Spinor L-Functions, Theta Correspondence, and Bessel Coefficients

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In this paper we prove two seemingly unrelated theorems. First we establish the entireness of the spinor L-functions of certain automorphic cuspidal representations of the similitude symplectic group of order four over the rational numbers. We also prove a theorem related to the existence of Bessel models for generic discrete series representations of the same group over the real numbers. The two results are linked by the method of proof; in both cases it is based on the pull-back of an appropriately chosen global Bessel functional via the theta correspondence for the dual pair $(\mathrm{GO}(2,2), \mathrm{GSp}(4))$.

The first main theorem is related to analytic properties of spinor L-functions. We prove the entireness of the spinor L-function for those generic automorphic cuspidal representation which satisfy a condition at the archimedean place (see below). Our study of the spinor L-function is based on an integral representation which works for generic representations. These integrals which were introduced by M. Novodvorsky in the Corvallis conference [26] serve as one of the few available integral representations for the Spinor L-function of $\mathrm{GSp}(4)$. Some of the details missing in Novodvorsky's original paper have been reproduced in Daniel Bump's survey article [4]. Further details have been supplied by [40]. Novodvorsky's integral was first generalized by Ginzburg [10], and further generalized by Soudry [39], to orthogonal groups of arbitrary odd degree.

In light of the results of [40], it is sufficient to study the integral of Novodvorsky at the archimedean place. Archimedean computations are often forbidding, and unless one expects major simplifications due to the nature of the parameters, the resulting integrals are often quite hard to manage. In our case of interest, the work of Moriyama [25] benefits from exactly such simplifications when he treats the case of cuspidal representations with archimedean components in the generic (limit of) discrete series. In this work, we concentrate on those archimedean representations for which direct computations have yielded very little. For this reason, our methods are a bit indirect, in fact somewhat more indirect than what at first seems necessary. Our method is based on the theta correspondence. First we observe in Lemma 2.2 that Novodvorsky's integral is in fact a split Bessel functional. Then in 2.1 we pull the Bessel functional back via the theta correspondence for the dual reductive pair $(\mathrm{GO}(2,2), \mathrm{GSp}(4))$, and prove that the resulting functional on $\mathrm{GO}(2,2)$ is Eulerian. On the other hand, one can prove that the integral of Novodvorsky itself is Eulerian, with an Euler product involving the Whittaker functions. Next obvious step is to pull back the Whittaker function via the theta correspondence; we do this in 2.3. Now we have obtained two different Euler product expansions which represent the same object, but do not look the same. Then one uses the standard technique of twisting with highly ramified characters in 2.5 to isolate the archimedean place to obtain an identity expressing the local Novodvorsky integral at the archimedean place in terms of an expression which does not go through the local Whittaker functions for GSp(4). The advantage of using this expression is that, first it avoids Whittaker functions on a group of rank two, so it is effectively more elementary, and second one can devise a two complex variable zeta function to study its analytic properties (see 2.2). This identity, at first, is established only for those representations which appear as archimedean components of global theta lifts from $\mathrm{GO}(2,2)$. Then one uses various density arguments in 2.6 to extend the identity to a larger class of representations, namely the *special representations* (see 2.4). At this time, we have not yet been able to give a reasonable characterization of the class of all *special representations*; we do know, however, that it contains discrete series representations, and an infinite family of principal series representations. We have included some speculations in 2.7.

The next main theorem of the paper is concerned with the existence of Bessel models. It is well-known that automorphic representations associated to holomorphic Siegel modular forms are not generic; that is, they fail to have Whittaker models. It is also known that the genericity of such representations specifically fails at the archimedean place. For this reason it is desirable to determine when holomorphic discrete series representations posses Bessel models which seem to be the next best thing in applications to L-functions [6, 8]. The conjecture of Gross and Prasad (Conjecture 6.9 of [12]) predicts that the existence of Bessel models for holomorphic discrete series is intertwined with the existence of such models for other members of the Vogan L-packet of the given discrete series representation, in particular the generic discrete series. It will be clear from the method, however, that the interested mathematician will be able to derive the desired result for holomorphic representations. In order to make this more plausible we have kept the result in its naked form (see Theorem 3.1 for exact statement).

We now state our result. Let Π be a generic discrete series representation of $\mathrm{GSp}(4,\mathbb{R})$, with trivial central character. Then there is a pair (D_k, D_l) of discrete series representations of $\mathrm{GL}(2,\mathbb{R})$ with trivial central character such that Π is obtained by a theta lift from $\mathrm{GO}(2,2)$ by the representation that the pair (D_k, D_l) defines (see 1.6). In order to land in generic discrete series, we need to assume that $k, l \geq 2$ satisfy $k \neq l$ and they have the same parity. Let n be an integer with $n > \max(k,l)$, and with different parity from k (or l). We set $\chi_n\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{in\theta}$. With these notations, we prove that Π has a $(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \chi_n, \psi)$-Bessel model.

A few remarks are in order. It is clear from our presentation of the theorem that our proof of Theorem 3.1 uses theta correspondence; in fact, we will use global theta correspondence, along with various substantial local and global results from the theory of automorphic forms [13, 23, 29, 43]. It may be desirable to find a direct local proof of the existence theorem as in [46]. Our attempts in this direction, however, have not been successful. Inspired by [36, 37], one is tempted to write down an integral and try to prove that the integral does not vanish for the correct choice of the data. There are convergence issues that one needs to deal with. In the Whittaker situation, what saves the day is the fact that one can do the analysis of the integrals "one root at a time"; we have no been able to successfully follow such an approach for the Bessel integrals. In order to establish the conjecture of Gross-Prasad for the pair $(\mathrm{SO}(5), \mathrm{SO}(2))$ for discrete series packets, one needs to study generic discrete series representations of PGSp(4), holomorphic discrete series representations of PGSp(4), and related representations of $\mathrm{SO}(4,1)$. The case of $\mathrm{SO}(4,1)$ is simpler as the group in question has rank one. Here we have considered the representations of the group PGSp(4). Thanks to Wallach's recent paper [46], the case of holomorphic representations is much better understood. This is the reason why we can concentrated our efforts on the generic case. Shalika has informed the author that he can prove the converse statement of our Theorem 3.1 using local methods based on [21]. Consequently, the "if" in the theorem may be replaced by "if and only if." Perhaps, it should also be pointed out here that, in light of Theorem 3.4 of [45], our results automatically extend to generic limits of discrete series.

As mentioned above, the main contribution of this work, if any, is the archimedean analysis. Some of the results of this paper, especially in the case of discrete series representations, were announced in [41]. As stated above, the appearance of [25] has made our results for discrete series representations obsolete; Moriyama has obtained better and more explicit results for generic (limits of) discrete series, and some other representations, using more direct methods. Also we have recently learned that Asgari and Shahidi have prepared two manuscripts [1, 2] which contain, among other things, the functorial transfer of generic automorphic forms from spinor groups to general linear groups; these results have trivialized our theorem on the entireness of the L-function, as GSp(4) is nothing but GSpin_5. With this in mind, our results on the entireness of L-functions are certainly not new; our result on the existence of Bessel functionals, however, seems to be new. At any rate, we hope that the methods of our paper would be of interest. For example, it may be possible to use our results to explicitly compute the Γ-factors at the archimedean place; our attempts in this direction, however, have yielded very little. Brooks Roberts has used methods very similar to ours in [31] to study various non-archimedean questions; Roberts had also, independently of us and around the same time, discovered Lemma 2.2 and had in fact done at least the computations of 2.1 and 2.3. It seems to me that both of us were influenced by Masaaki Furusawa, and communication with Furusawa and Shalika was our common source of inspiration. I learned about Bessel functionals and theta correspondence from J. A. Shalika while a graduate

student at Johns Hopkins. The idea of pulling back global Bessel functionals via theta correspondence came up in a conversation with Shalika while trying to understand a paper of Böcherer and Schulze-Pillot ([3]). Here we thank Shalika for continued support and encouragement over the past few years. Most of preliminary computations that led to the writing of this paper were also performed at Johns Hopkins under his supervision. I would like to thank Shalika for suggesting the problems that motivated this research, for useful conversations, and for lending us his notes on Bessel models. The author has benefited from conversations with Jeffrey Adams, Mahdi Asgari, Philippe Michel, Peter Sarnak, Freydoon Shahidi, Akshay Venkatesh, and especially Brooks Roberts. Comments by Tonomori Moriyama, Ralf Schmidt, and particularly the anonymous referee on an earlier draft of this paper were quite helpful. The author wishes to thank the Park City Mathematical Institute where he first met Michel, and learned of the work of Kowalski, Michel, and Vanderkam on the non-vanishing of the Rankin-Selberg L-functions at the center of critical strip. He also wishes to thank the Clay Mathematical Institute and the National Security Agency for partial support of the project.

Contents

1 Preliminaries on $\mathrm{GSp}(4)$ **5**

1.1 The group $\mathrm{GSp}(4)$ . . . 5
1.2 Discrete series . . . 8
1.3 Whittaker models . . . 9
1.4 Bessel functionals . . . 10
1.5 Theta correspondence . . . 12
1.6 Theta correspondence for $(\mathrm{Sp}(4,\mathbb{R}), \mathrm{O}(2,2))$ . . . 17
1.7 The Spinor L-function for $\mathrm{GSp}(4)$ . . . 17

2 Entireness of the spinor *L*-function 19

2.1 The pull-back . . . 19
2.2 The zeta integral of two complex variables; Euler product . . . 25
2.3 The Whittaker function . . . 34
2.4 Local Whittaker functions . . . 40
2.5 Archimedean Zeta function . . . 42
2.6 Analytic continuation . . . 44
2.7 Special representations . . . 46

3 Existence of Bessel functionals for generic discrete series 48

3.1 Non-vanishing of period integrals . . . 49
3.2 Proof of the theorem . . . 50

1 Preliminaries on GSp(4)

1.1 The group GSp(4)

In this paper, the group GSp(4) over an arbitrary field K is the group of all matrices $g \in \mathrm{GL}_4(K)$ that satisfy the following equation for some scalar $\nu(g) \in K$:

$$ {}^t g J g = \nu(g) J, $$

where $J = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$. It is a standard fact that $G = \mathrm{GSp}(4)$ is a reductive group. The map $(F^\times)^3 \longrightarrow G$, given by

$$ (a, b, \lambda) \mapsto \mathrm{diag}(a, b, \lambda a^{-1}, \lambda b^{-1}) $$

gives a parameterization of a maximal torus T in G. The Weyl group is a dihedral group of order eight. We have three standard parabolic subgroups: The Borel subgroup B, The Siegel subgroup P, and the Klingen subgroup Q with the following Levi decompositions:

$$ B = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1}\lambda & \\ & & & b^{-1}\lambda \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}, $$

$$ P = \left\{ \begin{pmatrix} g & \\ & \alpha\, {}^t g^{-1} \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix} ; g \in \mathrm{GL}(2) \right\}, $$

and finally Q is the maximal parabolic subgroup with non-abelian unipotent radical associated to the long simple root. If ψ is an additive character of the field K, we define a character θ_ψ of the unipotent radical $N(B)$ of the Borel subgroup by the following:

$$ \theta(\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix}) = \psi(x + t). $$

When K is a local field, we take always take ψ to be an unramified Tate character.

We define various subgroups of the group $G = \mathrm{Sp}(4)$ over the real numbers. We have

$$ G(\mathbb{R}) = \{ g \in \mathrm{GL}_4(\mathbb{R}) \mid {}^t g J g = J \}, $$

where as before $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Then the Lie algebra $\mathfrak{g}$ of G will be the set of matrices $X \in \mathfrak{sl}_4(\mathbb{R})$ such that ${}^tXJ + JX = 0$. The Cartan involution is given by $\theta(X) = -{}^tX$. Then we let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigen-spaces of θ, respectively. We have

$$\mathfrak{k} = \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A + iB \in U(2)\},$$

and

$$\mathfrak{p} = \{\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A = {}^tA, B = {}^tB\}.$$

Let K be the analytic subgroup defined by $\mathfrak{k}$. Next let

$$T = \{\begin{pmatrix} \cos\theta_1 & & \sin\theta_1 & \\ & \cos\theta_2 & & \sin\theta_2 \\ -\sin\theta_1 & & \cos\theta_1 & \\ & -\sin\theta_2 & & \cos\theta_2 \end{pmatrix} \mid \theta_1, \theta_2 \in \mathbb{R}\}. \quad (1)$$

We have $T \subset K$. The Lie algebra of T, denoted by $\mathfrak{t}$, is a Cartan subalgebra. We now describe the root spaces associated with T. Set

$$E_\alpha = \frac{1}{2}\begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_\beta = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 \end{pmatrix},$$

$$E_\gamma = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{pmatrix},$$

$$E_\delta = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & -i \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -1 \\ -i & 0 & -1 & 0 \end{pmatrix}.$$

Then $E_\alpha, E_\beta, E_\gamma$ and E_δ are elements of $\mathfrak{g}^{\mathbb{C}}$. Then we have

$$\mathrm{Ad}(t)E_\alpha = e^{2i\theta_1}E_\alpha,$$

$$\mathrm{Ad}(t)E_\beta = e^{2i\theta_2}E_\beta,$$

$$\mathrm{Ad}(t)E_\gamma = e^{i(\theta_1-\theta_2)}E_\gamma,$$

$$\mathrm{Ad}(t)E_\delta = e^{i(\theta_1+\theta_2)}E_\delta.$$

One way to verify these identity is to use the Cayley transform. For this, let

$$\tilde{C} = \begin{pmatrix} iI_2 & -iI_2 \\ I_2 & I_2 \end{pmatrix}.$$

Note that $\tilde{C} \in \mathrm{GSp}_4(\mathbb{C})$. One can then verify that

$$(\tilde{C})^{-1}t\tilde{C} = \begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & e^{-i\theta_1} & \\ & & & e^{-i\theta_2} \end{pmatrix}$$
$$= \begin{pmatrix} z & & & \\ & w & & \\ & & z^{-1} & \\ & & & w^{-1} \end{pmatrix},$$

for obvious choices of z and w. Next we set for each index α, $E_{-\alpha} = -{}^T\bar{E}_\alpha$. We will then have

$$E_{-\alpha} = \frac{1}{2}\begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{-\beta} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 1 \end{pmatrix},$$

$$E_{-\gamma} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{pmatrix},$$

$$E_{-\delta} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & 1 & 0 \end{pmatrix}.$$

If for each index $\pm\alpha$, we set $X_{\pm\alpha} = \tilde{C}^{-1}E_{\pm\alpha}\tilde{C}$, then $X_{\pm\alpha}$ will be a root vector for the with respect to the diagonal Cartan subgroup. The correspondence is the following

$$\alpha \longleftrightarrow z^2$$
$$\beta \longleftrightarrow w^2$$
$$\gamma \longleftrightarrow z/w$$
$$\delta \longleftrightarrow zw$$

Let X_α be a typical root vector. Then

$$X_\alpha = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$X_\delta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

One can easily verify that the normalization of X_α's and E_α's as above matches the one in the paper of [21].

Next, we observe that $E_\alpha, E_\beta, E_\delta \in \mathfrak{p}$, whereas $E_\gamma \in \mathfrak{k}$. This implies that

$$\Delta_n = \{\pm\alpha, \pm\beta, \pm\delta\},$$

$$\Delta_c = \{\pm\gamma\}$$

It is clear that $W(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) = \{1, w\}$, with $w = (12)(34)$.

It will then be easy to see that

$$\Delta = \{\pm(2,0), \pm(1,-1), \pm(1,1), \pm(0,2)\},$$

and

$$\Delta_K = \{\pm(1,-1)\}.$$

1.2 Discrete series

Analytically integral elements of $(\mathfrak{b}^{\mathbb{C}})'$ are given by pairs (a,b), and since the action of W_K induced equivalence of discrete series, we can assume that $a \geq b$. Since, we are interested in non-singular pairs, we need to assume $a \neq 0$, $b \neq 0$, $a \neq b$. There are four cases to be considered:

I. $a > b > 0$. In this case, we have

$$\Delta_\lambda^+ = \{(2,0), (1,-1), (1,1), (0,2)\},$$

and

$$\Lambda = (a+1, b+2), \qquad \lambda + \delta_G = (a+2, b+1).$$

II. $a > -b > 0$. In this case, we have

$$\Delta_\lambda^+ = \{(2,0), (1,-1), (1,1), (0,-2)\},$$

and

$$\Lambda = (a+1, b), \qquad \lambda + \delta_G = (a+2, b-1).$$

III. $-b > a > 0$. In this case, we have

$$\Delta_\lambda^+ = \{(2,0), (1,-1), (-1,-1), (0,-2)\},$$

and

$$\Lambda = (a, b-1), \qquad \lambda + \delta_G = (a+1, b-2).$$

IV. $-b > -a > 0$. In this case, we have

$$\Delta_\lambda^+ = \{(-2,0), (1,-1), (-1,-1), (0,-2)\},$$

and

$$\Lambda = (a-2, b-1), \qquad \lambda + \delta_G = (a-1, b-2).$$

If $\lambda = (a, b)$, with say $a > b > 0$, then the L-packet of π_λ consists of all $\pi_{\lambda'}$ with λ' in the orbit of λ under W_G. Let $\Phi(\pi_\lambda)$ be the L-packet of π_λ. Note that for each $J \in \{I, II, III, IV\}$ as above we have

$$|\Phi(\pi_\lambda) \cap J| = 1.$$

For the case of PSp(4), we will need the parameter to be trivial at $-I_4$. This would imply $a \equiv b \mod 2$. If we start from a discrete series representation of GSp(4) and restrict it to Sp(4), the resulting representation will decompose as the sum of two representations, either I+ IV, or II+III. The I+IV corresponds to the generic discrete series, and II+III corresponds to the holomorphic (and anti-holomorphic at the Sp(4) level).

1.3 Whittaker models

As we will primarily be dealing with representations which have Whittaker models, we take a moment to review basic definition and properties of such models.

Let π be an automorphic cuspidal representation of the group G. For each $\phi \in \pi$, we set

$$W_\phi(g) = \int_{(\mathbb{Q}\backslash\mathbb{A})^4} \phi\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & & x_4 & x_3 \\ & 1 & x_3 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix} g \right) \times \psi^{-1}(x_1 + x_2)\, dx_1\, dx_2\, dx_3\, dx_4$$

Let N be the unipotent radical of the Borel subgroup. For each place v of $\mathbb{Q}$, the restriction of θ to $N(\mathbb{Q}_v)$ is denoted by θ_v. Consider the representation of G induced from the character θ_v of $N(\mathbb{Q}_v)$:

$$C^{\infty}_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)) := \left\{ W : G(\mathbb{Q}_v) \to \mathbb{C} \mid \begin{smallmatrix} \text{smooth,} \\ W(ng)=\theta_v(n)W(g), \\ n\in N(\mathbb{Q}_v), g\in G(\mathbb{Q}_v) \end{smallmatrix} \right\}. \tag{2}$$

The action of $G(\mathbb{Q}_v)$ on $C^{\infty}_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v))$ is by right translation.

If v is a finite place of $\mathbb{Q}$, then for any irreducible admissible representation π_v of $G(\mathbb{Q}_v)$, the intertwining space $\mathrm{Hom}_{G(\mathbb{Q}_v)}(\pi_v, C^{\infty}_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)))$ is at most one dimensional ([32], Theorem 3). If there is a non-zero intertwining operator

$$\Psi \in \mathrm{Hom}_{G(\mathbb{Q}_v)}(\pi_v, C^{\infty}_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v))) \tag{3}$$

then we say that π_v is generic, and call the image $W_u := \Psi(u)$ of $u \in \pi_v$ the *local Whittaker function corresponding to* $u \in \pi_v$. The space of all W_u $(u \in \pi_v)$ is called the *Whittaker model of* π_v *with respect to* θ_v.

