Spinning strings in $AdS_5 \times S^5$: New integrable system relations

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A general class of rotating closed string solutions in $AdS_5 \times S^5$ is shown to be described by a Neumann-Rosochatius one-dimensional integrable system. The latter represents an oscillator on a sphere or a hyperboloid with an additional "centrifugal" potential. We expect that the reduction of the $AdS_5 \times S^5$ sigma model to the Neumann-Rosochatius system should have further generalizations and should be useful for uncovering new relations between integrable structures on two sides of the AdS/conformal field theory (CFT) duality. We find, in particular, new circular rotating string solutions with two AdS_5 and three S^5 spins. As in other recently discussed examples, the leading large-spin correction to the classical energy turns out to be proportional to the square of the string tension or the 't Hooft coupling λ, suggesting that it can be matched onto the one-loop anomalous dimensions of the corresponding "long" operators on the super-Yang-Mills side of the AdS/CFT duality.

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I. INTRODUCTION AND SUMMARY

Integrability of the spin chain Hamiltonian representing the planar one-loop dilatation operator of $\mathcal{N}=4$ super Yang-Mills theory [1–3] has recently made possible, following a proposal in [4,5], a number of remarkable and striking tests of AdS conformal field theory (CFT) duality [6–11]. This generalizes the near Bogomol'nyi-Prasad-Sommerfield (BPS) correspondence of [12] to non-BPS cases.

The AdS/CFT correspondence predicts that the energy of a given physical string state (in global AdS_5 coordinates) should match the scaling dimension of the corresponding operator in gauge theory. While the full energy spectrum of the quantum string in $AdS_5 \times S^5$ is hard to determine, some of its parts can be probed by considering the semiclassical string configurations [13,14]. In certain cases with large quantum numbers (such as angular momenta J_i in S^5), one finds that the energy of the string solution is given by its classical expression, i.e., quantum sigma model corrections appear to be suppressed [5].

On the gauge theory side, the (one-loop) scaling dimensions of gauge-invariant composite operators can be found by solving the eigenvalue problem for the Hamiltonian of an associated spin chain. This is achieved by means of algebraic Bethe ansatz techniques. In general, the Bethe ansatz leads to a complicated system of algebraic equations. However, in the thermodynamic limit (of large quantum numbers or "long" operators) the algebraic equations turn into integral ones and with some natural assumptions about the density distribution of Bethe roots the explicit solutions can be found. Remarkably, the Bethe solutions obtained in the thermodynamic limit turn out to be related to semiclassical string configurations in a precise way.

In general, one can classify strings moving on S^5 with three "R charges" [SO(6) spins] defining the highest weight state (J_1, J_2, J_3) of an SO(6) representation. For a simpler case of two nonvanishing spins (J_1, J_2) the string evolution equations are solved in terms of elliptic functions; the corresponding string configurations can have folded [7] or circular [4,8] profiles, giving rise to two different expressions for the space-time energy. On the gauge theory side, the relevant Bethe solutions and the associated scaling dimensions have been found in [6,9], and shown to agree with their string counterparts for both folded [6,7,9] and circular [6,8,9] type configurations. Other surprising examples of a perfect agreement between string energies and scaling dimensions of gauge theory operators include [11] a simple circular string solution with three spins [4] and a pulsating string solution [15].

Even more remarkably, in the recent work [10] the entire Bethe resolvent (corresponding either to the circular or to the folded string type thermodynamic density distributions) was reproduced from the classical string sigma model. This agreement goes beyond comparing just the string energies with the scaling dimensions: it involves matching the *infinite* towers of commuting conserved charges on the gauge and string sides of the AdS/CFT correspondence. In fact, the Bethe resolvent is nothing else but a generating function of local conserved commuting charges in string theory properly restricted to the leading [$O(\lambda)$ or "one-loop"] level.

The matching of higher local commuting string charges

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[10][1] and recent advances in study of integrability of the dilatation operator at higher loops in $\mathcal{N}=4$ super Yang-Mills theory [3,30] (and in its S^3 reduced matrix model version [31]) provide strong support that the same integrable structure should be underlying the two sides of the duality.

Still, our understanding of the gauge/string duality, even in the "semiclassical" (large quantum number) sector of states, is far from complete. More detailed analysis of different physical configurations in both gauge and string theories is required to elucidate how the duality works. While recent papers [9–11,27] shed some light on how the integrability of the $AdS_5\times S^5$ string theory is related to that of the planar SYM theory, many details are missing.

In view of the general problem of establishing correspondence between various integrable subsectors of string and gauge theories it is of interest to obtain a systematic picture of reductions of the two-dimensional integrable $O(4,2)\times O(6)$ sigma model describing propagation of the classical string in the $AdS_5\times S^5$ space-time to various one-dimensional integrable models. In [8] we have shown that for a natural rotating string ansatz the $AdS_5\times S^5$ string sigma model reduces to an integrable Neumann model [32] describing an oscillator on a 2-sphere.

The aim of the present paper is to make further progress in this direction. We will consider a more general integrable subsector in string theory which arises from a rotating string ansatz extending the one in [8]. In this case the 2D sigma model reduces to the Neumann-Rosochatius (NR) [33] integrable system describing a particle on a sphere in the $\Sigma_i(w_i^2 r_i^2 + v_i^2 r_i^{-2})$ potential (in the previous case [8] we had $v_i=0$). While, as in [8], the general solutions of this system are given by theta-functions on a genus 2 hyperelliptic curve its new feature is the existence of a very simple new class of solutions corresponding to circular strings with constant radii $r_i(\sigma)=\text{const}$. These solutions generalize the ones of [4] (which had two equal spins) to the case when all 2+3 $AdS_5\times S^5$ spins may be different. The corresponding energy has a very simple dependence on the spins and winding numbers. Understanding its SYM scaling dimension counterpart should help, in particular, to clarify the issue of how the winding numbers of circular strings are reflected in the Bethe root distributions (cf. [6,8,11]).

Let us now summarize the contents of the paper. In Sec. II A we shall present the generalized rotating string ansatz for a closed string fixed at the origin of AdS_5 and rotating in 3 orthogonal planes in S^5 and explain the reduction of the $O(6)$ invariant sigma model to the NR system for the 3 radial directions of the string. In Sec. II B we will list the corresponding integrals of motion and the Virasoro constraints allowing one to express the AdS_5 energy as a function of the three S^5 spins. In Sec. II C we shall mention that a "2D-dual" version of the rotating string ansatz (with roles of τ and σ interchanged) describes a general pulsating string solution with radii oscillating in time which is thus also described by an NR integrable model (some special cases of pulsating solutions were previously discussed in [11,13,15,21,34]). In Sec. II D we shall clarify how the integrability of the NR system follows from its relation to the integrable $O(6)$ sigma model by deriving its Lax representation. We shall also explain how higher commuting charges can be computed from the sigma-model monodromy function.

In Sec. III we shall study a very simple special class of NR solutions on S^5 which has a similarity with rotating string solutions in flat space and generalizes the circular 2-spin and 3-spin rotating string solutions in [4]. As will be shown in Sec. III B, the corresponding energy has a regular large-spin expansion in λ/J^2. In Sec. III C we shall find the spectrum of quadratic fluctuations near these circular solutions (extending and simplifying the discussion in [5] for the special solutions of [4]). We shall determine the stability conditions and mention some straightforward applications.

In Sec. IV we shall study more general solutions of the NR system with a nontrivial dependence on the world-sheet coordinate σ. We shall consider, in particular, a two-spin solution which is expressed in terms of the elliptic functions. The resulting system of equations relating energy and two-spins turns out to be more involved that in the previously discussed elliptic (sine-Gordon) limit of the Neumann model [7–9], but we expect that it might be possible to directly match an appropriate "one-loop" limit of this system onto the corresponding Bethe ansatz equations on the SYM side (as was done in the Neumann model case in [6,9]).

Finally, in Sec. V we shall generalize the discussion of Secs. II and III to the case when the string can rotate in both AdS_5 and S^5. Here we get a combination of the two NR systems (an AdS_2 and S^2 one) coupled by the Virasoro constraints. We again consider the simplest solution with constant radii parametrized by 2+3 spins (S_a, J_i) and 2+3 winding numbers. If the string rotates only in AdS_5 the corresponding energy does not have a regular large-spin expansion (Sec. V A), but it does if there is at least one large spin in S^5 (Sec. V B). For example, the simplest (S,J) string solution which is a circle in both AdS_5 and S^5 is stable, and it should be possible to match the leading large J correction to its energy with a particular anomalous dimension on the SYM side by identifying the corresponding distribution of Bethe roots in the associated $XXX_{-1/2}$ spin chain [2] (as was done for other folded and circular (S,J) string solutions in [9]).

[1]The integrability of the O(n) invariant sigma models was discussed, e.g., in [16–18]. Classical solutions for strings in constant curvature spaces were studied in [19–22] and refs. there (see also [23,24] for other similar solutions in $AdS_5\times S^5$ and its generalizations). More recent AdS/CFT motivated discussions concerning integrability, higher local and nonlocal charges and Yangian structures of related sigma models are in [25–29].

II. REDUCTION OF O(6) SIGMA-MODEL TO THE NEUMANN-ROSOCHATIUS SYSTEM

A. Generalized rotating string ansatz

Here we shall generalize the rotation ansatz in [8] which allowed us to reduce the classical string sigma-model equations to those of a 1D integrable model. That will lead to new interesting simple classes of rotating string solutions.

Let us consider the bosonic part of the classical closed string propagating in the $AdS_5\times S^5$ space-time. The world-sheet action in the conformal gauge is

$$I=-\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma[G^{(AdS_5)}_{mn}(x)\partial_a x^m\partial^a x^n + G^{(S^5)}_{pq}(y)\partial_a y^p\partial^a y^q],$$
$$\sqrt{\lambda}\equiv\frac{R^2}{\alpha'}. \tag{2.1}$$

It is convenient to represent Eq. (2.1) as an action for the $O(6)\times SO(4,2)$ sigma-model (we follow the notation of [4])

$$I=\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma(L_S+L_{AdS}), \tag{2.2}$$

where

$$L_S=-\frac{1}{2}\partial_a X_M\partial^a X_M+\frac{1}{2}\Lambda(X_M X_M-1), \tag{2.3}$$

$$L_{AdS}=-\frac{1}{2}\eta_{MN}\partial_a Y_M\partial^a Y_N + \frac{1}{2}\tilde{\Lambda}(\eta_{MN}Y_M Y_N+1). \tag{2.4}$$

Here X_M, $M=1,\ldots,6$ and Y_M, $M=0,\ldots,5$ are the embedding coordinates of R^6 with the Euclidean metric in L_S and with $\eta_{MN}=(-1,+1,+1,+1,+1,-1)$ in L_{AdS} respectively. Λ and $\tilde{\Lambda}$ are the Lagrange multipliers. The action (2.2) is to be supplemented with the usual conformal gauge constraints. The embedding coordinates of $AdS_5\times S^5$ can be parametrized in terms of angles of AdS_5 and S^5 as in [4,8]

$$X_1+iX_2=\sin\gamma\cos\psi e^{i\varphi_1},$$
$$X_3+iX_4=\sin\gamma\sin\psi e^{i\varphi_2},$$
$$X_5+iX_6=\cos\gamma e^{i\varphi_3}, \tag{2.5}$$

$$Y_1+iY_2=\sinh\rho\sin\theta e^{i\phi_1},$$
$$Y_3+iY_4=\sinh\rho\cos\theta e^{i\phi_2},$$
$$Y_5+iY_0=\cosh\rho e^{it}. \tag{2.6}$$

In this section we will be discussing the case when the string is located at the center of AdS_5 and rotating in S^5, i.e. is trivially embedded in AdS_5 as $Y_5+iY_0=e^{it}$, with the global time of AdS_5 being $t=\kappa\tau$ and with $Y_1,\ldots,Y_4=0$.

The S^5 metric has three commuting translational isometries in φ_i in Eqs. (2.5) which give rise to three global commuting integrals of motion (spins) J_i. Since we are interested in a periodic motion with $J_i\neq 0$ it is natural to choose the following ansatz for X_M:

$$\mathrm{X}_1\equiv X_1+iX_2=z_1(\sigma)e^{iw_1\tau},$$
$$\mathrm{X}_2\equiv X_3+iX_4=z_2(\sigma)e^{iw_2\tau},$$
$$\mathrm{X}_3\equiv X_5+iX_6=z_3(\sigma)e^{iw_3\tau}. \tag{2.7}$$

In contrast to our earlier work [8] here we shall not assume that z_i are real, i.e. in general

$$z_k=r_k(\sigma)e^{i\alpha_k(\sigma)},\quad k=1,2,3. \tag{2.8}$$

In order to find the relevant closed string solutions we need also to impose the periodicity conditions on X_M or z_i:

$$r_i(\sigma+2\pi)=r_i(\sigma),$$
$$\alpha_i(\sigma+2\pi)=\alpha_i+2\pi m_i,$$
$$m_i=0,\pm 1,\pm 2,\ldots. \tag{2.9}$$

Thus r_k are real periodic functions of σ, while real phases α_k are periodic only up to $2\pi m_k$ shift.

