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Spin systems on hierarchical lattices. Introduction and thermodynamic limit

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A number of exactly soluble models in statistical mechanics can be produced with the use of spins interacting with nearest neighbors on a hierarchical lattice. A general definition and several examples of such lattices are given, and the topological properties of one of these, the "diamond" lattice, are discussed in detail. It is shown that the free energy has a well-defined thermodynamic limit for a large class of discrete spin models on hierarchical lattices.

I. INTRODUCTION

It is difficult to overestimate the importance of exactly soluble models for the development of the statistical mechanics of phase transitions. One has only to note Onsager's solution[1] of the two-dimensional Ising model or Baxter's solution[2] of the eight-vertex model to realize that exact solutions can have a profound effect on theoretical ideas. To be sure, few exactly soluble models have been as influential as those just mentioned, but even the less important examples have played a very significant role, both in terms of explicit results and as a guide to and a test of approximate methods.

In this paper we shall be concerned with classical "spin" models, such as Ising and Potts models, on hierarchical lattices.[3] These constitute a large and quite diverse class of exactly soluble models exhibiting a wide variety of phase transitions, only a few of which have thus far been studied in any detail. Examples of hierarchical lattices will be found in Sec. II below, and a general definition is given in Sec. V. (It is important not to confuse soluble models on hierarchical *lattices* with Dyson's soluble *hierarchical model* and its generalizations[4]; as far as we know, the two are quite distinct.) We shall discuss certain topological properties of hierarchical lattices in Sec. III, and present a proof of the existence of a thermodynamic limit for the free energy of spin models on a class of such lattices in Sec. IV.

Cayley trees ("Bethe lattice with a boundary") form an important subclass of hierarchical lattices, and the properties of the corresponding spin systems have been studied extensively.[5–11] Another important subclass, of which the "diamond" hierarchical lattice (Sec. II) is a simple example, has developed from the important observation[12,13] that various *approximate* real-space renormalization-group schemes applied to spin systems on Bravais lattices are actually *exact* if the spins are, instead, placed on an appropriate hierarchical lattice. Several papers treating models in this subclass as objects of interest in and of themselves (and not simply as approximations for the corresponding Bravais lattices) have now appeared.[14–17] Still another subclass consists of various models with no phase transitions at finite temperature, though transitions can occur in the limit of zero temperature.[18,19]

It is worth emphasizing that hierarchical lattices have very different geometrical and topological properties from Bravais lattices, as is intuitively obvious in the case of Cayley trees, and less obvious but nevertheless true (see the discussion of the "diamond" hierarchical lattice in Sec. III) in other cases. This means that the results one obtains for soluble models on hierarchical lattices may be misleading in terms of Bravais lattices (and vice versa). Nevertheless, the former may well be a valuable source of insights and ideas which can be applied to, or at least tried out on the latter. Furthermore, an exploration of the ways in which hierarchical and Bravais lattice models differ from each other could yield additional understanding of the basic statistical mechanics of phase transitions.

As noted above, there is a close connection between hierarchical lattices and certain approximate real-space renormalization-group methods. Those approximate methods which are "realizable", in the sense that there is some hierarchical lattice for which the recursion formulas are exact, can be guaranteed to lead to "sensible" thermodynamic properties[12] in the sense that if there is a thermodynamic limit for the free energy, this limit possesses appropriate convexity properties[20]: positive heat capacities, susceptibilities, etc. (Not all ap-

proximate real-space methods are "realizable" in this sense, and it is perhaps worth noting that there are cases[21] in which the approximation would be realizable *except* that the recursion equations for the free energy violates the relationship $B=b^d$ of Sec. III below.) In addition, in the case of those realizable real-space methods which do give rise to surprising results, an examination of the corresponding hierarchical lattice may provide some insight into the source of the difficulties.[15] This is of some importance because, despite the considerable practical success of the renormalization-group approach to phase transitions, there remain some serious doubts[22] as to the mathematical validity of these procedures, and their range of validity.

Among the more interesting recent developments[16] in the theory of spin systems on hierarchical lattices is the study of cases with competing interactions giving rise to "frustration." The corresponding nonlinear recursion relations can exhibit period doubling and chaotic flows. Hierarchical lattices have also been used in the study of random conductance and percolation.[23]

The contents of the remainder of this paper is as follows. Section II gives selected examples indicating the general nature of hierarchical lattices and suggesting something of the enormous variety which is possible. (A few of these will be treated in more detail in a subsequent paper.) In Sec. III there is a discussion of the topology of hierarchical lattices, regarded as finite or infinite graphs, with special emphasis on the "diamond" lattice as an illustration of the very inhomogeneous structure which can be present. A proof of the existence of a thermodynamic limit for a fairly general class of discrete spin models on hierarchical lattices will be found in Sec. IV. Those features which seem (to us) most essential for characterizing hierarchical lattices in an abstract sense are written down as a definition of a general hierarchical lattice in Sec. V.

II. EXAMPLES OF HIERARCHICAL LATTICES

We begin with the case discussed by Berker and Ostlund,[12] in which the lattice is generated in an iterative manner as shown in Fig. 1. This figure may be interpreted in two different ways. The first, which we call "aggregation," is that four of the primitive or order zero bonds sketched in 1(a) are assembled to form a unit shown in 1(b), a bond of order 1. Then four of these bonds of order 1 are assembled in precise analogy with the step leading

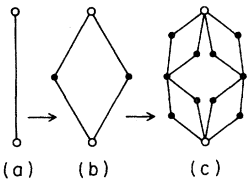

FIG. 1. Construction of the diamond hierarchical lattice.

from 1(a) to 1(b) to form a bond of order 2, shown in 1(c). The process can be iterated an arbitrary number of times to form a bond of arbitrarily large order. We shall call this a "diamond" hierarchical lattice; the name is suggested by Fig. 1(b).