Now let $v = \infty$ be the archimedean place. We say that a $\mathbb{C}$-valued function W on $G(\mathbb{R})$ is of moderate growth if there exists $C > 0$ and $M > 0$ such that $|W(g)| \le C\|g\|^M$ for all $g \in G(\mathbb{R})$. The form $\|g\|$ of $g = (g_{ij})$ is defined by $\|g\| := \max\{|g_{ij}|, |(g^{-1})_{ij}\}$. The space of functions $W \in C^{\infty}_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v))$ which is of moderate growth is denoted by $\mathcal{A}_{\theta_\infty}(N(\mathbb{R})\backslash G(\mathbb{R}))$. Improving Shalika's local multiplicity one theorem ([38], Theorem 3.1), Wallach ([44], Theorem 8.8 (1)) showed that for an arbitrary $(\mathfrak{g}, K)$-module π_∞ the intertwining space $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\infty, \mathcal{A}_{\theta_\infty}(N(\mathbb{R})\backslash G(\mathbb{R})))$ is at most one-dimensional. Again, if there is a non-zero intertwining operator

$$\Psi \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\infty, \mathcal{A}_{\theta_\infty}(N(\mathbb{R})\backslash G(\mathbb{R}))), \tag{4}$$

then we say π_∞ is generic and call the image $W_u := \Psi(u)$ of $u \in \pi_\infty$ the *local Whittaker function corresponding to* u.

1.4 Bessel functionals

We recall the notion of Bessel model introduced by Novodvorsky and Piatetski-Shapiro [27]. We follow the exposition of [6]. Let $S \in M_2(\mathbb{Q})$ be such that $S = {}^tS$. We define the discriminant $d = d(S)$ of S by $d(S) = -4\det S$. Let us define a subgroup $T = T_S$ of $\mathrm{GL}(2)$ by

$$T = \{g \in \mathrm{GL}(2) \mid {}^tgSg = \det g . S\}.$$

Then we consider T as a subgroup of $\mathrm{GSp}(4)$ via

$$t \mapsto \begin{pmatrix} t & \\ & \det t . {}^tt^{-1} \end{pmatrix},$$

$t \in T$.

Let us denote by U the subgroup of GSp(4) defined by

$$U = \{u(X) = \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix} \mid X = {}^tX\}.$$

Finally, we define a subgroup R of GSp(4) by $R = TU$.

Let ψ be a non-trivial character of $\mathbb{Q}\backslash\mathbb{A}$. Then we define a character ψ_S on $U(\mathbb{A})$ by $\psi_S(u(X)) = \psi(\mathrm{tr}(SX))$ for $X = {}^tX \in \mathsf{M}_2(\mathbb{A})$. Usually when there is no danger of confusion, we abbreviate ψ_S to ψ. Let Λ be a character of $T(\mathbb{Q})\backslash T(\mathbb{A})$. Denote by $\Lambda \otimes \psi_S$ the character of $R(\mathbb{A})$ defined by $(\Lambda \otimes \psi)(tu) = \Lambda(t)\psi_S(u)$ for $t \in T(\mathbb{A})$ and $u \in U(\mathbb{A})$.

Let π be an automorphic cuspidal representation of $\mathrm{GSp}_4(\mathbb{A})$ and V_π its space of automorphic functions. We assume that

$$\Lambda|_{\mathbb{A}^\times} = \omega_\pi. \tag{5}$$

Then for $\varphi \in V_\pi$, we define a function B_φ on $\mathrm{GSp}_4(\mathbb{A})$ by

$$B_\varphi(g) = \int_{Z_\mathbb{A} R_\mathbb{Q} \backslash R_\mathbb{A}} (\Lambda \otimes \psi_S)(r)^{-1}.\varphi(rh)\, dh. \tag{6}$$

We say that π has a global Bessel model of type (S, Λ, ψ) for π if for some $\varphi \in V_\pi$, the function B_φ is non-zero. In this case, the $\mathbb{C}$-vector space of functions on $\mathrm{GSp}_4(\mathbb{A})$ spanned by $\{B_\varphi \mid \varphi \in V_\pi\}$ is called the space of the global Bessel model of π.

Similarly, one can consider local Bessel models. Fix a local field $\mathbb{Q}_v$. Define the algebraic groups T_S, U, and R as above. Also, consider the characters Λ, ψ, ψ_S, and $\Lambda \otimes \psi_S$ of the corresponding local groups. Let (π, V_π) be an irreducible admissible representation of the group GSp(4) over $\mathbb{Q}_v$, when v is finite, or a $(\mathfrak{g}, K)$-module when v is archimedean. Then we say that the representation π has a local Bessel model of type (S, Λ, ψ) if there is a non-zero map in

$$\mathrm{Hom}(\pi_v, Ind(\Lambda \otimes \psi | R, G)). \tag{7}$$

Here the Hom space is the collection of $G(\mathbb{Q}_v)$-intertwining maps when v is finite, and the collection of all $(\mathfrak{g}, K)$-maps when v is archimedean. Also in the archimedean case, as in the Whittaker case, we work with that subspace of Ind which consists of functions of moderate growth.

In this work, we will be interested in two different types of Bessel models corresponding to two choices of the symmetric matrix S. The two choices of S are:

1. $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$,

2. $S = \begin{pmatrix} 1 & \\ & d \end{pmatrix}$, with d a positive square-free rational number.

Below, we will determine the subgroups T_S, and R, and explicitly write down the corresponding global Bessel functionals. We fix an irreducible automorphic cuspidal representation π of $\mathrm{GSp}_4(\mathbb{A})$ and a unitary character ψ of $\mathbb{A}$ throughout.

(1) $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. This is the case of interest for us in this work. In this case, the subgroup T_S is equal to the subgroup consisting of diagonal matrices. A straightforward analysis then shows that for every character Λ of $T_S(\mathbb{Q})\backslash T_S(\mathbb{A})$ subject to (5), there is a Hecke character of $\mathbb{A}^\times$ such that the global Bessel functional (6) is given by

$$B_\chi^{\mathrm{split}}(g;\varphi) = \int_{F^\times\backslash\mathbb{A}^\times} \varphi^U(\begin{pmatrix} y & & & \\ & 1 & & \\ & & 1 & \\ & & & y \end{pmatrix})\chi(y)\, d^\times y.$$

Here when ϕ is a cusp form on GSp(4), we have set

$$\phi^U(g) = \int_{(F\backslash\mathbb{A})^3} \phi(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g)\psi^{-1}(w)\, du\, dv\, dw.$$

(2) $S = \begin{pmatrix} 1 & \\ & d \end{pmatrix}$. In this case, the subgroup T_S is equal to a non-split torus. Then there is a Hecke character of the torus T_S, say χ, in such a way that

$$B_\chi(g;\varphi) = \int_{T_S(F)\mathbb{A}^\times\backslash T_S(\mathbb{A})} \varphi^U(\begin{pmatrix} \alpha & \\ & \det\alpha.\, {}^t\alpha^{-1} \end{pmatrix})\chi(\alpha)\, d\alpha,$$

with ϕ^U defined as before. The case of immediate interest is the case where $d = 1$, in which case,

$$\begin{aligned} T_S &= \{g \in \mathrm{GL}_2 \mid {}^tg.g = \det g\} \\ &= \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 \in \mathrm{GL}_1\}. \end{aligned}$$

The problems of existence of Bessel functionals for this choice of the matrix S seem to be more delicate.

1.5 Theta correspondence

In this section we collect various results on theta correspondence that we will use in the sequel. In fact, this section is a rough review of [29]. We have adapted the results of that paper to the case of our interest, split orthogonal spaces of signature $(2,2)$. Other references of interest are [14, 15].

Let V be the vector space M_2, of the two by two matrices, equipped with the quadratic form det. Let $(,)$ be the associated non-degenerate inner product, and $H = \mathrm{GO}(V,(,))$ be the group of orthogonal similitudes of V, $(,)$.

The group $\mathrm{GL}(2)\times\mathrm{GL}(2)$ has a natural involution t defined by $t(g_1,g_2) = ({}^tb_2^{-1}, {}^t b_1^{-1})$, where the superscript t stands for the transposition. Let $\tilde{H} = (\mathrm{GL}(2)\times\mathrm{GL}(2))\rtimes < t >$ be the semi-direct product of $\mathrm{GL}(2)\times\mathrm{GL}(2)$ with the group of order two generated by t. There is a sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1, \tag{8}$$

where the homomorphism $\rho : \tilde{H} \to H$ is defined by $\rho(g_1,g_2)(v) = g_1 v g_2^{-1}$, and $\rho(t)v = {}^tv$, for all $g_1, g_2 \in \mathrm{GL}(2)$ and $v \in V$. Also, $\mathbb{G}_m \to \tilde{H}$ is the natural map $z \mapsto (z,z)\times 1$. It follows that the image of the subgroup $\mathrm{GL}(2)\times\mathrm{GL}(2) \subset \tilde{H}$ under ρ is the connected component of the identity of H.

Let F be a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Fix a non-trivial unitary character ψ of F. The Weil representation ω of $\mathrm{Sp}(4,F)\times \mathrm{O(V,F)}$ defined with respect to ψ is the unitary representation on $L^2(V^2)$ given by

$$\begin{aligned}
\omega(1,h)\varphi(x) &= \varphi(h^{-1}x),\\
\omega\left(\begin{pmatrix} a & \\ & {}^ta^{-1}\end{pmatrix}\right)\varphi(x) &= |\det a|^2\,\varphi(xa),\\
\omega\left(\begin{pmatrix} 1 & b \\ & 1\end{pmatrix}\right)\varphi(x) &= \psi\left(\frac{1}{2}\mathrm{tr}(bx,x)\right)\varphi(x),\\
\omega\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right)\varphi(x) &= \gamma\hat{\varphi}(x).
\end{aligned}$$

Here, $\hat{\varphi}$ is the Fourier transform defined by

$$\varphi(x) = \int_{V^2} \varphi(x')\psi(\mathrm{tr}(x,x'))\,dx'$$

with dx' self-dual, and γ is a certain fourth root of unity on ψ. If $h \in \mathrm{O}(V,F)$, $a \in \mathrm{GL}(2,F)$, $b \in \mathsf{M}_n(F)$ with ${}^tb = b$ and $x = (x_1,x_2), x' = (x_1',x_2') \in V^2$, we write $h^{-1}x = (h^{-1}x_1, h^{-1}x_2)$, $xa = (x_1,x_2)(a_{ij})$, $(x,x') = ((x_i,x_j'))$, $bx = b\,{}^t(x_1,x_2)$.

If F is non-archimedean, ω preserves the space $\mathcal{S}(V^2)$; by ω we mean ω acting on the latter space. When $F = \mathbb{R}$, we will work with Harish-Chandra modules of real reductive groups. Fix $K_1 = \mathrm{Sp}(4,\mathbb{R})\cap \mathrm{O}(4,\mathbb{R})$ as a maximal compact subgroup of $\mathrm{Sp}(4,\mathbb{R})$. We denote the Lie algebra of $\mathrm{Sp}(4,\mathbb{R})$ by $\mathfrak{g}_1 = \mathfrak{s}p(4,\mathbb{R})$. Let V^+ and V^- be positive and negative definite subspaces of X, respectively, such that $V = V^+ \perp V^-$. Then a maximal compact subgroup of $\mathrm{O}(V,\mathbb{R})$ is $\mathrm{O}(V^+,\mathbb{R})\times \mathrm{V}(V^-,\mathbb{R}) \simeq \mathrm{O}(2,\mathbb{R})\times\mathrm{O}(2,\mathbb{R})$. The Lie algebra of $\mathrm{O}(V,\mathbb{R})$ is $\mathfrak{h}_1 = \mathfrak{o}(V,\mathbb{R})$. Let $\mathcal{S}(V^2) = \mathcal{S}_\psi(V^2)$ be the subspace of $L^2(V^2)$ consisting of the functions

$$p(x)\exp\left[-\frac{1}{2}|c|\left(\mathrm{tr}(x^+,x^+) - \mathrm{tr}(x^-,x^-)\right)\right].$$

Here p is a polynomial, and (x^+,x^+) and (x^-,x^-) are 2×2 matrices with (i,j)-th entries (x_i^+,x_j^+) and (x_i^+,x_j^+) respectively, where $x_i = x_i^+ + x_i^-$ corresponding to the decomposition of V; $c \in \mathbb{R}^\times$ is such that $\psi(t) = \exp(ict)$. Then $\mathcal{S}(V^2)$ is a $(\mathfrak{g}_1 \times \mathfrak{h}_1, K_1, J_1)$ module under ω; this is the Harish-Chandra module we will work with throughout. Often, for the sake of uniformity in presentation, one uses the notation and terminology of genuine representations for archimedean places as well. The reader has to keep on mind, however, that this is just a matter of convenience.

Let $\mathcal{R}(\mathrm{O(V,F)})$ be the set of elements of $\mathrm{Irr}\,(\mathrm{O(V,F)})$ which are non-zero quotients of ω, and define $\mathcal{R}(\mathrm{Sp}(4,F))$ similarly. Again, the reader will have to keep in mind that at the archimedean place, we are working with underlying Harish-Chandra modules. Suppose F is real or non-archimedean of odd residual characteristic. Then the set

$$\left\{(\pi,\sigma) \in \mathcal{R}(\mathrm{Sp}(4,F)) \times \mathcal{R}(\mathrm{O(V,F)}) \,|\, \mathrm{Hom}_{\mathrm{Sp(4,F)}\times\mathrm{O(V,F)}}(\omega, \pi\otimes\sigma) \neq 0\right\}$$

is the graph of a bijection, denoted by θ in either direction, between the corresponding sets. When F is non-archimedean of even residual characteristic, one can establish the same for tempered representations. We refer the reader to [29], section 1, for more information.

We now recall the extended Weil representation for similitude groups. Define

$$R_V(F) = \{(g,h) \in \mathrm{GSp}(4,F) \times \mathrm{GO}(V,F) \,|\, \nu(g) = \nu(h)\}\,.$$

The Weil representation of $\mathrm{Sp}(4,F) \times \mathrm{O(V,F)}$ on $L^2(V^2)$ extends to a unitary representation of $R_V(F)$ via

$$\omega(g,h)\varphi = |\nu(h)|^{-2}\,\omega(g\begin{pmatrix}1 & \\ & \nu(g)\end{pmatrix}^{-1},1)(\varphi\circ h^{-1}).$$

We would still like to consider the action of $R_V(F)$ on $\mathcal{S}(V^2)$, but one has to take some care when considering the archimedean place, as in this case $\mathcal{S}(V^2)$ is preserved only at the level of Harish-Chandra modules; we refer the reader to [29] for details. We denote the resulting genuine representation of R_V, in the non-archimedean case, or the $(\mathfrak{r}_\infty, L_\infty)$ Harish-Chandra module, in the archimedean case, again by ω.

In analogy with the isometry case, one can ask when $\mathrm{Hom}_{R_V}(\omega, \pi\otimes\sigma) \neq 0$ for $\pi \in \mathrm{Irr}\,(\mathrm{GSp(4,F)})$ and $\sigma \in \mathrm{Irr}(\mathrm{GO(V,F)})$. Here $\mathcal{R}$ for each group is the collection of representations of the similitude group which when restricted to the corresponding isometry group have a non-zero component in $\mathcal{R}$. Then by theorem 1.8 of [29], parts 1, 3, 5, $\mathrm{Hom}_{R_V}(\omega, \pi\otimes\sigma) \neq 0$ defines a bijection between $\mathcal{R}(\mathrm{GSp}(4,F))$ and $\mathcal{R}(\mathrm{GO}(V,F))$. Again, over a non-archimedean field of even residual characteristic one has to restrict to an appropriate class of representations. Again, one denotes the resulting bijection by θ. Proposition 1.11 of [29] states that θ maps unramified representations to unramified representations.

Let (π_1,π_2) be a pair of representations of GL_2 over the local field F with $\omega_{\pi_1}.\omega_{\pi_2} = 1$. Roberts [29] has associated to (π_1,π_2) an L-packet in $\mathrm{GSp}(4)$. Essentially, the idea is to consider the representation $\pi = \pi_1\otimes\pi_2$ of $\mathrm{GSO}(V,F)$ and then consider all possible extensions of π to $\mathrm{GO}(V,F)$; then consider the theta lifts of all such extended representations to $\mathrm{GSp}(4, F)$. We describe the L-parameter giving this packet in the archimedean situation. If $g_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$, $i = 1, 2$, we set

$$S(g_1, g_2) = \begin{pmatrix} \alpha_1 & & \beta_1 & \\ & \alpha_2 & & \beta_2 \\ \gamma_1 & & \delta_1 & \\ & \gamma_2 & & \delta_2 \end{pmatrix}.$$

For $i = 1, 2$, let $\rho_i : W_{\mathbb{R}} \to \mathrm{GL}_2(\mathbb{C})$ be the L parameter of π. Then define an L-parameter $\varphi(\rho_1, \rho_2) : W_{\mathbb{R}} \to \mathrm{GSp}(4, \mathbb{C})$ by

$$\varphi(\rho_1, \rho_2)(z) = S(\rho_1(z), \rho_2(z)^{-1}), \tag{9}$$

$z \in W_{\mathbb{R}}$. We take for granted the fact that the L packet defined by Roberts in the archimedean situation is the L packet associated to $\varphi(\rho_1, \rho_2)$ by Langlands. We refer the reader to section 4 of [29], in particular pages 283-285 for basic properties of the L packets.

We now turn our attention to global theta correspondence for the similitude groups [29], section 5. In order to define global theta correspondence we need a global Weil representation. Fix a non-trivial unitary character of $\mathbb{A}$ trivial on $\mathbb{Q}$. For a place v of $\mathbb{Q}$, let ω_v be the representation defined above. Let $x_1, \ldots, x_4$ be a vector space basis of $\mathsf{M}_2(\mathbb{Q})$ over $\mathbb{Q}$. Let $(g, h) \in R_V(\mathbb{A})$. Then for almost all places v, $\omega_v(g_v, h_v)$ fixes the characteristic function of $\mathrm{O}_v x_1 + \cdots + \mathrm{O}_v x_4$. Let $\mathcal{S}(V(\mathbb{A})^2)$ be the restricted algebraic direct product $\otimes_v \mathcal{S}(V(\mathbb{Q}_v)^2)$ which is naturally an $R_V(\mathbb{A}_f) \times (\mathfrak{r}_\infty, L_\infty)$-module. For $\varphi \in \mathcal{S}(V(\mathbb{A})^2)$ and $(g, h) \in R_V(\mathbb{A})$, define

$$\theta(g, h; \varphi) = \sum_{x \in V(\mathbb{Q})^2} \omega(g, h)\varphi(x).$$

This series converges absolutely and is left $R(\mathbb{Q})$ invariant. Fix a right invariant quotient measure on $\mathrm{O}(V, \mathbb{Q})\backslash\mathrm{O}(V, \mathbb{A})$. Let f be a cusp form on $\mathrm{GO}(V, \mathbb{A})$. For $g \in \mathrm{GSp}(4, \mathbb{A})$ define

$$\theta(f, \varphi)(g) = \int_{\mathrm{O}(V,\mathbb{Q})\backslash\mathrm{O}(V,\mathbb{A})} \theta(g, h_1 h; \varphi) f(h_1 h)\, dh_1,$$

where $h \in \mathrm{GO}(V, \mathbb{A})$ is any element such that $(g, h) \in R_V(\mathbb{A})$. This integral converges absolutely, does depend on the choice of h, and the function $\theta(f, \varphi)$ on $\mathrm{GSp}(4, \mathbb{A})$ is left $\mathrm{GSp}(4, \mathbb{Q})$ invariant. The function $\theta(f, \varphi)$ is an automorphic function on $\mathrm{GSp}(4, \mathbb{A})$ of central character equal to the central character of f. If V is a $\mathrm{GO}(V, \mathbb{A}) \times (h_\infty, J_\infty)$ subspace of the space of cusp forms on $\mathrm{GO}(V, \mathbb{A})$ of central character χ, then we denote by $\Theta(V)$ the $\mathrm{GSp}(4, \mathbb{A}_f) \times (g_\infty, K_\infty)$ subspace of the space of automorphic forms on $\mathrm{GSp}(4, \mathbb{A})$ of central character χ generated by all the $\theta(f, \varphi)$ for $f \in V$ and $\varphi \in \mathcal{S}(V(\mathbb{A})^2)$.

