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Spinors, Inflation, and Non-Singular Cyclic Cosmologies

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Spinors, Inflation, and Non-Singular Cyclic Cosmologies

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We consider toy cosmological models in which a *classical*, homogeneous, spinor field provides a dominant or sub-dominant contribution to the energy-momentum tensor of a flat Friedmann-Robertson-Walker universe. We find that, if such a field were to exist, appropriate choices of the spinor self-interaction would generate a rich variety of behaviors, quite different from their widely studied scalar field counterparts. We first discuss solutions that incorporate a stage of cosmic inflation and estimate the primordial spectrum of density perturbations seeded during such a stage. Inflation driven by a spinor field turns out to be unappealing as it leads to a blue spectrum of perturbations and requires considerable fine-tuning of parameters. We next find that, for simple, quartic spinor self-interactions, non-singular cyclic cosmologies exist with reasonable parameter choices. These solutions might eventually be incorporated into a successful past- and future-eternal cosmological model free of singularities. In an Appendix, we discuss the classical treatment of spinors and argue that certain quantum systems might be approximated in terms of such fields.

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I. INTRODUCTION

At least since the advent of the first inflationary models [1], cosmologies containing classical scalar fields have received widespread attention in the literature. From a purely phenomenological point of view, such scalar fields are general enough to accommodate a rich variety of behaviors. From a theoretical point of view, their invariable appearance in various theories of nature makes them natural candidates for cosmological applications. In spite of these facts, one might wonder to what extent scalar fields are singled out by the former considerations. Could other classical homogeneous fields play a significant role in cosmology?

In the present paper, we consider the possibility that *classical*, homogeneous, spinor fields might play a role in cosmology. By a classical spinor field, we simply mean a set of four complex-valued spacetime functions that transform according to the spinor representation of the Lorentz group. Although the existence of spin-1/2 fermions is both theoretically and experimentally undisputed, these are described by *quantum* spinor fields. It is unclear when fermionic quantum fields might be consistently treated as classical spinors. It is generally held that there exists no classical limit for fundamental quantum Fermi fields; however, one can imagine classical spinors as arising from an effective description of a more complex quantum system. We address possible justifications for the existence of classical spinors in an Appendix. For the bulk of this paper, we will simply presuppose their existence.

We find that classical spinors are mathematically perfectly consistent. Physically, one might object that spinors violate Lorentz invariance and isotropy. Without being explicit about the precise nature of Lorentz transformations in a general gravitational background, let us point out that Lorentz invariance is broken in any Friedmann-Robertson-Walker cosmology, regardless of whether a spinor has a non-vanishing expectation value or not. On the other hand, we shall see that eventual violations of isotropy caused by the spinor do not prevent consistent solutions of Einstein's equations, and might actually remain undetectable.

Compared to scalar fields, spinor fields have attracted little attention in cosmology. One of the first papers about the subject was Taub's study of the Dirac equation in various cosmological spaces [2]. Brill and Wheeler dealt with neutrinos in gravitational fields [3]. The quantization of a spinor field in an expanding universe was considered by Parker [4], and the quantization of gravity coupled to a spinor was addressed in [5, 6]. Explicit solutions of the Dirac equation in an open Friedmann-Robertson-Walker spacetime have been considered in [7]. Solutions of the Einstein equations coupled to a spinor in Bianchi Type I spaces have been extensively studied by Saha and Shikin [8].

The structure of this paper is the following. In Section II we discuss how to couple a spinor to gravity. The reader familiar with the formalism might want to skip

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to the next section and eventually refer back for notational details. Section III deals with the basic cosmological equations and general solutions in terms of an arbitrary spinor self-interaction term. In Section IV we study inflation driven by a spinor field and compute the spectrum of primordial density perturbations. Section Section V presents a cyclic, non-singular model of the universe that critically relies on the properties of a spinor field. Finally, in Section VI we summarize our results and draw our conclusions. The Appendix comments on the meaning and properties of classical spinors.

II. FORMALISM

In this section we briefly review how a spinor field is coupled to gravity. For complete discussions about spinors in curved spacetimes, see [9, 10, 11].

Because the group of diffeomorphisms does not admit spinor representations, in order to couple a spinor to gravitation one introduces the Lorentz group (which does actually have spinor representations) as a local symmetry group of the theory. Under diffeomorphisms $x^\mu \to \tilde{x}^\mu(x^\nu)$, a spinor ψ is a scalar, $\psi \to \tilde{\psi} = \psi$, but under a local Lorentz transformation with parameters $\lambda_{ab}(x)$ a spinor transforms according to

$$\psi \to \tilde{\psi} = \exp\left[\frac{1}{2}\lambda_{ab}(x)\Sigma^{ab}\right]\psi, \tag{1}$$

where $\Sigma^{ab} \equiv \frac{1}{4}[\gamma^a, \gamma^b]$ are the generators of the spinor representation of the Lorentz group, and the 4×4 matrices γ^a satisfy the Clifford algebra $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. We shall choose the Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2}$$

where the σ^i are the conventional 2×2 Pauli matrices. Then, $\gamma^0 = (\gamma^0)^\dagger$ is Hermitean, and the $\gamma^i = -(\gamma^i)^\dagger$ are anti-Hermitean. For later convenience, we shall define the additional (Hermitean) gamma matrix $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$.

A fermion is coupled to gravitation with the aid of a "vierbein" $e^\mu{}_a$, a set of four contravariant vector fields that satisfy the orthonormality condition

$$g_{\mu\nu} e^\mu{}_a e^\nu{}_b = \eta_{ab}\,, \tag{3}$$

where $g_{\mu\nu}$ is the spacetime metric and η_{ab} is the Minkowski metric $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$. Latin indices enumerate each of the vectors in the vierbein while Greek indices enumerate the spacetime components of each of these vectors. Spacetime and Lorentz indices are raised and lowered with the spacetime and Minkowski metrics respectively, leading to associated sets of vectors such as $e_{a\mu}$ and $e_\mu{}^a$.

Local Lorentz transformations $\Lambda(x)$ are just local "reshufflings" of the vierbein vectors

$$e^\mu{}_a \to \tilde{e}^\mu{}_a = \Lambda_a{}^b e^\mu{}_b, \tag{4}$$

that preserve the orthonormality relation (3) at each point. Thus, the spacetime metric only determines the vierbein up to such local Lorentz transformations. For this reason, one must ensure that any Lagrangian formed with the aid of the vierbein is invariant under the Lorentz group acting as a local symmetry. Invariant terms containing derivatives of a spinor can be constructed through the covariant derivative

$$D_\mu\psi = (\partial_\mu + \Omega_\mu)\psi, \tag{5}$$

which transforms as a (covariant) vector under diffeomorphisms and as a spinor under local Lorentz transformations. The 4×4 matrix Ω_μ is the spin connection

$$\Omega_\mu = \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}, \quad \omega_{\mu ab} = e^\nu{}_a \nabla_\mu e_{\nu b}\,. \tag{6}$$

The coefficients $\omega_{\mu ab}$ are the Ricci rotation (or spin) coefficients.