Comparing Eq. (2.7) to Eq. (2.5) we conclude that for this general "complex" ansatz the angles φ_i depend on both τ and σ,

$$\varphi_i=w_i\tau+\alpha_i(\sigma). \tag{2.10}$$

The integers m_i that will label different solutions thus play the role of "winding numbers" in the linear isometry directions φ_i.

As a consequence of $X_M^2=1$, r_k must lie on a two-sphere:

$$\sum_{i=1}^{3} r_i^2=1. \tag{2.11}$$

The space-time energy E of the string [related to the generator of a compact SO(2) "05" subgroup of SO(4,2)] here is simply

$$E=\sqrt{\lambda}\kappa\equiv\sqrt{\lambda}\mathcal{E}. \tag{2.12}$$

The spins $J_1=J_{12}$, $J_2=J_{34}$, $J_3=J_{56}$ forming a Cartan subalgebra of SO(6) are

$$J_i=\sqrt{\lambda}w_i\int_0^{2\pi}\frac{d\sigma}{2\pi}r_i^2(\sigma)\equiv\sqrt{\lambda}\mathcal{J}_i, \tag{2.13}$$

and thus satisfy

$$\sum_{i=1}^{3}\frac{\mathcal{J}_i}{w_i}=1. \tag{2.14}$$

As discussed in [4], to have a consistent semiclassical string state interpretation of these configurations one should look for solutions for which all other components of the SO(6) angular momentum tensor J_{MN} vanish. This is automatically the case if all w_i are different [8], but are to be checked in other cases. The nonvanishing Cartan components (J_1,J_2,J_3) would specify in quantum theory the highest weight state of the SO(6) irrep. with the Dynkin labels $[J_2-J_3,J_1-J_2,J_2+J_3]$ [these are the Dynkin labels describing the SO(6) representation content of the corresponding composite operator in the dual gauge theory].

The Virasoro constraints that need to be imposed on a sigma model solution of Eq. (2.3) are (dot and prime are derivatives over τ and σ)

$$\kappa^2=\dot{X}_M\dot{X}_M+X'_MX'_M=\sum_{i=1}^{3}(r_i'^2+r_i^2\alpha_i'^2+w_i^2r_i^2), \tag{2.15}$$

$$0=\dot{X}_MX'_M=2\sum_{i=1}^{3}w_ir_i^2\alpha_i'. \tag{2.16}$$

B. Integrals of motion and constraints

In general, starting with

$$X_i(\tau,\sigma)=r_i(\tau,\sigma)e^{i\varphi_i(\tau,\sigma)} \tag{2.17}$$

we get from Eq. (2.3) the Lagrangian

$$L_S=\frac{1}{2}\sum_{i=1}^{3}[\dot{r}_i^2-r_i'^2+r_i^2(\dot{\varphi}_i^2-\varphi_i'^2)] + \frac{1}{2}\Lambda\left(\sum_{i=1}^{3}r_i^2-1\right). \tag{2.18}$$

One can easily check that the ansatz

$$r_i=r_i(\sigma),\quad \varphi_i=w_i\tau+\alpha_i(\sigma) \tag{2.19}$$

is indeed consistent with the equations of motion.

Substituting the ansatz (2.19) or Eq. (2.7) into the SO(6) Lagrangian (2.3) we get the following effective 1D "mechanical" system for a particle on a 5D sphere (we change the sign of L since now σ plays the role of 1D time)

$$L=\frac{1}{2}\sum_{i=1}^{3}(z_i'z_i'^*-w_i^2z_iz_i^*)-\frac{1}{2}\Lambda\left(\sum_{i=1}^{3}z_iz_i^*-1\right). \tag{2.20}$$

If we set $z_k=x_k+ix_{k+3}$, this is recognized as a special case of the standard integrable $n=6$ Neumann model (harmonic oscillator on a 5-sphere) where three of the six frequencies are equal to the other three. This relation implies integrability of the (2.20) model, i.e. determines integrals of motion.

Equivalently, in the "planar" coordinates (2.8) we get from Eq. (2.18)

$$L=\frac{1}{2}\sum_{i=1}^{3}(r_i'^2+r_i^2\alpha_i'^2-w_i^2r_i^2)-\frac{1}{2}\Lambda\left(\sum_{i=1}^{3}r_i^2-1\right). \tag{2.21}$$

Equations for the angles α_i can be integrated once

$$\alpha_i'=\frac{v_i}{r_i^2},\quad v_i=\text{const}, \tag{2.22}$$

where v_i are three integrals of motion. Eliminating α_i' with the help of Eq. (2.22) we note that the equations of motion for the remaining three real radial coordinates r_i can be derived from the following effective Lagrangian:

$$L=\frac{1}{2}\sum_{i=1}^{3}\left(r_i'^2-w_i^2r_i^2-\frac{v_i^2}{r_i^2}\right)-\frac{1}{2}\Lambda\left(\sum_{i=1}^{3}r_i^2-1\right). \tag{2.23}$$

When the new integration constants v_i vanish, i.e. α_i are constant, we go back to the previously studied [8] example of the $n=3$ Neumann model. For nonzero v_i the Lagrangian (2.23) describes the so called Neumann-Rosochatius (NR) integrable system (see, e.g., [33]). Its integrability follows already from the fact that it is a special case of the 6-dimensional Neumann system.

Finding the integrals of the "radial" system (2.23) is straightforward using the relation to the Neumann model: the $n=6$ Neumann system with coordinates x_M has, in general, the following six integrals of motion:

$$F_M=x_M^2+\sum_{M\neq N}^{6}\frac{(x_Mx_N'-x_Nx_M')^2}{w_M^2-w_N^2},\qquad \sum_{M=1}^{6}F_M=1. \tag{2.24}$$

However, in our case there are equalities between frequencies $(w_1=w_4,w_2=w_5,w_3=w_6)$ so one should be careful to avoid singularities. The integrals of the Neumann-Rosochatius model are obtained as the following combinations $I_i=F_i+F_{i+3}$ $(i=1,2,3)$ in which singular terms cancel. Explicitly, we find [using Eq. (2.22)]

$$I_i=r_i^2+\sum_{j\neq i}^{3}\frac{1}{w_i^2-w_j^2}\left[(r_ir_j'-r_jr_i')^2+\frac{v_i^2}{r_i^2}r_j^2+\frac{v_j^2}{r_j^2}r_i^2\right],\qquad \sum_{i=1}^{3}I_i=1. \tag{2.25}$$

This gives us two independent integrals of motion (which we shall denote b_a) in addition to the three other integrals (v_i) we found already.

The constraints (2.15), (2.16) can be written as

$$\kappa^2=\sum_{i=1}^{3}\left(r_i'^2+w_i^2r_i^2+\frac{v_i^2}{r_i^2}\right), \tag{2.26}$$

$$\sum_{i=1}^{3} w_i v_i = 0. \tag{2.27}$$

As a consequence of Eq. (2.27) only two of the three integrals of motion v_i are independent of w_i.

As discussed in [8], the periodicity condition in Eq. (2.9) on r_i implies that the integrals of motion b_a can be traded for two integers n_a labeling different types of solutions. Imposing the periodicity condition in Eq. (2.9) on α_i gives, in view of Eq. (2.22), the following constraint:

$$v_i \int_0^{2\pi} \frac{d\sigma}{r_i^2(\sigma)} = 2\pi m_i. \tag{2.28}$$

It implies that v_i should be expressible in terms of the integers m_i, frequencies w_i and the ''radial'' integrals b_a or n_a.[2] The moduli space of solutions will thus be parametrized by $(w_1, w_2, w_3; n_1, n_2; m_1, m_2, m_3)$. The constraint (2.27) will give one relation between these 3+2+3 parameters. As a consequence, trading w_i for the angular momenta, the energy of the solutions as determined by Eqs. (2.12),(2.26) will be a function of the SO(6) spins and the ''topological'' numbers n_a and m_i

$$\mathcal{E} = \mathcal{E}(\mathcal{J}_i; n_a, m_i), \quad E = \sqrt{\lambda}\,\mathcal{E}\left(\frac{J_i}{\sqrt{\lambda}}; n_a, m_i\right). \tag{2.29}$$

The constraint (2.27) will provide one additional relation between J_i and n_a, m_i.

In the following sections of this paper we shall consider several special solutions of the above system (2.21). We shall start in Sec. III with a discussion of the simplest possible solution with constant r_i (for which $n_a = 0$) and which represent an interesting new class of circular 3-spin solutions generalizing the circular solution of [4].

C. ''2D-dual'' NR system for pulsating solutions

It is of interest to consider a ''2D-dual'' version of the rotation ansatz (2.7),(2.8) where τ and σ are interchanged (but still keeping the AdS_5 time as $t = \kappa\tau$), i.e.

$$X_i = z_i(\tau) e^{i m_i \sigma} = r_i(\tau) e^{i\alpha_i(\tau) + i m_i \sigma},$$
$$\sum_{i=1}^{3} r_i^2(\tau) = 1. \tag{2.30}$$

In this case the radial directions depend on τ instead of σ and the ''frequencies'' m_i must take integer values in order to satisfy the closed string periodicity condition. In general, in order to have the zero non-Cartan components of the O(6) angular momentum tensor one is to assume that $m_i \neq m_j$.

[2]Note that since the integral in Eq. (2.28) is of a positive function, $m_i = 0$ implies $v_i = 0$.

This ansatz describes an ''oscillating'' or ''pulsating'' S^5 string configuration, special cases of which (with motion in both AdS_5 and S^5) were discussed previously in [11,13,15,21,34].

Since the sigma model Lagrangian (2.3) is formally invariant under $\sigma \leftrightarrow \tau$, the resulting 1D effective Lagrangian will have essentially the same form as Eqs. (2.20),(2.21) (here we do not invert the sign of the Lagrangian)

$$L = \frac{1}{2}\sum_{i=1}^{3} (\dot{z}_i \dot{z}_i^* - m_i^2 z_i z_i^*) + \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} z_i z_i^* - 1\right). \tag{2.31}$$

Solving for $\dot{\alpha}_i$ as in Eq. (2.22) we get $r_i^2 \dot{\alpha}_i = \mathcal{J}_i = \text{const}$, where the counterparts of the integration constants v_i are, in fact, the angular momenta in Eq. (2.13). Then we end up with the following analogue of Eq. (2.23):

$$L = \frac{1}{2}\sum_{i=1}^{3}\left(\dot{r}_i^2 - m_i^2 r_i^2 - \frac{\mathcal{J}_i^2}{r_i^2}\right) + \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} r_i^2 - 1\right). \tag{2.32}$$

Thus pulsating solutions (carrying also 3 spins $\mathcal{J}_i$) are again described by a special Neumann-Rosochatius integrable system.

Since (the S^5 part of) the corresponding conformal gauge constraints are also $\tau \leftrightarrow \sigma$ symmetric, they take a form similar to Eqs. (2.15),(2.16) or Eqs. (2.26),(2.27)

$$\kappa^2 = \sum_{i=1}^{3}\left(\dot{r}_i^2 + m_i^2 r_i^2 + \frac{\mathcal{J}_i^2}{r_i^2}\right), \tag{2.33}$$

$$\sum_{i=1}^{3} m_i \mathcal{J}_i = 0. \tag{2.34}$$

One may look for periodic solutions of the above NR system (2.32) subject to the constraint (2.33), i.e. having finite 1D energy (equal to $\frac{1}{2}\kappa^2$). In the simplest (''elliptic'') case reducing to a sine-Gordon type system we may follow [15,21,35] and introduce, as for any periodic solitonic solution, an oscillation ''level number'' N. This may be achieved by considering a semiclassical (WKB) quantization of the action (2.32).

Here we shall not go into detailed study of the resulting pulsating string solutions. Let us only mention that a special $r_i = \text{const}$ solution of the above system (when, in fact, there is no oscillation of the radii) is essentially the same as the special circular solution with $r_i = \text{const}$ of the system (2.23) discussed below in Sec. III.

In the case of the S^5 pulsating solution in [11,15] the expansion of the energy at large level $N \gg 1$ appears to be regular in λ/N^2 (this is not the case for pulsating string in AdS_5 [15]) and, indeed, the leading λ/N^2 term in E can then be matched onto the SYM anomalous dimensions as was shown in [6,11].

D. Lax representation for the NR system induced from the O(6) sigma model

Having in mind further generalizations, it is useful to understand how the integrability of the NR system (e.g., the Lax representation) follows from the fact that this system is embedded into a much more general integrable [16] O(6) sigma model. Here we will clarify this issue and point out some related open problems.