An alternative interpretation of Fig. 1 which is often useful is that of "miniaturization": What appear in 1(a) is actually a bond of order $N(\geq 2)$, but the drawing shows only its *surface* sites, the two vertices at the top and bottom, and not its internal structure. The latter begins to appear in 1(b), where we see that the original bond is actually a composite formed of four smaller bonds of order $N-1$. To obtain 1(c), we start with 1(b) and expand each of the bonds in the latter in a manner precisely analogous to the step from 1(a) to 1(b). Again, this process can be iterated an arbitrary number of times.

An Ising model on the diamond hierarchical lattice is constructed by associating an Ising spin variable $\sigma_i=\pm 1$ with the ith vertex of the lattice, and assigning a dimensionless interaction $(-\mathcal{H}/kt)$ of the form

$$H^{(0)}=K_0\sigma_i\sigma_j+\tfrac{1}{2}h_0(\sigma_i+\sigma_j) \tag{2.1}$$

to the primitive bond which joins sites i and j. Here K_0 is the dimensionless exchange interaction and h_0 is proportional to the magnetic field. The dimensionless Hamiltonian for a composite unit is obtained by adding up the contributions from the primitive bonds of which it is composed. Thus for the bond of order 1 in Fig. 1(b) we have

$$H^{(1)}=K_0(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1) +h_0(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\,, \tag{2.2}$$

where h_0 appears without the $\frac{1}{2}$ found in (2.1) because at each site there is a contribution from two primitive bonds.

Of course it is possible to place other classical (non-quantum-mechanical) "spin" systems, such as Potts models, XY models, etc., on a diamond or other hierarchical lattice. The essential feature which makes such models soluble by iterative procedures is the requirement that the interactions extend no

further than between the sites at the two ends of a primitive bond.

Generating procedures for various other examples of hierarchical lattices are shown in Figs. 2, 3, and 4. In Fig. 2 we show, as in Fig. 1, two steps of the aggregation or miniaturization process in order to remove all ambiguity as to what is meant by the first step. The interpretation of Figs. 2(a) and 2(b) is quite similar to that of Fig. 1. However, 2(c) displays a new feature. The dashed line represents an interaction for the spins at its ends of the form (2.1), but with a different choice of constants. However, during the iteration procedure this bond is neither "aggregated" nor "miniaturized"; it remains invariant. The constants which are associated with this interaction, however, can be made to depend on the state of aggregation in the sense that when four order-n bonds are put together to form one of order $n+1$, the new "dashed" interactions added at this stage can depend on n.

Figures 2(d) and 2(e) show two different ways of producing what is, in effect, very similar to a finite portion of a Bethe lattice or Cayley tree of coordination number 3. Indeed, with appropriate choices of constants (in this case, those associated with the dashed lines do not depend on n), one obtains the problem studied by Eggarter[6] and Müller-Hartmann and Zittartz.[7] Finally, Fig. 2(f) indicates how one can, if desired, include closed loops.

There is no reason why the objects which are aggregated (or miniaturized) need be bonds. Figure 3

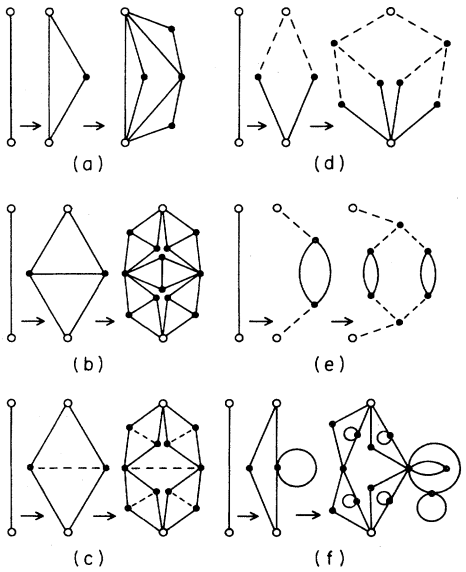

FIG. 2. Examples of hierarchical lattices.

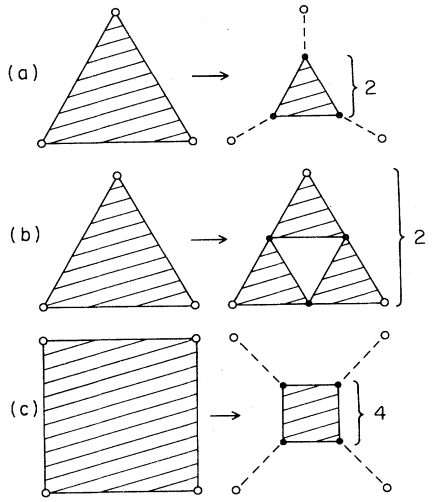

FIG. 3. Additional examples of hierarchical lattices.

shows some examples of generating procedures for other units. Owing to the added complexity, a second stage in the iteration is not shown. The unit on the right side indicated by the bracket actually occurs a number of times given by the integer, with the units "glued together" at the vertices indicated by solid circles. Thus in 3(a) one is to imagine two separate triangles connected at each of the three inner vertices; repeated iteration then proceeds to produce a structure whose elements resemble portions of a Cayley tree with coordination number 3. In 3(b) the right side consists of six triangles in two sheets, one above the other, with the open vertices (of which there are three), but not the solid vertices (of which there are six) common to both sheets. Finally, Fig. 3(c) shows schematically what results if Kadanoff's lower bound bond-shifting approximation[24] for the Ising square lattice is regarded as the exact solution for the problem on a hierarchical lattice. The dashed lines represent exchange interactions equal to Kadanoff's p parameter, and the dashed circle a set of two-spin and four-spin interactions which depend on p.