The vierbein and the flat-space gamma matrices allow one to define a new set of gamma matrices

$$\Gamma^\mu \equiv e^\mu{}_a\gamma^a \tag{7}$$

that satisfy[1] the algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$. These can be used to write down a generalization of the Dirac action in a curved spacetime background,

$$S_\psi = \int d^4x\, e\, \left[\frac{i}{2}\left(\bar{\psi}\Gamma^\mu D_\mu\psi - D_\mu\bar{\psi}\Gamma^\mu\psi\right) - V\right], \tag{8}$$

which we have written in a symmetrized form. Here, e is the determinant of the vierbein $e_\mu{}^a$, and the Dirac adjoint $\bar{\psi}$ is given by $\psi^\dagger\gamma^0$. The covariant derivative acting on the adjoint is $D_\mu\bar{\psi} = \partial_\mu\bar{\psi} - \bar{\psi}\,\Omega_\mu$. By an integration by parts, the kinetic term of the spinor can be cast in the "conventional" form $i\bar{\psi}\Gamma^\mu D_\mu\psi$. The term V stands for any scalar function of ψ, $\bar{\psi}$ and possibly additional matter fields. When a particular form of V is later needed, we will assume that V only depends on the scalar bilinear $\bar{\psi}\psi$. It turns out that this choice is general enough for our purposes. More general interactions in Bianchi Type I spacetimes have been considered in the series of papers [8].

The Lagrangian (8) describes how the spinor is coupled to gravity, but it does not specify the dynamics of gravity. We shall assume that the latter is governed by the Einstein-Hilbert action. Hence, we consider a spinor minimally coupled to general relativity,

$$S = S_\psi + S_m - \frac{1}{6}\int d^4x\sqrt{-g}R, \tag{9}$$

where R is the scalar curvature, S_ψ is given by Eq. (8) and S_m describes additional matter fields, such as scalar

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[1] Note that, while the γ^a's do not transform under local Lorentz transformations or diffeomorphisms, the Γ^μ's do. They inherit their transformation properties from the vierbein.

fields or gauge fields. The symmetries we have postulated up to now, diffeomorphism and local Lorentz invariance, certainly allow for the presence of additional terms in the action. For example, we could have written down a non-minimal term like $\bar{\psi}\psi R$. However, as we are going to see, in an expanding universe $\bar{\psi}\psi$ decays at least as fast as $1/a^3$. Therefore, during cosmic expansion such a term would quickly become negligible. This situation is in sharp contrast with the case of a scalar field ϕ, where *a priori* there is no reason to expect a term proportional to ϕR to be negligible (see however [12, 13]). Eventually this fact could be relevant in models where a spinor field drives late time cosmic acceleration.

Varying the action (9) with respect to the vierbein $e^\mu{}_a$ leads to the Einstein equations

$$G_{\mu\nu} = 3T_{\mu\nu}, \tag{10}$$

where the energy momentum tensor $T_{\mu\nu}$ is given by the variation of the matter action,

$$T_{\mu\nu} = \frac{e_{\mu a}}{e}\frac{\delta S}{\delta e^\nu{}_a}. \tag{11}$$

Note that we work in units where $8\pi G/3 = \hbar = c = 1$. Substituting the action (8) into Eq. (11) we obtain, after an integration by parts, the energy momentum tensor of the spinor (on-shell),

$${}^{(\psi)}T_{\mu\nu} = \frac{i}{2}\left[\bar{\psi}\Gamma_{(\mu}D_{\nu)}\psi - D_{(\nu}\bar{\psi}\Gamma_{\mu)}\psi\right] - g_{\mu\nu}L_\psi\,. \tag{12}$$

We have used a relation that follows from the Lorentz invariance of the spinor Lagrangian,

$$\begin{aligned} &D_\mu(\bar{\psi}\{\Gamma^\mu, \Sigma^{\rho\sigma}\}\psi) = \\ &= \bar{\psi}\Gamma^\rho D^\sigma\psi - \bar{\psi}\Gamma^\sigma D^\rho\psi - (D^\sigma\bar{\psi})\Gamma^\rho\psi + (D^\rho\bar{\psi})\Gamma^\sigma\psi, \end{aligned} \tag{13}$$

to rewrite the result of the spinor variation. On the other hand, varying the action with respect to the field $\bar{\psi}$ yields the equation of motion of the spinor, a generalization of the Dirac equation to a curved spacetime,

$$i\Gamma^\mu D_\mu\psi - \frac{\partial V}{\partial\bar{\psi}} = 0\,. \tag{14}$$

If the action is real, the variation of the action with respect to ψ yields the adjoint of the previous equation.

III. COSMOLOGICAL SOLUTIONS

Because we are interested in cosmology, in this paper we deal with homogeneous and isotropic FRW spacetimes. Current observations favor a flat universe [14], so we assume the spacetime metric to be spatially flat,

$$ds^2 = dt^2 - a^2(t)\,d\vec{x}^2. \tag{15}$$

For these isotropic solutions of the Einstein equations to exist, the energy-momentum tensor of the spinor must be compatible with the symmetries of the metric (15), homogeneity and isotropy. At this point, note that homogeneity of a spinor is not a gauge-invariant concept; by a local Lorentz transformation (1), it is always possible to transform a homogeneous (space-independent) spinor $\psi(t)$ into an inhomogeneous (space-dependent) one $\tilde{\psi}(t, \vec{x})$. We are going to look for spinor solutions of the Dirac equation that can be written as a gauge-transformed homogeneous spinor. If that is the case, there exists a vierbein where the Dirac equation allows space-independent solutions. In our case such a vierbein is given by[2]

$$e^\mu{}_0 = \delta^\mu{}_0 \quad , e^\mu{}_i = \frac{1}{a}\delta^\mu{}_i. \tag{16}$$

In the gauge (16) the equation of motion of a space-independent spinor (14) reads

$$\dot{\psi} + \frac{3}{2}H\psi + i\gamma^0 V'\psi = 0, \tag{17}$$

where a dot ($\dot{}$) denotes a time derivative, a prime ($'$) denotes a derivative with respect to $\bar{\psi}\psi$, and $H = d(\log a)/dt$ is the Hubble parameter. The equation manifestly admits space-independent solutions, and hence, spinor observables like the energy momentum tensor are also homogeneous.

One should also verify whether spinors are compatible with the isotropy of the FRW metric. The ${}_0{}^i$ Einstein Eqs. $0 \equiv G_0{}^i = T_0{}^i$ are satisfied only if $T_0{}^i$ vanishes. This is possible for conventional matter forms (perfect fluids and homogeneous scalars), but it is not generally true for a spinor. In fact, in spatially open or closed universes, it is not possible to satisfy the constraint $T_0{}^i = 0$ unless $\bar{\psi}\psi$ is zero [5, 6]. In a spatially flat universe however, the equation of motion (17) automatically implies the vanishing of $T_0{}^i$, so that the presence of the spinor is consistent with the isotropy of the metric.