For computational purposes, we need to make the above considerations explicit. Here the notation may be slightly different from above. Suppose π_1 and

π_2 are two irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ satisfying
$$\omega_{\pi_1}.\omega_{\pi_2} = 1.$$
Then for φ_1 and φ_2 cusp forms in the spaces of π_1 and π_2, respectively, one can think of
$$\varphi(h_1, h_2) = \varphi_1(h_1)\varphi_2(h_2),$$
as a cusp form on the algebraic group $\rho(\tilde{H})$. We extend the definition of φ to H by defining it to be right invariant under the compact totally disconnected group $< t > (\mathbb{A}) = \prod_v < t >$.

Define the subgroup H_1 consisting of elements (h_1, h_2) satisfying
$$\det(h_1) = \det(h_2).$$
Then if π_1 and π_2 are two automorphic cuspidal representations of the group GL(2) with
$$\omega_{\pi_1}.\omega_{\pi_2} = 1,$$
and
$$\pi_1 \neq \tilde{\pi}_2,$$
then one can naturally think of the pair (π_1, π_2) as an automorphic cuspidal representation of the group H. If φ_1 and φ_2 are cusp forms on $\mathrm{GL}_2(\mathbb{A})$, belonging to the spaces of the representations π_1 and π_2, respectively, we define a cuspidal function $\theta(\varphi_1, \varphi_2; f)$ on $\mathrm{GSp}(4, \mathbb{A})$ by
$$\theta(\varphi_1, \varphi_2; f)(g) = \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(g; h_1h^1, h_2h^2; f)\varphi_1(h_1h^1)\varphi_2(h_2h^2)\, d(h_1, h_2),$$
where the pair (h^1, h^2) is chosen such that
$$\det h^1(\det h^2)^{-1} = \nu(g).$$
Here f is a Bruhat-Schwartz function on $\mathsf{M}_2(\mathbb{A}) \times \mathsf{M}_2(\mathbb{A})$, and
$$\theta(g; h_1h^1, h_2h^2; f) = \sum_{M_1, M_2 \in \mathsf{M}_2(F)} \omega(g; h_1h^1, h_2h^2) f(M_1, M_2),$$
where ω is the Weil representation of [14]. We note this is different from the definition given earlier. Let $\Theta(\pi_1, \pi_2)$ be the vector space generated by the functions $\theta(\varphi_1, \varphi_2; f)$ for all choices of φ_1, φ_2, and f as above. Then $\Theta(\pi_1, \pi_2)$ is an irreducible generic automorphic cuspidal representation of GSp(4). In fact, this is the generic element of the global L packet defined by Roberts [29]. If $\Theta(\pi_1, \pi_2) = \otimes_v \Theta_v(\pi_1, \pi_2)$, then $\Theta_v(\pi_1, \pi_2)$ depends only on the v components of π_1, π_2, and is the generic element of corresponding local L packet.

1.6 Theta correspondence for $(\mathrm{Sp}(4,\mathbb{R}),\mathrm{O}(2,2))$

The result of this subsection is taken from [14]. Let $G=\mathrm{Sp}(4,\mathbb{R})$, $H=\mathrm{O}(2,2)$, $K=U(2)$, and $L=\mathrm{O}(2)\times\mathrm{O}(2)$. Next we have the following:

Proposition 1.1 *Let $\pi=\pi_{\Lambda+\rho}$ be the generic discrete series representation with Harish-Chandra parameter*

$$\Lambda+\rho=\begin{cases}(a+2,-b-1)\\(b+1,-a-2)\end{cases}$$

of G. Then π occurs in the theta correspondence for (G,H), and

$$\theta(\pi)=\begin{cases}\pi(a+b+4,b-a-2)\\\pi(a+b+4,a-b+2)\end{cases}$$

1.7 The Spinor L-function for $\mathrm{GSp}(4)$

In this section, we review the integral representation given by Novodvorsky [26] for $G=\mathrm{GSp}(4)$. The details of the material in the following paragraphs appear in [4, 40].

Let φ be a cusp form on $\mathrm{GSp}(4,\mathbb{A})$, belonging to the space of an irreducible cuspidal automorphic representation π. Consider the integral

$$Z_N(s,\phi,\mu)=\int_{\mathbb{A}^\times/\mathbb{Q}^\times}\int_{(\mathbb{A}/\mathbb{Q})^3}\phi\Big(\begin{pmatrix}1&x_2&x_4&\\&1&&\\&&1&\\&z&-x_2&1\end{pmatrix}\begin{pmatrix}y&&&\\&y&&\\&&1&\\&&&1\end{pmatrix}\Big)$$
$$\times\,\psi(-x_2)\mu(y)|y|^{s-\frac{1}{2}}\,dz\,dx_2\,dx_4\,d^\times y.$$

Since ϕ is left invariant under the matrix

$$w=\begin{pmatrix}&&&1\\&&-1&\\&1&&\\-1&&&\end{pmatrix},$$

this integral has a functional equation $s\to1-s$. Observe that this choice of w corrects an inaccuracy in [40]; we thank Brooks Roberts for pointing out this error. A usual unfolding process as sketched in [4] then shows that

$$\mathbb{Z}_N(s,\phi,\mu)=\int_{\mathbb{A}^\times}\int_{\mathbb{A}}W_\phi\begin{pmatrix}y&&&\\&y&&\\&&1&\\&x&&1\end{pmatrix}\mu(y)|y|^{s-\frac{3}{2}}\,dx\,d^\times y. \qquad (10)$$

Here the Whittaker function W_φ is given by

$$W_\phi(g) = \int_{(\mathbb{A}/\mathbb{Q})^4} \phi\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & & x_4 & x_3 \\ & 1 & x_3 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix} g \right) \times \psi^{-1}(x_1 + x_2)\, dx_1\, dx_2\, dx_3\, dx_4$$

Equation (10) implies that, in order for $Z_N(\varphi, s)$ to be non-zero, we need to assume that W_φ is not identically equal to zero. A representation satisfying this condition is called "generic." Every irreducible cuspidal representation of GL(2) is generic. On other groups, however, there may exist non-generic cuspidal representations. In fact, those representations of GSp(4) which correspond to holomorphic cuspidal Siegel modular forms are not generic.

From this point on, we assume that all the representations of GSp*(4), local or global, which appear in the text are generic.*

If φ is chosen appropriately, the Whittaker function may be assumed to decompose locally as $W(g) = \prod_v W_v(g_v)$, a product of local Whittaker functions. Hence, for $\Re s$ large, we obtain

$$\mathcal{Z}(\varphi, s) = \prod_v \mathcal{Z}(W_v, s), \tag{11}$$

where

$$Z_N(W_v, s) = \int_{F_v^\times} \int_{F_v} W_v\left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & x & & 1 \end{pmatrix} \right) |y|^{s-\frac{3}{2}}\, dx\, d^\times y. \tag{12}$$

As usual, we have a functional equation: There exists a meromorphic function $\gamma(\pi_v, \psi_v, s)$ (rational function in $\mathbb{N}v^{-s}$ when $v < \infty$) such that

$$Z_N(W_v, s) = \gamma(\pi_v, \psi_v, s) \tilde{\mathcal{Z}}(W_v^w, 1-s), \tag{13}$$

with w as above,

$$\tilde{\mathcal{Z}}(W_v, s) = \int_{F_v^\times} \int_{F_v} W_v\left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & x & & 1 \end{pmatrix} \right) \chi_v^{-1}(y) |y|^{s-\frac{3}{2}}\, dx\, d^\times y,$$

and χ_v the central character of π_v.

We also consider the unramified calculations. Suppose v is any nonarchimedean place of F such that W_v is right invariant by $\mathrm{GSp}(4, \mathrm{O}_v)$ and such that the largest fractional ideal on which ψ_v is trivial is O. Then the Casselman-Shalika formula [5] allows us to calculate the last integral (cf. [4]). The result is the following:

$$\mathcal{Z}(W_v, s) = L(s, \pi_v, \mathrm{Spin}). \tag{14}$$

Let us explain the notation. The connected L-group ${}^LG^0$ is $\mathrm{GSp}(\mathbb{C})$. Let ${}^LT$ be the maximal torus of elements of the form

$$t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix},$$

where $\alpha_1\alpha_4 = \alpha_2\alpha_3$. The fundamental dominant weights of the torus are λ_1 and λ_2 where

$$\lambda_1 t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1,$$

and

$$\lambda_2 t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1\alpha_3^{-1}.$$

The dimensions of the representation spaces associated with these dominant weights are four and five, respectively. In our notation, Spin is the representation of $\mathrm{GSp}(4, \mathbb{C})$ associated with the dominant weight λ_1, i.e. the standard representation of $\mathrm{GSp}(4, \mathbb{C})$ on $\mathbb{C}^4$. The L-function $L(s, \pi, \mathrm{Spin})$ is called the Spinor, or simply the Spin, L-function of $\mathrm{GSp}(4)$.

Next step is to use the integral introduced above to extend the definition of the Spinor L-function to ramified non-archimedean and archimedean places.

Corollary 1.2 *Let π be an irreducible generic representation of* $\mathrm{GSp}(4)$ *over a non-archimedean local field K. Let μ be a quasi-character of $K^\times$. If μ is highly ramified, we have*

$$L(s, \pi \otimes \mu) = 1.$$

2 Entireness of the spinor L-function

The purpose of this section is to prove the following theorem:

Theorem 2.1 *Let $\pi = \otimes_v \pi_v$ be a generic automorphic cuspidal representation of* $\mathrm{GSp}(4)$ *over $\mathbb{Q}$. Let π_∞ be* special *as defined in 2.4. Then $L(s, \pi, \mathrm{Spin})$ is entire.*

The proof of this theorem covers paragraphs 2.1 through 2.6.

2.1 The pull-back

In the global situation, there is a simple relationship between the integral representation of the previous section and split Bessel functionals. The following simple observation which for the ease of reference we separate as a lemma forms the fundamental idea of the proof of Theorem 2.1:

Lemma 2.2 *We have*

$$B^{\mathrm{split}}_{\mu|\,.\,|^{s-\frac{1}{2}}}(I_4;\phi) = \int_{\mathbb{A}^\times}\int_{\mathbb{A}} W_\phi\left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & x & & 1 \end{pmatrix} w^{-1}\right)\mu(y)|y|^{s-\frac{3}{2}}\,dx\,d^\times y,$$

with

$$w = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix}.$$

This lemma should be compared to equation (16) of [7]. The lemma motivates the following definition.

Definition 2.3 *For φ_1, φ_2, and f as above and μ a Hecke character, we define*

$$\begin{aligned}\mathcal{Z}(\varphi_1,\varphi_2,f;\mu) &= B^{\mathrm{split}}_{\mu|\,.\,|^{-\frac{1}{2}}}(I_4;\theta(\varphi_1,\varphi_2;f)) \\ &= \int_{F^\times\backslash\mathbb{A}^\times} \theta(\varphi_1,\varphi_2;f)^U(\begin{pmatrix} y & & & \\ & 1 & & \\ & & 1 & \\ & & & y \end{pmatrix})\mu(y)|y|^{-\frac{1}{2}}\,d^\times y.\end{aligned}$$

Here if ϕ is a cusp form on GSp(4), we have set

$$\phi^U(g) = \int_{(F\backslash\mathbb{A})^3} \phi(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g)\psi^{-1}(w)\,du\,dv\,dw.$$

We prove that the above integral is an infinite product of local integrals. We do so by finding an expression relating our function $\mathcal{Z}(\varphi_1,\varphi_2,f;s)$ to the Jacquet-Langlands zeta functions of φ_1, and φ_2.

Before stating our proposition, we recall a notation from [17]. If ϕ is a cusp form on $\mathrm{GL}_2(\mathbb{A}_F)$, in the space of a representation π, μ a Hecke character, and $h \in \mathrm{GL}_2(\mathbb{A}_F)$, we set

$$Z(\phi,h,\mu) = \int_{F^\times\backslash\mathbb{A}^\times} \phi(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h)\mu(a)|a|^{-\frac{1}{2}}\,d^\times a,$$

and

$$\tilde{Z}(\phi,h,\mu) = \int_{F^\times\backslash\mathbb{A}^\times} \phi(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h)\omega_\pi(a)^{-1}\mu(a)|a|^{-\frac{1}{2}}\,d^\times a$$

Then, we have the following proposition:

Proposition 2.4 *For φ_1, φ_2, and f as above, we have*

$$\mathcal{Z}(\varphi_1, \varphi_2, f; \mu) = \int_{D(\mathbb{A})\backslash H_1(\mathbb{A})} Z(\varphi_1, h_1, \mu) Z(\varphi_2, h_2, \mu^{-1}|\,.\,|) \\ L(h_1, h_2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_1\, dh_2$$

Proof. First, we obtain an expression for $\theta(\varphi_1, \varphi_2; f)^U$. We start by the following:

$$\theta(\varphi_1, \varphi_2; f)^U(g) \\ = \int_{(F\backslash\mathbb{A})^3} \theta(\varphi_1, \varphi_2; f)(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g) \psi^{-1}(w)\, du\, dv\, dw$$

$$= \int_{(F\backslash\mathbb{A})^3} \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g; h_1 h^1, h_2 h^2; f) \\ \varphi_1(h_1 h^1) \varphi_2(h_2 h^2)\, d(h_1, h_2) \psi^{-1}(w)\, du\, dv\, dw,$$

where h^1 and h^2 are chosen in such a way that

$$\det h^1 . (\det h^2)^{-1} = \nu(g).$$

Next, it follows from the definition of θ that

$$\theta(\varphi_1, \varphi_2; f)^U(g) = \\ \int_{H_1(F)\backslash H_1(\mathbb{A})} \varphi_1(h_1 h^1) \varphi_2(h_2 h^2) G_f(h_1 h^1, h_2 h^2; g)\, dh_1\, dh_2, \tag{15}$$

where

$$G_f(h_1 h^1, h_2 h^2; g) = \\ \sum_{M_1, M_2} \int_{(F\backslash\mathbb{A})^3} \omega(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g, h_1 h^1, h_2 h^2) f(M_1, M_2) \\ \psi^{-1}(w)\, du\, dv\, dw.$$

Next, for fixed M_1 and M_2 we have

$$\int_{(F\backslash\mathbb{A})^3} \omega(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g, h_1h^1, h_2h^2) f(M_1, M_2)\psi^{-1}(w)\, du\, dv\, dw$$
$$= \omega(g, h_1h^1, h_2h^2) f(M_1, M_2)$$
$$\int_{(F\backslash\mathbb{A})^3} \psi(tr \begin{pmatrix} u & w \\ w & v \end{pmatrix} \begin{pmatrix} \det M_1 & B(M_1, M_2) - \frac{1}{2} \\ B(M_2, M_1) - \frac{1}{2} & \det M_2 \end{pmatrix})\, du\, dv\, dw.$$

Next, we have the following straightforward lemma:

Lemma 2.5 *For any 2×2 matrix $A \in \mathsf{M}_2(\mathbb{A})$, we have*

$$\int_{(F\backslash\mathbb{A})^3} \psi(tr \begin{pmatrix} u & w \\ w & v \end{pmatrix} A)\, du\, dv\, dw = 0,$$

unless $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, in which case the value of the integral is equal to 1.

The lemma implies that

$$G_f(h_1h^1, h_2h^2; g) = \sum_{(M_1, M_2) \in \mathcal{S}} \omega(g, h_1h^1, h_2h^2) f(M_1, M_2),$$

where

$$\mathcal{S} = \{(X, Y) \in \mathsf{M}_2(F) \times \mathsf{M}_2(F) \mid \det X = 0, \det Y = 0, \det(X + Y) = 1\}.$$

Lemma 2.6 *The set $\mathcal{S}$ consists of a single orbit under the action of $H_1(F)$. The point $P = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$ belongs to $\mathcal{S}$. The stabilizer of P in $H_1(F)$ is the subgroup $D(F)$.*

Consequently,

$$G_f(h_1h^1, h_2h^2; g) = \sum_{\gamma \in D(F)\backslash H_1(F)} \omega(1, \gamma)\omega(g, h_1h^1, h_2h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}).$$

Inserting the right hand side of this expression for G_f in equation (15) gives

$$\theta(\varphi_1, \varphi_2; f)^U(g) = \int_{D(F)\backslash H_1(\mathbb{A})} \varphi_1(h_1h^1)\varphi_2(h_2h^2)\omega(g, h_1h^1, h_2h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\, dh_1\, dh_2, \tag{16}$$

We now turn our attention to $\mathcal{Z}(\varphi_1,\varphi_2,f;s)$. For this purpose, we need to first simplify $\omega(g,h_1h^1,h_2h^2)f(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix})$, when $g=\begin{pmatrix}y&&&\\&1&&\\&&1&\\&&&y\end{pmatrix}$, $h^1=\begin{pmatrix}y&\\&1\end{pmatrix}$, and $h^2=$ identity, say. We have

$$\omega(\begin{pmatrix}y&&&\\&1&&\\&&1&\\&&&y\end{pmatrix},h_1\begin{pmatrix}y&\\&1\end{pmatrix},h_2)f(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix})$$

$$=\omega(\begin{pmatrix}y&&&\\&1&&\\&&1&\\&&&y\end{pmatrix}\begin{pmatrix}1&&&\\&1&&\\&&y^{-1}&\\&&&y^{-1}\end{pmatrix})L(h_1\begin{pmatrix}y&\\&1\end{pmatrix},h_2)f(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix})$$

$$=|y|^2L(h_1\begin{pmatrix}y&\\&1\end{pmatrix},h_2)f(\begin{pmatrix}y&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix})$$

$$=f(\begin{pmatrix}y^{-1}&0\\0&1\end{pmatrix}h_1^{-1}\begin{pmatrix}y&0\\0&0\end{pmatrix}h_2,\begin{pmatrix}y^{-1}&0\\0&1\end{pmatrix}h_1^{-1}\begin{pmatrix}0&0\\0&1\end{pmatrix}h_2)$$

$$=f(\begin{pmatrix}y^{-1}&0\\0&1\end{pmatrix}h_1^{-1}\begin{pmatrix}y&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}h_2,\begin{pmatrix}y^{-1}&0\\0&1\end{pmatrix}h_1^{-1}\begin{pmatrix}y&0\\0&1\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}h_2).$$

Hence, for the choices of g, h^1, and h^2 as above, we have

$$\omega(g,h_1h^1,h_2h^2)f(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix})=$$
$$L(\begin{pmatrix}y^{-1}&0\\0&1\end{pmatrix}h_1\begin{pmatrix}y&0\\0&1\end{pmatrix},h_2)f(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}).$$

This equation combined with equation (16) gives

$$\theta(\varphi_1,\varphi_2;f)^U(\begin{pmatrix}y&&&\\&1&&\\&&1&\\&&&y\end{pmatrix})=\int_{D(F)\backslash H_1(\mathbb{A})}\varphi_1(h_1\begin{pmatrix}y&\\&1\end{pmatrix})\varphi_2(h_2)$$
$$L(\begin{pmatrix}y^{-1}&0\\0&1\end{pmatrix}h_1\begin{pmatrix}y&0\\0&1\end{pmatrix},h_2)f(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix})\,dh_1\,dh_2.$$