We start with describing the zero-curvature representation for the O(6) sigma-model in terms of 4×4 matrices. Let X_i be the 3 complex embedding fields (2.7) of the O(6) model. Let us introduce the following skew-symmetric matrix S:

$$S=\begin{pmatrix} 0 & X_1 & -X_2 & \bar{X}_3 \\ -X_1 & 0 & X_3 & \bar{X}_2 \\ X_2 & -X_3 & 0 & \bar{X}_1 \\ -\bar{X}_3 & -\bar{X}_2 & -\bar{X}_1 & 0 \end{pmatrix}. \tag{2.35}$$

The matrix S is also unitary, $SS^\dagger=1$, provided $X_i\bar{X}_i=1$. Let us also introduce the su(4)-valued current A with components

$$A_\tau=S\partial_\tau S^\dagger,\quad A_\sigma=S\partial_\sigma S^\dagger,$$
$$A_\pm=\frac{1}{2}(A_\tau\pm A_\sigma). \tag{2.36}$$

This current can be used to construct the following matrices U and V [16]:

$$U=\frac{1}{1+\ell}A_- - \frac{1}{1-\ell}A_+$$
$$V=-\frac{1}{1+\ell}A_- - \frac{1}{1-\ell}A_+ . \tag{2.37}$$

Here ℓ is a spectral parameter, and by construction U and V have simple poles at $\ell=\pm1$. They obey the zero-curvature condition

$$\partial_\tau U-\partial_\sigma V+[U,V]=0, \tag{2.38}$$

which is a crucial device for demonstrating the integrability of the sigma models. Quite generally, one can associate to Eq. (2.38) the transition matrix $T(\sigma,\ell)$ (see, e.g., [36]) defined through the path-ordered exponent,

$$T(\sigma,\ell)=P\exp\int_0^\sigma U(\sigma',\ell)d\sigma', \tag{2.39}$$

and show that the trace of the monodromy matrix (the parallel transport along the period of the zero-curvature connection)

$$\mathcal{Q}(\ell)=\mathrm{Tr}\,T(2\pi,\ell) \tag{2.40}$$

generates (when expanded as $\mathcal{Q}=\sum_{n=0}^\infty\mathcal{Q}_n\ell^n$) an infinite tower of commuting integrals of motion.[3]

Consider now the generalized rotation (or "Neumann") ansatz for the sigma model variables X_i in Eq. (2.7), i.e.

$$X_i=z_i(\sigma)e^{iw_i\tau},\qquad \sum_{i=1}^3|z_i|^2=1. \tag{2.41}$$

Remarkably, the current (2.36) evaluated on X_i of the form (2.41) admits the following factorization:

$$A_\tau=Q(\tau)\mathcal{A}_\tau Q^\dagger(\tau),\qquad A_\sigma=Q(\tau)\mathcal{A}_\sigma Q^\dagger(\tau). \tag{2.42}$$

Here $Q(\tau)$ is the diagonal matrix

$$Q(\tau)=\mathrm{diag}(e^{-iw_3\tau},\ e^{-iw_2\tau},\ e^{-iw_1\tau},\ e^{-i(w_1+w_2+w_3)\tau}),$$

while the matrices $\mathcal{A}_\tau$ and $\mathcal{A}_\sigma$ are independent of τ and given by

$$\mathcal{A}_\tau=i\begin{pmatrix} 2w_3z_3z_3^*-w_iz_iz_i^* & (w_2+w_3)z_2z_3^* & (w_1+w_3)z_1z_3^* & (w_1-w_2)z_1z_2 \\ (w_2+w_3)z_2^*z_3 & 2w_2z_2z_2^*-w_iz_iz_i^* & (w_1+w_2)z_1z_2^* & -(w_1-w_3)z_1z_3 \\ (w_1+w_3)z_1^*z_3 & (w_1+w_2)z_1^*z_2 & 2w_1z_1z_1^*-w_iz_iz_i^* & (w_2-w_3)z_2z_3 \\ (w_1-w_2)z_1^*z_2^* & -(w_1-w_3)z_1^*z_3^* & (w_2-w_3)z_2^*z_3^* & w_iz_iz_i^* \end{pmatrix},$$

and

$$\mathcal{A}_\sigma=\begin{pmatrix} z_1z_1'^*+z_2z_2'^*+z_3^*z_3' & z_3^*z_2'-z_3'^*z_2 & z_3^*z_1'-z_3'^*z_1 & z_2z_1'-z_2'z_1 \\ z_2^*z_3'-z_2'^*z_3 & z_1z_1'^*+z_3z_3'^*+z_2^*z_2' & z_2^*z_1'-z_2'^*z_1 & z_1z_3'-z_1'z_3 \\ z_1^*z_3'-z_1'^*z_3 & z_1^*z_2'-z_1'^*z_2 & z_2z_2'^*+z_3z_3'^*+z_1^*z_1' & z_3z_2'-z_3'z_2 \\ z_1^*z_2'^*-z_1'^*z_2^* & z_3^*z_1'^*-z_3'^*z_1^* & z_2^*z_3'^*-z_2'^*z_3^* & z_i'z_i^* \end{pmatrix}.$$

[3]Derivation of the Poisson algebra satisfied by matrix elements of the transition matrix and the proof of commutativity of the integrals generated by the monodromy matrix can be found in [37].

As the consequence, one finds

$$U=Q(\tau)\mathcal{U}(\sigma)Q^\dagger(\tau),\quad V=Q(\tau)\mathcal{V}(\sigma)Q^\dagger(\tau), \tag{2.43}$$

where $\mathcal{U}$ and $\mathcal{V}$ depend only on σ. The zero-curvature condition (2.38) reduces to

$$\partial_\sigma\mathcal{V}=[Q^\dagger\partial_\tau Q-\mathcal{V},\mathcal{U}]. \tag{2.44}$$

Next, we note that the diagonal matrix $Q^\dagger\partial_\tau Q$ is σ-independent and, therefore, one can introduce the following L and M-operators:

$$L\equiv\mathcal{V}-Q^\dagger\partial_\tau Q,\quad M\equiv-\mathcal{U}, \tag{2.45}$$

which furnish the Lax representation for the NR system,

$$\partial_\sigma L=[L,M]. \tag{2.46}$$

This is a *new* Lax representation for the NR system; the previously known examples include the formulation of the Lax equations in terms of 3×3 [38] or 2×2 [39] matrices. Thus, the O(6) sigma model indices the Lax pair for the NR system in terms of traceless anti-hermitian 4×4 matrices. An interesting open problem is to construct the classical r-matrix corresponding to the Lax system (2.45), (2.46).

As was discussed in the previous subsection, the NR system has the (σ-independent) integrals precisely in number which is required for its Liouville integrability. Regarding now σ as a (periodic) time variable, the integrals of motion of the NR system can be constructed, e.g., as $F_n=\mathrm{Tr}\,L^n$. However, being embedded into the more general two-dimensional integrable system it inherits an *infinite* number of conserved (i.e. τ-independent) integrals of motion. One possible way to exhibit this infinite commuting family is to compute the monodromy (2.40) for the Neumann connection $U(\sigma,\ell)$. In general, this is a difficult problem, but it can be simplified by considering the special (simplest) solutions of the NR system.

A significant simplification of the Lax pair occurs if we restrict ourselves to the two-spin solutions, which are obtained by setting $X_3=0$. In this case we have effectively the SO(4) sigma model that is isomorphic to two copies of SU(2) models. Indeed, one can show that by a similarity transformation the matrices $\mathcal{A}_\tau$ and $\mathcal{A}_\sigma$ can be brought to the form

$$\mathcal{A}_\tau=i\begin{pmatrix} w_2z_2z_2^*-w_1z_1z_1^* & (w_1+w_2)z_1z_2^* & 0 & 0\\ (w_1+w_2)z_1^*z_2 & w_1z_1z_1^*-w_2z_2z_2^* & 0 & 0\\ 0 & 0 & -w_1z_1z_1^*-w_2z_2z_2^* & (w_1-w_2)z_1z_2\\ 0 & 0 & (w_1-w_2)z_1^*z_2^* & w_1z_1z_1^*+w_2z_2z_2^*\end{pmatrix}$$

and

$$\mathcal{A}_\sigma=\begin{pmatrix} z_1z_1'^*+z_2^*z_2' & z_2^*z_1'-z_2'^*z_1 & 0 & 0\\ z_1^*z_2'-z_1'^*z_2 & z_2z_2'^*+z_1^*z_1' & 0 & 0\\ 0 & 0 & z_1z_1'^*+z_2z_2'^* & z_2z_1'-z_2'z_1\\ 0 & 0 & z_1^*z_2'^*-z_1'^*z_2 & z_1'z_1^*+z_2'z_2^*\end{pmatrix},$$

which exhibits factorization into two SU(2) sectors. It is easy to see that the NR evolution equations arise already from a single SU(2) sector, e.g., from the upper left conner of the Lax matrices. Schematically, the corresponding L-operator reads as

$$L=L_0+\frac{L_1}{1-\ell}+\frac{L_{-1}}{1+\ell}, \tag{2.47}$$

were $L_0=\mathrm{diag}(-iw_2,-iw_1)$.

Let us recall that the Neumann model admits two different kinds of two-spin solutions corresponding to string configurations of the folded or circular type respectively [8]. For instance, the two-spin circular type solution can be written in terms of the standard Jacobi elliptic functions as follows $(z_3=0)$:

$$z_1(\sigma)=\mathrm{sn}(a\sigma,\mathrm{t}),\quad z_2(\sigma)=\mathrm{cn}(a\sigma,\mathrm{t}),$$

$$a\equiv\sqrt{\frac{w_{12}^2}{\mathrm{t}}}=\frac{2}{\pi}\mathrm{K}(\mathrm{t}), \tag{2.48}$$

where $w_{12}^2=w_1^2-w_2^2$ is related to the elliptic modulus t through the closed string periodicity condition.[4] The modulus t is related to the spins $\mathcal{J}_1$ and $\mathcal{J}_2$ by a transcendental equation (see [8] for details). On this particular solution the matrices U and V projected on the first SU(2) sector are

[4] K is the standard complete elliptic integral of the first kind.

$$U=\frac{1}{1-\ell^2}\begin{pmatrix} i\ell(w_1\mathrm{sn}^2 a\sigma - w_2\mathrm{cn}^2 a\sigma) & -a\,\mathrm{dn}a\sigma - i\ell(w_1+w_2)\mathrm{sn}a\sigma\mathrm{cn}a\sigma \\ a\,\mathrm{dn}a\sigma - i\ell(w_1+w_2)\mathrm{sn}a\sigma\mathrm{cn}a\sigma & -i\ell(w_1\mathrm{sn}^2 a\sigma - w_2\mathrm{cn}^2 a\sigma)\end{pmatrix}$$

and

$$V=\frac{1}{1-\ell^2}\begin{pmatrix} i(w_1\mathrm{sn}^2 a\sigma - w_2\mathrm{cn}^2 a\sigma) & -a\ell\,\mathrm{dn}a\sigma - i(w_1+w_2)\mathrm{sn}a\sigma\mathrm{cn}a\sigma \\ a\ell\,\mathrm{dn}a\sigma - i(w_1+w_2)\mathrm{sn}a\sigma\mathrm{cn}a\sigma & -i(w_1\mathrm{sn}^2 a\sigma - w_2\mathrm{cn}^2 a\sigma)\end{pmatrix}.$$

Using these matrices and applying the (recurrent) Abelianization procedure of Zakharov and Shabat [40] one can compute the monodromy function $\mathcal{Q}(\ell)$ in Eq. (2.40) and, as a consequence, the corresponding higher commuting charges.

In a recent work [10] the higher commuting (local) charges were obtained for both the folded and the circular two-spin solutions of the Neumann model and linked with those of the one-loop planar $\mathcal{N}=4$ SYM theory. The approach of [10] was based on finding the form of Bäcklund transformations, which also provides a way of generating the commuting conserved charges (see, e.g., [18]). It would be of interest to understand better a relation between the Bäcklund transformations and the monodromy approach in our stringy context.

III. SPECIAL CIRCULAR SOLUTIONS: CONSTANT Λ CASE

A very simple special class of solutions of the system (2.20) or (2.21) which has a similarity with rotating string solutions in flat space and generalizes the circular rotating string solutions in [4,5] has the property that the Lagrange multiplier is constant, i.e. $\Lambda=\mathrm{const}$.