Figure 4 shows a case in which the lattice involves pieces which are iterated separately. That is to say, at each stage of aggregation not one but two new units are formed by employing the two types of units available at the earlier stage.

The examples we have discussed are only a small fraction of the possible hierarchical structures which lead to statistical problems which can (in principle) be solved by iterative methods. In each of the cases considered above the lattices are *self-similar* in the sense that the aggregation procedure

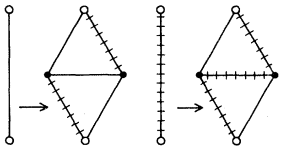

FIG. 4. Example of a nonuniform hierarchical lattice.

is identical each time units of the next higher order are assembled, and *uniform* in that (with the exception of Fig. 4) identical units are assembled at each step. Neither of these properties is essential though both of them facilitate the process of determining the thermodynamic and other properties of the corresponding statistical model. (For a discussion of what we do regard as essential properties, see Sec. V.)

We have deliberately used the term "uniform" rather than "homogeneous" because several of the examples in Fig. 1 to 3 are rather more inhomogeneous as assembled lattices than the uniformity of the individual units might at first suggest, as will be discussed in detail for the diamond lattice in Sec. III. This inhomogeneity of the lattice leads (in general) to inhomogeneous local statistical properties of the corresponding Ising (or other) models.

III. TOPOLOGY OF HIERARCHICAL LATTICES

The purpose of this section is to discuss certain geometrical or topological properties of hierarchical lattices, regarded as graphs, which are of importance for the thermodynamic and statistical properties of Ising and similar lattice models. The discussion is limited to self-similar and uniform lattices, as defined at the end of Sec. II, though some of the definitions and procedures have a broader application. A labeling procedure for the diamond hierarchical lattice is given in some detail because of its utility for discussing the inhomogeneous character of this lattice.

The *aggregation number* B of a hierarchical lattice is the number of subunits assembled at each step to form a new unit. For the diamond lattice, Fig. 1., $B=4$, and for the six cases shown in Fig. 2, B is 3, 5, 4, 2, 2, and 5, respectively. Note that the number of noniterated "dashed" bonds is not counted when determining B, nor is the number of additional vertices [as in Fig. 2(d) or 3(a)] which must be added to those already present in the subunits. For the cases shown in Fig. 3, B is 2, 6, and 4. (In a lattice which is not self-similar, B can vary from step to step.)

The quantity B is often denoted by b^d when a hierarchical lattice is used as a renormalization-group approximation for a Bravais lattice of dimension d, where b is the factor by which the linear dimension of the Bravais lattice is decreased at each step. However, it is not clear how to define d or b for a hierarchical lattice (at least in general), and for the purpose of discussing thermodynamic critical exponents it is B rather than d or b which is important. Thus we shall not refer to d (or b) again in this paper.

A unit in a hierarchical lattice will, in general, contain sites or vertices of several different sorts. We shall use the term *bound sites* for sites which are part of one or more subunits, and *free sites* for those which are not. That is to say, when a unit is assembled out of subunits, the free sites must be added to those already present in the subunit. The *surface sites* of a unit of a given order are those at which units of this order are attached to other units of the same order—either by making sites coincide, or adding noniterated bonds between sites—to form units of higher order. That is to say, they play a definite role in the aggregation procedure by which units of higher order are formed. Sites which are not surface sites are *interior* sites. In our diagrammatic representation, these are the sites which make their appearance during "miniaturization." The surface sites are denoted by open circles in Figs. 1, 2, and 3. Sometimes, as in Fig. 1, 2(a), 3(b), etc., these surface sites are bound sites, while in Fig. 2(e), 3(a), and 3(c) they are free sites. For the examples shown in Figs. 1—3 the free sites are also surface sites, but this need not be the case in general. We shall always assume that a unit of finite order contains only a finite number of sites (of whatever type).

It is also convenient to employ the terms "surface" and "interior" for the subunits (of whatever order) which make up a particular unit. If at least one surface site of a subunit coincides with a surface site of the unit under consideration, the subunit is a *surface subunit* or "on the surface"; otherwise it is an *interior subunit* or "in the interior." Inspection of Fig. 1(c) shows that in the diamond lattice all bonds of order $N-1$, but only half the bonds of order $N-2$, are on the surface of a bond of order N.

The terms surface and interior suggest a rough but useful analogy between a single unit of large but finite order in a hierarchical lattice and a finite portion of a Bravais lattice, such as that part within a

large cube. Proofs of the thermodynamic limit for spin systems on a Bravais lattice make use of the fact that in a large cube or other "thick" object the fraction of sites on or near the surface is small compared with the fraction in the interior. Certain hierarchical lattices possess a similar property, which we shall now discuss.