Next, we make a change of variables

$$(h_1,h_2)\mapsto(\begin{pmatrix}y&\\&1\end{pmatrix}h_1\begin{pmatrix}y^{-1}&\\&1\end{pmatrix},h_2)$$

to obtain

$$\theta(\varphi_1,\varphi_2;f)^U(\begin{pmatrix} y &&& \\ & 1 && \\ && 1 & \\ &&& y \end{pmatrix}) =$$
$$\int_{D(F)\backslash H_1(\mathbb{A})} \varphi_1(\begin{pmatrix} y & \\ & 1 \end{pmatrix} h_1)\varphi_2(h_2)L(h_1,h_2)f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\, dh_1\, dh_2.$$

Next,

$$\begin{aligned}
&\mathcal{Z}(\varphi_1,\varphi_2,f;\mu)\\
&\quad= \int_{F^\times\backslash\mathbb{A}^\times} \theta(\varphi_1,\varphi_2;f)^U(\begin{pmatrix} y &&& \\ & 1 && \\ && 1 & \\ &&& y \end{pmatrix})\mu(y)|y|^{-\frac{1}{2}}\, d^\times y\\
&\quad= \int_{F^\times\backslash\mathbb{A}^\times}\int_{D(F)\backslash H_1(\mathbb{A})} \varphi_1(\begin{pmatrix} y & \\ & 1 \end{pmatrix} h_1)\varphi_2(h_2)\\
&\qquad L(h_1,h_2)f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\mu(y)|y|^{-\frac{1}{2}}\, dh_1\, dh_2\, d^\times y.
\end{aligned}$$

At this stage, we use the obvious isomorphism

$$F^\times\backslash\mathbb{A}^\times \longrightarrow D(F)\backslash D(\mathbb{A}),$$

given by

$$a \mapsto (\begin{pmatrix} a & \\ & 1 \end{pmatrix}, \begin{pmatrix} a & \\ & 1 \end{pmatrix})$$

to obtain

$$\begin{aligned}
&\mathcal{Z}(\varphi_1,\varphi_2,f;\mu)\\
&\quad= \int_{F^\times\backslash\mathbb{A}^\times}\int_{D(\mathbb{A})\backslash H_1(\mathbb{A})}\int_{F^\times\backslash\mathbb{A}^\times} \varphi_1(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix} h_1)\varphi_2(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h_2)\\
&\qquad L(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h_1, \begin{pmatrix} y & \\ & 1 \end{pmatrix} h_2)f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\mu(y)|y|^{-\frac{1}{2}}\, d^\times a\, dh_1\, dh_2\, d^\times y\\
&\quad= \int_{F^\times\backslash\mathbb{A}^\times}\int_{D(\mathbb{A})\backslash H_1(\mathbb{A})}\int_{F^\times\backslash\mathbb{A}^\times} \varphi_1(\begin{pmatrix} ya & \\ & 1 \end{pmatrix} h_1)\varphi_2(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h_2)\\
&\qquad L(h_1,h_2)f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\mu(y)|y|^{-\frac{1}{2}}\, d^\times a\, dh_1\, dh_2\, d^\times y\\
&\quad= \int_{F^\times\backslash\mathbb{A}^\times}\int_{D(\mathbb{A})\backslash H_1(\mathbb{A})}\int_{F^\times\backslash\mathbb{A}^\times} \varphi_1(\begin{pmatrix} y & \\ & 1 \end{pmatrix} h_1)\varphi_2(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h_2)\\
&\qquad L(h_1,h_2)f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\mu(y)|y|^{-\frac{1}{2}}\mu^{-1}(a)|a|^{\frac{1}{2}}\, d^\times a\, dh_1\, dh_2\, d^\times y,
\end{aligned}$$

after a change of variable $y \mapsto ya^{-1}$. The proposition now follows from a simple re-arrangement of the last expression. $\square$

2.2 The zeta integral of two complex variables; Euler product

In order to study the zeta integral $\mathcal{Z}(\varphi_1, \varphi_2, f; \mu)$, we would have liked to introduce a function of two complex variables s_1, s_2 as follows: For φ_1, φ_2, and f as above, and μ Hecke character, we set

$$\mathcal{Z}(\varphi_1, \varphi_2, f; \mu, |\,.\,|^{s_1}, |\,.\,|^{s_2}) = \int_{D(\mathbb{A})\backslash H_1(\mathbb{A})} Z(\varphi_1, h_1, \mu|\,.\,|^{s_1}) Z(\varphi_2, h_2, \mu^{-1}|\,.\,|^{s_2}) \, L(h_1, h_2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_1 \, dh_2,$$

with $s_1, s_2 \in \mathbb{C}$. Unfortunately, however, this integral is not well-defined for $s_2 \neq 1 - s_1$. In order to circumvent this problem we proceed as follows.

If ϕ is a cusp form on $\mathrm{GL}_2(\mathbb{A}_F)$, we define its Whittaker function by

$$W_\phi(g) = \int_{F\backslash\mathbb{A}} \phi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) \psi(x)^{-1} \, dx,$$

for $g \in \mathrm{GL}_2(\mathbb{A}_F)$. Then, we have the Fourier expansion

$$\phi(g) = \sum_{\alpha \in F^\times} W_\phi(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g),$$

with the right hand side a uniformly convergent series on compact sets in $\mathrm{GL}_2(A)$. It is then a classical observation of [17] that for $\Re s$ large, we have

$$Z(\phi, h, \mu|\,.\,|^s) = \int_{\mathbb{A}} W_\phi(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h) \mu(a) |a|^{s - \frac{1}{2}} \, d^\times a.$$

We denote the right hand side of this equation by $Z(W_\phi, h, s)$.

We have a formal identity as follows:

$$\mathcal{Z}(\varphi_1, \varphi_2, f; \mu, |\,.\,|^{s_1}, |\,.\,|^{s_2}) = \int_{D(\mathbb{A})\backslash H_1(\mathbb{A})} Z(W_{\varphi_1}, h_1, \mu|\,.\,|^{s_1}) Z(W_{\varphi_2}, h_2, \mu^{-1}|\,.\,|^{s_2}) \, L(h_1, h_2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_1 \, dh_2.$$

Next, we consider the Euler product. We choose φ_i, for $i = 1, 2$, so that

$$W_{\varphi_i} = \otimes_{v \in \mathcal{M}_F} W_v^i.$$

Also, we choose f to be a pure tensor of the form

$$\otimes_{v \in \mathcal{M}_F} f_v,$$

with f_v unramified for almost all v.

With this choice of the data, we have yet another formal identity

$$\mathcal{Z}(\varphi_1, \varphi_2, f; \mu, |\,.\,|^{s_1}, |\,.\,|^{s_2}) = \prod_{v \in \mathcal{M}_F} \mathcal{Z}_v(W_v^1, W_v^2, f_v; \mu_v, |\,.\,|_v^{s_1}, |\,.\,|_v^{s_2}). \tag{17}$$

Here, we have set

$$\mathcal{Z}_v(W_v^1, W_v^2, f_v; \mu_v, |\,.\,|^{s_1}, |\,.\,|^{s_2}) = \int_{D(F_v)\backslash H_1(F_v)} Z(W_v^1, h_1, \mu_v|\,.\,|_v^{s_1}) Z(W_v^2, h_2, \mu_v^{-1}|\,.\,|_v^{s_2}) \\ L(h_1, h_2) f_v(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_1\, dh_2.$$

Also, for W_v a Whittaker function on a local group $\mathrm{GL}_2(F_v)$, and $h \in \mathrm{GL}_2(F_v)$, we have used the notation $Z(W_v, h, \mu_v)$ to denote

$$\int_{F_v^\times} W_v(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h) \mu_v(a) |a|^{-\frac{1}{2}}\, d^\times a.$$

The idea is to make sense out of the expression for

$$\mathcal{Z}_v(W_v^1, W_v^2, f_v; \mu_v, |\,.\,|^{s_1}, |\,.\,|^{s_2})$$

for $\Re s_1, \Re s_2$ large. For this we use the following lemma:

Lemma 2.7 *Let $v \in \mathcal{M}_F$, and Ψ a continuous function of compact support on $D(F_v)\backslash H_1(F_v)$. Choose an arbitrary lift Φ' of Φ to $\mathrm{GL}_2(F_v) \times \mathrm{GL}_2(F_v)$. The functional $\mu(\Phi)$ defined by*

$$\int_{K_v} \int_{F_v^2} \int_{F_v^\times} \Phi'(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} k_1, \begin{pmatrix} \epsilon & \\ & \epsilon^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix} k_2) |\epsilon|^{-1}\, d^\times \epsilon\, du\, dv\, dk_1\, dk_2,$$

for an appropriate choice of a local maximal compact (and open for v non-archimedean), defines an invariant measure on $D(F_v)\backslash H_1(F_v)$. Furthermore, this measure has the following property: Fix a Haar measure μ_D on $D(F_v)$, and for any continuous function of compact support Ψ on $H_1(F_v)$, set

$$\Psi_D(x) = \int_{D(F_v)} \Psi(yx)\, d\mu_1(y),$$

for $x \in D(F_v)\backslash H_1(F_v)$. Then the functional μ_2 defined by

$$\mu_2(\Psi) = \mu(\Psi_D),$$

with Ψ as above defines a Haar measure on $H_1(F_v)$.

Definition 2.8 *We set*

$$\begin{aligned}
&\mathcal{Z}_v(W_v^1, W_v^2, f_v; \mu_v, |\,.\,|^{s_1}, |\,.\,|^{s_2}) \\
&= \int_{u,v\in F_v}\int_{\epsilon\in F_v^\times}\int_{K_v^2} f(k_1^{-1}\begin{pmatrix}\epsilon^{-1} & \epsilon^{-1}v\\ 0 & 0\end{pmatrix}k_2, k_1^{-1}\begin{pmatrix}0 & -u\epsilon\\ 0 & \epsilon\end{pmatrix}k_2) \\
&\omega_{\pi_2}(\epsilon)|\epsilon|^{2s_2-2}(\int_{F_v^\times} W_1(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}k_1)\mathbf{e}(u\alpha)\mu(\alpha)|\alpha|^{s_1-\frac{1}{2}}\,d^\times\alpha) \\
&(\int_{F_v^\times} W_2(\begin{pmatrix}\beta & \\ & 1\end{pmatrix}k_2)\mathbf{e}(v\beta)\mu^{-1}(\beta)|\beta|^{s_2-\frac{1}{2}}\,d^\times\beta)\,du\,dv\,d^\times\epsilon\,dk_1\,dk_2.
\end{aligned}$$

We immediately observe that if the integral is convergent, it is well-defined.

Proposition 2.9 *Suppose W_1, W_2 are two Whittaker functions of $\mathrm{GL}_2(F_v)$ belonging to the spaces of representations π_1, π_2, respectively, with $\omega_{\pi_1}.\omega_{\pi_2} = 1$. Then the integral $\mathcal{Z}(W_1, W_2, f; \mu_v, |\,.\,|_v^{s_1}, |\,.\,|_v^{s_2})$ converges absolutely for $\Re s_1, \Re s_2 \gg 0$.*

Proof. We give a complete proof only for the case where v is a real place, the proof of the non-archimedean statement being identical. Also it is clear that we may assume that the quasi-character μ_v is trivial. By definition, we need to show that the integral

$$\begin{aligned}
&\int_{u,v\in\mathbb{R}}\int_{\epsilon\in\mathbb{R}_+^\times}\int_{K_v^2} f(k_1^{-1}\begin{pmatrix}\epsilon^{-1} & \epsilon^{-1}v\\ 0 & 0\end{pmatrix}k_2, k_1^{-1}\begin{pmatrix}0 & -u\epsilon\\ 0 & \epsilon\end{pmatrix}k_2) \\
&\omega_{\pi_2}(\epsilon)|\epsilon|^{2s_2-2}(\int_{\mathbb{R}^\times} W_1(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}k_1)\mathbf{e}(u\alpha)|\alpha|^{s_1-\frac{1}{2}}\,d^\times\alpha) \\
&(\int_{\mathbb{R}^\times} W_2(\begin{pmatrix}\beta & \\ & 1\end{pmatrix}k_2)\mathbf{e}(v\beta)|\beta|^{s_2-\frac{1}{2}}\,d^\times\beta)\,du\,dv\,d^\times\epsilon\,dk_1\,dk_2.
\end{aligned}$$

converges absolutely. By lemma 8.3.3 of [18], there are gauge functions ξ_1, ξ_2 such that

$$|W_1| \le \xi_1, \text{ and } |W_2| \le \xi_2.$$

This implies that

$$\int_{\mathbb{R}^\times} |W_1(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}k_1)\mathbf{e}(u\alpha)|\alpha|^{s_1-\frac{1}{2}}|\,d^\times\alpha \le \int_{\mathbb{R}^\times}\xi_1(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix})|\alpha|^{\sigma_1-\frac{1}{2}}\,d^\times\alpha,$$

and

$$\int_{\mathbb{R}^\times} |W_2(\begin{pmatrix}\beta & \\ & 1\end{pmatrix}k_2)\mathbf{e}(v\beta)|\beta|^{s_2-\frac{1}{2}}|\,d^\times\beta \le \int_{\mathbb{R}^\times}\xi_2(\begin{pmatrix}\beta & \\ & 1\end{pmatrix})|\beta|^{\sigma_2-\frac{1}{2}}\,d^\times\beta.$$

The latter integrals converge absolutely for σ_1, σ_2 large. In order to conclude the proof, we need to study the convergence of

$$\begin{aligned}
&\int_{u,v\in\mathbb{R}}\int_{\epsilon\in\mathbb{R}_+^\times}\int_{K_v^2} f(k_1^{-1}\begin{pmatrix}\epsilon^{-1} & \epsilon^{-1}v\\ 0 & 0\end{pmatrix}k_2, k_1^{-1}\begin{pmatrix}0 & -u\epsilon\\ 0 & \epsilon\end{pmatrix}k_2) \\
&\omega_{\pi_2}(\epsilon)|\epsilon|^{2s_2-2}\,du\,dv\,d^\times\epsilon\,dk_1\,dk_2.
\end{aligned}$$

We claim that this integral converges absolutely for all values of s_2. In fact, if $f \in \mathcal{S}(\mathsf{M}_2(\mathbb{R}) \times \mathsf{M}_2(\mathbb{R}))$, the function g defined by

$$g(X,Y) = \int_{K_v^2} f(k_1^{-1} X k_2, k_1^{-1} Y k_2)\, dk_1\, dk_2$$

is in $\mathcal{S}(\mathsf{M}_2(\mathbb{R}) \times \mathsf{M}_2(\mathbb{R}))$. Thus, we must show that

$$\int_{u,v\in\mathbb{R}} \int_{\epsilon\in\mathbb{R}_+^\times} f(\begin{pmatrix} \epsilon^{-1} & \epsilon^{-1}v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -u\epsilon \\ 0 & \epsilon \end{pmatrix}) \omega_{\pi_2}(\epsilon)|\epsilon|^{2s_2-2}\, du\, dv\, d^\times \epsilon$$

converges absolutely for all s_2. The first observation, due to Weil, is that the absolute value of a Schwartz-Bruhat function is bounded by a Schwartz-Bruhat function. Consequently, we can assume that f is a positive Schwartz-Bruhat function. But now it is clear that the function Ξ defined by

$$\Xi(\epsilon) = \int_{u,v\in\mathbb{R}} f(\begin{pmatrix} \epsilon^{-1} & \epsilon^{-1}v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -u\epsilon \\ 0 & \epsilon \end{pmatrix})\, du\, dv$$

is in the space $\mathcal{S}(\mathbb{R}^\times)$. Since our original integral is bounded by

$$\int_{\mathbb{R}} \Xi(\epsilon)\omega_{\pi_2}(\epsilon)|\epsilon|^{2\sigma_2-2}\, d^\times \epsilon,$$

the proposition is immediate. □
Then we have the following proposition:

Proposition 2.10 *Let v be a non-archimedean place. Let W_1 and W_2 be given. Then there is a choice of f such that*

$$\mathcal{Z}(W_1, W_2, f; \mu, |\,.\,|_v^{s_1}, |\,.\,|_v^{s_2}) = Z(W_1, \mu|\,.\,|_v^{s_1}) Z(W_2, \mu^{-1}|\,.\,|_v^{s_2}).$$

Proof. Let M be a very large positive integer. Let $f = g \otimes h$ be a Schwartz function such that

$$\text{Support}\, g \subset \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} + \begin{pmatrix} \mathfrak{p}^M & \mathfrak{p}^M \\ \mathfrak{p}^M & \mathfrak{p}^M \end{pmatrix},$$

and

$$\text{Support}\, h \subset \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \mathfrak{p}^M & \mathfrak{p}^M \\ \mathfrak{p}^M & \mathfrak{p}^M \end{pmatrix}.$$

Then upon setting,

$$h_1 = \begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1},$$

$$h_2 = \begin{pmatrix} \epsilon & \\ & \epsilon^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get

$$f\left(\begin{pmatrix} \alpha\epsilon(a+vc) & \alpha\epsilon(b+vd) \\ \gamma\epsilon(a+vc) & \gamma\epsilon(b+vd) \end{pmatrix}, \begin{pmatrix} c\epsilon^{-1}(\alpha u+\beta) & d\epsilon^{-1}(\alpha u+\beta) \\ c\epsilon^{-1}(\gamma u+\delta) & d\epsilon^{-1}(\gamma u+\delta) \end{pmatrix}\right) \neq 0.$$

With the choice of f, it is not hard to draw the following conclusions:

1. $\gamma, c \in \mathfrak{p}^M$,
2. u, v are integral,
3. ϵ is a unit,
4. $b + vd, \alpha u + \beta \in \mathfrak{p}^M$,
5. $\alpha\epsilon a, d\epsilon^{-1}\delta \in 1 + \mathfrak{p}^M$.