A. Constant radii solution

Let us start with the Lagrangian (2.20) in terms of the complex coordinates z_i. Then the equations of motion are

$$z_i''+m_i^2 z_i=0, \quad m_i^2\equiv w_i^2+\Lambda, \quad \sum_{i=1}^{3}|z_i|^2=1, \tag{3.1}$$

$$\Lambda=\sum_{i=1}^{3}(|z_i'|^2-w_i^2|z_i|^2). \tag{3.2}$$

Equation (3.1) can be easily integrated if one assumes that $\Lambda=\mathrm{const}$,

$$z_i=a_i e^{im_i\sigma}+b_i e^{-im_i\sigma}, \tag{3.3}$$

where a_i, b_i are complex coefficients. The periodicity condition $z_i(\sigma+2\pi)=z_i(\sigma)$ implies that m_i must be integer. To satisfy the constancy of Λ in Eq. (3.2) we need to impose

$$\sum_{i=1}^{3}(|a_i|^2+|b_i|^2)=1,$$

$$\sum_{i=1}^{3}(m_i^2+w_i^2)(a_i^* b_i e^{2im_i\sigma}+a_i b_i^* e^{-2im_i\sigma})=0. \tag{3.4}$$

In addition, we need to impose $\Sigma_{i=1}^3|z_i|^2=1$, i.e.

$$\sum_{i=1}^{3}(|a_i|^2+|b_i|^2)=1,$$

$$\sum_{i=1}^{3}(a_i^* b_i e^{2im_i\sigma}+a_i b_i^* e^{-2im_i\sigma})=0. \tag{3.5}$$

It is easy to show that modulo the global SU(3) [subgroup of SO(6)] invariance of the system (2.20) or Eqs. (3.1),(3.2) the only nontrivial solution of Eqs. (3.4),(3.5) is $b_i=0$ *or* $a_i=0$. In the former case [m_i may be positive or negative and a_i can be made real by $U(1)$ rotations]

$$z_i=a_i e^{im_i\sigma}, \quad \sum_{i=1}^{3}a_i^2=1. \tag{3.6}$$

It may seem that one may get a new solution if two of the windings m_i are equal while the third is zero, i.e. (this is, in fact, the circular solution of [4]) if

$$z_1=a\cos m\sigma, \quad z_2=a\sin m\sigma, \quad z_3=\sqrt{1-a^2}, \tag{3.7}$$

but it can be transformed back into the form (3.6) by a global SU(2) rotation.

It is useful also to rederive the solution (3.6) in a slightly different way using real coordinates r_i,α_i, i.e. starting with Eqs. (2.23),(2.22). The potential $w_i r_i^2+(v_i^2/r_i^2)$ in Eq. (2.23) has a minimum, and that suggests that $r_i=\mathrm{const}$ may be a solution. That needs to be checked since r_i are constrained to be on S^2. The equations of motion that follow from Eq. (2.23) are

$$r_i''=-w_i^2 r_i+\frac{v_i^2}{r_i^3}-\Lambda r_i, \tag{3.8}$$

$$\Lambda=\sum_{j=1}^{3}\left(r_j'^2-w_j^2 r_j^2+\frac{v_j^2}{r_j^2}\right), \quad \sum_{j=1}^{3}r_j^2=1. \tag{3.9}$$

They indeed have a solution if

$$r_i(\sigma)=a_i=\text{const},\quad w_i^2-\frac{v_i^2}{a_i^4}=\nu^2=\text{const},\qquad(3.10)$$

where ν is an arbitrary constant (which may be positive or negative). Then it follows that the Lagrange multiplier in Eq. (2.23) is thus constant on this solution

$$\Lambda=-\nu^2.\qquad(3.11)$$

As a result, we obtain an interesting 3-spin *generalization* of the circular string solution found in [4] (where two out of three spins were equal).

Equation (3.10) implies

$$a_i^2=\frac{|v_i|}{\sqrt{w_i^2-\nu^2}},$$

$$\alpha_i'=\frac{v_i}{a_i^2}=\frac{v_i}{|v_i|}\sqrt{w_i^2-\nu^2}\equiv m_i,\qquad(3.12)$$

i.e.

$$\alpha_i=\alpha_{0i}+m_i\sigma,\qquad(3.13)$$

where m_i must be integers to satisfy the periodicity condition (2.9) and α_{0i} can be set to zero by SO(2) rotations. Then

$$w_i^2=m_i^2+\nu^2,\quad v_i=a_i^2m_i.\qquad(3.14)$$

The constraints (2.15),(2.16) or (2.26),(2.27) give

$$\kappa^2=\sum_{i=1}^3 a_i^2(w_i^2+m_i^2)=2\sum_{i=1}^3 a_i^2w_i^2-\nu^2,$$

$$\sum_{i=1}^3 a_i^2=1,\quad \sum_{i=1}^3 a_i^2w_im_i=0,\qquad(3.15)$$

or, equivalently, in terms of the energy and spins [cf. Eqs. (2.12),(2.13),(2.14)]

$$\mathcal{E}^2=2\sum_{i=1}^3 w_i\mathcal{J}_i-\nu^2,\quad\text{i.e.}\quad \mathcal{E}^2=2\sum_{i=1}^3\sqrt{m_i^2+\nu^2}\,\mathcal{J}_i-\nu^2,\qquad(3.16)$$

$$\sum_{i=1}^3\frac{\mathcal{J}_i}{w_i}=1,\quad\text{i.e.}\quad\sum_{i=1}^3\frac{\mathcal{J}_i}{\sqrt{m_i^2+\nu^2}}=1,\qquad(3.17)$$

$$\sum_{i=1}^3 m_i\mathcal{J}_i=0.\qquad(3.18)$$

We shall assume for definiteness that all w_i and thus all $\mathcal{J}_i$ are non-negative. Then Eq. (3.18) implies that one of the three m_i's must have opposite sign to the other two.

One can check directly that the only nonvanishing components of the SO(6) angular momentum tensor $J_{MN}=\sqrt{\lambda}\int_0^{2\pi}(d\sigma/2\pi)(X_M\dot X_N-X_N\dot X_M)$ on this solution are indeed the Cartan ones $J_1=J_{12}$, $J_2=J_{34}$, $J_3=J_{56}$.

Since our aim is to express $\mathcal{E}$ in terms of $\mathcal{J}_i$ and m_i as in Eq. (2.29) the strategy is then to first solve the condition (3.17) in terms of ν^2, determining it as a function of $\mathcal{J}_i$ and m_i and then substitute the result into Eq. (3.16). The condition (3.18) may then be imposed at the very end.

Let us first consider the special case of $\nu^2=0$ (or $\Lambda=0$) which corresponds to a flat-space solution which can be embedded into S^5 by choosing the free radial parameters of a circular string to satisfy the condition $\Sigma_{i=1}^3a_i^2=1$. As follows from Eq. (3.14) for $\nu^2=0$ we find that all frequencies must be integer $w_i=|m_i|$, e.g.,

$$w_1=-m_1>0,\quad w_2=m_2>0,\quad w_3=m_3>0,\qquad(3.19)$$

so that the solution is a combination of the left and right moving waves [here we use complex combinations of coordinates in Eq. (2.7)][5]

$$X_1=a_1e^{im_1(\sigma-\tau)},\quad X_2=a_2e^{im_2(\sigma+\tau)},$$

$$X_3=a_3e^{im_3(\sigma+\tau)},\quad\sum_{i=1}^3 a_i^2=1.\qquad(3.20)$$

In the case of Eq. (3.19) we get from Eqs. (3.16)–(3.18)

$$\mathcal{E}^2=2\sum_{i=1}^3|m_i|\mathcal{J}_i,\quad\sum_{i=1}^3\frac{\mathcal{J}_i}{|m_i|}=1,\quad\sum_{i=1}^3 m_i\mathcal{J}_i=0.\qquad(3.21)$$

This corresponds to a very special point in the moduli space of solutions. For fixed m_i, we get two constraints on $\mathcal{J}_i$, and the energy is given by the standard flat-space linear Regge relation. For the choice (3.19) we end up with $|m_1|\mathcal{J}_1=m_2\mathcal{J}_2+m_3\mathcal{J}_3$ [where $\mathcal{J}_2$ and $\mathcal{J}_3$ are related via $\Sigma_{i=1}^3(\mathcal{J}_i/|m_i|)=1$] and thus $\mathcal{E}^2=4|m_1|\mathcal{J}_1$. Clearly, the energy of this "flat" solution does not have a regular expansion in $1/\mathcal{J}^2$ (cf. [4]) and thus it cannot be directly compared to some anomalous dimension on the SYM side.

B. Energy as function of the spins

Now let us turn to the genuinely "curved" ($\nu\neq0$) solutions which will have indeed a regular expansion of the energy for large spins, as was the case of the circular solution of [4].

In the 3-spin case one is first to solve Eq. (3.17) to determine ν. The solution of this algebraic equation cannot be

[5]This may look like an example of a "flat" or "chiral" solution of the $O(N)$ sigma model that trivially satisfies the equations of motion following from Eq. (2.3) $\partial_+\partial_-X^M+\partial_+X_N\partial_-X_NX^M=0$ since $\partial_+X^M=0$ or $\partial_-X^M=0$. But one still needs to impose the Virasoro constraints, and that implies that we need a particular combination of left and right moving modes.

written down explicitly for generic $\mathcal{J}_i$ but one can find it as a power series in the large $\mathcal{J}=\Sigma_{i=1}^{3}\mathcal{J}_i$ expansion as in [4,8] ($\mathcal{J}\gg 1$)

$$\nu^2=\mathcal{J}^2-\sum_{i=1}^{3} m_i^2\frac{\mathcal{J}_i}{\mathcal{J}}+\dots,$$

$$\mathcal{E}^2=\mathcal{J}^2+\sum_{i=1}^{3} m_i^2\frac{\mathcal{J}_i}{\mathcal{J}}+\dots, \tag{3.22}$$

where $\mathcal{J}\equiv\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3\gg 1$, and thus

$$\mathcal{E}=\mathcal{J}+\frac{1}{2\mathcal{J}}\sum_{i=1}^{3} m_i^2\frac{\mathcal{J}_i}{\mathcal{J}}+\dots. \tag{3.23}$$

As in the previous examples in [4,7,8], here the energy thus admits a regular expansion in $1/\mathcal{J}^2=\lambda/J^2$

$$E=J+\frac{\lambda}{2J}\sum_{i=1}^{3} m_i^2\frac{J_i}{J}+\dots, \quad \sum_{i=1}^{3} m_iJ_i=0. \tag{3.24}$$

Hence it should be possible to match, as in [6,8,9], the coefficient of the $O(\lambda)$ term in Eq. (3.24) with the 1-loop anomalous dimensions of the corresponding SYM operators determined by a special 3-spin case of the integrable SU(2,2|4) spin chain of [2]. The simplicity of the expression (3.24) suggests that one may be able to establish the correspondence with particular solutions of the Bethe ansatz equations in a relatively direct way, as was the case in [6] for the $J_1=J_2$, $J_3=0$ and in [11] for the $J_1=J_2$, $J_3\neq 0$ circular solutions of [4].

Let us now look at some special cases. If $\mathcal{J}_2=\mathcal{J}_3=0$, $a_2=a_3=0$, i.e. in the one-spin case, we have a solution if $w_1^2=\nu^2$, i.e. $m_1=0$ and $J_1=w_1$, and then $\mathcal{E}=\mathcal{J}_1$. This is simply the point-like geodesic case: for $m_1=0$ there is no σ-dependence in X_i.

In the two-spin case $\mathcal{J}_3=0$, $a_3=0$ Eq. (3.17) for ν^2 becomes a quartic equation

$$\frac{\mathcal{J}_1}{\sqrt{m_1^2+\nu^2}}+\frac{\mathcal{J}_2}{\sqrt{m_2^2+\nu^2}}=1. \tag{3.25}$$

Its simple explicit solution is found in the case when $\mathcal{J}_1=\mathcal{J}_2$, i.e. $a_1=a_2=1/\sqrt{2}$, $m_2=-m_1\equiv m>0$,

$$\nu^2=\mathcal{J}^2-m^2, \quad \mathcal{J}\equiv\mathcal{J}_1+\mathcal{J}_2=2\mathcal{J}_1, \tag{3.26}$$

so that

$$\mathcal{E}^2=\mathcal{J}^2+m^2. \tag{3.27}$$

This is the same $\mathcal{E}(\mathcal{J})$ relation as for the 2-spin circular solution of [4]. In fact, as was already mentioned above, the two solutions are *equivalent*: here we have

$$\mathrm{X}_1=\frac{1}{\sqrt{2}}e^{iw\tau-im\sigma}, \quad \mathrm{X}_2=\frac{1}{\sqrt{2}}e^{iw\tau+im\sigma}, \tag{3.28}$$

which is related to the solution in [4] by an SO(4) rotation,

$$X_1'=\frac{1}{\sqrt{2}}(X_1+X_2), \quad X_2'=\frac{1}{\sqrt{2}}(-X_1+X_2). \tag{3.29}$$

In the general case of two unequal spins we can again solve Eq. (3.25) in the limit of large $\mathcal{J}_1,\mathcal{J}_2$ (for fixed m_1,m_2), getting the special case of Eq. (3.24) with $m_1\mathcal{J}_1+m_2\mathcal{J}_2=0$, $\mathcal{J}_3=0$, i.e.

$$\mathcal{E}=\mathcal{J}+\frac{m_2(m_2+|m_1|)\mathcal{J}_2}{2\mathcal{J}^2}+\dots$$
$$=J-\frac{m_1m_2}{2\mathcal{J}}+\dots. \tag{3.30}$$

In another special case when two out of three nonvanishing spins are equal, e.g., $\mathcal{J}_2=\mathcal{J}_3$, and with $m_1=0$, $m_2=-m_3=m$ we get from Eq. (3.24)

$$\mathcal{E}=\mathcal{J}+\frac{m^2\mathcal{J}_2}{\mathcal{J}^2}+\dots=\mathcal{J}+\frac{m^2}{2\mathcal{J}}+\dots. \tag{3.31}$$

This is the same as the expression for the circular 3-spin solution ($\mathcal{J}_1\neq 0$, $\mathcal{J}_2=\mathcal{J}_3$) in [4]. Indeed, for any values of $J_1, J_2=J_3$ the two solutions are related by a global rotation in X_2,X_3 directions as in Eq. (3.29), converting $e^{im_2\sigma}$ into $\cos m_2\sigma$ and $\sin m_2\sigma$.