We shall call l the *separation index* of a hierarchical lattice provided a unit of order $N \geq l$ has at least one unit of order $N-l$ in its interior, and provided l is the smallest integer for which this is the case. For the diamond hierarchical lattice, $l=2$. For the case shown in Fig. 2, $l=2$ for 2(a) and 2(c)—note that the dashed line in the latter does not constitute a "unit" as we use the term—$l=1$ for 2(b), 2(e), and 2(f); 2(d) does not possess a separation index ($l=\infty$), since the subunits are never in the interior. For Fig. 3, $l=1$ for 3(a) and 3(c), and 2 for 3(b).

Provided a hierarchical lattice possesses a separation index, it is "thick" in the sense that the ratio of the number of subunits of a given *fixed* order n on the surface of a unit of order $N \geq n$ to the number of similar subunits in the interior tends to zero as N tends to infinity. That this is true can be seen as follows. There is at least one subunit of order $N-l$ in the interior, so at most B^l-1 such subunits are on the surface of a unit of order N. But subunits of order $N-2l$ are themselves subunits of the units of order $N-l$, and as the former can only be on the surface of the unit of order N if they are on the surfaces of units of order $N-l$ which are on the surface of the unit of order N, it follows that no more than $(B^l-1)^2$ units of order $N-2l$, or a fraction $(1-B^{-l})^2$ of the total, can be on the surface of the unit of order N. This argument may be continued, and of course $(1-B^{-l})^p$ goes to zero as $p \to \infty$.

In order to provide a systematic description of a hierarchical lattice, it is useful to introduce labels for the sites (vertices) and edges or other components of the lattice. A convenient way of doing this is to assign to a unit of order N one of B symbols in a manner which indicates its location ("address") within the unit of order $N+1$ which contains it. The *label* for the unit of order N consists of this symbol followed on the right by the corresponding symbol for the unit of order $N+1$ which contains it, followed by the symbol for the unit of order $N+2$ which contains the unit of order $N+1$, etc. The label is of finite or infinite length depending on whether the lattice is finite or infinite, and may be preceded by something which designates the order of the unit under consideration, when this is not clear from the context. Similarly, a site can be labeled with a symbol indicating its location within a unit of a particular order, followed by the label for this unit.

This procedure as applied to the diamond lattice is illustrated in Fig. 5, where the symbols attached to the four bonds are $+r$, $-r$, $+s$, and $-s$. For convenience in making the labels, the $+$ signs have been omitted and the $-$ signs placed above the symbols: $\bar{r}$, $\bar{s}$. We use the convention that within a bond of order N the symbols $+r$ and $-r$ are used for the bonds of order $N-1$ attached to that surface site of the bond of order N which has the higher coordination number in the larger structure of which this bond is a part, and $+s$ and $-s$ are used for the remaining two bonds of order $N-1$. That the coordination number of the two surface bonds will be different is, unfortunately, not apparent in Fig. 5(a), nor even in 5(b) in which four of the diamonds have been assembled to form a diamond of one higher order. However, from the fact that the surface sites in Fig. 5(b)—the open circles—will have a coordination number of at least 8 when this is incorporated in a still larger structure, one sees that each of the four smaller diamonds extends from a site of coordination number 4 to one of coordination number 8 or more. As a consequence the first symbol labeling each of the (primitive) edges in 5(b) is r or $\bar{r}$ if that edge is adjacent to one of the open circles, and otherwise it is s or $\bar{s}$. By contrast with r and s, the use of $+$ or $-$ is a matter of indifference, apart from the convention, evident in Fig. 5(a), that the edges $\bar{r}$ and $\bar{s}$ are adjacent to the same vertex, while $\bar{r}$ and s are not.

Thus a bond or order N in the diamond lattice contains 4^N primitive bonds labeled by a string of symbols $\beta_1 b_1, \beta_2 b_2 \ldots$, or $[\beta, b]$ for short, in which β_j, the jth element of β, is $+1$ or -1, and b_j, the jth element of b, is r or s. We shall say that two primitive bonds are related to each other by or are equivalent under symmetry if one can be mapped onto the other by an element of the automorphism group of the bond of order N, regarded as a graph. This group consists of one-to-one maps of the set of

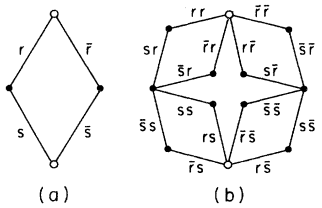

FIG. 5. Labels for the bonds of a diamond lattice.

vertices onto itself in a manner which preserves the structure of edges (primitive bonds), and which carry each surface vertex onto itself. (The two surface vertices are treated in a special way because they are obviously inequivalent to all the other vertices and to each other when the bond of order N is imbedded in a larger structure.) In the same way, two bonds of order $n<N$ are equivalent under symmetry if one can be mapped onto the other by an element of the automorphism group.