A convenient combination of the remaining Einstein Eqs. (10), the ${}_0{}^0$ and the ${}_i{}^j$, involves the energy density ρ_k and the pressure p_k of the different constituents of the universe,

$$\begin{aligned} H^2 &= \rho_\psi + \rho_m & (18)\\ \ddot{a} &= -\frac{1}{2}[\rho_\psi + \rho_m + 3(p_\psi + p_m)]a. & (19) \end{aligned}$$

Here, the sub-index ψ stands for the spinor and m for any additional matter, such as dust, radiation, or even dark energy. The spinor's energy density and pressure

[2] In open or closed FRW universes, one can construct a vierbein that allows homogeneous spinor solutions. These are formed from the Killing vectors of the homogeneous 3-dimensional spaces of Bianchi Type V and Type IX, respectively.

are given by the corresponding components of the energy momentum tensor (12),

$$\rho_\psi \equiv {}^{(\psi)}T_0{}^0 = V \tag{20}$$
$$p_\psi \equiv -{}^{(\psi)}T_i{}^i = V'\bar{\psi}\psi - V. \tag{21}$$

The equation of state of the spinor w_ψ is the ratio of its pressure to energy density, and hence, it is given by

$$w_\psi \equiv \frac{p_\psi}{\rho_\psi} = \frac{V'\bar{\psi}\psi - V}{V}. \tag{22}$$

The equation of state is not restricted to be within the interval $-1 \leq w \leq 1$. For a conventional massive fermion, $V = m\bar{\psi}\psi$, the equation of state agrees with that of a fluid of dust, $w_\psi = 0$. For more general choices of V, w_ψ may acquire any real value.

It is possible to directly integrate the spinor equation of motion if V only depends on $\bar{\psi}\psi$. It follows directly from the Dirac equation (17) that

$$\bar{\psi}\psi = \frac{A}{a^3}, \tag{23}$$

where A is a time-independent constant. Note that this result is valid for any time dependence of the background geometry, $a(t)$, and thus, is valid regardless of the dominant energy component of the universe. In an expanding universe, the value of $\bar{\psi}\psi$ monotonically decreases, whereas in a contracting universe the value of $\bar{\psi}\psi$ monotonically increases. These facts do not imply however that the energy density of the spinor follows the same behavior. The Dirac equation can be cast as a continuity equation

$$\dot{\rho} + 3H\rho(1+w) = 0. \tag{24}$$

Integrating Eq. (24) or directly from Eq. (23) it is possible to find ρ_ψ as a function of the scale factor a for arbitrarily given $V(\bar{\psi}\psi)$,

$$\rho_\psi = V\big|_{\bar{\psi}\psi = A/a^3}. \tag{25}$$

Conversely, given an arbitrary function $\rho_\psi(a)$ one can always find a $V(\bar{\psi}\psi)$ such that $\rho(a)$ is a solution of the equation of motion (24),

$$V(\bar{\psi}\psi) = \rho_\psi\big|_{a=(A/\bar{\psi}\psi)^{1/3}}. \tag{26}$$

In conclusion, a spinor field can accommodate any desired behavior of its energy density by an appropriate choice of V. In that respect, a spinor field is completely different from a scalar field. A (canonical) scalar field cannot violate the null[3] energy condition [16], whereas a spinor field can violate any desired—weak[4], null, strong[5] or dominant[6]—one. Even non-canonical scalar fields—"k-fields" [17]—cannot reproduce the behavior of a spinor. In the former there are barriers that prevent a transition from $\rho + p > 0$ to $\rho + p < 0$, whereas such barriers are nonexistent for spinors. The converse is however true. A spinor can reproduce the behavior of a scalar field. In particular, it can drive inflation and late time cosmic acceleration.

Although these solutions of Einstein equations sourced by a spinor are perfectly valid and consistent, they might break isotropy. By that we mean that the spatial components of certain vector quantities that involve the spinor do not necessarily vanish, and hence, are not invariant under spatial rotations. For instance, it turns out that for non-trivial homogeneous solutions of the Dirac Eq. (17) the spatial components of the vector $j^\mu \equiv \bar{\psi}\Gamma^\mu\psi$ are generically non-zero. If the action (9) does not include a coupling between j^μ and any other observable vector quantity, such a violation would be undetectable. On the other hand, if the action contained such a coupling, there still exist some spinors for which $j^i = 0$, such as, for instance,

$$\psi = (\psi_1, 0, 0, 0), \tag{27}$$

Note that this form of the spinor is compatible with the equation of motion (17). In other cases, no choice of a spinor prevents isotropy violations. There is no non-trivial spinor such that the "pseudo-vector" $\bar{\psi}\gamma_5\gamma^\mu\psi$ has non-vanishing spatial components [5, 6]. But again, if there is no coupling between the latter vector and any other observable component (say, because of parity conservation), such a violation would remain undetectable.

Although in this section we have mainly assumed that V only depends on the scalar bilinear $\bar{\psi}\psi$, some of the results can be easily generalized for rather arbitrary choices of V. Consider, for example, any V that is invariant under the global transformation $\psi \to e^{i\alpha}\psi$, for arbitrary constant α. Such a symmetry means that the ψ flavor is conserved, and hence, there is a conserved current $\nabla_\mu(\bar{\psi}\Gamma^\mu\psi) = 0$. Then, for a homogeneous spinor

$$\bar{\psi}\gamma^0\psi = \frac{\tilde{A}}{a^3}, \tag{28}$$

which already suggests that Eq. (23) is not just a consequence of our choice of V. In fact, writing down the 4-spinor in terms of two 2-spinors, $\psi = (u, v)$, it follows from the identity

$$(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma_5\psi)^2 + (\bar{\psi}\gamma^0\gamma_5\psi)^2 = (\bar{\psi}\gamma^0\psi)^2 - 4\left[(u^\dagger u)(v^\dagger v) - (u^\dagger v)(v^\dagger u)\right] \tag{29}$$

and the Cauchy-Schwarz inequality that

$$(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma_5\psi)^2 + (\bar{\psi}\gamma^0\gamma_5\psi)^2 \leq \frac{\tilde{A}^2}{a^6}. \tag{30}$$

[3] $\rho + p \geq 0$
[4] $\rho + p \geq 0$ and $\rho \geq 0$
[5] $\rho + p \geq 0$ and $\rho + 3p \geq 0$
[6] $\rho \geq |p|$

Therefore, spinor bilinears generically decay during the expansion of the universe, without regard to the precise form of the spinor interaction. In particular, in an expanding universe, there are no non-trivial solutions of the spinor equations of motion with constant ψ.

IV. INFLATION

In this section, we investigate the possibility that a classical spinor field could drive inflation. A sufficiently long stage of inflation [1] explains many of the features of our universe that remain unexplained otherwise (see [18] for diverging claims). Nevertheless, the nature of the component that was responsible for inflation remains unknown. Most inflationary scenarios rely on a homogeneous scalar field rolling down an appropriate potential; however, at present there is no direct experimental evidence for the existence of fundamental scalar fields in nature. Hence, a natural question is whether a another type of field could have driven a stage of inflation in the early universe. Inflation driven by a vector field has been considered by Ford [23], and inflation driven by a spinning fluid has been discussed by Obukhov [20].