To summarize, we have shown that the constant-radius solutions of the NR system represent a simple generalization of the circular 2-spin and 3-spin solutions of [4]. This opens up a possibility of a direct comparison to SYM one-loop anomalous dimensions in the (i) 2-spin sector with unequal spins (cf. [6]) and (ii) general 3-spin sector (cf. [11]).

C. Quadratic fluctuations and stability

Let us now study small fluctuations near the solutions of Sec. III A. This will generalize (and simplify) the discussion in [5] in the case of the special $J_1=J_2$ 3-spin solution and will clarify the conditions of stability of our new solutions. One application of this analysis would be to compute the 1-loop sigma-model correction to the classical energy (3.24) and to show that it is indeed suppressed by an extra power of $1/J$ as in the special case considered in [5]. Another would be to find the spectrum of excited string states carrying the same charges as the "ground-state" classical solution as these may be possible to compare to the corresponding spectrum of anomalous dimensions on the SYM side (as was done for the special $J_1=J_2$, $J_3=0$ case in [6]).

It is straightforward to find the quadratic fluctuation Lagrangian by expanding near the solution (3.6) or Eqs. (3.10)–(3.14). We shall follow the discussion in Sec. 2 of [5] where the special case of circular solution with two equal spins was considered. Using three complex combinations of coordinates in Eq. (2.7) and expanding $\mathrm{X}_i\to\mathrm{X}_i+\tilde{\mathrm{X}}_i$ the sigma model action (2.3) near the classical solution

$$X_i = a_i e^{iw_i\tau + im_i\sigma}, \quad w_i^2 = m_i^2 + \nu^2,$$

$$\sum_{i=1}^{3} a_i^2 = 1, \quad \sum_{i=1}^{3} a_i^2 w_i m_i = 0, \tag{3.32}$$

we find the following Lagrangian for the quadratic fluctuations (see [5]):

$$\tilde{L} = -\frac{1}{2}\partial_a \tilde{X}_i \partial^a \tilde{X}_i^* + \frac{1}{2}\Lambda \tilde{X}_i \tilde{X}_i^*, \tag{3.33}$$

where $\Lambda = -\nu^2$ and $\tilde{X}_i$ are subject to the constraint[6]

$$\sum_{i=1}^{3} (X_i \tilde{X}_i^* + X_i^* \tilde{X}_i) = 0. \tag{3.34}$$

To solve this constraint we set

$$\tilde{X}_i = e^{iw_i\tau + im_i\sigma} Z_i(\tau,\sigma), \quad Z_i = g_i + i f_i, \tag{3.35}$$

so that Eq. (3.34) becomes

$$\sum_{i=1}^{3} a_i g_i = 0. \tag{3.36}$$

After integrating by parts, Eq. (3.33) takes the form (cf. [5])

$$\tilde{L} = \sum_{i=1}^{3} \left[\frac{1}{2}(\dot{f}_i^2 + \dot{g}_i^2 - f_i'^2 - g_i'^2) - 2w_i f_i \dot{g}_i + 2m_i f_i g_i' \right]. \tag{3.37}$$

To solve Eq. (3.36) we may apply a global rotation to g_i, $\overline{g_i} = M_{ij}(a) g_j$, that transforms $\Sigma_{i=1}^3 a_i g_i$ into $\overline{g_1}$ and set the latter to zero in the resulting Lagrangian (3.37). Equivalently, we may solve Eq. (3.36) for g_1 and substitute the result into Eq. (3.37).

For simplicity, let us first consider the 2-spin case when [cf. Eq. (3.25)]

$$a_1^2 + a_2^2 = 1, \quad a_3 = 0, \quad a_1^2 |m_1| w_1 - a_2^2 m_2 w_2 = 0,$$

$$w_1^2 - m_1^2 = w_2^2 - m_2^2 = \nu^2. \tag{3.38}$$

We shall assume that $m_1 < 0$, $m_2 > 0$. These relations allow us to express a_1 and a_2 in terms of m_1, m_2 and ν

$$a_1^2 = \frac{m_2\sqrt{m_2^2+\nu^2}}{|m_1|\sqrt{m_1^2+\nu^2} + m_2\sqrt{m_2^2+\nu^2}},$$

$$a_2^2 = \frac{|m_1|\sqrt{m_1^2+\nu^2}}{|m_1|\sqrt{m_1^2+\nu^2} + m_2\sqrt{m_2^2+\nu^2}}. \tag{3.39}$$

In this case the fluctuations in the $i=3$ direction decouple, and we find the following Lagrangian for the remaining 3 fluctuations g_2, f_1, f_2 [e.g. solving Eq. (3.36) for g_1 and rescaling g_2]:

$$\tilde{L} = \frac{1}{2}(\dot{f}_1^2 + \dot{f}_2^2 + \dot{g}_2^2 - f_1'^2 - f_2'^2 - g_2'^2) + 2(a_2 w_1 f_1 - a_1 w_2 f_2)\dot{g}_2 - 2(a_2 m_1 f_1 - a_1 m_2 f_2) g_2'. \tag{3.40}$$

Solving the resulting equations of motion for $F_q = (f_1, f_2, g_2)$ using the ansatz (see [5]) $F_q = \Sigma_{s,n} A_{q,s,n} e^{i\omega_s\tau + in\sigma}$ we find the following characteristic equation of the frequencies ω:

$$(\omega^2 - n^2)^2 - 4a_2^2(w_1\omega - m_1 n)^2 - 4a_1^2(w_2\omega - m_2 n)^2 = 0. \tag{3.41}$$

This is a quartic equation for ω, and the stability condition is that all four roots should be real. The solutions are obviously real for $n=0$ so instability may appear only for $n = \pm 1, \ldots$. In the special case of the 2-spin circular solution of [4], i.e. $w_1 = w_2 = w$, $-m_1 = m_2 = m$, $a_1^2 = a_2^2 = \frac{1}{2}$ we get

$$(\omega^2 - n^2)^2 - 4w^2\omega^2 - 4m^2 n^2 = 0, \tag{3.42}$$

which implies instability when $n^2 - 4m^2 < 0$, i.e. for the modes with $n = \pm 1, \ldots, \pm(2m-1)$ [5].

For generic a_1, a_2, m_1, m_2 and small enough n one finds that two of the four roots are complex (with nonzero real part).[7] In spite of the instability it is useful to work out the spectrum of frequencies and the stability condition in the limit of large spins (i.e. large ν) since the resulting energies may be compared to SYM theory. First, let us consider the case of equal spins $(-m_1 = m_2 = m)$. Equation (3.42) implies that [5]

$$\omega_\pm^2 = n^2 + 2\nu^2 + 2m^2 \pm 2\sqrt{(\nu^2+m^2)^2 + n^2(\nu^2 + 2m^2)}, \tag{3.43}$$

so that the large ν expansion gives (for the lower-energy modes)

$$\omega_-^{(\pm)} = \pm \frac{1}{2\nu} n\sqrt{n^2 - 4m^2} + O\left(\frac{1}{\nu^3}\right). \tag{3.44}$$

Then the "one-loop" contribution to the energy of rotating string from (a pair of) such modes is (here $\kappa^2 = \nu^2 + 2m^2$, $J = J_1 + J_2 = \sqrt{\lambda}\sqrt{\nu^2+m^2}$; $m = k$ in the notation of [5])

[6] Note that imposition of Virasoro constraints on the fluctuations is not necessary in order to determine the non-trivial part of the fluctuation spectrum [5].

[7] For example, setting $m_1 = -1$, $m_2 = 2$, $n = 1$ one gets complex solutions for ν from 0 to 1000. This implies instability of "asymmetric" solutions with $|m_1| \neq m_2$.

$$\Delta E_n = \frac{1}{\kappa} 2|\omega_-| = \frac{1}{\nu^2} n\sqrt{n^2-4m^2} + O\left(\frac{1}{\nu^4}\right)$$
$$= \frac{\lambda}{J^2} n\sqrt{n^2-4m^2} + O\left(\frac{\lambda^2}{J^4}\right). \qquad (3.45)$$

This expression was indeed reproduced in [6] (for $m=1$) as the 1-loop anomalous dimension of excited states on the SYM side (corresponding to a particular Bethe root distribution for the Heisenberg spin chain).

In the general (m_1,m_2) case, expanding Eq. (3.41) at large ν assuming[8] $\omega = O(1/\nu)$ we find the following generalization of Eq. (3.44):

$$\omega_- = \frac{1}{2\nu} n[2a_2^2 m_1 + 2a_1^2 m_2 \pm \sqrt{n^2 - 4a_1^2 a_2^2 (m_1-m_2)^2}] + O\left(\frac{1}{\nu^3}\right), \qquad (3.46)$$

where $a_1^2 + a_2^2 = 1$. This reduces to Eq. (3.44) in the equal-spin case when $a_1^2 = a_2^2 = \frac{1}{2}$, $m_1 = -m_2$. Stability condition is $n^2 \geqslant 4a_1^2(1-a_1^2)(m_1-m_2)^2$. If we recall that we have the constraint $m_1 J_1 + m_2 J_2 = 0$ where $J_i = a_i^2\sqrt{m_i^2+\nu^2}$ one may wish to solve it in the large ν limit getting $a_1^2 m_1 + (1-a_1^2)m_2 = 0$, i.e. $a_1^2 = m_2/(m_2-m_1)$, $1-a_1^2 = -m_1/(m_2-m_1)$, giving the condition $n^2 \geqslant 4|m_1 m_2|$, which implies the existence of unstable modes with $n^2 < 4|m_1 m_2|$.

One should be able to reproduce the analog of Eq. (3.45) in the case of Eq. (3.46), i.e. (here we assume $|m_1| > m_2$)

$$\Delta E_n = \frac{\lambda}{J^2} n|2(|m_1| - m_2) - \sqrt{n^2 - 4|m_1|m_2|}| + O\left(\frac{\lambda^2}{J^4}\right) \qquad (3.47)$$

on the gauge theory side.

It is straightforward to extend the above discussion to the 3-spin case, i.e. when a_3 is nonzero. This will give a generalization of the spectrum found in the $(J_1, J_2 = J_3)$ case in [5]; as in that special case, there should then be a range of parameters for which the solution is stable. The generalization of Eq. (3.41) to the 3-spin case is[9]

$$(\omega^2-n^2)^4 - (\omega^2-n^2)^2[(a_2^2+a_3^2)\Omega_1^2 + (a_2^2+a_3^2)\Omega_2^2 + (a_1^2+a_2^2)\Omega_3^2] + a_3^2\Omega_1^2\Omega_2^2 + a_2^2\Omega_1^2\Omega_3^2 + a_1^2\Omega_2^2\Omega_3^2 = 0, \qquad (3.48)$$

where

$$\Omega_i \equiv 2(w_i\omega - m_i n), \qquad w_i = \sqrt{m_i^2+\nu^2}. \qquad (3.49)$$

Setting $\Omega_3 = 0$, $a_3 = 0$ we indeed go back to Eq. (3.41). This equation gives 8 characteristic frequencies. Solving the equations for a_2, a_3 in terms of a_1 and $w_i = \sqrt{m_i^2+\nu^2}$ we get the following generalization of Eq. (3.39):

$$a_2^2 = -\frac{m_3 w_3(1-a_1^2) + m_1 w_1 a_1^2}{m_2 w_2 - m_3 w_3},$$
$$a_3^2 = \frac{m_2 w_2(1-a_1^2) + m_1 w_1 a_1^2}{m_2 w_2 - m_3 w_3}. \qquad (3.50)$$

Concentrating then on those frequencies that scale as

$$\omega = \frac{\bar{\omega}}{\nu} + O\left(\frac{1}{\nu^2}\right), \qquad \nu \gg 1 \qquad (3.51)$$

we get the following equation for the leading part of Eq. (3.48):

$$A + Ba_1^2 = 0, \qquad (3.52)$$

$$A = [4(\bar{\omega} - nm_3)^2 - n^4][4[\bar{\omega} - n(m_2+m_3)]^2 - n^2(n^2+4m_2m_3)],$$

$$B = 4(m_1-m_2)(m_1-m_3)n^2[12\bar{\omega}^2 - 8n(m_1+m_2+m_3)\bar{\omega} + 4n^2(m_1m_2+m_1m_3+m_2m_3) + n^4].$$

Stable solutions arise in the range of the parameters m_1, m_2, m_3 such that Eq. (3.52) has real roots $\bar{\omega}$ for all integers n. The general stability condition on m_1, m_2, m_3 and a_1^2 appears to be complicated, but one can find particular values of m_1, m_2, m_3 for which the solution is stable.