In fact, two primitive bonds (β,b) and (β',b') are related by symmetry if and only if $b=b'$. To see this, we note that if $b=b'$ and $\beta=\beta'$ except $\beta_1\neq\beta'_1$, the two bonds can be mapped into each other by a "rotation" of the bond of order 1 which contains both of them (see Fig. 5). Since the lattice is self-similar, the same argument shows that bonds of order one can be mapped onto each other if they differ only in β_2, i.e., in the sign of the symbol which gives their location within the bond of order two which contains them, by a rotation of this bond. This procedure can be continued so as to generate an appropriate map whenever $\beta\neq\beta'$ but $b=b'$. On the other hand, suppose that $b_1\neq b'_1$. Then one of the bonds (s) connects vertices of coordination number 2 and 4, the other (r) vertices of coordination number 2 and 8 or more. Thus they are obviously inequivalent. If $b_1=b'_1$ but $b_2\neq b'_2$, the bonds of order one containing each of the primitive bonds cannot be mapped onto each other as their surface sites do not have the same coordination numbers, and thus the primitive bonds in one are inequivalent to the primitive bonds in the other. A similar argument can be applied if $b_j\neq b'_j$ for any j.

A large diamond hierarchical lattice is actually a very inhomogeneous structure, despite the initial impression given by Fig. 1. The preceding discussion shows that the class of primitive bonds equivalent to one another under symmetry in a bond of order N numbers 2^N, which is a fraction 2^{-N} of the total number of primitive bonds. Similarly, it is easy to show that the $2(4^{N-n})$ vertices of coordination number 2^n for $n\leq N-1$ in a bond of order N divide up into equivalence classes of $2(2^{N-n})$ elements, so that the fraction in each equivalence class tends to zero as N becomes infinite. (This is in marked contrast to a Bravais lattice in which all sites are equivalent under translational symmetry.) Thus one must in general anticipate a corresponding inhomogeneity in the Gibbs probability distribution for an Ising (or similar) model on a diamond lattice, even when the same interactions are associated with all of the primitive bonds.

The inhomogeneity of finite hierarchical diamond lattices has an importance consequence for the infinite lattice, which is not unique. By an infinite lattice we shall mean a collection of primitive bonds labeled $[\beta,b]$, where β and b are now infinite sequences of appropriate symbols, which form a connected graph. The requirement of connectivity is a natural one for statistical mechanics, because the probability distribution for a disconnected system is a product of the distributions of its connected components, as there are no interactions between disconnected pieces. Two primitive bonds $[\beta,b]$ and $[\beta',b']$ are in the same (connected) lattice provided they are both contained in some larger composite bond of finite order, that is, if there is some $k<\infty$ (which of course depends on the pair of primitive bonds one is considering) such that

$$\beta'_j=\beta_j,\quad b'_j=b_j \tag{3.1}$$

for all $j\geq k$.

Consequently the infinite graph consisting of primitive bonds corresponding to all possible symbols $[\beta,b]$ splits up into an infinite collection of connected components, each of which is an infinite lattice. Our previous discussion of symmetries then shows that two of these lattices are equivalent, in the sense of a one-to-one map of one onto the other which preserves the edge structure, if and only if they contain precisely the same set of b symbols for their primitive bonds (i.e., strings of r and s ignoring the $+$ or $-$ sign). Thus there are an (uncountably) infinite number of *inequivalent* infinite connected lattices which can be generated by the procedure indicated in Fig. 1. For example, the primitive bonds $rsrsrs\cdots$ and $rrsrrsrrs\cdots$ (repeating periodically) belong to inequivalent infinite lattices. While these inequivalent lattices lead to the same thermodynamic properties (under appropriate restrictions discussed in Sec. IV below), the corresponding Gibbs states will, in general, be different. Such a state of affairs is quite different from what encounters in the case of models with the translational invariance of a Bravais lattice.

Many other hierarchical lattices are extremely inhomogeneous in the sense just discussed. In particular this is true of the cases shown in Figs. 2(a)–2(c) and 2(f), and in Fig. 3(b). On the other hand, those corresponding to Figs. 2(d) and 2(e), and Figs. 3(a) and 3(c) are much more symmetrical, in the sense that while not all sites are equivalent, the number in a particular equivalence class is a finite fraction of the total as $N\rightarrow\infty$.

IV. THERMODYNAMIC LIMIT FOR THE FREE ENERGY

In this section we shall examine conditions for the existence of a thermodynamic limit for the free energy for Ising and similar models associated with uniform and self-similar hierarchical lattices of the sort described in Sec. II. The essential results of this section are contained in theorems 1 and 2. We shall illustrate various points of the argument with reference to an Ising model on a diamond hierarchical lattice, but of course the theorems apply in much more general cases (though hardly the most general case in which one could imagine a well-defined free energy).

We shall assume that at each step in the iteration process a collection of $B \geq 2$ identical units of order $N-1$ are assembled along with a certain number of free sites to form a unit of order N. (Self-similarity implies that the number of free sites added is independent of N.) As a matter of *terminological convenience, we shall throughout this section consider all free sites to be surface sites,* whether or not they are actually on the surface of the unit in the sense defined in Sec. III.

With each site of the lattice we associate a "spin" variable which can take a finite number of values, the same number at each site. The letters σ and τ^{α} will be used to denote the *collection* of spin variables associated with the surface (i.e., surface or free) sites of a unit of a particular order, where the superscript α will distinguish variables associated with units of the same order. Since different units will in general have certain vertices in common, the different collections are not independent. Keeping track of this "overlap" is a notational inconvenience, and for this reason we shall actually assign separate (i.e., independent) variables to the different subunits, and introduce Kronecker delta functions of an appropriate sort in the Y_N functions (defined below) in order to enforce the identity produced by common vertices. Since we have assumed (Sec. III) that the number of surface sites is finite, and as this number cannot depend on the unit in a uniform, self-similar lattice, the total number of possible values for σ or τ^{α} is always the same finite number λ. (For an Ising model on the diamond lattice, each unit has two sites, and thus σ takes on $\lambda=4$ values.)