Next,

$$Z(W_1, h_1, \mu_1|\,.\,|_v^{s_1}) = \int_{\mathbb{Q}_v^\times} W_1\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1}\right)\mu(x)|x|^{s_1-\frac{1}{2}}\, d^\times x;$$

but

$$\begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^{-1} & \\ & \alpha(\alpha\delta - \beta\gamma)^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & -(\beta + u\alpha)\alpha(\alpha\delta - \beta\gamma)^{-1} \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ -\alpha^{-1}\gamma & 1 \end{pmatrix},$$

implying that for M large, we have

$$\begin{aligned} Z(W_1, h_1, \mu|\,.\,|_v^{s_1}) &= \int_{\mathbb{Q}_v^\times} W_1\left(\begin{pmatrix} x\alpha^{-1} & \\ & \alpha(\alpha\delta - \beta\gamma)^{-1} \end{pmatrix}\right)\mu_1(x)|x|^{s_1-\frac{1}{2}}\, d^\times x \\ &= (\omega_{\pi_1}\mu)(\alpha^2(\alpha\delta - \beta\gamma)^{-1})Z(W_1, \mu|\,.\,|_v^{s_1}). \end{aligned}$$

Similarly, for M large,

$$Z(W_1, h_2, \mu^{-1}|\,.\,|_v^{s_2}) = \mu^{-1}(\epsilon^{-1}d(ad - bc)^{-1})(\omega_{\pi_2}\mu^{-1})(\epsilon^{-1}d)Z(W_2, \mu^{-1}|\,.\,|_v^{s_2}).$$

The proposition is now immediate. □

Corollary 2.11 *There is a choice of W_1, W_2, f such that*

$$\mathcal{Z}(W_1, W_2, f; \mu, |\,.\,|_v^{s_1}, |\,.\,|_v^{s_2}) \equiv 1.$$

When W_1, W_2 are spherical, the situation is particularly nice:

Proposition 2.12 *Suppose v is a non-archimedean place, and π_1, π_2 are spherical representations of $\mathrm{GL}_2(F_v)$ with $\omega_{\pi_1}.\omega_{\pi_2} = 1$. Also, suppose that $W_i \in \mathcal{W}(\pi_i, \psi)$, $i = 1, 2$, is the normalized K_v-fixed vector. Furthermore, let f be the characteristic function of $\mathsf{M}_2(\mathrm{O}_v) \times \mathsf{M}_2(\mathrm{O}_v)$. Then for unramified quasi-character μ we have*

$$\mathcal{Z}(W_1, W_2, f; \mu, |\,.\,|_v^{s_1}, |\,.\,|_v^{s_2}) = L_v(s_1, \pi_1, \mu)L(s_2, \pi_2, \mu^{-1}).$$

Proof. In order to see this, we need to verify that if

$$L(h_1,h_2)f(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}) \neq 0,$$

for $(h_1,h_2)\in H_1(F_v)$, we must have $(h_1,h_2)\in D(F_v)(\mathrm{GL}_2(\mathrm{O}_v)\times\mathrm{GL}_2(\mathrm{O}_v))$. For this, we start by the observation that one can take as a set $\mathcal{R}$ of representatives for

$$D(F_v)\backslash H_1(F_v)/(\mathrm{GL}_2(\mathrm{O}_v)\times\mathrm{GL}_2(\mathrm{O}_v)),$$

the set of pairs of the form

$$(\begin{pmatrix}1 & u\\ & 1\end{pmatrix},\begin{pmatrix}\epsilon & \\ & \epsilon^{-1}\end{pmatrix}\begin{pmatrix}1 & v\\ & 1\end{pmatrix}).$$

Hence, we need to verify our claim only for elements (h_1,h_2) of the above form. We have

$$L(h_1,h_2)f(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}) = f(\begin{pmatrix}\epsilon & \epsilon v\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & -u\epsilon^{-1}\\ 0 & \epsilon^{-1}\end{pmatrix}).$$

Since f is the characteristic function of $\mathsf{M}_2(\mathrm{O}_v)\times\mathsf{M}_v(\mathrm{O}_v)$, for this last expression to be non-zero, we must have $\epsilon^{\pm 1}\in\mathrm{O}_v$, $\epsilon v\in\mathrm{O}_v$, and $\epsilon^{-1}u\in\mathrm{O}_v$. This in turn implies that $\epsilon\in\mathrm{O}_v^\times$, and $u,v\in\mathrm{O}_v$. Now an application of lemma 2.7 gives the result. □

We can now proceed to collect information about the analytic properties of our two variable zeta function. we prove the following proposition:

Proposition 2.13 *For W_1, W_2 Whittaker functions, and f as above, the function $\mathcal{Z}(W_1,W_2,f;\mu,|\,.\,|^{s_1},|\,.\,|^{s_2})$ has an analytic continuation to a meromorphic function on $\mathbb{C}^2$. Furthermore, the ratio*

$$\Psi(W_1,W_2,f;\mu,|\,.\,|_v^{s_1},|\,.\,|_v^{s_2}) = \frac{\mathcal{Z}(W_1,W_2,f;\mu,|\,.\,|_v^{s_1},|\,.\,|_v^{s_2})}{L(s_1,\pi_1,\mu)L(s_2,\pi_2,\mu^{-1})}$$

extends to an entire function on the entire $\mathbb{C}^2$. There is a choice of W_1, W_2, and f such that the above ratio is a nowhere vanishing entire function.

Proof. We prove only the analyticity statement; the non-vanishing follows from proposition 2.10 and the corresponding GL(2) statement. We write out the details for the archimedean place. For simplicity, we will assume that π_1 and π_2 are irreducible principal series representations. Also we will assume that the quasi-character μ is trivial. By lemma 2.7, we need to consider the integral

$$\begin{aligned}&\int_{u,v\in\mathbb{R}}\int_{\epsilon\in\mathbb{R}_+^\times}\int_{K_v^2} f(k_1^{-1}\begin{pmatrix}\epsilon^{-1} & \epsilon^{-1}v\\ 0 & 0\end{pmatrix}k_2, k_1^{-1}\begin{pmatrix}0 & -u\epsilon\\ 0 & \epsilon\end{pmatrix}k_2)\\ &\omega_{\pi_2}(\epsilon)|\epsilon|^{2s_2-2}(\int_{\mathbb{R}^\times}W_1(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}k_1)\mathbf{e}(u\alpha)|\alpha|^{s_1-\frac{1}{2}}\,d^\times\alpha)\\ &(\int_{\mathbb{R}^\times}W_2(\begin{pmatrix}\beta & \\ & 1\end{pmatrix}k_2)\mathbf{e}(v\beta)|\beta|^{s_2-\frac{1}{2}}\,d^\times\beta)\,du\,dv\,d^\times\epsilon\,dk_1\,dk_2.\end{aligned} \tag{18}$$

For this purpose, we use the description of the Whittaker model of a principal series representation from [17], page 101-102. Suppose $\pi_1 = \pi(\mu_1, \mu_2)$, and $\pi_2 = \pi(\mu_3, \mu_4)$. Then there is a Schwartz function $P_i(x, y)$, $i = 1, 2$, such that $W_1 = W_{P_i}$ by the following recipe. Let

$$f_1(g) = (\mu_1 \nu^{\frac{1}{2}})(\det g) \int_{\mathbb{R}^\times} P_1[(0,1)\gamma g](\mu_1\mu_2^{-1}\nu)(\gamma)\, d^\times\gamma,$$

and

$$f_2(g) = (\mu_3 \nu^{\frac{1}{2}})(\det g) \int_{\mathbb{R}^\times} P_2[(0,1)\delta g](\mu_3\mu_4^{-1}\nu)(\delta)\, d^\times\delta,$$

when the integrals converge. Next, we set for $i = 1, 2$

$$W_{P_i}(g) = \int_{\mathbb{R}} f_{P_i}(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g)\mathbf{e}(x)\, dx.$$

In particular,

$$W_{P_1}(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} k_1) = \int_{\mathbb{R}} \int_{\mathbb{R}^\times} (\mu_1 v^{\frac{1}{2}})(\alpha)(\mu_1\mu_2^{-1}\nu)(\gamma) P_1((-\alpha\gamma, -x\gamma)k_1)\mathbf{e}(x)\, dx\, d^\times\gamma,$$

and

$$W_{P_2}(\begin{pmatrix} \beta & \\ & 1 \end{pmatrix} k_2) = \int_{\mathbb{R}} \int_{\mathbb{R}^\times} (\mu_3 v^{\frac{1}{2}})(\beta)(\mu_3\mu_4^{-1}\nu)(\delta) P_2((-\beta\delta, -y\delta)k_2)\mathbf{e}(y)\, dy\, d^\times\delta.$$

These integrals may not converge, but they have analytic continuations to entire functions of the characters μ_i, $i = 1, \ldots, 4$.

We need a lemma/notation:

Lemma 2.14 *Suppose P_1, P_2, and f are Schwartz-Bruhat functions as above. Then the function Γ whose value at*

$$(X, Y, m, n, p, q) \in \mathsf{M}_2(\mathbb{R}) \times \mathsf{M}_2(\mathbb{R}) \times \mathbb{R}^4$$

is given by

$$\Gamma(X,Y,m,n,p,q) = \int_{K^2} f(k_1^{-1}Xk_2, k_1^{-1}Yk_2) P_1((m,n)k_1) P_2((p,q)k_2)\, dk_1\, dk_2$$

is a Schwartz-Bruhat function.

The integral (18) is now equal to

$$\int_{\alpha\in\mathbb{R}^\times}\int_{\beta\in\mathbb{R}^\times}\int_{\gamma\in\mathbb{R}^\times}\int_{\delta\in\mathbb{R}^\times}\int_{\epsilon\in\mathbb{R}_+^\times}\int_{u\in\mathbb{R}}\int_{v\in\mathbb{R}}\int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}}$$
$$\Gamma(\begin{pmatrix}\epsilon^{-1} & \epsilon^{-1}v\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & -u\epsilon\\ 0 & \epsilon\end{pmatrix},-\alpha\gamma,-x\gamma,-\beta\delta,-y\delta)$$
$$\omega_{\pi_2}(\epsilon)|\epsilon|^{2s_2-2}\mathbf{e}(u\alpha)|\alpha|^{s_1-\frac{1}{2}}\mathbf{e}(v\beta)|\beta|^{s_2-\frac{1}{2}}(\mu_1 v^{\frac{1}{2}})(\alpha)$$
$$(\mu_1\mu_2^{-1}\nu)(\gamma)\mathbf{e}(x)(\mu_3 v^{\frac{1}{2}})(\beta)(\mu_3\mu_4^{-1}\nu)(\delta)\mathbf{e}(y)$$
$$dy\,dx\,dv\,du\,d^\times\epsilon\,d^\times\delta\,d^\times\gamma\,d^\times\beta\,d^\times\alpha.$$

$$=\int_{\alpha\in\mathbb{R}^\times}\int_{\beta\in\mathbb{R}^\times}\int_{\gamma\in\mathbb{R}^\times}\int_{\delta\in\mathbb{R}^\times}\int_{\epsilon\in\mathbb{R}_+^\times}\int_{u\in\mathbb{R}}\int_{v\in\mathbb{R}}\int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}}$$
$$\Gamma(\begin{pmatrix}\epsilon^{-1} & \epsilon^{-1}v\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & -u\epsilon\\ 0 & \epsilon\end{pmatrix},-\alpha\gamma,-x\gamma,-\beta\delta,-y\delta)$$
$$\omega_{\pi_2}(\epsilon)|\epsilon|^{2s_2-2}\mathbf{e}(u\alpha)|\alpha|^{s_1}\mathbf{e}(v\beta)|\beta|^{s_2}(\mu_1)(\alpha) \tag{19}$$
$$(\mu_1\mu_2^{-1}\nu)(\gamma)\mathbf{e}(x)(\mu_3)(\beta)(\mu_3\mu_4^{-1}\nu)(\delta)\mathbf{e}(y)$$
$$dy\,dx\,dv\,du\,d^\times\epsilon\,d^\times\delta\,d^\times\gamma\,d^\times\beta\,d^\times\alpha.$$

We will abbreviate the inner Γ-expression appearing above to

$$\Gamma(\epsilon^{-1},\epsilon^{-1}v,-u\epsilon,\epsilon,-\alpha\gamma,-x\gamma,-\beta\delta,-y\delta).$$

Next we consider the integral

$$\int_{u\in\mathbb{R}}\int_{v\in\mathbb{R}}\int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}}\Gamma(\epsilon^{-1},\epsilon^{-1}v,-u\epsilon,\epsilon,-\alpha\gamma,-x\gamma,-\beta\delta,-y\delta)$$
$$\mathbf{e}(x)\mathbf{e}(y)\mathbf{e}(u\alpha)\mathbf{e}(v\beta)\,dy\,dx\,dv\,du$$
$$=|\gamma|^{-1}|\delta|^{-1}\int_{u\in\mathbb{R}}\int_{v\in\mathbb{R}}\int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}}\Gamma(\epsilon^{-1},v,u,\epsilon,-\alpha\gamma,x,-\beta\delta,y)$$
$$\mathbf{e}(-\frac{x}{\gamma})\mathbf{e}(-\frac{y}{\delta})\mathbf{e}(-u\frac{\alpha}{\epsilon})\mathbf{e}(v\beta\epsilon)\,dy\,dx\,dv\,du$$
$$=|\gamma|^{-1}|\delta|^{-1}\widetilde{\Gamma}(\epsilon^{-1},-\beta\epsilon,\alpha\epsilon^{-1},\epsilon,-\alpha\gamma,\gamma^{-1},-\beta\delta,\delta^{-1}),$$

where $\widetilde{\Gamma}$ is the appropriate Fourier transform of Γ.

Going back to (19), we obtain

$$\int_{\alpha\in\mathbb{R}^\times}\int_{\beta\in\mathbb{R}^\times}\int_{\gamma\in\mathbb{R}^\times}\int_{\delta\in\mathbb{R}^\times}\int_{\epsilon\in\mathbb{R}_+^\times}$$
$$|\gamma|^{-1}|\delta|^{-1}\widetilde{\Gamma}(\epsilon^{-1},-\beta\epsilon,\alpha\epsilon^{-1},\epsilon,-\alpha\gamma,\gamma^{-1},-\beta\delta,\delta^{-1})$$
$$\omega_{\pi_2}(\epsilon)|\epsilon|^{2s_2-2}|\alpha|^{s_1}|\beta|^{s_2}\mu_1(\alpha)(\mu_1\mu_2^{-1}\nu)(\gamma)\mu_3(\beta)(\mu_3\mu_4^{-1}\nu)(\delta)$$
$$d^\times\epsilon\,d^\times\delta\,d^\times\gamma\,d^\times\beta\,d^\times\alpha.$$

$$
\begin{aligned}
&= \int_{\alpha\in\mathbb{R}^\times}\int_{\beta\in\mathbb{R}^\times}\int_{\gamma\in\mathbb{R}^\times}\int_{\delta\in\mathbb{R}^\times}\int_{\epsilon\in\mathbb{R}_+^\times} \\
&\widetilde{\Gamma}(\epsilon^{-1}, -\beta\epsilon, \alpha\epsilon^{-1}, \epsilon, -\alpha\gamma^{-1}, \gamma, -\beta\delta^{-1}, \delta) \\
&\omega_{\pi_2}(\epsilon)|\epsilon|^{2s_2-2}|\alpha|^{s_1}|\beta|^{s_2}\mu_1(\alpha)(\mu_1\mu_2^{-1})(\gamma^{-1})\mu_3(\beta)(\mu_3\mu_4^{-1})(\delta^{-1}) \\
&d^\times\epsilon\, d^\times\delta\, d^\times\gamma\, d^\times\beta\, d^\times\alpha.
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\alpha\in\mathbb{R}^\times}\int_{\beta\in\mathbb{R}^\times}\int_{\gamma\in\mathbb{R}^\times}\int_{\delta\in\mathbb{R}^\times}\int_{\epsilon\in\mathbb{R}_+^\times} \\
&\widetilde{\Gamma}(\epsilon^{-1}, -\beta\delta\epsilon, \alpha\gamma\epsilon^{-1}, \epsilon, -\alpha, \gamma, -\beta, \delta) \\
&\omega_{\pi_2}(\epsilon)|\epsilon|^{2s_2-2}|\alpha|^{s_1}|\gamma|^{s_1}|\beta|^{s_2}|\delta|^{s_2}\mu_1(\alpha)\mu_2(\gamma)\mu_3(\beta)\mu_4(\delta) \\
&d^\times\epsilon\, d^\times\delta\, d^\times\gamma\, d^\times\beta\, d^\times\alpha
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\alpha\in\mathbb{R}^\times}\int_{\beta\in\mathbb{R}^\times}\int_{\gamma\in\mathbb{R}^\times}\int_{\delta\in\mathbb{R}^\times}\int_{\epsilon\in\mathbb{R}_+^\times} \\
&\widetilde{\Gamma}(\epsilon^{-1}, -\beta\delta\epsilon, \alpha\gamma\epsilon^{-1}, \epsilon, -\alpha, \gamma, -\beta, \delta) \\
&(\mu_1\nu^{s_1})(\alpha)(\mu_2\nu^{s_1})(\gamma)(\mu_3\nu^{s_2})(\beta)(\mu_4\nu^{s_2})(\delta)(\omega_{\pi_2}\nu^{2s_2-2})(\epsilon) \\
&d^\times\epsilon\, d^\times\delta\, d^\times\gamma\, d^\times\beta\, d^\times\alpha
\end{aligned} \tag{20}
$$

after obvious changes of variables, and simple re-arrangement of terms.

Our result now follows from the following standard lemma:

Lemma 2.15 *Let Φ be a Schwartz-Bruhat function on $\mathbb{R}^n$. Suppose $\gamma_1, \ldots, \gamma_n$ are quasi-characters. Define the function $Z(s_1, \ldots, s_n) = Z(\Phi; \gamma_1, \ldots, \gamma_n; s_1, \ldots, s_n)$ of the complex variables $s_1, \ldots, s_n$ by*

$$
Z(s_1, \ldots, s_n) = \int_{(\mathbb{R}^\times)^n} \Phi(\alpha_1, \ldots, \alpha_n) \prod_i \gamma_i(\alpha_i)|\alpha_i|^{s_i}\, d^\times\alpha_i,
$$

whenever the integral converges. Then the integral converges for $\Re s_i$ large enough, for $i = 1, \ldots, n$. The ratio

$$
\frac{Z(\Phi; \gamma_1, \ldots, \gamma_n; s_1, \ldots, s_n)}{\prod_{i=1}^n L(s_i, \gamma_i)}
$$

extends to an entire function. If $\Phi \in \mathcal{S}(\mathbb{R}^\times \times \mathbb{R}^{n-1})$, then the ratio

$$
\frac{Z(\Phi; \gamma_1, \ldots, \gamma_n; s_1, \ldots, s_n)}{\prod_{i=2}^n L(s_i, \gamma_i)}
$$

has an analytic continuation to an entire function.

□

Corollary 2.16 *Let v be a non-archimedean place. Then in the above situation for μ highly ramified $\mathcal{Z}(W_1, W_2, f; \mu, |\,.\,|_v^{s_1}, |\,.\,|_v^{s_2})$ extends to an entire function of s_1, s_2.*

Corollary 2.17 *Let W_1, W_2 be flat sections of Whittaker spaces as in the last section. Then the function $\Psi(W_1, W_2, f; \mu, |\,.\,|_v^{s_1}, |\,.\,|_v^{s_2})$ is holomorphic in the parameters of W_1, W_2.*

Summarizing,

Proposition 2.18 *Let the data be as above. Let S a finite collection of places containing the archimedean place such that for $v \notin S$, the local data at v is unramified. Then we have*

$$\mathcal{Z}(\varphi_1, \varphi_2, \mu|\,.\,|^s) = L(s, \pi_1, \mu) L(1-s, \pi_2, \mu^{-1}) \left\{ \prod_v \Psi(W_1, W_2, f; \mu_v, |\,.\,|_v^s, |\,.\,|_v^{1-s}) \right\}$$

where by lemmas 2.12 and 2.13 the expression in curly braces is a finite product and is entire.

2.3 The Whittaker function

In this section, we aim to relate the local Euler factor of the integral of Novodvorsky at the archimedean place to the corresponding Euler factor of the integral considered in Section 1.5. For this purpose, we start by studying the Whittaker function associated to $\theta(\varphi_1, \varphi_2; f)$, and from that we derive formulae for the corresponding local Whittaker functions.