For example, setting $m_1 = 0$, $m_2 = -m_3 = m$, so that $a_1 \equiv a$, $a_3^2 = a_2^2 = \frac{1}{2}(1-a^2)$, which is the case of the 3-spin solution of [4], $\mathcal{J}_1 = a^2\nu$, $\mathcal{J}_2 = \mathcal{J}_3 = \frac{1}{2}(1-a^2)\sqrt{m^2+\nu^2}$, we find, in agreement with [5][10]

$$\bar{\omega}^2 = \frac{1}{4} n^2 m^2 \left[\frac{n^2}{m^2} - 2 + 6a^2 \pm 2\sqrt{(3a^2-1)^2 + 4a^2\left(\frac{n^2}{m^2} - 1\right)}\right]. \qquad (3.53)$$

[8]There are also two other frequencies for which $\omega^2 \to 4\nu^2$ at large ν.

[9]It can be found, e.g., by adding the constraint (3.36) to the Lagrangian (3.37) and solving the corresponding equations of motion.

[10]In the notation of [5] $a = \cos\gamma_0$, $m = \mathrm{k}$.

The condition of stability, i.e. $\overline{\omega^2}\geqslant 0$ is obtained by demanding that $(q^2-4)(q^2-4a^2)\geqslant 0$ and $(3a^2-1)^2+4a^2(q^2-1)\geqslant 0$, where $q\equiv n/m$. The stability condition is satisfied if $q^2\geqslant 1$ and $a^2\geqslant \frac{1}{4}$, which applies to all modes if $m=1$ as in [5]. For $m=2$ the potentially unstable mode is $n=\pm 1$ having $q^2=\frac{1}{4}$. Then to have stability we need to demand $a^2\geqslant\frac{1}{16}$ as well as $\frac{1}{16}\leqslant a^2\leqslant\frac{1}{6}(3-\sqrt{5})$ or $\frac{1}{6}(3+\sqrt{5})a^2<1$. Similar conditions on a are found for higher values of m.

If instead we set $m_1=m_2$ (or $m_1=m_3$) in Eq. (3.52) we find

$$\bar{\omega}=n\left(m_3\pm\frac{1}{2}n\right),$$

$$\bar{\omega}=n\left[(m_2+m_3)\pm\frac{1}{2}\sqrt{n^2-4|m_2m_3|}\right] \tag{3.54}$$

implying that modes with $n^2<4|m_2m_3|$ are unstable irrespective of the value of a_1, just like in the 2-spin case (3.46).

IV. MORE GENERAL "NONCONSTANT" SOLUTIONS OF THE NEUMANN-ROSOCHATIUS SYSTEM

A. NR equations in ellipsoidal coordinates

Analogously to the case of the Neumann system in [8] we can rewrite the equations of motion following from Eq. (2.23) in the ellipsoidal coordinates (ζ_1,ζ_2) which are introduced as

$$r_i=\sqrt{\frac{(w_i^2-\zeta_1)(w_i^2-\zeta_2)}{\prod_{j\neq i} w_{ij}^2}},\qquad w_{ij}^2=w_i^2-w_j^2. \tag{4.1}$$

Expressing the integrals of motion (2.25) in terms of ζ_a one finds the following separable system of equations:

$$\left(\frac{d\zeta_1}{d\sigma}\right)^2=-4\frac{P(\zeta_1)}{(\zeta_1-\zeta_2)^2},\qquad \left(\frac{d\zeta_2}{d\sigma}\right)^2=-4\frac{P(\zeta_2)}{(\zeta_1-\zeta_2)^2}, \tag{4.2}$$

where $P(\zeta)$ is

$$\begin{aligned}P(\zeta)=&(\zeta-b_1)(\zeta-b_2)(\zeta-w_1^2)(\zeta-w_2^2)(\zeta-w_3^2)\\&+v_1^2(\zeta-w_2^2)^2(\zeta-w_3^2)^2\\&+v_2^2(\zeta-w_1^2)^2(\zeta-w_3^2)^2\\&+v_3^2(\zeta-w_1^2)^2(\zeta-w_2^2)^2.\end{aligned}$$

Here $b_{1,2}$ are the constants of motion which can be expressed in terms of the original integrals I_i in Eq. (2.25). The Hamiltonian of the NR system reduces then to

$$H=\frac{1}{2}\left[\sum_{i=1}^{3}(w_i^2+v_i^2)-b_1-b_2\right]. \tag{4.3}$$

As in the Neumann case, $P(\zeta)$ is the fifth order polynomial which defines a hyperelliptic curve $s^2+P(\zeta)=0$. However, with nonzero v_i the positions of the roots get shifted. The general solution of Eqs. (4.2) can be given in terms of theta-functions associated to the Jacobian of the hyperelliptic curve.

We will not consider the problem of solving Eqs. (4.2) in full generality, rather we will treat the simplest case of the vanishing integral v_3. As one can see, for $v_3=0$ the value $\zeta=w_3^2$ is a root of $P(\zeta)$ and then the NR system can be solved in terms of elliptic functions.

B. Two-spin solution of the NR system

If $v_3=0$ we may set $\alpha_3=0$ and further assume that $r_3=0$ [see Eq. (2.22)] which brings us to the two-spin case. In terms of the ellipsoidal coordinates the two-spin solution arises in the limit $b_2\rightarrow w_3^2$.[11] It is convenient to perform the following change of variables $\zeta_a\rightarrow\xi_a$ (see [8] for details):

$$\zeta_1\rightarrow w_2^2-(w_2^2-b_1)\xi_1,\quad \zeta_2\rightarrow w_3^2-(w_3^2-b_2)\xi_2. \tag{4.4}$$

Then we find that the first equation in Eqs. (4.2) reduces to

$$\begin{aligned}(\xi')^2=&4w_{21}^2\xi(1-\xi)(1-\mathrm{t}\xi)\\&-4v_1^2\xi^2-4v_2^2\left(\frac{1}{\mathrm{t}}-\xi\right)^2,\\ \xi\equiv&\,\xi_1.\end{aligned} \tag{4.5}$$

Here $\mathrm{t}\equiv(w_2^2-b_1)/w_{21}^2$ is the modulus of the elliptic curve. The variable v_2 can be eliminated using $v_2^2=v_1^2w_1^2/w_2^2$. Thus we get a one-parameter family of solutions (parametrized by the additional parameter v_1).

It is possible to reduce the elliptic curve corresponding to Eq. (4.5) to the standard Jacobian form, but the new modulus k appears to be a rather complicated function of t,w_1,w_2,v_1. Indeed, we get

$$(\xi')^2=4w_{21}^2\mathrm{t}(\xi-e_0)(\xi-e_1)(\xi-e_2), \tag{4.6}$$

where

$$\begin{aligned}v_1^2=&w_{21}^2\mathrm{t}^3e_0e_1e_2,\\ v_1^2-v_2^2+\kappa^2-w_2^2&\\=&-w_{21}^2\mathrm{t}^2(e_0e_1+e_0e_2+e_1e_2).\end{aligned} \tag{4.7}$$

After the change of variable $\xi=e_{10}\eta^2+e_0$, where $e_{nm}\equiv e_n-e_m$, Eq. (4.6) becomes

$$(\eta')^2=w_{21}^2\mathrm{t}e_{20}(1-\eta^2)(1-k\eta^2),\quad k=\frac{e_{10}}{e_{20}}. \tag{4.8}$$

Thus, a solution obeying the condition $\eta(0)=0$ reads

[11]To be specific we will treat the case of the folded string (cf. [8]), analysis of the circular string solution is very similar.

$$\eta(\sigma)=\mathrm{sn}(\sigma\sqrt{w_{21}^2 \mathrm{t} e_{20}},k). \tag{4.9}$$

The radii of the embedding coordinates in Eq. (2.8) then are

$$r_1^2(\sigma)=1-\mathrm{t}(e_0+e_{10}\eta^2), \quad r_2^2(\sigma)=\mathrm{t}(e_0+e_{10}\eta^2). \tag{4.10}$$

Note that this is the most general two-spin solution of the NR system. In the present case, we require in addition that η should be periodic, $\eta(\sigma+2\pi)=\eta(\sigma)$, which gives

$$\frac{\pi}{2}\sqrt{w_{21}^2 \mathrm{t} e_{20}}=\mathrm{K}(k), \tag{4.11}$$

where $\mathrm{K}(k)$ (and E and Π appearing below) are the standard elliptic functions defined, e.g., in [8].

Since for the periodic solutions $\varphi_{1,2}(\sigma+2\pi)=\varphi_{1,2}(\sigma)+2\pi m_{1,2}$ we have also the condition (2.28), we can trade the parameters v_1,v_2 for the two integers m_1,m_2. Using the explicit solution (4.9) one can compute the integrals in Eq. (2.28) with the result

$$m_1=\frac{v_1}{(1-\mathrm{t}e_0)\mathrm{K}(k)}\Pi\left(\frac{\mathrm{t}e_{10}}{1-\mathrm{t}e_0},k\right),$$

$$m_2=\frac{v_2}{\mathrm{t}e_0\mathrm{K}(k)}\Pi\left(\frac{e_{01}}{e_0},k\right). \tag{4.12}$$

For given nonzero integers m_i these are highly transcendental equations on v_1,v_2. Computing the spins we get

$$\mathcal{J}_1=w_1-w_1e_0\left(1+k-\frac{\mathrm{E}(k)}{\mathrm{K}(k)}\right), \tag{4.13}$$

$$\mathcal{J}_2=w_2e_0\left(1+k-\frac{\mathrm{E}(k)}{\mathrm{K}(k)}\right). \tag{4.14}$$

Finally, the energy is given by

$$\mathcal{E}^2=\kappa^2=w_1^2+\mathrm{t}w_{21}^2-v_1^2\left(1+\frac{w_1^2}{w_2^2}\right). \tag{4.15}$$

Note that due to the extra condition (2.27), i.e. $v_1w_1=-v_2w_2$, the solution exists only if J_1 and J_2 are related in a certain way.

The above system of Eqs. (4.11)–(4.15), determines the energy E parametrically as a function of the R-charges $J_1=\sqrt{\lambda}\mathcal{J}_1$, $J_2=\sqrt{\lambda}\mathcal{J}_2$ and winding numbers m_1,m_2. This system is rather complicated to allow for an explicit formula for $E=\sqrt{\lambda}\mathcal{E}(J_1/\sqrt{\lambda},J_2/\sqrt{\lambda};m_1,m_2)$. Nevertheless, we hope that it might be possible to directly match this system [its leading $O(\lambda)$ or the "one-loop" approximation] onto the corresponding equations governing the algebraic Bethe ansatz (for a particular choice of the Bethe root distribution) for the anomalous dimensions of the corresponding operators on the gauge theory side, as was done in the $v_i=0$ case in [6,9].