The dimensionless $(-\mathcal{H}/kT)$ Hamiltonian $H^{(N)}$ for a unit of order N is given by

$$H^{(N)}=\sum_{\alpha} H^{(0)}(\tau^{\alpha})+\hat{H}^{(N)} . \tag{4.1}$$

The sum is over the primitive $(n=0)$ subunits of which this unit is composed, with $H^{(0)}$ the same function for each subunit. Thus, for example, in (2.1) the constants K_0 and h_0 are independent of the primitive bond considered. The term $\hat{H}^{(N)}$ is the sum of all of the noniterated interactions.

The restricted partition function $Z_N(\sigma)$ associated with a unit of order N is the sum of $\exp H^{(N)}$ over the variables associated with the interior sites of the unit while those associated with the surface sites, denoted by σ, are held fixed. The partition function Z_N and the dimensionless free energy per primitive unit f_N are given by

$$Z_N=\exp B^N f_N=\sum_{\sigma} Z_N(\sigma) . \tag{4.2}$$

The *effective Hamiltonian* $H_N(\sigma)$, which should be carefully distinguished from $H^{(N)}$, is defined by

$$Z_N(\sigma)=\exp[H_N(\sigma)+C_N] , \tag{4.3}$$

and by the requirement that it have zero trace,

$$\sum_{\sigma} H_N(\sigma)=0 , \tag{4.4}$$

a condition which fixes the value of the constant C_N. For $N=0$, H_0 is identical with $H^{(0)}$ apart from constant ψ_0 (which is zero if $H^{(0)}$ has zero trace).

The restricted partition function for a unit of order N can be expressed in terms of the restricted partitions function for a unit of order N can be expressed in terms of the restricted partition functions of the subunits of order $N-1$ which it contains. In the case of an Ising model on a diamond lattice we have

$$Z_N(\sigma_1,\sigma_2)=\sum_{\sigma_3}\sum_{\sigma_4} Z_{N-1}(\sigma_1,\sigma_3)Z_{N-1}(\sigma_1,\sigma_4) \times Z_{N-1}(\sigma_2,\sigma_3)Z_{N-1}(\sigma_2,\sigma_4) , \tag{4.5}$$

where σ_1 and σ_2 are associated with the top and bottom, and σ_3 and σ_4 with the left and right sites in Fig. 1(b). In the general case the analog of (4.5) can be written in the form

$$Z_N(\sigma)=\sum_{\tau} Y_N(\sigma;\tau)\prod_{\alpha=1}^{B} Z_{N-1}(\tau^{\alpha}) , \tag{4.6}$$

where τ^{α} denotes the collection of surface variables associated with the αth subunit of order $N-1$. These variables (as explained above) are treated as independent and are collectively denoted by τ, which thus takes on λ^B different values. The fact that various surface sites are identified with one

another during the assembly process, and some of these become surface sites of the larger unit, is taken account of by setting $Y_N=0$ whenever its arguments are inconsistent with this identification. [In the case of (4.5) the required identification has been achieved by using the same variables as arguments of more than one restricted partition function.]

In addition, the Y_N contain the Boltzmann weights associated with the noniterated interactions (e.g., the dashed lines in Figs. 1–3). If these interactions are the same at each step of the assembly process, the subscript N can be deleted from Y, as Y is always the same function. We shall retain the possibility that these noniterated interactions can vary, both because such situations are of interest in practice [as when p is allowed to vary in Kadanoff's procedure corresponding to Fig. 3(c)], and because this additional generality causes no particular complications in the following proofs.

If (4.6) is iterated a finite number of times, the result is as follows:

$$Z_N(\sigma)=\sum_{\tau} Y_N^{(l)}(\sigma;\tau)\prod_{\alpha} Z_{N-l}(\tau^{\alpha}) , \tag{4.7}$$

where α is now a label consisting of a string of l symbols (see Sec. III), and τ again denotes the collection of all the τ^{α}. Of course, $Y_N^{(1)}$ is the same as Y_N.

Equation (4.6) may be rewritten using the effective Hamiltonians introduced in (4.2):

$$\exp[H_N(\sigma)+\psi_N]=\sum_{\tau} Y_N(\sigma;\tau)\exp\left[\sum_{\alpha=1}^{B} H_{N-1}(\tau^{\alpha})\right] . \tag{4.8}$$

The constant ψ_N is unique provided (4.3) is satisfied, and is related to C_N in (4.2) through

$$C_N=\psi_N+BC_{N-1} . \tag{4.9}$$

We shall define ψ_0 to be the same as C_0.

A comparison of (4.2), (4.3), and (4.9) shows that the thermodynamic limit for the free energy

$$f=\lim_{N\to\infty} f_N \tag{4.10}$$

is given by

$$f=\sum_{N=0}^{\infty} B^{-N}\psi_N , \tag{4.11}$$

provided the sum converges and provided

$$\lim_{N\to\infty} B^{-N}\ln\left[\sum_{\sigma}\exp H_N(\sigma)\right]=0 . \tag{4.12}$$

The existence of the limit (4.10) and the validity of (4.11) depend entirely on the properties of $H^{(0)}$ and of the Y_N. We now list the conditions which will be used in theorems 1 and 2:

(i) The Hamiltonian $H^{(0)}(\sigma)$ of a primitive unit takes on *finite* (real) values.