In the sequel, we first compute the Whittaker function of a cuspidal function $\theta(\varphi_1, \varphi_2; f)$. Fix a non-trivial character ψ of $F\backslash\mathbb{A}$. Define a character, again denoted by ψ, of the unipotent radical of the Borel subgroup of GSp(4) by the following

$$\psi(\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix}) = \psi(v+t).$$

Then we set

$$W(g) = \int_{N(F)\backslash N(\mathbb{A})} \theta(\varphi_1, \varphi_2; f)(ng) \psi^{-1}(n)\, dn.$$

The h^1 and h^2 above can be taken to be $\begin{pmatrix} v(g) & \\ & 1 \end{pmatrix}$ and the identity matrix, respectively. Then we have

Theorem 2.19 *If $\tilde{\pi}_1 \neq \bar{\pi}_2$, we have*

$$W(g) = \int_{\hat{N}(\mathbb{A})\backslash H_1(\mathbb{A})} W_1(\epsilon h_1 h^1) W_2(h_2 h^2) \omega(g, h_1 h^1, h_2 h^2) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I_{2\times 2})\, dh_1\, dh_2,$$

where

$$\hat{N} = \left\{ \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right) \mid x \in \mathbb{G}_a \right\}.$$

Proof. We start by

$$W(g) = \int_{H_1(F)\backslash H_1(\mathbb{A})} \varphi_1(h_1 h^1)\varphi_2(h_2 h^2)$$
$$(\sum_{M_1,M_2} \int_{N(F)\backslash N(\mathbb{A})} \omega(ng; h_1 h^1, h_2 h^2) f(M_1, M_2)\psi^{-1}(n)\, dn)$$
$$d(h_1, h_2).$$

Therefore, we have to study the expression

$$I(M_1, M_2) = \int_{N(F)\backslash N(\mathbb{A})} \omega(ng; h_1 h^1, h_2 h^2) f(M_1, M_2)\psi^{-1}(n)\, dn.$$

For this we have

$$I(M_1, M_2) = \int_{F\backslash\mathbb{A}} (\int_{(F\backslash\mathbb{A})^3} \omega(\begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix}, I_2, I_2)$$
$$\omega(\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} g, h_1 h^1, h_2 h^2) f(M_1, M_2)\psi^{-1}(t)\, dr\, ds\, dt)$$
$$\psi^{-1}(v)\, dv$$
$$= \int_{F\backslash\mathbb{A}} \omega(\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} g, h_1 h^1, h_2 h^2) f(M_1, M_2)$$
$$(\int_{(F\backslash\mathbb{A})^3} \psi(tr(\begin{pmatrix} s & r \\ r & t \end{pmatrix} \begin{pmatrix} \det M_1 & B(M_1, M_2) \\ B(M_2, M_1) & \det M_2 - 1 \end{pmatrix}))\, dr\, ds\, dt)$$
$$\psi^{-1}(v)\, dv.$$

But the inner most integral

$$\int_{(F\backslash\mathbb{A})^3} \psi(tr(\begin{pmatrix} s & r \\ r & t \end{pmatrix} \begin{pmatrix} \det M_1 & B(M_1, M_2) \\ B(M_2, M_1) & \det M_2 - 1 \end{pmatrix}))\, dr\, ds\, dt = 0$$

unless $\det M_1 = 0$, $\det M_2 = 1$, and $B(M_1, M_2) = 0$, in which case it is equal to 1.

Lemma 2.20 *Under the action of $H_1(F)$, the set $\mathcal{S}$ consisting of the pairs of matrices (M_1, M_2) satisfying the conditions just mentioned is the union of the following two orbits:*

1. *The orbit of* (O, I). *The stabilizer of this element is the diagonal embedding of* $\mathrm{PGL}(2)$ *into* H_1.

2. *The orbit of* $(\begin{pmatrix} 1 & \\ & \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix})$. *The stabilizer of this element is the subgroup* $\tilde{N}$ *of* H_1 *consisting of pairs of matrices of the form*

$$(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, w \begin{pmatrix} 1 & x \\ 1 & \end{pmatrix} w^{-1}).$$

Proof. Since $\det M_1 = 0$, there are two cases to be considered:

1. $M_1 = 0$,
2. $M_1 \neq 0$.

It's obvious that the first case corresponds to the first orbit in the statement of the lemma. Also the statement regarding the stabilizer is immediate. Next we consider the case when $M_1 \neq 0$. It is clear that under the action of H_1, M_1 is equivalent to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Next suppose $M_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $B(M_1, M_2) = 0$ and $\det M_1 = 0$, we obtain that $\det(M_1 + M_2) = 1$. This then implies that $d = 0$. But then since $\det M_2 = 1$, we obtain $c = -b^{-1}$. Hence $M_2 = \begin{pmatrix} a & b \\ -b^{-1} & \end{pmatrix}$. Next consider the element

$$h = (\begin{pmatrix} 1 & \\ & b^{-1} \end{pmatrix} \begin{pmatrix} b^{-1} & a \\ & b \end{pmatrix}, \begin{pmatrix} b^{-1} & \\ & 1 \end{pmatrix}) \in H_1(F).$$

Then it is easy to check that

$$h.(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ -b^{-1} & \end{pmatrix}) = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}).$$

The statement regarding the stabilizer is straightforward. □

Next we study the contribution of each orbit to the Whittaker integral. Corresponding to the two orbits obtained above, we have the following two integrals:

$$I_1(g) = \int_{G(F)\backslash H_1(\mathbb{A})} \int_{F\backslash\mathbb{A}} \omega(\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} g, h_1 h^1, h_2 h^2)$$

$$f(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix})\varphi_1(h_1 h^1)\varphi_2(h_2 h^2)\psi^{-1}(v)\, dv\, d(h_1, h_2),$$

and

$$I_2(g) = \int_{\tilde{N}(F)\backslash H_1(\mathbb{A})} \int_{F\backslash\mathbb{A}} \omega(\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} g, h_1 h^1, h_2 h^2)$$

$$f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix})\varphi_1(h_1 h^1)\varphi_2(h_2 h^2)\psi^{-1}(v)\, dv\, d(h_1, h_2).$$

Then it is clear that

$$W(g) = I_1(g) + I_2(g).$$

Lemma 2.21 *We have*

$$I_1(g) = 0,$$

except when $\tilde{\pi}_1 = \bar{\pi}_2$.

Proof. By [14], we have

$$I_1(g) = \int_{G(F)\backslash H_1(\mathbb{A})} \int_{F\backslash\mathbb{A}} \omega(\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} g \begin{pmatrix} I & \\ & \nu(g)^{-1} I \end{pmatrix})$$
$$L(h_1 h^1, h_2 h^2) f(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}) \varphi_1(h_1 h^1) \varphi_2(h_2 h^2)$$
$$\psi^{-1}(v)\, dv\, d(h_1, h_2)$$

$$= \int_{G(\mathbb{A})\backslash H_1(\mathbb{A})} \int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})} \int_{F\backslash\mathbb{A}} \omega(\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} g \begin{pmatrix} I & \\ & \nu(g)^{-1} I \end{pmatrix})$$
$$L(\gamma h_1 h^1, \gamma h_2 h^2) f(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}) \varphi_1(\gamma h_1 h^1) \varphi_2(\gamma h_2 h^2)$$
$$\psi^{-1}(v)\, dv\, d\gamma\, d(h_1, h_2)$$

$$= \int_{G(\mathbb{A})\backslash H_1(\mathbb{A})} \int_{F\backslash\mathbb{A}} \omega(\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} g \begin{pmatrix} I & \\ & \nu(g)^{-1} I \end{pmatrix})$$
$$L(h_1 h^1, h_2 h^2) f(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}) \psi^{-1}(v)$$
$$\left(\int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})} \varphi_1(\gamma h_1 h^1) \varphi_2(\gamma h_2 h^2)\, d\gamma \right) dv\, d(h_1, h_2).$$

The inner most integral

$$\int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})} \varphi_1(\gamma h_1 h^1) \varphi_2(\gamma h_2 h^2)\, d\gamma =$$
$$< \pi_1(h_1 h^1)\varphi_1, \overline{\pi_2(h_2 h^2)\varphi_2} >_{L^2(\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A}))}.$$

The statement of the lemma is now obvious. □

Next we study the contribution of the second orbit.

Lemma 2.22 *We have*

$$I_2(g) = \int_{\hat{N}(\mathbb{A})\backslash H_1(\mathbb{A})} W_{\varphi_1}(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h_1 \begin{pmatrix} \nu(g) & \\ & 1 \end{pmatrix}) W_{\varphi_2}(h_2)$$
$$\omega(g, h_1 \begin{pmatrix} \nu(g) & \\ & 1 \end{pmatrix}, h_2) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I)\, d(h_1, h_2).$$

In this lemma, $\hat{N}$ is the diagonal embedding of the unipotent upper triangular matrices in GL(2) in H_1. Also if φ is a cuspidal automorphic function on $\mathrm{GL}_2(\mathbb{A})$, we have set

$$W_\varphi(g) = \int_{F\backslash\mathbb{A}} \varphi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) \psi^{-1}(x)\, dx.$$

Proof. The proof consists of simple manipulations of the original expression for $I_2(g)$. We have

$$I_2(g) = \int_{\tilde{N}(F)\backslash H_1(\mathbb{A})} \int_{F\backslash\mathbb{A}} \omega(\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} g, h_1 h^1, h_2 h^2)$$
$$f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}) \varphi_1(h_1 h^1) \varphi_2(h_2 h^2) \psi^{-1}(v)\, dv\, d(h_1, h_2).$$

We recall that $\tilde{N}(F) = \{(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} w^{-1})\}$, and also that $h^1 = \begin{pmatrix} \nu(g) & \\ & 1 \end{pmatrix}$ and $h^2 = I$. Using the formulae in [14], we have

$$\omega(\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} g, h_1 h^1, h_2 h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}) =$$
$$\omega(g, h_1 h^1, h_2 h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -v \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}).$$

Hence

$$I_2(g) = \int_{\tilde{N}(F)\backslash H_1(\mathbb{A})} \int_{F\backslash\mathbb{A}} \omega(g, h_1 h^1, h_2 h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -v \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix})$$
$$\varphi_1(h_1 h^1) \varphi_2(h_2 h^2) \psi^{-1}(v)\, dv\, d(h_1, h_2)$$
$$= \int_{\tilde{N}(\mathbb{A})\backslash H_1(\mathbb{A})} \int_{F\backslash\mathbb{A}} \int_{F\backslash\mathbb{A}} \omega(g, \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} h_1 h^1, w \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} w^{-1} h_2 h^2)$$
$$f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -v \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}) \varphi_1(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} h_1 h^1) \varphi_2(w \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} w^{-1} h_2 h^2)$$
$$\psi^{-1}(v)\, du\, dv\, d(h_1, h_2)$$

Next by definition and Lemma 5.1.2 of [14]

$$\omega(g, \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} h_1 h^1, w \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} w^{-1} h_2 h^2)$$

$$= \omega(g \begin{pmatrix} I & \\ & \nu(g)^{-1} I \end{pmatrix}) L(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} h_1 h^1, w \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} w^{-1} h_2 h^2)$$
$$= L(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} h_1 h^1, w \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} w^{-1} h_2 h^2) \omega(\begin{pmatrix} I & \\ & \nu(g)^{-1} I \end{pmatrix} g).$$

This identity implies that

$$\omega(g, \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} h_1 h^1, w \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} w^{-1} h_2 h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -v \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix})$$

$$= L(h_1 h^1, h_2 h^2) \omega(\begin{pmatrix} I & \\ & \nu(g)^{-1} I \end{pmatrix} g) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -v \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix})$$
$$= L(h_1 h^1, \begin{pmatrix} 1 & -v \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} h_2 h^2) \omega(\begin{pmatrix} I & \\ & \nu(g)^{-1} I \end{pmatrix} g) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I)$$
$$= \omega(g, h_1 h^1, \begin{pmatrix} 1 & -v \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} h_2 h^2) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I).$$

Going back to $I_2(g)$, we obtain

$$I_2(g) = \int_{\tilde{N}(\mathbb{A}) \backslash H_1(\mathbb{A})} \int_{F \backslash \mathbb{A}} \int_{F \backslash \mathbb{A}} \omega(g, h_1 h^1, \begin{pmatrix} 1 & -v \\ & 1 \end{pmatrix} w h_2 h^2) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I)$$
$$\varphi_1(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} h_1 h^1) \varphi_2(w \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} w^{-1} h_2 h^2) \psi^{-1}(v) \, du \, dv \, d(h_1, h_2)$$

Next we make a change of variables $(h_1, h_2) \mapsto (h_1, w^{-1} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix} h_2)$, to obtain

$$I_2(g) = \int_{\hat{N}(\mathbb{A}) \backslash H_1(\mathbb{A})} \int_{F \backslash \mathbb{A}} \int_{F \backslash \mathbb{A}} \omega(g, h_1 h^1, h_2 h^2) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I)$$
$$\varphi_1(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} h_1 h^1) \varphi_2(\begin{pmatrix} 1 & u+v \\ & 1 \end{pmatrix} h_2 h^2) \psi^{-1}(v) \, du \, dv \, d(h_1, h_2).$$

Now a change of variables $v \mapsto v - u$ and re-arranging the order of integrals gives the result. □

Combining everything finishes the proof of the theorem. □

2.4 Local Whittaker functions

In this paragraph, we study the integrals of the previous section in some detail.

Suppose π_1 and π_2 are two generic irreducible admissible representations of the group GL(2) over a local field, such that $\tilde{\pi}_1 \neq \pi_2, \bar{\pi}_2$, and $\omega_{\pi_1}.\omega_{\pi_2} = 1$. For $W_i \in \mathcal{W}(\pi_i, \psi)$, for $i = 1, 2$, set

$$\mathbb{W}_v(W_1, W_2; f)(g) = \int_{\hat{N}(F_v)\backslash H_1(F_v)} W_1(\epsilon h_1 \begin{pmatrix} \nu(g) & \\ & 1 \end{pmatrix}) W_2(h_2)$$
$$\omega(g, h_1 \begin{pmatrix} \nu(g) & \\ & 1 \end{pmatrix}, h_2) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I_{2\times 2})\, dh_1\, dh_2.$$

Proposition 2.23 *For all $W_i \in \mathcal{W}(\pi_i, \psi)$, $i = 1, 2$, K-finite f in the space of Schwartz-Bruhat functions, and $g \in \mathrm{GSp}_4(F_v)$, the integral defining $\mathbb{W}(W_1, W_2; f)(g)$ is absolutely convergent.*

Proof. As usual we prove the proposition for the archimedean place. It is clear that we only need to prove the absolute convergence for $g = I_{4\times 4}$. In order to do this, we start by identifying a measurable set of representatives for $\hat{N}(\mathbb{R})\backslash H_1(\mathbb{R})$, and identifying the corresponding measure. On $H_1(\mathbb{R})$, we have the following natural set of representatives

$$(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k_1, \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix} k_2),$$

with $x, y \in \mathbb{R}$, $\epsilon \in \mathbb{R}^\times$, and $k_1, k_2 \in \mathrm{SO}(2)$. Also the corresponding measure is

$$|\eta|^{-2}\, dx\, dy\, d^\times \eta\, dk_1\, dk_2.$$

This statement implies that the set of elements of the form

$$(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k_1, \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix} k_2),$$

constitutes a measurable set of representatives for $\hat{N}(\mathbb{R})\backslash H_1(\mathbb{R})$. Also with this normalization the measure is

$$|\eta|^{-2}\, dx\, d^\times \eta\, dk_1\, dk_2.$$

Hence we are reduced to proving the convergence of the following integral:

$$\int_K \int_K \int_{\mathbb{R}} \int_{\mathbb{R}^\times} \left| W_1(\epsilon \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k_1) W_2(\begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix} k_2) \right| .$$
$$\left| f(k_1^{-1} \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix} k_2, k_1^{-1} \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix} k_2) \right|$$
$$d^\times \eta\, dx\, dk_1\, dk_2.$$

Next we observe that in order to prove the absolute convergence of this integral, we just need to prove the absolute convergence of the integral over η and x. Also since

$$W_1(\epsilon \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k_1) = \psi(-x) W_1(\epsilon k_1),$$

we obtain

$$\left| W_1(\epsilon \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k_1) \right| = |W_1(\epsilon k_1)|\,.$$

Hence we are reduced to proving the convergence of the following integral:

$$I = \int_{\mathbb{R}} \int_{\mathbb{R}^\times} \left| W_2(\begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix}) \right| \cdot \left| f(\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix}, \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix}) \right| \, d^\times \eta \, dx.$$

But this integral is equal to

$$I = \int_{\mathbb{R}} \int_{\mathbb{R}^\times} \left| \omega_{\pi_2}(\eta^{-1}) W_2(\begin{pmatrix} \eta^2 & \\ & 1 \end{pmatrix}) f(\begin{pmatrix} 0 & -\eta^{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \eta & -x\eta^{-1} \\ 0 & \eta^{-1} \end{pmatrix}) \right| \, d^\times \eta \, dx$$

Now we write

$$f(\begin{pmatrix} 0 & -\eta^{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \eta & -x\eta^{-1} \\ 0 & \eta^{-1} \end{pmatrix}) = q(\eta, \eta^{-1}, x\eta^{-1}),$$

where q is some Schwartz-Bruhat function in three variables. We then need to prove the convergence of the integral

$$I = \int_{\mathbb{R}} \int_{\mathbb{R}^\times} \left| \omega_{\pi_2}(\eta^{-1}) W_2(\begin{pmatrix} \eta^2 & \\ & 1 \end{pmatrix}) q(\eta, \eta^{-1}, x\eta^{-1}) \right| \, d^\times \eta \, dx,$$

which after a change of variables $x \mapsto x\eta$ and integration over x is equivalent to the convergence of an integral of the form

$$\int_{\mathbb{R}_+^\times} \left| W(\begin{pmatrix} \eta & \\ & 1 \end{pmatrix} \xi(\eta) \right| \eta^\sigma \, d^\times \eta$$

for $\xi \in \mathcal{S}(\mathbb{R}^\times)$. Such an integral always converges by the moderate growth of the Whittaker function. □

We denote by $W_\theta(\pi_1, \pi_2)$ the collection of all Whittaker functions $\mathbb{W}(W_1, W_2; f)$ for all choices of W_1, W_2, f. In the archimedean situation, this is a $(\mathfrak{g}, K)$-module. We call a representation of π of $\mathrm{GSp}_4(\mathbb{R})$ *special* if it is irreducible generic, and its Whittaker model is isomorphic as a $(\mathfrak{g}, K)$-module to a $W_\theta(\pi_1, \pi_2)$ for π_1, π_2 with $\tilde{\pi}_1 \neq \pi_2, \bar{\pi}_2$, and $\omega_{\pi_1}.\omega_{\pi_2} = 1$; notice that this is not standard terminology.

Going back to the global situation, we choose φ_i, for $i = 1, 2$, so that

$$W_{\varphi_i} = \otimes_{v \in \mathcal{M}_F} W_v^i.$$

We also choose f to be a pure tensor of the form $\otimes_v f_v$. Then theorem 2.19 can be written in the form

$$W(g) = \prod_v \mathbb{W}_v(W_v^1, W_v^2; f_v)(g_v).$$

under appropriate conditions. This implies that for each local place v, if W_v is a K_v-finite vector in the local Whittaker model, there is a choice of the data such that $W_v = \mathbb{W}_v(W_v^1, W_v^2; f_v)$. It is clear from the construction that, in the archimedean situation, the space of all such $\mathbb{W}$'s forms a $(\mathfrak{g}, K)$-module.