V. ROTATING STRINGS IN $\mathrm{AdS}_5\times\mathrm{S}^5$

Let us now generalize the discussion of Secs. II and III to the case when the string can rotate in both AdS_5 and S^5. For that we need to supplement the S^5 rotating string ansatz (2.7) by the similar AdS_5 one

$$\mathrm{Y}_0\equiv Y_5+iY_0=\mathrm{z}_0(\sigma)e^{i\omega_0\tau},$$

$$\mathrm{Y}_1\equiv Y_1+iY_2=\mathrm{z}_1(\sigma)e^{i\omega_1\tau},$$

$$Y_2\equiv Y_3+iY_4=\mathrm{z}_2(\sigma)e^{i\omega_2\tau}, \tag{5.1}$$

where now (generalizing the ansatz considered in [4,8]) $\mathrm{z}_r=(\mathrm{z}_0,\mathrm{z}_1,\mathrm{z}_2)$ are complex, and because of the condition $\eta_{MN}Y^MY^N=-1$, their real radial parts lie on a hyperboloid $[\eta_{rs}=(-1,1,1)]$

$$\mathrm{z}_r=\mathrm{r}_re^{i\beta_r}, \quad \eta^{rs}\mathrm{r}_r\mathrm{r}_s\equiv-\mathrm{r}_0^2+\mathrm{r}_1^2+\mathrm{r}_2^2=-1. \tag{5.2}$$

In the previous sections we had $\mathrm{r}_0=1$, $\mathrm{r}_1=\mathrm{r}_2=0$, $\beta_r=0$. To satisfy the closed string periodicity conditions we need, as in Eq. (2.9),

$$\mathrm{r}_r(\sigma+2\pi)=\mathrm{r}_r(\sigma), \quad \beta_r(\sigma+2\pi)=\beta_r(\sigma)+2\pi k_r, \tag{5.3}$$

where k_r are integers. Comparing Eq. (5.1) to Eq. (2.5) we conclude that the AdS_5 time t and the angular coordinates ϕ_1,ϕ_2 are related to β_r by

$$t=\omega_0\tau+\beta_0(\sigma), \phi_1=\omega_1\tau+\beta_1(\sigma), \quad \phi_2=\omega_2\tau+\beta_2(\sigma). \tag{5.4}$$

We shall require the time coordinate t to be single-valued (we are considering a universal cover of AdS_5), i.e. we ignore windings in time direction and we will also rename ω_0 into κ, i.e.

$$k_0=0, \quad \omega_0\equiv\kappa. \tag{5.5}$$

The three O(2,4) Cartan generators (spins) here are $[S_0=E,\ \omega_r=(\omega_0,\omega_1,\omega_2)]$

$$S_r=\sqrt{\lambda}\omega_r\int_0^{2\pi}\frac{d\sigma}{2\pi}\mathrm{r}_r^2(\sigma)\equiv\sqrt{\lambda}\mathcal{S}_r. \tag{5.6}$$

In view of Eq. (5.2), they satisfy the relation

$$\sum_{s,r}\eta^{sr}\frac{\mathcal{S}_r}{\omega_s}=-1, \quad \text{i.e.} \quad \frac{\mathcal{E}}{\kappa}-\frac{\mathcal{S}_1}{\omega_1}-\frac{\mathcal{S}_2}{\omega_2}=1. \tag{5.7}$$

Substituting the above rotational ansatz into the AdS_5 Lagrangian (and changing overall sign) we find the analog of the 1D Lagrangian (2.20) in the S^5 case

$$\tilde{L}=\frac{1}{2}\eta^{rs}(\mathrm{z}_r'\mathrm{z}_s^{*\prime}-\omega_r^2\mathrm{z}_r\mathrm{z}_s^*)-\frac{1}{2}\tilde{\Lambda}(\eta^{rs}\mathrm{z}_r\mathrm{z}_s^*+1). \tag{5.8}$$

Like its S^5 counterpart (2.20), this 1D Lagrangian is a special case of an $n=6$ Neumann system now with signature

$(-+++++-)$, and thus represents again an integrable system (being related, as in [8], to a special Euclidean-signature Neumann model by an analytic continuation). The reduction of the total $AdS_5\times S^5$ Lagrangian on the rotation ansatz is then given by the sum of Eqs. (2.20) and (5.8). Writing Eq. (5.8) in terms of r_rr and β_r we find as in Eq. (2.22)

$$\beta_r'=\frac{u_r}{\mathrm{r}_r^2}, \quad u_r=\text{const}, \tag{5.9}$$

so that finally we end up with

$$\tilde{L}=\frac{1}{2}\eta^{rs}\left(\mathrm{r}_r'\mathrm{r}_s'-\omega_r^2\mathrm{r}_r\mathrm{r}_s-\frac{u_ru_s}{\mathrm{r}_r\mathrm{r}_s}\right)-\frac{1}{2}\tilde{\Lambda}(\eta^{rs}\mathrm{r}_r\mathrm{r}_s+1), \tag{5.10}$$

where, as above, we assume summation over r,s. Comparing this to the NR Lagrangian (2.23), we conclude that Eq. (5.10) describes a system which is similar to the Neumann-Rosochatius integrable system, but with an indefinite signature, i.e. δ_{ij} replaced by η_{rs}.

While the equations for r_i and r_r following, respectively, from Eqs. (2.23) and (5.10) are decoupled, the variables of the two NR systems are mixed in the conformal gauge constraints (2.15),(2.16) which now take the form [generalizing Eqs. (2.26),(2.27) where we had $\mathrm{r}_0=1$, $u_r=0$, $\mathrm{r}_a=0$]

$$\mathrm{r}_0'^2+\kappa^2\mathrm{r}_0^2+\frac{u_0^2}{\mathrm{r}_0^2}=\sum_{a=1}^{2}\left(\mathrm{r}_a'^2+\omega_a^2\mathrm{r}_a^2+\frac{u_a^2}{\mathrm{r}_a^2}\right)+\sum_{i=1}^{3}\left(r_i'^2+w_i^2r_i^2+\frac{v_i^2}{r_i^2}\right), \tag{5.11}$$

$$\kappa u_0=\sum_{a=1}^{2}\omega_au_a+\sum_{i=1}^{3}w_iv_i, \tag{5.12}$$

where $\mathrm{r}_0^2-\Sigma_{a=1}^2\mathrm{r}_a^2=1$, and $\Sigma_{i=1}^3r_i^2=1$. We should also require the periodicity condition analogous to Eq. (2.28)

$$u_r\int_0^{2\pi}\frac{d\sigma}{\mathrm{r}_r^2(\sigma)}=2\pi k_r. \tag{5.13}$$

Then k_0 implies that we should set $u_0=0$ as a consequence of single-valuedness of the AdS_5 time t.

One can then repeat the discussion of Secs. II and III in the present case, classifying general solutions of the resulting NR system. The resulting solutions generalize those of Sec. IV B in [4] where the integrals v_i and u_r were zero.

A. Simple circular strings in AdS_5

Let us first assume that the string is not rotating in S^5 (i.e. $w_i,v_i=0$, $r_i=$const) and consider the AdS_5 analog of the simplest circular solution of Sec. III by demanding that $\tilde{\Lambda}=$const. The discussion is then exactly the same as (a special case of that) in Sec. III with few signs reversed. As in Sec. III A, finding solutions with $\tilde{\Lambda}=$const turns out to be equivalent to looking for constant radii ($\mathrm{r}_r=$const) configurations. Then [cf. Eqs. (3.10),(3.14)]

$$\mathrm{r}_r=\text{const}, \quad \beta_a=k_a\sigma, \quad k_0=0, u_0=0, \quad u_a=\mathrm{r}_ak_a, \tag{5.14}$$

$$\omega_0^2\equiv\kappa^2=\tilde{\Lambda}, \quad \omega_a^2=k_a^2+\kappa^2, \quad a=1,2. \tag{5.15}$$

The energy as a function of spins is then obtained by solving the system of the two equations that follow from the definition of the charges (5.6) and the constraints (5.11),(5.12) with κ as a parameter [cf. Eqs. (3.17), (3.18)]

$$\frac{\mathcal{E}}{\kappa}-\frac{\mathcal{S}_1}{\sqrt{k_1^2+\kappa^2}}-\frac{\mathcal{S}_2}{\sqrt{k_2^2+\kappa^2}}=1, \tag{5.16}$$

$$\kappa\mathcal{E}-\frac{1}{2}\kappa^2=\sqrt{k_1^2+\kappa^2}\mathcal{S}_1+\sqrt{k_2^2+\kappa^2}\mathcal{S}_2,$$

$$k_1\mathcal{S}_1+k_2\mathcal{S}_2=0. \tag{5.17}$$

This implies

$$\frac{k_1^2\mathcal{S}_1}{\sqrt{k_1^2+\kappa^2}}+\frac{k_2^2\mathcal{S}_2}{\sqrt{k_2^2+\kappa^2}}=\frac{1}{2}\kappa^2. \tag{5.18}$$

Considering the limit of large spins $\mathcal{S}_a\gg1$, with k_a being fixed, we conclude that $\kappa=(2k_1^2\mathcal{S}_1+2k_1^2\mathcal{S}_1)^{1/3}+\dots$ and then

$$\mathcal{E}=\mathcal{S}_1+\mathcal{S}_2+\frac{3}{4}(2k_1^2\mathcal{S}_1+2k_2^2\mathcal{S}_2)^{1/3}+\dots, \tag{5.19}$$

or, in view of $k_1\mathcal{S}_1=-k_2\mathcal{S}_2$ (treating $\mathcal{S}_1,\mathcal{S}_2$ and k_1 as independent data)

$$\mathcal{E}=\mathcal{S}+\frac{3}{4}\left(2k_1^2\mathcal{S}\frac{\mathcal{S}_1}{\mathcal{S}_2}\right)^{1/3}+\dots, \quad \mathcal{S}\equiv\mathcal{S}_1+\mathcal{S}_2. \tag{5.20}$$

Using Eq. (5.6) this can be rewritten also as

$$E=S+\frac{3}{4}(\lambda S)^{1/3}\left(2k_1^2\frac{S_1}{S_2}\right)^{1/3}+\dots. \tag{5.21}$$

The case of $k_1=-k_2=k$ when the two spins are equal $\mathcal{S}_1=\mathcal{S}_2=\frac{1}{2}\mathcal{S}$ is that of the circular solution found in [4] for which we get

$$\mathcal{E}=\mathcal{S}+\frac{3}{4}(2k\mathcal{S})^{1/3}+\dots. \tag{5.22}$$

As was shown in [4], this $k_1=-k_2$ solution is stable only for small enough $\mathcal{S}$.

The "nonperturbative" scaling of the subleading term in Eq. (5.21) with λ precludes one from entertaining a possibility of a direct comparison to anomalous dimensions of the corresponding operators [4], i.e. $\bar{\Phi} D^{S_1}_{1+i2} D^{S_2}_{3+i4}\Phi$, in SYM theory, in contrast to what was found in the S^5 case.

Let us now see how this conclusion changes when we consider "hybrid" solutions where the circular string rotates both in AdS_5 and S_5.

B. Constant radii circular strings in $AdS_5\times S^5$

It is straightforward to combine the solutions of Secs. V A and III A to write down the most general circular constant-radii solution in $AdS_5\times S^5$. It will be parametrized by the 3+3 frequencies ($a=1,2$; $i=1,2,3$)

$$\omega_0=\kappa,\quad \omega_a^2=k_a^2+\kappa^2,$$
$$w_i^2=m_i^2+\nu^2,$$
$$\kappa^2=\tilde{\Lambda},\quad \nu^2=-\Lambda, \tag{5.23}$$

related to the energy $\mathcal{E}$ and 2+3 spins $\mathcal{S}_a$ and $\mathcal{J}_i$, and by the topological numbers k_a and m_i. These will be related by Eqs. (3.17) and (5.7) as well as by the conformal gauge constraints (5.11) and (5.12). Explicitly, we get the following generalization of both Eqs. (3.16)–(3.18) and Eqs. (5.16),(5.17)

$$\sum_{i=1}^{3}\frac{\mathcal{J}_i}{\sqrt{m_i^2+\nu^2}}=1,\quad \frac{\mathcal{E}}{\kappa}-\sum_{a=1}^{2}\frac{\mathcal{S}_a}{\sqrt{k_a^2+\kappa^2}}=1, \tag{5.24}$$

$$2\kappa\mathcal{E}-2\sum_{a=1}^{2}\sqrt{k_a^2+\kappa^2}\,\mathcal{S}_a-\kappa^2 =2\sum_{i=1}^{3}\sqrt{m_i^2+\nu^2}\,\mathcal{J}_i-\nu^2, \tag{5.25}$$

$$\sum_{a=1}^{2}k_a\mathcal{S}_a+\sum_{i=1}^{3}m_i\mathcal{J}_i=0. \tag{5.26}$$

For given (integer or half-integer, in quantum theory) spins S_a and J_i the solution exists only for such integers k_a and m_i that satisfy Eq. (5.26). Assuming that all spins are of the same order and large $\mathcal{S}_a\sim\mathcal{J}_i\gg 1$ we find

$$\kappa=\mathcal{J}+\frac{1}{2\mathcal{J}^2}\left(\sum_{i=1}^{3}m_i^2\mathcal{J}_i+2\sum_{a=1}^{2}k_a^2\mathcal{S}_a\right)+O\left(\frac{1}{\mathcal{J}^2}\right),$$
$$\mathcal{J}\equiv\sum_{i=1}^{3}\mathcal{J}_i, \tag{5.27}$$

$$\nu=\mathcal{J}-\frac{1}{2\mathcal{J}^2}\sum_{i=1}^{3}m_i^2\mathcal{J}_i+O\left(\frac{1}{\mathcal{J}^2}\right), \tag{5.28}$$

and thus

$$E=J+\frac{\lambda}{2J^2}\left(\sum_{i=1}^{3}m_i^2J_i+\sum_{a=1}^{2}k_a^2S_a\right)+O\left(\frac{\lambda^2}{J^3}\right). \tag{5.29}$$

This expression is a direct generalization of Eq. (3.23) in the $\mathcal{S}_a=0$ case. The energy is minimal if m_i^2 and k_a^2 have minimal possible values (0 or 1). We may also look at a different limit when $\mathcal{J}\gg\mathcal{S}\gg 1$ (corresponding to $k_1^2\gg m_i^2$). In this case we get a "BMN-type" (single J rotation type) asymptotics with the leading term still given by Eq. (5.27), i.e. $\Delta\mathcal{E}\sim(1/2\mathcal{J}^2)\mathcal{S}$.