(ii) $0\le Y_N(\sigma,\tau)<\infty$.

(iii) For every N and for every value of σ there is at least one value of τ for which $Y_N(\sigma;\tau)$ is greater than zero.

(iv) The sum $\sum_{N=1}^{\infty} B^{-N}G_N$ converges, where

$$G_N=\max_{\sigma,\tau}\ln Y_N(\sigma;\tau) . \tag{4.13}$$

(v) The sum $\sum_{N=1}^{\infty} B^{-N}g_N$ converges, where

$$g_N=\min_{\sigma,\tau}{}^{*}\ln Y_N(\sigma;\tau) , \tag{4.14}$$

and min* denotes the minimum over those choices of σ and τ for which $Y_N(\sigma;\tau)>0$.

(vi) There is a finite l such that for every $N\ge l$ and for every value of σ there is at least one α (which may depend on σ) such that for any value of τ^{α} there is some τ (i.e., some choices for the τ^{β} with $\beta\neq\alpha$) for which $Y_N^{(l)}(\sigma;\tau)>0$.

As these conditions are somewhat abstract, the following should be noted. Ordinarily the zeros of Y_N will arise from the fact that certain variables are, in fact, identical, i.e., correspond to the same site in a hierarchical lattice. When its arguments satisfy this consistency condition, Y_N will be strictly positive, with a value determined by the noniterated interactions (assumed to be finite). In this situation (ii) and (iii) are automatically satisfied. If in addition the hierarchical lattice has a finite separation index (Sec. III), (vi) is satisfied with l equal to the separation index. When there are no iterated interactions, Y_N is either 0 or 1, so that $G_N=g_N=0$, and (iv) and (v) are satisfied. When there are iterated interactions, (iv) and (v) are, in effect, a demand that these not grow too rapidly with N.

Thus in most applications of interest, (i), (ii), and (iii) will be satisfied, while (iv) and (v) provide some (not very restrictive) limits on the way in which the noniterated interactions grow with N. It is only (vi) which reflects an interesting topological property of the hierarchical lattice. [In theorem 1, (iii) and (v) are actually only used to establish that f_N has a lower bound independent of N, so that if an alternative argument for such a lower bound is available, it can replace (iii) and (v).]

Theorem 1. If conditions (i)–(iii), (iv), and (v) are satisfied, the limit (4.10) for f (the free energy per primitive unit) exists and is finite.

Theorem 2. If in addition to (i)—(v), (vi) is satisfied, f is given by the sum (4.11).

In proving these theorems we shall make use of the quantities

$$f_N(\sigma)=B^{-N}\ln Z_N(\sigma)\ , \tag{4.15}$$

$$f_N^M=\max_\sigma f_N(\sigma)\ , \tag{4.16}$$

$$f_N^m=\min_\sigma f_N(\sigma)\ . \tag{4.17}$$

An immediate consequence of (4.2) is

$$f_N^M\le f_N\le f_N^M+B^{-N}\ln\lambda\ , \tag{4.18}$$

so that if f_N^M has a limit as $N\to\infty$, f_N has the same limit. A second inequality,

$$f_N^M\le B^{-N}G_N+B^{1-N}\ln\lambda+f_{N-1}^M\ , \tag{4.19}$$

follows from (4.13) and the fact that τ in (4.6) can take on only λ^B values.

Another consequence of (4.6) is the inequality

$$f_N(\sigma)\ge B^{-N}g_N+B^{-1}\sum_{\alpha=1}^{B} f_{N-1}(\tau^\alpha)\ , \tag{4.20}$$

which holds at least for those choices of the τ^α, and thus τ, for which $Y_N(\sigma;\tau)$ does not vanish; by condition (iii) there is at least one such τ for every σ. Consequently,

$$f_N^m\ge B^{-N}g_N+f_{N-1}^m\ . \tag{4.21}$$

From (4.19) and (4.21) we see that the sequences

$$f_N^M+\sum_{n=N+1}^{\infty} B^{-N}G_n+B^{1-N}(1-B)^{-1}\ln\lambda \tag{4.22}$$

and

$$f_N^m+\sum_{n=N+1}^{\infty} B^{-n}g_n\ , \tag{4.23}$$

whose existence is guaranteed by conditions (iv) and (v), are monotone decreasing and monotone increasing, respectively, as N increases. Since for each N, (4.23) is evidently smaller than (4.22), both sequences must have well-defined limits, which we denote by f^M and f^m, as N tends to infinity. These are, of course, the same as the limits of f_N^M and f_N^m as $N\to\infty$, by conditions (iv) and (v) and f^M is, in view of (4.18), the same as f. This completes the proof of theorem 1.

To prove theorem 2, we iterate (4.20) to obtain

$$f_N(\sigma)\ge\sum_{t=0}^{l-1} B^{t-N}g_{N-t}+B^{-l}\sum_\alpha f_{N-l}(\tau^\alpha)\ , \tag{4.24}$$

where the α notation is the same as in (4.7), and the sum is over B^l different values. If σ has the value which minimizes $f_N(\sigma)$ and l is the integer which appears in conditions (vi), that condition implies that (4.24) holds for at least one τ with the property that for some α, $f_{N-1}(\tau^\alpha)$ is equal to its maximum value. Consequently, we conclude that

$$f_N^m\ge\sum_{t=0}^{l-1} B^{t-N}g_{N-t} +B^{-l}[(B^l-1)f_{N-l}^m+f_{N-l}^M]\ . \tag{4.25}$$

By taking the limit of this inequality as $N\to\infty$, we see that f^m cannot be less than $f^M=f$, so it must be equal to f.