A. Background

By definition, inflation is a stage of accelerated expansion of the universe, $\ddot{a} > 0$. It follows from Eq. (19) that any component driving inflation has an equation of state that obeys $w < -1/3$. (We assume ρ to be positive.) Three types of inflation are mainly considered in the literature: pole-like inflation with $w < -1$, de Sitter inflation with $w = -1$ and power-law inflation with $-1 < w < -1/3$. It is easy to verify from Eq. (22) that inflation (or simply expansion) with constant equation of state w results from a "potential"

$$V = (\bar{\psi}\psi)^{1+w}. \tag{31}$$

If $w < -1$ the expansion runs into a future singularity, while if $w > -1$ the expansion runs into a past singularity. However, by an appropriate choice of V, the universe could pole-like inflate in the past and power-like inflate in the future.

If $w = -1$, the formula (31) implies that V is constant, as for a cosmological term. It is possible to relax the condition on the function V by looking for a stage of nearly de Sitter inflation, $w \approx -1$. In terms of the function V, the condition for nearly de Sitter inflation is

$$\left|\frac{d\log V}{d\log\bar{\psi}\psi}\right| \ll 1. \tag{32}$$

Note that the latter condition alone suffices to guarantee quasi de Sitter inflation. This is to be compared with conventional scalar-field driven inflation, where two slow-roll conditions are needed. In general, any V that asymptotes to a positive constant at large $\bar{\psi}\psi$ satisfies Eq. (32). Examples of such functions V are $\log[1+(\bar{\psi}\psi)^n]$, $\tanh n\cdot\bar{\psi}\psi$ and $(\bar{\psi}\psi)^n/(1+\bar{\psi}\psi)^n$ for arbitrary positive n and sufficiently large $\bar{\psi}\psi$.

Although nothing prevents a spinor field from driving inflation, certain facts make this possibility unappealing. Inflation solves the homogeneity problem if it lasts for about 60 e-foldings. Let us assume that V is such that $w < -1/3$ for $\bar{\psi}\psi > (\bar{\psi}\psi)_e$ and $w = -1/3$ for $\bar{\psi}\psi = (\bar{\psi}\psi)_e$ (Fig. 1). The end of inflation is determined by $(\bar{\psi}\psi)_e$, the point where the equation of state w crosses the "critical" value $-1/3$. For instance, for $V = (\bar{\psi}\psi)^n/(1+\bar{\psi}\psi)^n$ inflation ends once $\bar{\psi}\psi$ reaches $3n/2-1$. If the initial value of the scalar bilinear is $(\bar{\psi}\psi)_i$, then inflation lasts a number of e-foldings N given by

$$N = \frac{1}{3}\log\frac{(\bar{\psi}\psi)_i}{(\bar{\psi}\psi)_e}. \tag{33}$$

It follows that during 60 e-foldings $\bar{\psi}\psi$ changes by eighty orders of magnitude! This is to be compared with a conventional chaotic model, where the scalar field changes by just an order of magnitude. This fact is particularly important in nearly de Sitter inflation, since the "flatness" condition (32) has to be satisfied for a range of values of $\bar{\psi}\psi$ that encompasses eighty orders of magnitude.

An important difference between inflation driven by a spinor and the conventional scenarios is the reheating mechanism after the end of inflation. In the conventional scenarios, the universe is reheated when the scalar field starts oscillating around the bottom of its potential and decays into particles [1, 21, 22]. If inflation is driven by a spinor field, the quantity $\bar{\psi}\psi$ evolves according to Eq. (23) and hence does not oscillate. Nevertheless, there are several mechanisms to reheat the universe. One of them is gravitational particle production at the end of inflation [23, 24]; more efficient ways have been suggested in [25].

B. Perturbations

One of the most appealing features of many of the conventional inflationary models is their prediction of an adiabatic, nearly scale invariant spectrum of primordial density perturbations, in agreement with current observations. Our goal in this section is to compute the power spectrum of density perturbations generated during a stage of nearly de Sitter inflation driven by the spinor field ψ. In a proper treatment of the problem, we would perturb both metric and spinor and solve the linearized Einstein equations. The nature of the spinor makes this path cumbersome, so we shall rely on a simplified analysis, where we only perturb the spinor in a given, fixed, spacetime background (de Sitter spacetime).

We shall characterize density perturbations $\delta\rho$ by the variable

$$\zeta \equiv \frac{\delta\rho}{\rho + p}. \tag{34}$$

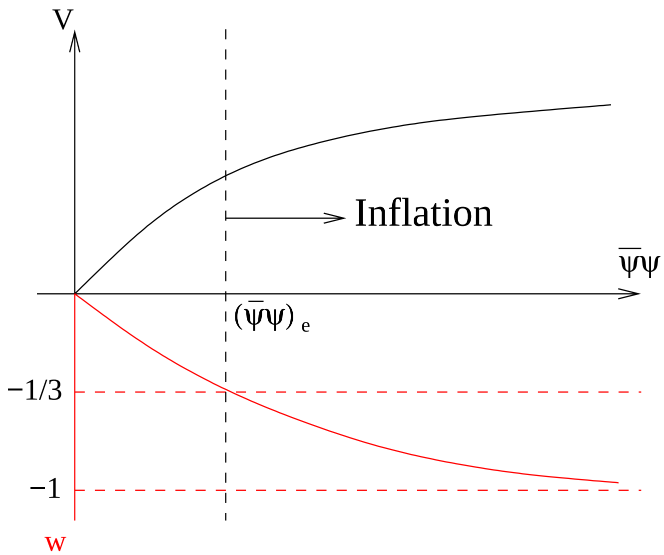

FIG. 1: A plot of a generic interaction that yields inflation. The corresponding equation of state is also shown in the diagram. For large values of $\bar\psi\psi$, the interaction is flat, Eq. (32), allowing nearly de Sitter inflation . At the critical value $(\bar\psi\psi)$ the equation of states reaches $-1/3$ and inflation ceases to be possible.

This quantity is somewhat analogous to the Bardeen variable, which is conserved on large scales in the absence of entropy perturbations, and which can be directly related to the cosmic microwave background temperature fluctuations. The source of the density perturbations $\delta\rho$ are the fluctuations $\delta\psi$ of the spinor field around its homogeneous background value ψ_0. We treat ψ_0 as a classical field, and the fluctuations $\delta\psi$ as a quantum field in an expanding universe [4],

$$\delta\psi = \frac{1}{(2\pi)^{3/2}}\int d^3k \sum_\sigma \Big[u(t,\vec{k},\sigma)a(k,\sigma)e^{i\vec{k}\vec{x}} + \\ + v(t,\vec{k},\sigma)b^\dagger(k,\sigma)e^{-i\vec{k}\vec{x}}\Big]. \quad (35)$$

The index σ runs over the two spin states of the spinor, and the operators a and b are particle and antiparticle annihilation operators, $\{a(\vec{k},\sigma), a^\dagger(\vec{k'},\sigma')\} = \{b(\vec{k},\sigma), b^\dagger(\vec{k'},\sigma')\} = \delta^{(3)}(\vec{k}-\vec{k'})\delta_{\sigma\sigma'}$.

