5\cdot 3$
$1P\ Nd;\ 1d\ (N-2)\ h;\ 3$	$20(2N+5)(N-1)(N-2)/7^2\cdot 3$
$1D\ Nd;\ 1d\ \ Nd\ ;\ 0$	$4\{(N-1)^2-(N-1)+1\}^2/9$
$1D\ Nd;\ 1d\ \ Nd\ ;\ 1$	$*4(N-1)^2$

$1D\ Nd\,;\ 1d\quad Nd\quad ;\ 2$	$16\{(N-1)^2-(N-1)+7\}^2/7^2\cdot 3^2$
$1D\ Nd\,;\ 1d\ (N+1)\ \ s\ ;\ 2$	$(2N+3)(2N-9)^2N/9\cdot 7\cdot 5$
$1D\ Nd\,;\ 1d\ (N-1)\ \ g\ ;\ 2$	$8(2N+5)(N-1)(N-2)^2/7^2\cdot 5$
$1D\ Nd\,;\ 1d\quad Nd\quad ;\ 3$	$*16(N-1)^2/7^2$
$1D\ Nd\,;\ 1d\ (N-1)\ \ g\ ;\ 3$	$*20(2N+5)(N-1)/7^2$
$1D\ Nd\,;\ 1d\quad Nd\quad ;\ 4$	$\{4(N-1)^2-4(N-1)-21\}^2/7^2\cdot 3^2$
$1D\ Nd\,;\ 1d\ (N-1)\ \ g\ ;\ 4$	$80(2N+5)(N-1)(N-2)^2/11\cdot 7^2\cdot 3^2$
$1D\ Nd\,;\ 1d\ (N-2)\ \ i\ ;\ 4$	$10(2N+7)(2N+5)(N-1)(N-2)/11\cdot 9\cdot 7$
$2S\ Nd\,;\ 1d\ (N+1)\ \ s\ ;\ 2$	$(2N+3)(2N-3)^2N/9\cdot 5\cdot 4$
$2S\ Nd\,;\ 1d\quad Nd\quad ;\ 2$	$\{2(N-1)^2+(N-1)-7\}^2/9\cdot 7$
$2S\ Nd\,;\ 1d\ (N-1)\ \ g\ ;\ 2$	$2(2N+5)(N-1)(N-4)^2/7\cdot 5$

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