To establish (4.11) we note that (4.4) implies that

$$\min_\sigma H_N(\sigma)\le 0,\quad \max_\sigma H_N(\sigma)\ge 0\ . \tag{4.26}$$

Consequently (4.2) and (4.3) along with (4.9) imply that

$$f_N^m\le B^{-N}C_N=\sum_{j=0}^{N} B^{-j}\psi_j\le f_N^M\ . \tag{4.27}$$

The limit of these inequalities as $N\to\infty$ yields (4.11). This completes the proof of theorem 2.

If in addition one has appropriate estimates for g_n and G_n, it is possible to use (4.19), (4.25), and (4.27) to estimate the rate of convergence of (4.11). In particular, if g_n and G_n are bounded as $n\to\infty$, the error in using the first N terms of (4.11) decreases exponentially (proportional to $e^{-\beta N}$ for some $\beta>0$) with N.

Note that the argument given above for the convergence of (4.11) will also work in certain other cases in which a condition other than (4.4) is used to determine the ψ_N in (4.8). The proof of theorem 2 only uses (4.26), and it is clear that even weaker constraints are possible.

The importance of (vi) for theorem 2 is evident from considering an Ising model on the lattice in Fig. 2(d), which does not possess a finite separation index. When the primitive bonds (corresponding to the *solid* lines) have an interaction (2.1) with $h_0\neq 0$, one can show that the sum in (4.11) converges—but to the wrong answer.

V. GENERAL HIERARCHICAL LATTICE

In this section we propose a definition of a general hierarchical lattice which seems to include the basic features, apart from symmetry, of the examples discussed in Sec. II. The important property of

such lattices from the viewpoint of statistical mechanics is that the partition function for a set of classical spins on the lattice sites, with interactions associated with the edges, can be obtained from an iterative procedure of summing over spin variables at selected sites, with each such summation corresponding to a "bounded problem" in the following sense. When certain spin variables have been summed over, the result is an effective Hamiltonian for the remaining spins, and in this effective Hamiltonian the next spin to be summed over interacts with at most n other spins, where n is a fixed bound which does not change as the summation proceeds.

The definition of a general hierarchical lattice given below is designed to ensure the property just mentioned, while at the same time allowing for multiple-spin interactions and not simply pair interactions. Of course the defnition is a "geometrical" or "graphical" one which makes no mention of a Hamiltonian or interactions. Its motivation can, however, be understood by assuming that a term in the Hamiltonian involving an interaction among spins at a certain set of sites is associated with an "edge" identified with this subset of sites.

We shall define a generalized graph $(V,\mathscr{E})$ as a set of vertices V and a collection of "edges" $\mathscr{E}$, each edge E in $\mathscr{E}$ being a subset of V. We shall use the notation that $|A|$ is the number of elements in a set A. In an ordinary graph $|E|=2$ for every edge.

We define a hierarchical lattice to be a generalized graph $(V,\mathscr{E})$ satisfying the following conditions. We assume at the outset that $|V|$ is finite or countably infinite, and that $|E|$ is finite for any edge E. Next we require that for each non-negative integer j there be a collection of subsets (units) U_α^j of V, with α some index, having the following properties.

(1). Let S_α^j, the "surface sites" of the unit U_α^j, be the smallest subset of U_α^j with the property that for every edge E, either

$$E\cap U_\alpha^j=\phi \text{ or } E\subset U_\alpha^j \text{ or } E\cap U_\alpha^j\subset S_\alpha^j \ . \quad (5.1)$$

We require that the number of surface sites be bounded:

$$|S_\alpha^j|\le m<\infty \ , \quad (5.2)$$

where m is independent of j and α.

(2) For $j=0$ it is the case that

$$|U_\alpha^0|\le m'<\infty \ , \quad (5.3)$$

independent of α.

(3). For $j\ge 1$, U_α^j can be written as a union of a finite number $m_{j\alpha}$ of (not necessarily disjoint) units U_β^{j-1}, and

$$m_{j\alpha}\le m''<\infty \quad (5.4)$$

independent of j and α. (Note that $m_{j\alpha}$ may be equal to 1; it is not necessary that units "grow" as j increases.)

(4). Given any pair of vertices v and w in V, there is some j and some α such that both v and w are contained in U_α^j. (Under some circumstances it is useful to weaken this condition and allow for a small number of sites at "infinite j" which are not contained in any U_α^j.)

With the definition complete, let us see why it yields the property discussed in the first paragraph of this section. The summation is carried out in stages with $j=0,1,2,\ldots$. When j has a particular value, the sum is carried out over all the sites in each U_α^j which are not surface sites (not in S_α^j) and which have not been previously summed over. For a given α this number cannot exceed mm'', as the sites in question must be surface sites of the units of order $j-1$ which make up the unit in question, and from the definition of S_α^j is follows that in the effective Hamiltonian these spins only interact with one another and with the spins on the sites in S_α^j, thus with at most $m(m''+1)$ other spins. Condition 2 ensures that the process of summation can be started at $j=0$, and condition 4 that it eventually includes the spins at all sites [apart from those "at infinity" in the weakened version of condition (4)]. Note that in the terminology employed here, the "free sites" introduced in Sec. III are to be regarded as units, or parts of units, of the next lower order.

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