The power spectrum $\mathcal{P}(k)$ is a measure of the fluctuations of the variable ζ on comoving scales of size $1/k$, and it is implicitly defined by the relation [26]

$$\langle\zeta(t,\vec{x})\zeta(t,\vec{x}+\vec{r})\rangle = \int \frac{dk}{k}\frac{\sin kr}{kr}\mathcal{P}(k). \quad (36)$$

Here, $\langle\,\rangle$ denotes the expectation value in an appropriately chosen vacuum state, $a|0\rangle = b|0\rangle = 0$. Using expressions (20) and (21) for the energy density and pressure of the spinor field respectively, we find that

$$\zeta = \frac{\delta\bar\psi\,\psi + \bar\psi\,\delta\psi}{\bar\psi\psi}, \quad (37)$$

where we have dropped the subindex 0 that denotes background quantities. Substituting Eq. (37) into the left hand side of Eq. (36) we obtain (for $\vec{r}=0$)

$$\langle\zeta(t,\vec{x})\zeta(t,\vec{x})\rangle = \frac{2\langle\delta\bar\psi(t,\vec{x})\psi(t)\cdot\bar\psi(t)\delta\psi(t,\vec{x})\rangle}{(\bar\psi\psi)^2}, \quad (38)$$

where we have used the fact than only terms with equal number of creation/annihilation operators have a non-vanishing expectation value.

Using the Pauli-Fierz identities [27] we can express the previous four spinor expectation value in terms of perturbation bilinears,

$$\langle\zeta\,\zeta\rangle = \frac{\langle\delta\bar\psi\,\delta\psi\rangle}{2\bar\psi\psi} + \frac{(\bar\psi\,\gamma_a\psi)\langle\delta\bar\psi\,\gamma^a\delta\psi\rangle}{2(\bar\psi\psi)^2} + \cdots. \quad (39)$$

Note that the second term in the right hand side introduces violations of isotropy in the power spectrum unless $\bar\psi\gamma_a\psi$ vanishes[7]. Because we are only interested in an estimate of the amplitude and the k dependence of the spectrum, we can concentrate on the first term on the right hand side. Substituting the expansion (35) into that term we finally obtain that the power spectrum is of the order

$$\mathcal{P}(k) \sim \frac{k^3}{4\pi^2}\sum_\sigma \frac{\bar{v}(t,\vec{k},\sigma)\,v(t,\vec{k},\sigma)}{(\bar\psi\psi)}. \quad (40)$$

The time evolution of v is dictated by the equation of motion of $\delta\psi$. The field $\delta\psi$ itself satisfies the linearized Dirac equation $i\Gamma^0 D_0\delta\psi + i\Gamma^i D_i\delta_\psi - m\delta\psi = 0$, where we assume that $m \equiv V'$ is small but non-zero and V'' is negligible. It is convenient to work with the rescaled field, $\tilde{\delta\psi} = a^{3/2}\delta\psi$ instead of $\delta\psi$. The rescaled field behaves as a spinor with a time-dependent mass in flat space, and in particular, $\tilde{v}$ satisfies

$$i\gamma^0\frac{d\tilde{v}}{d\eta} + \gamma^i k_i\tilde{v} - am\tilde{v} = 0, \quad (41)$$

where η denotes conformal time, $d\eta = dt/a$. In de Sitter space, $\eta = -e^{-Ht}/H$ runs from $-\infty$ to 0, and $a = -1/(H\eta)$. Solutions of the Dirac equation (41) in a de Sitter background were studied by Taub [2]. The ansatz $\tilde{v} = (v_+V_+, v_-V_-)$, where V_+ and V_- are two time-independent two-component spinors, yields the second order differential equation

$$v''_\pm + [k^2 + a^2m^2 \pm i(am)']v_\pm = 0. \quad (42)$$

Different linear combinations of the solutions to Eq. (42) correspond to different choices of vacuum state. We

[7] The power spectrum is isotropic if the Fourier transform of the correlation function on the lhs of Eq. (36) only depends on $k \equiv |\vec{k}|$, and not on $\vec{k}$ itself. For simplicity, we have implicitly assumed isotropy in our definition of the power spectrum $\mathcal{P}$, in the rhs of Eq. (36).

choose the standard prescription where $v_\pm \propto e^{ik\eta}$ as $\eta \to -\infty$ [10]. The corresponding properly normalized spinor solutions are then

$$v(\eta, \vec{k}, \uparrow) = \sqrt{\frac{-\pi k\eta}{a^3}} \frac{e^{-\pi m/2H}}{2} \begin{pmatrix} H_\nu^{(2)}(-k\eta)k_3/k \\ H_\nu^{(2)}(-k\eta)(k_1+ik_2)/k \\ e^{\pi m/H} H_{\bar\nu}^{(2)}(-k\eta) \\ 0 \end{pmatrix},$$
$$v(\eta, \vec{k}, \downarrow) = \sqrt{\frac{-\pi k\eta}{a^3}} \frac{e^{-\pi m/2H}}{2} \begin{pmatrix} H_\nu^{(2)}(-k\eta)(k_1-ik_2)/k \\ -H_\nu^{(2)}(-k\eta)k_3/k \\ 0 \\ e^{\pi m/H} H_{\bar\nu}^{(2)}(-k\eta) \end{pmatrix}. \tag{43}$$

The functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the Hankel functions of the first kind and second kind [28], and $\nu = \frac{1}{2} - im/H$. Up to the factor $a^{-3/2}$, the previous spinors oscillate as $e^{ik\eta}$ for modes inside the horizon, $-k\eta \gg 1$. Using the asymptotic expansion for the Hankel function in the limit $-k\eta \ll 1$ (modes outside the horizon)

$$H_\nu^{(2)}(-k\eta) \approx \frac{i}{\pi}\Gamma(\nu)\left(\frac{-k\eta}{2}\right)^{-\nu}, \tag{44}$$

it is straightforward to verify that the power spectrum "freezes" on large scales and becomes equal to

$$\mathcal{P}(k) \sim -\frac{k^3}{2\pi^3 A}\,|\Gamma(\nu)|^2 \sinh\frac{\pi m}{H} \qquad (\text{for } -k\eta \ll 1). \tag{45}$$

Such a power spectrum has spectral index $n = 4$, in strong disagreement with experimental results consistent with a scale invariant spectrum with $n \approx 1$ [14]. The constant A is the quantity that appears in Eq. (23). Equation (45) can also be used to estimate the power spectrum of the density contrast,

$$\mathcal{P}_{\delta\rho/\rho} \sim \left(\frac{mA}{Va^3}\right)^2 \mathcal{P}. \tag{46}$$

Because during inflation V is nearly constant while a grows exponentially, spinor fluctuations are highly suppressed with respect to, say, scalar field density fluctuations.