The conclusion is that to have a regular (i.e. with analytic λ-dependence) large-spin expansion of the energy one needs to have at least one large component of the spin in the S^5 direction. This turns out to be the same also in the case of other spinning string solutions with more complicated σ-dependence.

As an explicit example, let us consider the simplest hybrid solution when only one of each two types of spin is nonzero, i.e. $\mathcal{J}_1=\mathcal{J}$, $\mathcal{S}_1=\mathcal{S}$, $\mathcal{S}_2=\mathcal{J}_2=\mathcal{J}_3=0$. The string then has $r_0^2-r_1^2=1$, $r_3=0$ and $r_1=1$, $r_2=r_3=0$, i.e. [cf. Eq. (2.5)]

$$Y_0=\cosh\rho_0 e^{i\kappa\tau},$$
$$Y_1=\sinh\rho_0 e^{i\omega\tau+ik\sigma},$$
$$X_1=e^{iw\tau+im\sigma}, \tag{5.30}$$

where $r_0=\cosh\rho_0$ determines the fixed radial coordinate in AdS_5 at which the circular string is located while it is spread and rotating in ϕ_1 (it is positioned at $\theta=\pi/2$ and $\phi_2=0$ in S^3 of AdS_5). Also, the string is a rotating circle along φ_1 in S^5 located at $\varphi_2=\varphi_3=0$, $\gamma=\pi/2$, $\psi=0$. Its energy for $\mathcal{J}\sim\mathcal{S}\gg 1$ is then[12]

$$\mathcal{E}=\mathcal{J}+\mathcal{S}+\frac{1}{2\mathcal{J}^2}(m^2\mathcal{J}+k^2\mathcal{S})+\dots$$
$$=\mathcal{J}+\mathcal{S}+\frac{1}{2\mathcal{J}}k^2\frac{\mathcal{S}}{\mathcal{J}}\left(1+\frac{\mathcal{S}}{\mathcal{J}}\right)+\dots, \tag{5.31}$$

where we used that $k\mathcal{S}+m\mathcal{J}=0$ and treat $\mathcal{S}$, $\mathcal{J}$ and k as independent data. Restoring λ dependence we thus have[13]

$$E=J+S+\frac{\lambda k^2}{2J}\frac{S}{J}\left(1+\frac{S}{J}\right)+\dots. \tag{5.32}$$

It should be possible to reproduce the same expression as a 1-loop anomalous dimension on the SYM side as was done for the folded (S,J) solution in [9].

One can easily analyze the small fluctuations near this solution as was done in Sec. III D. One finds 1 massless and 4 massive (mass ν) fluctuations in S^5 directions. In addition to 2 massive (mass κ) decoupled AdS_5 fluctuations there are

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[12]Here $\mathcal{J}=\sqrt{m^2+\nu^2}$, $2\kappa\mathcal{E}-\kappa^2=2\sqrt{k^2+\kappa^2}\mathcal{S}+\mathcal{J}^2+m^2$, $k\mathcal{S}+m\mathcal{J}=0$, $\mathcal{E}=\kappa+[\kappa\mathcal{S}/\sqrt{(k^2+\kappa^2)}]$.

[13]The "BMN-type" limit (cf. [14,41]) here corresponds to $\mathcal{S}/\mathcal{J}\ll 1$.

also 3 coupled ones with a Lagrangian similar to Eq. (3.40): to obtain it one is to do the following replacements in Eq. (3.40): $f_2\to f_1$, $g_2\to gm$, $f_1\to i f_0$, $w_1\to\kappa$, $w_2\to\omega_1=\sqrt{\kappa^2+n^2}$, $m_1=0$, $m_2=k$, $a_2\to i\mathrm{r}_1$, $a_1\to\mathrm{r}_0$, so that Eq. (3.41) for the characteristic frequencies ω becomes

$$(\omega^2-n^2)^2+4\mathrm{r}_1^2(\kappa\omega)^2 -4\mathrm{r}_0^2(\sqrt{k^2+\kappa^2}\omega-kn)^2=0. \tag{5.33}$$

The solutions of this equation are real. Indeed, the analog of Eq. (3.46) is found to be

$$\Omega_-=\frac{1}{2\kappa}n[2(1+\mathrm{r}_1^2)k \pm\sqrt{n^2+4\mathrm{r}_1^2(\mathrm{r}_1^2+1)k^2}]+O\left(\frac{1}{\kappa^3}\right). \tag{5.34}$$

We conclude that [in contrast to similar (S_1,S_2) and (J_1,J_2) circular solutions] this hybrid (S,J) solution is always *stable*.

It should be possible to match Eq. (5.32) with anomalous dimensions of particular $\mathrm{tr}(D^S\Phi^J)+\ldots$ operators on the SYM side by identifying the corresponding distribution of Bethe roots in the Bethe ansatz equations of the associated $XXX_{-1/2}$ Heisenberg spin chain [2], as was done for other folded and circular (S,J) string solutions in [9].

VI. CONCLUSIONS

To summarize, we have found, in particular, a solution of circular type with five spins (S_1,S_2,J_1,J_2,J_3) whose leading large-spin correction in the energy looks like a one-loop term from the viewpoint of SYM theory. Therefore, it is plausible that it can be matched onto the one-loop anomalous dimension corresponding to certain Bethe root distributions on the SYM side. The string prediction for this anomalous dimension is summarized in Eq. (5.29) [with Eq. (3.24) as a particular case]. Deriving it from the spin chain [2] Hamiltonian would clarify, in particular, how the winding numbers of circular string states are encoded in the Bethe root distributions.

One interesting special case is that of the solution with a single spin component S in AdS_5 and a single R-charge J. We have shown that this solution is stable for all values of spins and winding numbers. The corresponding energy formula in Eq. (5.32) is very simple; it should be possible to reproduce it on the SYM side as was done for other (S,J) solutions in [9].

For general solutions of the Neumann-Rosochatius system, the energy is a complicated implicit function of spins and topological numbers. For example, in the two-spin case of Sec. IV B the general solution of the NR system can be written in terms of elliptic functions but the energy is a solution of a parametric transcendental system of equations. It would be very interesting to find a more direct map between the NR system and Bethe equations for some properly chosen Bethe root distributions on the SYM side. It would also be important to find new pulsating solutions of the NR system mentioned in Sec. II C that may have simple SYM counterparts.

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[1] J.A. Minahan and K. Zarembo, J. High Energy Phys. **03**, 013 (2003).
[2] N. Beisert and M. Staudacher, Nucl. Phys. **B670**, 439 (2003).
[3] N. Beisert, C. Kristjansen, and M. Staudacher, Nucl. Phys. **B664**, 131 (2003); N. Beisert, *ibid.* **B676**, 3 (2004).
[4] S. Frolov and A.A. Tseytlin, Nucl. Phys. **B668**, 77 (2003).
[5] S. Frolov and A.A. Tseytlin, J. High Energy Phys. **07**, 016 (2003).
[6] N. Beisert, J.A. Minahan, M. Staudacher, and K. Zarembo, J. High Energy Phys. **09**, 010 (2003).
[7] S. Frolov and A.A. Tseytlin, Phys. Lett. B **570**, 96 (2003).
[8] G. Arutyunov, S. Frolov, J. Russo, and A.A. Tseytlin, Nucl. Phys. **B671**, 3 (2003).
[9] N. Beisert, S. Frolov, M. Staudacher, and A.A. Tseytlin, J. High Energy Phys. **10**, 037 (2003).
[10] G. Arutyunov and M. Staudacher, ''Matching higher conserved charges for strings and spins,'' hep-th/0310182.
[11] J. Engquist, J.A. Minahan, and K. Zarembo, J. High Energy Phys. **11**, 063 (2003).
[12] D. Berenstein, J.M. Maldacena, and H. Nastase, J. High Energy Phys. **04**, 013 (2002).
[13] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, Nucl. Phys. **B636**, 99 (2002).
[14] S. Frolov and A.A. Tseytlin, J. High Energy Phys. **06**, 007 (2002).
[15] J.A. Minahan, Nucl. Phys. **B648**, 203 (2003).
[16] K. Pohlmeyer, Commun. Math. Phys. **46**, 207 (1976).
[17] M. Luscher and K. Pohlmeyer, Nucl. Phys. **B137**, 46 (1978); H. Eichenherr and M. Forger, *ibid.* **B155**, 381 (1979).
[18] A.T. Ogielski, M.K. Prasad, A. Sinha, and L.L. Wang, Phys. Lett. **91B**, 387 (1980).
[19] B.M. Barbashov and V.V. Nesterenko, Commun. Math. Phys. **78**, 499 (1981); *Introduction to the Relativistic String Theory* (World Scientific, Singapore, 1990), p. 249.

[20] H.J. de Vega and N. Sanchez, Phys. Rev. D **47**, 3394 (1993).
[21] H.J. de Vega, A.L. Larsen, and N. Sanchez, Phys. Rev. D **51**, 6917 (1995).
[22] H.J. de Vega and I.L. Egusquiza, Phys. Rev. D **54**, 7513 (1996).
[23] D. Aleksandrova and P. Bozhilov, J. High Energy Phys. **08**, 018 (2003).
[24] D. Aleksandrova and P. Bozhilov, ''On the classical string solutions and string/field theory duality. II,'' hep-th/0308087.
[25] G. Mandal, N.V. Suryanarayana, and S.R. Wadia, Phys. Lett. B **543**, 81 (2002).
[26] I. Bena, J. Polchinski, and R. Roiban, Phys. Rev. D **69**, 046002 (2004).
[27] L. Dolan, C.R. Nappi, and E. Witten, J. High Energy Phys. **10**, 017 (2003).
[28] A. Gorsky, ''Spin chains and gauge/string duality,'' hep-th/0308182.
[29] L.F. Alday, J. High Energy Phys. **12**, 033 (2003).
[30] N. Beisert, J. High Energy Phys. **09**, 002 (2003); ''The su(2|3) dynamic spin chain,'' hep-th/0310252.
[31] T. Klose and J. Plefka, Nucl. Phys. **B679**, 127 (2004).
[32] C. Neumann, J. Reine Angew. Math. **56**, 46 (1859); O. Babelon and M. Talon, Nucl. Phys. **B379**, 321 (1992).
[33] T.S. Ratiu, *The Lie Algebraic Interpretation of the Complete Integrability of the Rosochatius System*, AIP Conf. Proc. No. **88** (AIP, New York, 1982), p. 109; L. Gagnon, J.P. Harnad, J. Hurtubise, and P. Winternitz, J. Math. Phys. **26**, 1605 (1985); C. Bartocci, G. Falqui, and M. Pedroni, ''A geometrical approach to the separability of the Neumann-Rosochatius system,'' nlin-sys/0307021; R. Kubo, W. Ogura, T. Saito, and Y. Yasui, ''Geodesic flows for the Neumann-Rosochatius systems,'' YITP-97-46; J. Moser, *Geometry of Quadrics*, Chern Symposium, Berkeley (1979), p. 147.
[34] A. Khan and A.L. Larsen, Phys. Rev. D **69**, 026001 (2004).
[35] R.F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **11**, 3424 (1975).
[36] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer-Verlag, Berlin, 1987).
[37] J.M. Maillet, Phys. Lett. **167B**, 401 (1986); **162B**, 137 (1985).
[38] M. Adler and P. van Moerbeke, Adv. Math. **38**, 267 (1980).
[39] J. Avan and M. Talon, Phys. Lett. B **268**, 209 (1991).
[40] V.E. Zakharov and A. B. Shabat, Funct. Anal. **12**, 3 (1978); V.E. Zakharov and A.V. Mikhailov, Sov. Phys. JETP **47**, 1017 (1978) [Zh. Eksp. Teor. Fiz. **74** 1953 (1978)].
[41] J.G. Russo, J. High Energy Phys. **06**, 038 (2002).