2.5 Archimedean Zeta function

In this section, we use the results of the previous paragraphs to obtain information about the archimedean zeta function. We have by lemma 2.2

$$B(\phi, \chi_s) = \int_{\mathbb{A}^\times} \int_{\mathbb{A}} W_\phi \left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & x & & 1 \end{pmatrix} w^{-1} \right) \mu(y)|y|^{s-\frac{3}{2}} \, dx \, d^\times y, \tag{21}$$

with

$$w = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix}$$

and

$$\chi_s(y) = \mu(y)|y|^{s-\frac{1}{2}}.$$

If we set $\phi = \theta(\varphi_1, \varphi_2; f)$, the left hand side of the above identity will be equal to what we have called $\mathcal{Z}(\varphi_1, \varphi_2, f; \mu|\,.\,|^s)$. We saw in 2.18 that

$$\mathcal{Z}(\varphi_1, \varphi_2, \mu|\,.\,|^s) = L(s, \pi_1, \mu) L(1-s, \pi_2, \mu^{-1}) \left\{ \prod_v \Psi(W_1^v, W_2^v, f; \mu_v|\,.\,|_v^s, \mu_v^{-1}|\,.\,|_v^{1-s}) \right\}.$$

If we choose our vectors appropriately, that is factorizable, the right hand side of (21) is now equal to

$$\begin{aligned} &\prod_v Z_{v,N}(s, \pi_v(w^{-1})\mathbb{W}_v(W_v^1, W_v^2; f_v), \mu_\infty) \\ &\quad = Z_{\infty,N}(s, \pi_\infty(w^{-1})\mathbb{W}_\infty(W_\infty^1, W_\infty^2; f_\infty), \mu_\infty) \\ &\qquad \times L_S(s, \Pi, \mu) \prod_{v \in S \setminus \{\infty\}} Z_{v,N}(s, \pi_v(w^{-1})\mathbb{W}_v(W_v^1, W_v^2; f_v), \mu_v) \end{aligned}$$

By the main result of [40], for each local place $v \in S\setminus\{\infty\}$, we can choose $W_v^{\mathsf{sp}} \in \mathcal{W}(\Pi_v)$ in such a way that

$$Z_{v,N}(s, \Pi_v^{-1}(w^{-1}) W_v^{\mathsf{sp}}, \mu_v) = L_v(s, \Pi_v, \mu_v).$$

By the remark at the end of 2.4, we can choose the local data such that

$$\mathbb{W}_v(W_v^1, W_v^2; f_v) = W_v^{\mathtt{sp}}.$$

With this choice of the local data, we have

$$\begin{aligned} &Z_{\infty,N}(s, \pi_\infty(w^{-1})\mathbb{W}_\infty(W_\infty^1, W_\infty^2; f_\infty), \mu_\infty) \\ &= \Phi_S^{\mathtt{finite}}(\pi_1, \pi_2, \mu, s; W_1, W_2, f) L_\infty(s, \pi_1, \mu) L_\infty(1-s, \pi_2, \mu^{-1}) \\ &\times \Psi(W_1^\infty, W_2^\infty, f_\infty; \mu_\infty|\,.\,|_\infty^s, \mu_\infty^{-1}|\,.\,|_\infty^{1-s}), \end{aligned} \tag{22}$$

with

$$\begin{aligned} &\Phi_S^{\mathtt{finite}}(\pi_1, \pi_2, \mu, s; W_1, W_2, f) \\ &\quad = \frac{L^\infty(s, \pi_1, \mu) L^\infty(1-s, \pi_2, \mu^{-1})}{L^\infty(s, \Pi, \mu)} \prod_{v \in S \setminus \{\infty\}} \Psi(W_1^v, W_2^v, f; \mu_v|\,.\,|_v^s, \mu_v^{-1}|\,.\,|_v^{1-s}) \\ &\quad = \prod_{v \in S \setminus \{\infty\}} \Psi(W_1^v, W_2^v, f; \mu_v|\,.\,|_v^s, \mu_v^{-1}|\,.\,|_v^{1-s}), \end{aligned}$$

if μ is chosen in such a way that for $v \in S\setminus\{\infty\}$, the local quasi-character μ_v is highly ramified. Combining everything proves the first statement of the following theorem. We observe that in the case of interest of [25] the proof of the corresponding theorem is quite technical.

Theorem 2.24 *In the above situation, for each K-finite $W \in \mathcal{W}(\Pi_\infty)$, the ratio*

$$\frac{Z(s, W, \mu_\infty)}{L_\infty(s, \pi_1^\infty, \mu_\infty) L_\infty(s, \tilde{\pi}_2^\infty, \mu_\infty)}$$

extends to an entire function of s. Furthermore, for each s, there is a choice of W such that the above expression does not vanish at s.

Proof. We only need to prove the second statement. In order to do this, we prove the existence of an entire function $\Phi(s)$ such that

$$\begin{aligned} &Z_{\infty,N}(s, \pi_\infty(w^{-1})\mathbb{W}_\infty(W_\infty^1, W_\infty^2; f_\infty), \mu_\infty) \\ &= \frac{1}{\Phi(s)} L_\infty(s, \pi_1, \mu) L_\infty(1-s, \pi_2, \mu^{-1}) \\ &\quad \times \Psi(W_1^\infty, W_2^\infty, f_\infty; \mu_\infty|\,.\,|_\infty^s, \mu_\infty^{-1}|\,.\,|_\infty^{1-s}). \end{aligned} \tag{23}$$

By proposition 2.13 there is a choice of the data with the required property. Again we assume that μ is highly ramified for non-archimedean $v \in S$, and unramified outside S. In order to show the existence of $\Phi(s)$ it is not hard to see that if we can show the existence of local non-archimedean data with the property that

$$L_v(s, \pi_1, \mu) L_v(1-s, \pi_2, \mu^{-1}) \Psi(W_1^v, W_2^v, f; \mu_v|\,.\,|_v^s, \mu_v^{-1}|\,.\,|_v^{1-s})$$

is a constant, then we can take

$$\Phi(s) = C \prod_{v \in S \setminus \{\infty\}} Z_{v,N}(s, \pi_v(w^{-1})\mathbb{W}_v(W_v^1, W_v^2; f_v), \mu_v),$$

with C the obvious non-zero constant. The existence of such data is the statement of Corollary 2.11.

We claim that the function $\Phi(s)$ is nowhere vanishing. To see this, we set

$$F_1(W_\infty^1, W_\infty^2, s) = \frac{Z_{\infty,N}(s, \pi_\infty(w^{-1})\mathbb{W}_\infty(W_\infty^1, W_\infty^2; f_\infty), \mu_\infty)}{L_\infty(s, \pi_1, \mu)L_\infty(1-s, \pi_2, \mu^{-1})}$$

$$F_2(W_\infty^1, W_\infty^2, s) = \Psi(W_1^\infty, W_2^\infty, f_\infty; \mu_\infty|\,.\,|_\infty^s, \mu_\infty^{-1}|\,.\,|_\infty^{1-s}).$$

So far we know that given any W_∞^1, W_∞^2, the complex functions $F_1(s), F_2(s)$ are both entire. Next, let s_0 be given and suppose $\Phi(s_0) = 0$; but,

$$F_2(s) = \Phi(s)F_1(s), \tag{24}$$

which would then imply that for all choices of data we must have $F_2(s_0) = 0$ which, by proposition 2.13, is not true. This finishes the proof of the theorem. □

Remark 2.25 *We observe that the function $\Phi(s)$ defined in the proof of the theorem does not depend on W_∞^1, W_∞^2, and its dependence on $\pi_1^\infty, \pi_2^\infty$ is merely through the non-archimedean components of the automorphic representations π_1, π_2. As $\Phi(s)$ is the product of polynomials of q_v^{-s}, for $v \in S$, and as it nowhere vanishing, it is a function of the form*

$$AB^{-s}$$

with B rational. Also prime numbers appearing in the decomposition of B are all from the set S. We will see later that $\Phi(s)$ is in fact a constant.

2.6 Analytic continuation

Let τ be a complex number with $\Re\tau > 0$. Then one can consider the archimedean principal series representation $\pi(\tau) = \mathrm{Ind}(|\,.\,|^\tau \otimes |\,.\,|^{-\tau})$. Let $\rho_\tau : W_\mathbb{R} \to \mathrm{GL}_2(\mathbb{C})$ be the L parameter associated with the representation $\pi(\tau)$. We observe that if $\pi(\tau)$ is irreducible, the corresponding L packet has a single element. Then as described in [5] one can consider a continuous map

$$P(\tau) : \mathcal{S}(\mathrm{GL}_2(\mathbb{R})) \longrightarrow \pi(\tau).$$

Also for $v \in \pi(\tau)$, we set

$$W(v, g) = \int_{N(\mathbb{R})} v(ng)\psi^{-1}(n)\, dn$$

when the integral converges. Fix a Schwartz function f, and set

$$W_\tau(f;g) := W(P(\tau)(f), g).$$

A theorem of Shahidi asserts that W_τ extends to an entire function of τ. Usually, suppressing f, we simply write W_τ. Fix two sections of W_τ, say W_{τ_1} and W_{τ_2}. Next, consider the function

$$\mathbb{W}_f(\tau_1,\tau_2) := \mathbb{W}(W_{\tau_1}, W_{\tau_2}; f)$$

as before. We write $F_i(\tau_1,\tau_2,s)$, $i = 1, 2$, instead of the functions of the previous paragraph.

Let $\mathbb{C}_{\text{aut}}$ be the collection of those complex numbers τ with the property that $\pi(\tau)$ occurs as the archimedean component of some automorphic cuspidal representation of the group GL(2). It is well-known that $\mathbb{C}_{\text{temp}} := \mathbb{C}_{\text{aut}} \cap i\mathbb{R}$ is dense in $i\mathbb{R}$.

The function $\mathbb{W}_f(\tau_1,\tau_2)$ is entire on $\mathbb{C}^2$, and for fixed $(\tau_1,\tau_2) \in \mathbb{C}^2$ defines a Whittaker function on $\mathrm{GSp}(4,\mathbb{R})$. Also by construction if $\tau_1,\tau_2 \in \mathbb{C}_{\text{temp}}$, the function $\mathbb{W}_f(\tau_1,\tau_2)$ will make up the K-finite Whittaker model of the unique element of the local L packet $\varphi(\rho_{\tau_1},\rho_{\tau_2})$. In fact, if we stay away from the points of reducibility, the unique element of the L packet given by $\varphi(\rho_{\tau_1},\rho_{\tau_2})$ is generic.

We have established the identity

$$F_1(\tau_1,\tau_2,s) = \Phi(s)F_2(\tau_1,\tau_2,s)$$

whenever $(\tau_1,\tau_2) \in \mathbb{C}_{\text{aut}} \times \mathbb{C}_{\text{aut}}$, and $\Re s > b(\tau_1,\tau_2)$. Presumably, the function $\Phi(s)$ depends on s, and, though we have suppressed the dependence, on τ_1,τ_2. We now show that for $\tau_1,\tau_2 \in \mathbb{C}_{\text{temp}}$, $\Phi(s)$ is an absolute constant independent of all variables. For this we follow the argument of lemma 5 of [42], which is in the spirit of Burger-Li-Sarnak. The proof of Lemma 5 of [42] implies that given $\tau \in i\mathbb{R}$ one can find an automorphic cuspidal representation of GL(2) with archimedean component arbitrarily close to $\pi(\tau)$ and ramified only at one prescribed place. This, applied to a pair of tempered representations of GL(2) considered as a representation of GO(2, 2), implies that given a tempered representation of $\mathrm{GO}(2,2)(\mathbb{R})$ one can find two automorphic cuspidal representations with disjoint sets S. This observation combined with remark 2.25 proves that $\Phi(s)$ must be a constant. Next, we have

$$F_1(\tau_1,\tau_2,s) = \Phi F_2(\tau_1,\tau_2,s)$$

whenever $\tau_1,\tau_2 \in \mathbb{C}_{\text{temp}}$ and $\Re s > b(s_1,s_2)$. The density of $\mathbb{C}_{\text{temp}}$ in $i\mathbb{R}$ then implies that the identity must hold for all τ_1,τ_2, whenever $\Re s > b(\tau_1,\tau_2)$. But we have seen that F_2 is entire as a function of three complex variables; consequently, as F_1 and F_2 agree on an open set, F_2 is the analytic continuation of F_1. Consequently, whatever we proved about F_2 carries over to F_1.

This finishes the proof of Theorem 2.1.

2.7 Special representations

In 2.4 we defined the so-called *special representations*. It seems that the class of special representations is the same as the collection of generic elements of L-packets defined by Roberts with parameters of the form (9). This is in part inspired by the above considerations, especially in 2.6, and the global results of [29]. If this speculation is correct, then the class of the representations covered by the above analysis is quite large. In order to explain by what we mean by "large" we proceed by studying the L-parameters of $\mathrm{GSp}(4,\mathbb{R})$ representations as follows. The following results, especially Proposition 2.26 and its proof, were kindly provided to us by Brooks Roberts ([30]).

In [29], Roberts defines two types of L-parameters for $\mathrm{GSp}(4,\mathbb{R})$. The first kind of parameter $\varphi(\eta,\rho)$ is associated to a two dimensional representation

$$\rho : W_{\mathbb{C}} \to \mathrm{GL}(2,\mathbb{C})$$

whose determinant is Galois invariant, i.e., invariant under conjugation by the element j of $W_{\mathbb{R}}$ (see [20]), and a continuous homomorphism

$$\eta : W_{\mathbb{C}} \to \mathbb{C}^{\times}$$

that extends $\det\rho$; observe that there are two such η. As a representation, $\varphi(\eta,\rho)$ is $Ind_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}\rho$; if V is the space of ρ and one regards as usual the space of $Ind_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}\rho$ as $V\oplus V$, via the map

$$f \mapsto f(1)\oplus f(j),$$

then the symplectic form is given by

$$\langle v_1\oplus v_2, v_1'\oplus v_2'\rangle = \eta(j)\langle v_1, v_1'\rangle + \langle v_2, v_2'\rangle.$$

Here we have fixed a symplectic form on the space of V; there is only one such symplectic form up to multiplication by nonzero scalars. The second kind of parameter $\varphi(\rho_1,\rho_2)$, which we already defined in (9), is associated to a pair ρ_1 and ρ_2 of two dimensional representations of $W_{\mathbb{R}}$. As a representation $\varphi(\rho_1,\rho_2)$ is $\rho_1\oplus\rho_2$; the symplectic form is given by

$$\langle v_1\oplus v_2, v_1'\oplus v_2'\rangle = \langle v_1, v_1'\rangle_1 + \langle v_2, v_2'\rangle_2,$$

where we have fixed symplectic forms $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ on the spaces of ρ_1 and ρ_2. We now have the following result:

Proposition 2.26 ([30]) *Every L-parameter $\varphi : W_{\mathbb{R}} \to \mathrm{GSp}(4,\mathbb{C})$ is equivalent to a parameter of the form $\varphi(\eta,\rho)$ or a parameter of the form $\varphi(\rho_1,\rho_2)$.*

Proof. Let V be the space of φ; it is equipped with a symplectic form. As all representations of $W_{\mathbb{R}}$ of dimension greater than two are reducible (see [20]), V admits a two dimensional $W_{\mathbb{R}}$-subspace $W\subset V$. As a first case, suppose some such W is non-degenerate as a symplectic space, i.e. not totally isotropic.

Then we can write $V = W \oplus W^\perp$ as $W_\mathbb{R}$ representations. The two dimensional representations W and $W^\perp$ have the same determinant $\lambda \circ \varphi$, and $\varphi \cong \varphi(\rho_1, \rho_2)$ with ρ_1 and ρ_2 the representations of $W_\mathbb{R}$ on W and $W^\perp$, respectively.

Next suppose that all two dimensional $W_\mathbb{R}$ subspaces $W \subset V$ are totally isotropic. Write $V = V_1 \oplus \cdots \oplus V_t$ as a sum of irreducible $W_\mathbb{R}$ subspaces; each subspace has dimension one or two. Using this, we can write $V = W \oplus W'$, where W and W' are two dimensional $W_\mathbb{R}$ subspaces, and by our assumption, totally isotropic. We can find a basis w_1, w_2 for W and a basis w_1', w_2' for W' such that w_1, w_2, w_1', w_2' is a symplectic basis for the symplectic form on V, i.e. the form has matrix

$$\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

with respect to this basis. Using this basis, for $x \in W_\mathbb{R}$ write

$$\varphi(x) = \begin{pmatrix} \pi(x) & 0 \\ 0 & \pi'(x) \end{pmatrix}$$

where $\pi(x), \pi'(x) \in \mathrm{GL}(2, \mathbb{C})$. We must have

$$\pi'(x) = \lambda(\varphi(x))^t \pi(x)^{-1}.$$

The representation π is irreducible; otherwise, V admits a nondegenerate two dimensional $W_\mathbb{R}$ subspace. Now define $\eta : W_\mathbb{R} \to \mathbb{C}^\times$ by $\eta = \lambda \circ \varphi$. Also let $\alpha : W_\mathbb{C} = \mathbb{C}^\times \to \mathbb{C}^\times$ be a continuous homomorphism such that

$$\pi = \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \alpha.$$

Define $\alpha' : W_\mathbb{C} = \mathbb{C}^\times \to \mathbb{C}^\times$ by $\alpha' = \alpha^{-1} \eta|_{W_\mathbb{C}}$. Set $\rho = \alpha \oplus \alpha'$. Then $\det \rho = \eta$ is invariant under conjugation by j, and η extends $\det \rho$. Then we claim that $\varphi \cong \varphi(\mu, \rho)$. To see this, let $V = \mathbb{C} \oplus \mathbb{C}$ be the space of ρ. Set $\varphi_1 = \varphi(\eta, \rho)$. As a model for φ_1 we take, as usual (see [29]), $V \oplus V$ with the action

$$\varphi_1(z)(v \oplus v') = \rho(z)v \oplus \rho(jzj^{-1})v', \quad z \in W_\mathbb{C} \tag{25}$$

$$\varphi_1(j)(v \oplus v') = v' \oplus \rho(j^2)v = v' \oplus \rho(-1)v. \tag{26}$$

The symplectic form on $V \oplus V$ is given by

$$\langle v_1 \oplus v_2, v_1' \oplus v_2' \rangle = \eta(j)\langle v_1, v_1' \rangle + \langle v_2, v_2' \rangle.$$

Here, the symplectic form on $V = \mathbb{C} \oplus \mathbb{C}$ is the standard one. Consider the following subspaces of φ_1:

$$W = \mathbb{C} \oplus 0 \oplus \mathbb{C} \oplus 0, \quad W' = 0 \oplus \mathbb{C} \oplus 0 \oplus \mathbb{C}.$$

These are both totally isotropic $W_\mathbb{R}$ subspaces, and $\varphi_1 = W \oplus W'$. Fix the following basis for W:

$$1 \oplus 0 \oplus 0 \oplus 0, \quad 0 \oplus 0 \oplus 1 \oplus 0.$$

With respect to this basis, the action of $W_{\mathbb{R}}$ is given by

$$\varphi(\eta,\rho)(z)|_W = \begin{pmatrix} \alpha(x) & 0 \\ 0 & \alpha(\bar{z}) \end{pmatrix}, z \in W_{\mathbb{C}}, \quad \varphi(\eta,\rho)(j)|_W = \begin{pmatrix} 0 & 1 \\ \alpha(-1) & 0 \end{pmatrix}.$$

This immediately gives that $W \cong \pi$ as representations of $W_{\mathbb{R}}$. As in the proof of the first part, then $\varphi_1 \cong \varphi$. □

If our speculation at the beginning of this paragraph is correct, then we have treated generic elements of packets of the form $\varphi(\rho_1, \rho_2)$. Observe that packets of the form $\varphi(\rho_1, \rho_2)$ may also come from GO(4), but such representations will not be generic. It remains to consider packets of the form $\varphi(\eta, \rho)$. This is naturally related to GO(3, 1). This case has been, for a different purpose, considered in [15]. Inspired by the computations of [15], we believe results analogous to ours can be obtained for packets of the form $\varphi(\eta, \rho)$.

3 Existence of Bessel functionals for generic discrete series

Let D_n be the irreducible representation of $\mathrm{GL}_2(\mathbb{R})$ with trivial central character whose restriction to $\mathrm{SL}_2(\mathbb{R})$ contains the discrete series representation with Blattner parameter $n \geq 2$. Suppose D_k and D_l are two such representations. As will be explained later in the text, one can view the representation $D_k \otimes D_l$ as a representation of the group SO(2, 2), and, extended trivially, as a representation of O(2, 2). Next, we consider the theta lift of the $D_k \otimes D_l$ to $\mathrm{Sp}(4, \mathbb{R})$, denoted by $\theta(D_k \times D_l)$. Let $\Pi_{k,l}$ be the representation of $\mathrm{GSp}(4, \mathbb{R})$ obtained from $\theta(D_k \times D_l)$ via the usual process. It is well-known that every generic discrete series representation of $\mathrm{GSp}(4, \mathbb{R})$ can be obtained as one such representation $\Pi_{k,l}$. Our main result is the following:

Theorem 3.1 *Let $\psi(x) = e^{2\pi i x}$ and $\chi_n\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{in\theta}$. Suppose $k, l \geq 2$ have the same parity, and $n > \max(k, l)$ has different parity. Then $\Pi_{k,l}$ has a $(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \chi_n, \psi)$-Bessel model.*

As is clear of our presentation of the theorem that our method of proof uses the theta correspondence for the dual reductive pair (O(2, 2), Sp(4)). The method of proof is roughly the following. We will use the theta correspondence to construct automorphic cusp forms on GSp(4) which have $\Pi_{k,l}$ as their archimedean component. The construction is the obvious one: almost by definition, there are holomorphic cusp forms on the upper half space which have D_k and D_l as their real components; since by Deligne's celebrated theorem such forms are globally tempered, the construction of Roberts [29] goes through and we obtain forms on $\mathrm{GSp}(4, \mathbb{R})$ that have $\Pi_{k,l}$ at the archimedean place. Then we consider all global Bessel functionals that have the χ_n-Bessel functional as their archimedean component. If we can show that one of these Bessel functionals evaluated at one of our automorphic forms is non-zero, our result will clearly follow. In order to prove the existence of such functionals and such functions, we pull back the Bessel functional via the global theta construction to the $\mathrm{GL}(2,\mathbb{A}) \times \mathrm{GL}(2,\mathbb{A})$ level. Here, we use the theorem of Waldspurger [43] to translate the non-vanishing problem to a statement regarding L-function at the center of critical strip. The desired non-vanishing statement then follows from a refinement of the results of [22] as explained in the appendix by P. Michel.