The k^3 dependence of the power spectrum (45) and the a^3 decay of the density contrast are to some extent an expression of the conformal triviality of the system. Indeed, a massless spinor is conformally invariant, and the power spectrum of a massless spinor in flat spacetime displays the same k^3 dependence. Our calculation shows that even the inclusion of a conformal symmetry-violating mass does not significantly alter this result. Note that although expression (45) vanishes for $m = 0$, this merely reflects the chiral asymmetry of the first term in the expansion (39). In the limit of zero mass, the discarded terms give the dominant contributions, which also are proportional to k^3.

In conclusion, at the level of our simplified preliminary analysis, it seems that a stage of (quasi) de Sitter inflation driven by a spinor cannot seed a scale invariant spectrum of primordial density perturbations by itself. Eventually, a light scalar field present during inflation (as in curvaton models [15]) may solve this problem.

V. NON-SINGULAR CYCLIC COSMOLOGIES

One of the most intriguing issues in cosmology is the ultimate origin of the universe and the character of its initial state. One of the attractions of cyclic cosmologies [29, 30, 31] is that—to the extent that they are truly cyclic, returning to the same state after each cycle—they dispense altogether with that problem. Since they are past eternal, there is no need to formulate initial conditions from which the universe is evolved into the future. Furthermore, the universe has always existed for the same reason, so that there is no need to ask where it originated from. However, many of the cyclic universe models that have been proposed so far suffer from singularities that prevent a continuous account of cosmic history. At a certain time, the universe evolves into a singular state where the conventional low-energy effective theory description of the universe breaks down. Furthermore, even if the singularity is regulated in some way, these models can still lead to inconsistencies [32].

In this section we describe a scenario which avoids this latter breakdown (see [33] for alternatives). For simple choices of the self-interaction term $V(\bar\psi\psi)$, cyclic cosmologies free of singularities exist. Here, we present a simple model illustrating this point. Though we make no claim that this simple model leads to an entirely satisfactory cosmology, we see no obstacle to refining the basic idea into a more realistic description of our universe.

Consider a spatially flat universe that contains "matter" (dark energy, dust and radiation) and a homogeneous spinor field ψ, and suppose that the interaction term V in Eq. (8) has the form given in Figure 2. The "potential" is negative for "small" values of $\bar\psi\psi$, and it becomes negative and decreases "fast enough" for large values of $\bar\psi\psi$. Such an interaction might be given for instance by

$$V(\bar\psi\psi) = \Lambda_\psi + m\bar\psi\psi - \lambda(\bar\psi\psi)^2. \tag{47}$$

Here Λ_ψ is a (negative) contribution to the total cosmological constant, m is a (positive) mass and λ is a (positive) coupling constant. Hence, such a model describes a conventional, self-interacting, massive spinor with a negative contribution to the vacuum energy.

In order to describe cosmic evolution in such a universe, let us arbitrarily start our description during expansion. Suppose that the matter energy density dominates over the energy density of the spinor at a time when the latter is positive (region II in Fig. 3.) In an expanding universe $\bar\psi\psi \propto a^{-3}$ is driven to values where the energy density of the spinor becomes negative, while the energy densities of radiation ($\propto a^{-4}$) and dust ($\propto a^{-3}$) are "redshifted away" and tend to zero. The only assumption we have to make at this point is that the energy density of

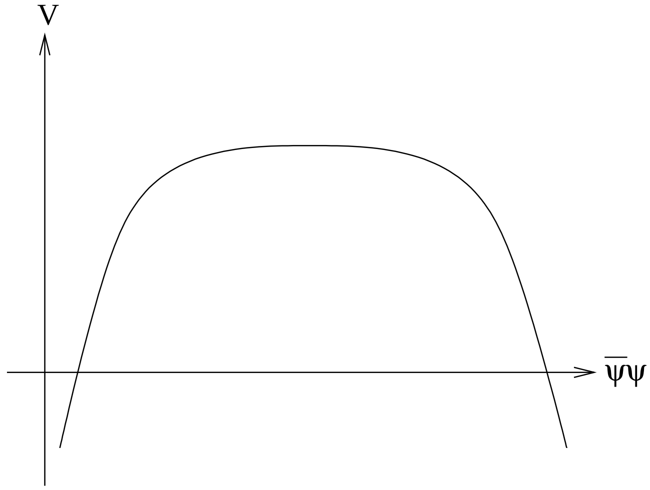

FIG. 2: Generic form of the potential V in a cyclic non-singular universe.

dark energy does not increase at late times[8]. Then, if Λ_ψ in Eq. (47) is large enough, there necessarily exists a value of the scale factor a_{max} where the total energy density ρ_{tot} becomes zero, $\rho_{tot} = 0$ (region I in Fig. 3.) It follows from Eq. (18) that at a_{max}, $\dot{a} = 0$. In addition, at a_{max} the right hand side of of Eq. (19),

$$\ddot{a} = -\frac{3a_{max}}{2}\rho_m(w_m - w_\psi), \tag{48}$$

is negative, since if ρ_{tot} reaches zero from a positive value, the combination $w_m - w_\psi$ has to be positive. Thus, at a_{max}, $\dot{a} = 0$ and $\ddot{a} < 0$, so the universe automatically starts contracting.

After the universe starts contracting, $\bar{\psi}\psi \propto a^{-3}$ reverses its motion and starts growing. At the same time, the energy densities in matter increase as the universe contracts. At sufficiently small a we can assume that the matter component of the universe mainly consists of radiation. Assume that for large values of $\bar{\psi}\psi$, V decreases faster than the rate at which the energy density of radiation increases, $w_\psi > 1/3$ (recall that the universe contracts). If V is well approximated by a power at large values of $\bar{\psi}\psi$, this implies, from Eq. (31), that $|V|$ grows faster than $(\bar{\psi}\psi)^{4/3}$, which is satisfied by the interaction (47). Then, the ratio of spinor to radiation energy densities steadily approaches -1 (region III in Fig. 3.) Again, there exists then a scale factor a_{min} where $\rho_{tot} = 0$. At a_{min}, $\dot{a} = 0$ and from Eq. (19) $\ddot{a} > 0$ since, when $\rho_{tot} = 0$ and $w > 1/3$ are satisfied, Eq. (19) reads

$$\ddot{a} \approx a_{min}\frac{\rho_m}{2}(3w_\psi - 1) > 0. \tag{49}$$

Hence, at a_{min} the universe bounces and starts expanding until it again reaches region II. From that point on cosmic history as described above repeats itself. Note that cosmic evolution is singularity-free throughout. After the bounce from contraction to expansion, the total equation of state of the universe evolves from $w_{tot} = -\infty$ towards $w_{tot} = 1/3$ during radiation domination. Hence, there is a stage of inflation between the bounce and radiation domination, and eventually a spectrum of nearly-scale invariant perturbations in the matter fields could be seeded.

It is possible to estimate the values of the parameters Λ_ψ, m and λ by imposing certain observational constraints on our model. Let us set today's value of the scale factor to 1 and denote the present value of the spinor bilinear by $(\bar{\psi}\psi)_0$. Requiring the energy density of the spinor to be subdominant today we find

$$|\Lambda_\psi + m(\bar{\psi}\psi)_0| \ll \rho_{crit} \approx 10^{-121}, \tag{50}$$

where we have assumed that the term proportional to λ is negligible today. The previous relation shows that our model requires a certain degree of fine tuning, but let us point us that the required fine tuning is of the same order as the one needed to explain late time cosmic acceleration. If condition (50) is satisfied, the spinor field remains subdominant all the way into the past, until the moment when its (negative) energy density exactly compensates the (positive) matter energy density and the universe bounces (Fig. 3). Primordial nucleosynthesis is the earliest epoch when conditions in the universe can be probed. In order for the bounce to occur before nucleosynthesis, and in order not to conflict with its standard predictions, the energy density of the spinor should be subdominant at that time as well. Since primordial nucleosynthesis occurs at $a \approx 10^{-10}$, and today $(\rho_r)_0 \approx 10^{-4}\rho_{crit}$, it follows that

$$\lambda(\bar{\psi}\psi)_0^2 \ll 10^{-145}. \tag{51}$$

Finally, in order for our field-theoretic classical description to remain valid throughout cosmic history, we impose that the energy density of radiation at the bounce be significantly below the Planckian energy density,

$$\lambda(\bar{\psi}\psi)_0^2 \gg 10^{-188}. \tag{52}$$

Due to the freedom in the dynamical variable $(\bar{\psi}\psi)_0$, there is a large set of parameters that satisfy the constraints (50), (51) and (52). It is easy to verify that a set of parameters that satisfies all the constraints is

$$\Lambda_\psi \approx -10^{-2}\cdot(10^{-3}\text{eV})^4,\ m \approx 10^{-3}\text{eV},\ \lambda \approx 10^{-5}\text{GeV}^{-2}, \tag{53}$$

where we have assumed $(\bar{\psi}\psi)_0 = 10^{-95}$, which implies that at the bounce $(\bar{\psi}\psi)_{min} \approx 10^{-43}$. Hence, throughout

[8] By a suitable modification of the interaction in Fig. 2, the spinor field could also account for dark energy. In that case one can drop the latter assumption and, up to the constraints on the parameters of our toy model, the rest of our discussion remains unaltered.

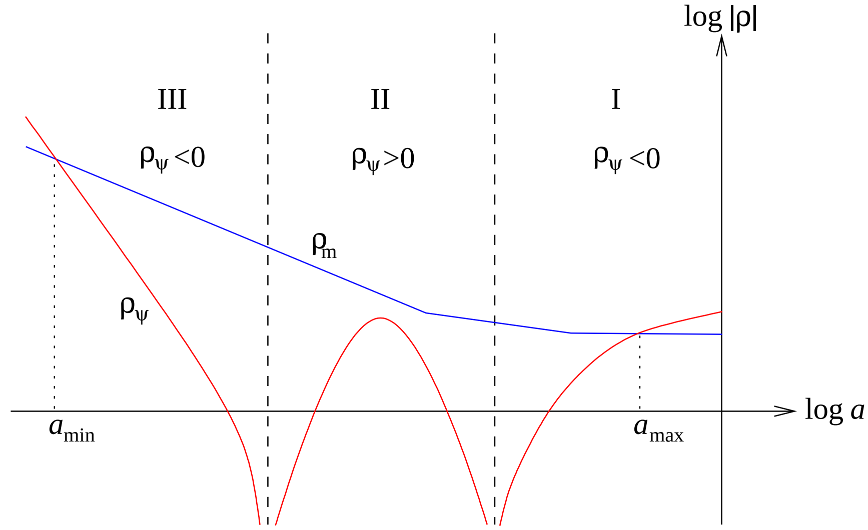

FIG. 3: Schematic plot of the energy densities of matter (ρ_m) and spinor (ρ_ψ) as a function of the scale factor a. In Region I the energy density of the spinor is negative. If the universe expands, it starts contracting at a_{max}, when spinor and matter energy densities add to zero. In Region II the spinor remains subdominant. Once its energy density becomes negative again in Region III, its importance starts growing until the moment when the total energy density is zero again and the universe bounces back.

cosmic history $\bar{\psi}\psi$ remains much smaller than 1. Notice that the cosmological term in (53) is of the same order as the component that is presently driving cosmic acceleration, the mass m agrees with common neutrino mass models, and the coupling constant λ is of the order of Fermi's constant. From this point of view, the parameter choices (53) do not appear to be unphysical.

Certainly, some aspects of our cyclic scenario still remain to be discussed. We have not addressed the issue of how to obliterate the inhomogeneous debris of previous cycles, and we have not taken into account particle production at the bounce. Eventually, both issues might be related. Finally, let us point out that even in the absence of matter, the spinor can support a cyclic universe, where a_{min} and a_{max} are simply determined by the values of $\bar{\psi}\psi$ where V becomes zero.

VI. SUMMARY AND CONCLUSIONS

In a flat FRW universe there are consistent solutions of Einstein's equations coupled to a homogeneous spinor field. For such solutions, the scalar bilinear $\bar{\psi}\psi$ is proportional to a^{-3}, as for the number density of a gas of non-relativistic particles. The energy density of the spinor is given by an (*a priori*) arbitrary self-interaction term V. For a given form of the spinor energy density $\rho_\psi(a)$, one can always find a self-interaction $V(\bar{\psi}\psi)$ that has $\rho_\psi(a)$ as a solution of the equations of motion. Thus, canonical, classical, homogeneous spinors can violate any desired energy condition, and their behavior in general cannot be reproduced by a minimally coupled, homogeneous scalar field.

A spinor field can also support a sufficiently long stage of inflation, provided the self-interaction term V satisfies a single condition on its slope for an exponentially large range of $\bar{\psi}\psi$. This condition is satisfied, for instance, if V asymptotes to a constant value at large values of $\bar{\psi}\psi$. The spectrum of primordial spinor density perturbations seeded during such a stage has a spectral index $n \approx 4$, and is hence strongly scale dependent. In addition, the power spectrum can be anisotropic, even though it is seeded within an FRW-universe. The existence of a "curvaton" field [15] during spinor-driven inflation might resolve these problems, ultimately resulting in the generation of an adiabatic, nearly scale-invariant spectrum of density perturbations.

Finally, for simple choices of the self-interaction V, there exist smooth cyclic cosmologies where the spinor energy density oscillates back and forth. The parameters needed to accommodate a realistic cosmology do not appear to be unphysical. Although the simplicity of the models gets somewhat distorted, by a straightforward modification of the self-interaction V, the spinor can also account for dark energy and still allow for realistic cyclic non-singular solutions.

Our approach has been to treat the spinor field as a complex valued, classical object obeying a simple relativistic equation of motion, a non-linear generalization of the Dirac equation in an expanding universe. In the Appendix we have addressed the validity and relevance of this assumption. Additionally, the validity of our results certainly depends on the stability of our homogeneous solutions against the growth of inhomogeneous fluctuations. We have left this question for future work.