3.1 Non-vanishing of period integrals

Before we get to the proof of the theorem we need some preparation. Let q be a prime number of the form $4k+1$, and suppose $f \in S_k(\Gamma_0(q))$ is a primitive new form. Let π_f be the automorphic cuspidal representation associated to f via the standard process. Let λ be a grössencharacter of $\mathbb{Q}(i)$. We identify $\mathbb{Q}(i)$ with a subgroup T of $\mathrm{GL}(2)/\mathbb{Q}$ via the following map

$$a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{27}$$

Denote by λ, again, the character of $T(\mathbb{A})$ obtained by transferring λ from $\mathbb{Q}(i)$ to T. Assume we have the following compatibility condition:

$$\lambda|_{Z(\mathbb{A})} \equiv 1. \tag{28}$$

Write

$$\lambda = \otimes_v \lambda_v, \quad \pi_f = \otimes_v \pi_v. \tag{29}$$

Here the restricted tensor products are over the set of places of $\mathbb{Q}$. Also, since

$$T(\mathbb{R}) = S^1 \times \mathbb{R}_+^\times, \tag{30}$$

the character λ_∞ will be the product of two group homomorphisms

$$\lambda_\infty^0 : S^1 \to \mathbb{C}^\times \tag{31}$$

and

$$\lambda_\infty^1 : \mathbb{R}_+^\times \to \mathbb{C}^\times. \tag{32}$$

Next we have the following lemma:

Lemma 3.2 *Suppose*

$$\dim_{\mathbb{C}} \mathit{Hom}_{\ T(\mathbb{R})}(\pi_\infty, \lambda_\infty) \neq 0.$$

Then there is a $\varphi \in V_{\pi_f}$ *satisfying*

$$\int_{T_{\mathbb{Q}} Z_{\mathbb{A}} \backslash T_{\mathbb{A}}} \varphi(t) \lambda(t)^{-1}\, dt \neq 0, \tag{33}$$

if and only if $L(\frac{1}{2}, BC_{\mathbb{Q}(i)/\mathbb{Q}}(\pi_f) \times \lambda^{-1}) \neq 0$.

Proof. By a theorem of Waldspurger ([43]), the period integral (33) is not identically zero, if and only if $L(\frac{1}{2}, BC_{\mathbb{Q}(i)/\mathbb{Q}}(\pi_f) \times \lambda^{-1}) \neq 0$, and for every place v, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}\ _{T(\mathbb{Q}_v)}(\pi_v, \lambda_v) \neq 0. \tag{34}$$

We will show that with our choices of π_f and λ, the local conditions (34) are always satisfied. For this we recall the following dichotomy theorem of Tunnell, as described in [13]: For every place v, and every irreducible admissible representation Π_v of $\mathrm{GL}_2(\mathbb{Q}_v)$, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{T(\mathbb{Q}_v)}(\Pi_v, \lambda_v) + \dim_{\mathbb{C}} \mathrm{Hom}_{T(\mathbb{Q}_v)}(\Pi_v^{JL}, \lambda_v) = 1. \tag{35}$$

Here Π_v^{JL} is the Jacquet-Langlands lift of Π_v to the unique quaternion algebra D_v at v. If Π_v^{JL} does not exists, we define it to be zero. Hence, $\dim_{\mathbb{C}} \mathrm{Hom}_{T(\mathbb{Q}_v)}(\Pi_v^{JL}, \lambda_v) = 0$ if Π_v is unramified, or $T(\mathbb{Q}_v)$ is split.

Applying Tunnell's theorem to our π_v implies that if $v \neq q, \infty$, since π_v is unramified, the local condition (34) is satisfied at v. Also, if $v = \infty$, the local condition is the assumption of the lemma.

If $v = q$, since $q \equiv 1 \mod 4$, the local torus $T(\mathbb{Q}_q)$ is split, and consequently by the observation following the statement Tunnell's theorem, we have the local condition. This finishes the proof of the lemma. □

We will use the result of the appendix in the following form:

Theorem 3.3 ([23]) *We can choose λ subject to the above conditions in such that for q large enough, there is a primitive $f \in S_k(\Gamma_0(q))$ such that*

$$L(\frac{1}{2}, BC_{\mathbb{Q}(i)/\mathbb{Q}}(\pi_f) \times \lambda^{-1}) \neq 0.$$

3.2 Proof of the theorem

Now we can present the proof of the main theorem:

Proof of Theorem 3.1. Let λ be a grössencharacter of $\mathbb{Q}(i)$ such that $\lambda_\infty^0 = \chi_n$ and $\lambda_\infty^1 \equiv 1$. Suppose that the grössen-character of $\mathbb{Q}$ given by $\lambda|_{Z(\mathbb{A})}$ is trivial. Then for p, q large enough, with $p, q \equiv 1 \mod 4$, there are primitive new form $f_1 \in S_k(\Gamma_0(p))$ and $f_2 \in S_l(\Gamma_0(q))$ with the following property: There are vectors $\varphi_i \in V_{\pi_{f_i}}$, $i = 1, 2$, satisfying

$$\int_{Z_{\mathbb{A}} T_{\mathbb{Q}} \backslash T_{\mathbb{A}}} \varphi_1(t) \lambda(t)^{-1} \, dt \neq 0$$

and

$$\int_{Z_{\mathbb{A}} T_{\mathbb{Q}} \backslash T_{\mathbb{A}}} \varphi_2(t) \lambda(t) \, dt \neq 0.$$

Also we observe that if $p \neq q$, then $\pi_{f_1} \not\cong \widetilde{\pi_{f_2}}$. We now fix a pair of vectors φ_1, φ_2, and we will show the existence of a Schwartz-Bruhat function f such that

$$B_\Phi(I_4) \neq 0, \tag{36}$$

with $\Phi = \theta_f(\varphi_1, \varphi_2)$. For this, we proceed as follows. First, we obtain an expression for $\theta(\varphi_1, \varphi_2; f)^U$. We start by the following:

$$\begin{aligned}
&\theta(\varphi_1, \varphi_2; f)^U(g) \\
&\quad = \int_{(F\backslash\mathbb{A})^3} \theta(\varphi_1, \varphi_2; f)(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g)\psi^{-1}(u+v)\, du\, dv\, dw \\
&\quad = \int_{(F\backslash\mathbb{A})^3} \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g; h_1h^1, h_2h^2; f) \\
&\quad\quad \varphi_1(h_1h^1)\varphi_2(h_2h^2)\, d(h_1,h_2)\psi^{-1}(u+v)\, du\, dv\, dw,
\end{aligned}$$

where h^1 and h^2 are chosen in such a way that

$$\det h^1.(\det h^2)^{-1} = \nu(g).$$

Next, it follows from the definition of θ that

$$\begin{aligned}
&\theta(\varphi_1, \varphi_2; f)^U(g) = \\
&\quad \int_{H_1(F)\backslash H_1(\mathbb{A})} \varphi_1(h_1h^1)\varphi_2(h_2h^2) G_f(h_1h^1, h_2h^2; g)\, dh_1\, dh_2,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
&G_f(h_1h^1, h_2h^2; g) = \\
&\quad \sum_{M_1, M_2} \int_{(F\backslash\mathbb{A})^3} \omega(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g, h_1h^1, h_2h^2) f(M_1, M_2) \\
&\quad \psi^{-1}(u+v)\, du\, dv\, dw.
\end{aligned}$$

Next, for fixed M_1 and M_2 we have

$$\begin{aligned}
&\int_{(F\backslash\mathbb{A})^3} \omega(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g, h_1h^1, h_2h^2) f(M_1, M_2)\psi^{-1}(u+v)\, du\, dv\, dw \\
&= \omega(g, h_1h^1, h_2h^2) f(M_1, M_2) \\
&\int_{(F\backslash\mathbb{A})^3} \psi(tr \begin{pmatrix} u & w \\ w & v \end{pmatrix} \begin{pmatrix} \det M_1 - 1 & B(M_1, M_2) \\ B(M_2, M_1) & \det M_2 - 1 \end{pmatrix})\, du\, dv\, dw.
\end{aligned}$$

Next, we have the following straightforward lemma:

Lemma 3.4 *For any* 2×2 *matrix* $A \in \mathsf{M}_2(\mathbb{A})$*, we have*

$$\int_{(F\backslash\mathbb{A})^3} \psi(tr \begin{pmatrix} u & w \\ w & v \end{pmatrix} A)\, du\, dv\, dw = 0,$$

unless $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$*, in which case the value of the integral is equal to 1.*

The lemma implies that

$$G_f(h_1h^1, h_2h^2; g) = \sum_{(M_1,M_2)\in\mathcal{S}} \omega(g, h_1h^1, h_2h^2) f(M_1, M_2),$$

where

$$\mathcal{S} = \{(X, Y) \in \mathsf{M}_2(F) \times \mathsf{M}_2(F) \mid \det X = 1, \det Y = 1, B(X, Y) = 0\}.$$

Lemma 3.5 *The set* $\mathcal{S}$ *consists of a single orbit under the action of* $H_1(F)$*. The point* $P = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$ *belongs to* $\mathcal{S}$*. The stabilizer of* P *in* $H_1(F)$ *is the subgroup* $D(F)$ *to be defined in the proof.*

Proof. Since $\det X = 1$, we have $X \sim \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ under the action of H_1. Next,

$$B(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}) = \frac{1}{2}(\alpha + \delta).$$

In particular, $B(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, Y) = 0$ implies that $\mathrm{tr}Y = 0$. Next the set of elements of H_1 that fix $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ is the diagonal subgroup $\Delta = \{(g, g) \mid g \in \mathrm{PGL}_2\}$. Next, our lemma will follow from the statement that any matrix Y subject to $\det Y = 1$ and $\mathrm{tr}Y = 0$ is in the orbit of $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ under H_1. For this, we recall the theorem of Skolem-Noether: Let A be a central simple algebra, and B a simple algebra. If f, g are algebra homomorphisms $B \to A$, then there exists an invertible element $s \in A$ such that $f(b) = s^{-1}g(b)s$, for all $b \in B$. To apply the theorem, consider the following two copies of $\mathbb{Q}(i)$ in $M_2(\mathbb{Q})$:

1. $1 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, i \mapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$,
2. $1 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, i \mapsto Y$.

Now it is an easy exercise to see that $D = \Delta \cap T \times T$ is the stabilizer of P. □

Consequently,

$$G_f(h_1h^1, h_2h^2; g) = \sum_{\gamma \in D(F)\backslash H_1(F)} \omega(1, \gamma)\omega(g, h_1h^1, h_2h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}).$$

Inserting the right hand side of this expression for G_f in equation (37) gives

$$\theta(\varphi_1,\varphi_2;f)^U(g) = \int_{D(F)\backslash H_1(\mathbb{A})} \varphi_1(h_1h^1)\varphi_2(h_2h^2)\omega(g,h_1h^1,h_2h^2)f(\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix}) \, dh_1\, dh_2. \tag{38}$$

We then obtain the following identity

$$\begin{aligned} B(I_4) = \int_{Z_{\mathbb{A}}T_{\mathbb{Q}}\backslash T_{\mathbb{A}}} \int_{D(\mathbb{Q})\backslash H_1(\mathbb{A})} &\varphi_1(h_1\begin{pmatrix}\det\alpha & \\ & 1\end{pmatrix})\varphi_2(h_2)\chi^{-1}(\alpha) \\ &\omega(\begin{pmatrix}\alpha & \\ & \det\alpha.\,{}^t\alpha^{-1}\end{pmatrix}, h_1\begin{pmatrix}\det\alpha & \\ & 1\end{pmatrix}, h_2)f(P)\, dh\, d\alpha. \end{aligned} \tag{39}$$

Next we simplify the integrand:

$$\begin{aligned} \omega(\begin{pmatrix}\alpha & \\ & \det\alpha.\,{}^t\alpha^{-1}\end{pmatrix}&, h_1\begin{pmatrix}\det\alpha & \\ & 1\end{pmatrix}, h_2)f(P) \\ &= \omega(\begin{pmatrix}\alpha & \\ & {}^t\alpha^{-1}\end{pmatrix})L(h_1\begin{pmatrix}\det\alpha & \\ & 1\end{pmatrix}, h_2)f(P) \\ &= |\det\alpha|^2 L(h_1\begin{pmatrix}\det\alpha & \\ & 1\end{pmatrix}, h_2)f(P.\alpha), \end{aligned} \tag{40}$$

where it is easily seen that

$$P.\alpha = (\alpha, \alpha.\begin{pmatrix} & -1\\ 1 & \end{pmatrix}).$$

Thus, (40) is equal to

$$\begin{aligned} |\det\alpha|^2 L(h_1\begin{pmatrix}\det\alpha & \\ & 1\end{pmatrix}, h_2)&f(\alpha, \alpha.\begin{pmatrix} & -1\\ 1 & \end{pmatrix}) \\ &= |\det(h_1h_2^{-1})|^{-2} f(\begin{pmatrix}\det\alpha & \\ & 1\end{pmatrix}^{-1} h_1^{-1}\alpha h_2, \\ &\qquad \begin{pmatrix}\det\alpha & \\ & 1\end{pmatrix}^{-1} h_1^{-1}\alpha\begin{pmatrix} & -1\\ 1 & \end{pmatrix} h_2). \end{aligned} \tag{41}$$

We then get

$$
\begin{aligned}
B(I_4) = &\int_{Z_{\mathbb{A}}T_{\mathbb{Q}}\backslash T_{\mathbb{A}}} \int_{D(\mathbb{Q})\backslash H_1(\mathbb{A})} \varphi_1(h_1 \begin{pmatrix} \det\alpha & \\ & 1 \end{pmatrix})\varphi_2(h_2)\chi^{-1}(\alpha) \\
&\quad |\det(h_1 h_2^{-1})|^{-2} f(\begin{pmatrix} \det\alpha & \\ & 1 \end{pmatrix}^{-1} h_1^{-1}\alpha h_2, \\
&\qquad \begin{pmatrix} \det\alpha & \\ & 1 \end{pmatrix}^{-1} h_1^{-1}\alpha \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} h_2)\, dh\, d\alpha \qquad (42) \\
= &\int_{Z_{\mathbb{A}}T_{\mathbb{Q}}\backslash T_{\mathbb{A}}} \int_{D(\mathbb{Q})\backslash H_1(\mathbb{A})} \varphi_1(\alpha h_1)\varphi_2(h_2)\chi^{-1}(\alpha) \\
&\quad |\det(h_1 h_2^{-1})|^{-2} f(h_1^{-1}h_2, h_1^{-1} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} h_2)\, dh\, d\alpha,
\end{aligned}
$$

after a change of variable

$$(h_1, h_2) \mapsto (\alpha h_1 \begin{pmatrix} \det\alpha & \\ & 1 \end{pmatrix}^{-1}, h_2).$$

Next, we have the natural isomorphisms

$$D(\mathbb{Q})\backslash H_1(\mathbb{A}) \cong (D(\mathbb{Q})\backslash D(\mathbb{A}))\backslash (D(\mathbb{A})\backslash H_1(\mathbb{A})),$$

and

$$D(\mathbb{Q})\backslash D(\mathbb{A}) \cong Z_{\mathbb{A}}T_{\mathbb{Q}}\backslash T_{\mathbb{A}}.$$

Hence,

$$
\begin{aligned}
B(I_4) = &\int_{Z_{\mathbb{A}}T_{\mathbb{Q}}\backslash T_{\mathbb{A}}} \int_{Z_{\mathbb{A}}T_{\mathbb{Q}}\backslash T_{\mathbb{A}}} \int_{D(\mathbb{A})\backslash H_1(\mathbb{A})} \varphi_1(\alpha\beta h_1)\varphi_2(\beta h_2)\chi^{-1}(\alpha)| \\
&\quad \det(h_1 h_2^{-1})|^{-2} f(h_1^{-1}h_2, h_1^{-1} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} h_2)\, dh\, d\alpha\, d\beta \\
= &\int_{Z_{\mathbb{A}}T_{\mathbb{Q}}\backslash T_{\mathbb{A}}} \int_{Z_{\mathbb{A}}T_{\mathbb{Q}}\backslash T_{\mathbb{A}}} \int_{D(\mathbb{A})\backslash H_1(\mathbb{A})} \varphi_1(\alpha h_1)\varphi_2(\beta h_2)\chi^{-1}(\alpha)\chi(\beta) \\
&\quad |\det(h_1 h_2^{-1})|^{-2} f(h_1^{-1}h_2, h_1^{-1} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} h_2)\, dh\, d\alpha\, d\beta,
\end{aligned} \qquad (43)
$$

after the change of variable $\alpha \mapsto \alpha\beta^{-1}$. After re-arrangement, we get

$$B(I_4) = \int_{D(\mathbb{A})\backslash H_1(\mathbb{A})} \Xi_1(h_1)\Xi_2(h_2) f(h_1^{-1}h_2, h_1^{-1} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} h_2)\, dh, \qquad (44)$$

with

$$\Xi_1(h_1) = \int_{Z_{\mathbb{A}}T_{\mathbb{Q}}\backslash T_{\mathbb{A}}} \varphi_1(\alpha h_1)\chi^{-1}(\alpha)\, d\alpha,$$

and

$$\Xi_2(h_2) = \int_{Z_{\mathbb{A}}T_{\mathbb{Q}}\backslash T_{\mathbb{A}}} \varphi_1(\beta h_2)\chi(\beta)\, d\beta.$$

By our choice of φ_1, φ_2, we know that $\Xi_i(I_2) \neq 0$, $i = 1, 2$. The theorem now follows from the following lemma:

Lemma 3.6 *Suppose* Ψ *is a function on* $D(\mathbb{A})\backslash H_1(\mathbb{A})$ *such that*

$$\int_{D(\mathbb{A})\backslash H_1(\mathbb{A})} \Psi(h_1, h_2) f(h_1^{-1}h_2, h_1^{-1} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} h_2)\, dh = 0, \tag{45}$$

for every K*-finite Schwartz function* f*. Then* $\Psi \equiv 0$*.*

Proof of the lemma. We have

$$\int_X \Psi(x) f(\gamma(x)\, dx = \int_{\gamma(X)} f(y) \left(\int_{\gamma^{-1}(y)} \Psi(x)\, dx \right) dy.$$

Then the claim is that if $(h_1, h_2), (g_1, g_2) \in H_1(\mathbb{A})$, and

$$(h_1^{-1}h_2, h_1^{-1} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} h_2) = (g_1^{-1}g_2, g_1^{-1} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} g_2),$$

then $(h_1, h_2) = (tg_1, tg_2)$ for some $t \in T_{\mathbb{A}}$. This claim is obvious, and implies the lemma.□

The theorem now follows.□

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