To conclude, we have shown that spinors can accommodate a large set of interesting cosmological solutions. Although bilinears generically decay during expansion, they could still be presently important if initially they were sufficiently displaced out of equilibrium (as for any interesting cosmological solution). Due to its nature, when dealing with a spinor field the question to ask is not whether certain behavior is possible, but rather, whether the corresponding self-interaction is natural.

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APPENDIX A: CLASSICAL SPINORS

A Dirac spinor is a four-component object ψ that transforms according to (1) and obeys Dirac's equation. As far as one is only concerned with solutions of equations of motion, the components of a spinor can be consistently regarded as being complex numbers, as we have done in this paper. However, our world is ultimately described by quantum-mechanical laws, and the question is: To what extent is a classical treatment a good approximation to the quantum-mechanical problem?

In the canonical approach to quantum field theory [34], spinors are operator-valued fields that act on an appropriately defined Hilbert space. The spinor operator $\hat{\psi}$ also satisfies the Dirac equation,

$$i\gamma^\mu \partial_\mu \hat{\psi} - m\hat{\psi} = 0, \tag{A1}$$

where for the purposes of illustration and simplicity we consider a massive fermion in flat spacetime. We work in the Heisenberg representation, where operators are time-dependent and states are time-independent. We would like to interpret a classical spinor as the expectation value of the spinor in an appropriate state $|s\rangle$,

$$\psi_{cl} \equiv \langle s|\hat{\psi}|s\rangle \equiv \langle\hat{\psi}\rangle. \tag{A2}$$

Taking the expectation value of equation (A1), we find that ψ_{cl} satisfies the equation

$$i\gamma^\mu \partial_\mu \psi_{cl} - m\psi_{cl} = 0, \tag{A3}$$

which simply states that the classical spinor ψ_{cl} obeys the conventional Dirac equation. Therefore, we already recover one of the main ingredients we have used in this paper.

Note that the expectation value of a spinor *in a physical state* is a complex number, not a Grassmann number. There exist in fact states $|c\rangle$ such that $\langle c|\hat{\psi}_a(x)|c\rangle = \psi_a(x)$, with the $\psi_a(x)$'s four Grassmann valued fields ($a = 1,..,4$ is the spin index). However, that such states are not part of the physical Fock space is easily seen by considering the energy density in such a state:

$$\rho_c = m\langle c|\hat{\bar{\psi}}\hat{\psi}|c\rangle = m\bar{\psi}\psi\,. \tag{A4}$$

The energy density in a physical state must be a real number. However, because $\psi_a\psi_b = -\psi_b\psi_a$, $\rho_c^n = 0$ for any $n > 4$. This is impossible for a non-zero real number.

Although the expectation value of the spinor obeys the classical Dirac equation (A1), large quantum fluctuations of ψ around its expectation value might invalidate the classical approximation. In our particular case, the only observable that enters the classical Einstein equations is the energy density, which in our classical treatment is

$$\rho_{cl} = m\,\bar{\psi}_{cl}\psi_{cl} \tag{A5}$$

We want to find out whether the expectation value of the energy density $\langle\rho\rangle = m\langle\bar{\psi}\psi\rangle$ is well approximated by (A5). At this stage, one has to face a well-known problem. The vacuum expectation value of $\bar{\psi}\psi$ is $\sum_k(-1) = -\infty$. The conventional way of dealing with this divergence is to replace expectation values by their "renormalized" counterparts,

$$\langle\ldots\rangle_{ren} \equiv \langle s|\ldots|s\rangle - \langle 0|\ldots|0\rangle. \tag{A6}$$

With this prescription, the expectation value of $\bar{\psi}\psi$ is zero for the vacuum, and n for a state containing n particles plus antiparticles per unit volume.

Then, in order for our classical approximation to be valid, the following relation should hold,

$$\left|\frac{\langle\bar{\psi}\psi\rangle_{ren} - \langle\bar{\psi}\rangle_{ren}\langle\psi\rangle_{ren}}{\langle\bar{\psi}\rangle_{ren}\langle\psi\rangle_{ren}}\right| \ll 1. \tag{A7}$$

In the bosonic case, the states that satisfy inequalities analogous to (A7) have large occupation numbers. As the largest occupation number of fermion modes is one, it is commonly believed that fermionic physical states cannot satisfy relations such as (A7). This turns out not to be the case.

Let A and B be two complex numbers and let $|s\rangle$ be the state

$$|s\rangle = A|0\rangle + B|1\rangle. \tag{A8}$$

Here, $|0\rangle$ is the vacuum and $|1\rangle = a_0^\dagger|0\rangle$ is a zero-momentum one-particle state. The state is normalized if

$$|A|^2 + |B|^2 = 1. \tag{A9}$$

The spinor operator can be expanded in creation and annihilation operators,

$$\psi = \sum_k \left(a_k u_k + b^\dagger{}_k v_k\right), \tag{A10}$$

where u_k and v_k are normalized complex-valued spinors, $\bar{u}_k u_k = -\bar{v}_k v_k = 1$ and $\bar{v}_k u_k = \bar{u}_k v_k = 0$. The reader can easily verify that for the state (A8)

$$\langle\psi\rangle_{ren} = A^* B u_0, \quad \langle\bar{\psi}\rangle_{ren} = B^* A \bar{u}_0, \quad \langle\bar{\psi}\psi\rangle_{ren} = |B|^2. \tag{A11}$$

Therefore, condition (A7) implies

$$|B|^2 = 1 - |A|^2 \ll 1. \tag{A12}$$

Obviously, the last condition can be easily met. It means that a spinor can be treated classically if its quantum state is "close" to the vacuum. In fact, this is what one is doing by setting the fermions to zero in a classical treatment of any theory that contains spinors. But even if there are departures from the vacuum, we have shown that treating a massive Dirac spinor classically is in some cases a good approximation.

Finally, let us comment on the fermion condensates that are often encountered in particle and condensed matter physics. In quantum theories with self-interacting fermions, it might happen that the spinor bilinear $\bar{\psi}\psi$ develops a non-zero vacuum expectation value. This is what occurs for instance in the BCS theory of superconductivity [35], where phonon-induced interactions cause electrons to form bound Cooper pairs. In the relativistic Nambu-Jona-Lasinio model [36] or its renormalizable counterpart, the Gross-Neveu model [37], self-interacting chiral fermions form a scalar condensate, spontaneously breaking chiral-symmetry and dynamically generating a fermion mass. Within the effective action formalism, the dynamics of the condensate is completely determined by a classical scalar field theory. The exact classical theory that reproduces the full variety of phenomena is extremely complicated. However, for certain states of the quantum system the effective action is well approximated by a simple, local, relativistic scalar field theory[9]. Just as a strongly coupled fermionic system can be effectively described by a classical scalar field theory, it is conceivable that certain strongly coupled systems might be described by a simple classical spinor field theory, as we consider in this paper.

[9] Note, however, that one generically expects fermion condensates to couple non-minimally to gravity [38].

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