This implies

$$(V_M)^n \subseteq (M+\tfrac{1}{2}\cdot V)^n \subseteq \sum_{k=0}^{n}\binom{n}{k}\cdot M^k\cdot\frac{1}{2^{n-k}}\cdot V$$

$$\subseteq \sum_{k=0}^{n}\binom{n}{k}\cdot C\cdot t^{-k}\cdot\frac{1}{2^{n-k}}\cdot V \subseteq C\cdot(t^{-1}+\tfrac{1}{2})^n\cdot V.$$

Since $(t^{-1}+\frac{1}{2})^n$ tends to zero if $n\to\infty$, it follows that $(V_M)^n \subseteq V$ if n is sufficiently large. By Lemma 1, A is an AE-algebra and the theorem is proved.

Remark. The metrizability of A is only used to prove m-convexity. We regard as an example the algebra $\mathcal{D}(\boldsymbol{R})$ of test functions, which is m-convex. Then our theorem shows that $\mathcal{D}(\boldsymbol{R})$ is an AE-algebra.

References

[1] H. Boseck, G. Czichowski, K.-P. Rudolph, *Analysis on Topological Groups — General Lie Theory*, Teubner Verlag Leipzig (to print).

[2] B. Mitjagin, S. Rolewicz, W. Żelazko, *Entire functions in B_0-algebras*, Studia Math. 21 (1962), 291–306.

[3] L. Waelbroeck, *Topological Vector Spaces and Algebras*, Lecture Notes in Mathematics 230 (1971).

[4] W. Żelazko, *Selected Topics in Topological Algebras*, Lecture Notes Series Aarhus University No. 31 (1971).

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Spline bases in classical function spaces on compact C^∞ manifolds

Part II

by

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Abstract. Using spline functions the desired Schauder bases in Sobolev and Besov spaces on cubes with boundary conditions are constructed. The combination of these results and of the decomposition of function spaces established in Part I permits to complete in Section 11 the proofs of the main results formulated in Part I. Section 11 contains also applications (e.g., improved Sobolev type embedding theorems, estimates for the eigenvalues of integral operators and asymptotic estimates for the Kolmogorov diameters in the class of Besov spaces).

In this part we complete the proofs of Theorems A and B formulated in the Introduction to Part I of this paper. In order to read this part it is necessary to know some definitions and results given in Section 2.

In Section 4 of Part I the proofs of Theorems A and B are reduced to constructing suitable bases in the spaces $W_p^k(Q)_Z$ and $B_{p,q}^s(Q)_Z$, where $k\geqslant 0$, $s>0$, $1\leqslant p, q\leqslant\infty$, introduced in Section 2.

The boundary conditions induced by a set Z of the form (2.37) have a "tensor product" nature. This allows us to reduce our problem essentially to good approximation of vector-valued (L_p-valued) functions on some intervals and to constructing special bases in L_p spaces on the interval $\langle 0,1\rangle$. All this is carried out in Sections 7–10 by means of vector-valued splines and spline bases.

Section 7 contains the basics on vector-valued splines.

In Section 8 we consider families of orthogonal projections onto increasing subspaces of splines corresponding to the various boundary conditions. We also study associated families of projections relevant for the Sobolev spaces. The most important results are here the exponential estimates for the kernels of these projections and for the basic functions (cf. Proposition 8.10, Lemmas 8.13 and 8.27).

In Section 9 special spline bases on the unit interval are defined. Their tensor products (in the rectangular ordering) are bases in the spaces $W_p^0(Q)_Z$ and $W_p^m(Q)_Z$ for $1\leqslant p\leqslant\infty$, and they are unconditional bases if

$1 < p < \infty$. Moreover, their biorthogonal sequences have analogous properties (with Z replaced by Z' defined in (2.47)). This is all we need to complete the proof of Theorem A.

The bases in $B^s_{p,q}(Q)_Z$ needed for Theorem B of Part I are defined in Section 10. The Schauder decomposition into dyadic blocks is here the same as in Section 9. However, in the finite-dimensional subspaces corresponding to the dyadic blocks we construct new bases such that Theorem 10.19 holds.

Section 11 indicates some applications of the main results and contains bibliographical comments.

7. Spline functions. We reduce the approximation problems with boundary conditions on d-dimensional cubes to approximation of vector-valued functions. Approximating by splines we are led naturally to vector-valued splines. The space of values is denoted by X and as in Section 2 it is equal either to $\boldsymbol{R}$ or to $W^0_p(Q)$, where Q is a parallelepiped in $\boldsymbol{R}^{d-1}$.

The real-valued B-splines supply the basic tool for our construction. We recall some of their definitions, fundamental properties and results rephrasing them in the vector-valued setting. Most of the proofs will be omitted as they follow by repeating step by step the argument applied in the corresponding real-valued case. For detailed discussion of the real-valued B-splines we refer to H. B. Curry–I. J. Schoenberg [18] and C. de Boor [6] and [7].

For given positive integer r and partition $\Pi = (t_j)$ such that $t_j \leqslant t_{j+1}$, $t_j < t_{j+r}$ for $j = 0, \pm 1, \ldots$, $A = \lim_{j\to-\infty} t_j < \lim_{j\to+\infty} t_j = B$, the B-spline $N^{(r)}_j$, $j = 0, \pm 1, \ldots$, is defined by the formula

$$N^{(r)}_j(s) = (t_{j+r} - t_j)[t_j, \ldots, t_{j+r}; (t-s)^{r-1}_+],$$

where the square bracket denotes the divided difference of $(\cdot - s)^{r-1}_+$ taken at $t_j, \ldots, t_{j+r}$. The function $N^{(r)}_j$ is of the class $C^{r-1-\alpha}$ at t_k, $j \leqslant k \leqslant j+r$ with $\alpha = \#\{i\colon t_i = t_k\}$, where C^{-1} denotes now the class of functions with discontinuities of the first kind.

For $I = \langle a, b\rangle \subset (A, B)$ the non-trivial restrictions $N^{(r)}_j|_I$ are linearly independent over I.

Moreover, the B-splines have the following properties:

(7.1) $$N^{(r)}_j \geqslant 0,$$

(7.2) $$\operatorname{supp} N^{(r)}_j = \langle t_j, t_{j+r}\rangle,$$

(7.3) $$\int_{t_j}^{t_{j+r}} N^{(r)}_j(t)\,dt = \frac{t_{j+r} - t_j}{r},$$

(7.4) $$\sum_j N^{(r)}_j(t) = 1, \quad t \in (A, B),$$

(7.5) if $M^{(r)}_j = r(t_{j+r} - t_j)^{-1} N^{(r)}_j$ and

$$t_j < t_{j+r-1}, \quad t_{j+1} < t_{j+r},$$

then

$$DN^{(r)}_j = M^{(r-1)}_j - M^{(r-1)}_{j+1},$$

(7.6) for $1 \leqslant p \leqslant \infty$ we have in the norm of $L_p(t_j, t_{j+r})$

$$r^{-1/q}\|N^{(r)}_j\|^{1/p}_1 \leqslant \|N^{(r)}_j\|_p \leqslant \|N^{(r)}_j\|^{1/p}_1.$$

Definition 7.7. A function $f\colon (A, B) \to X$ is called an *X-valued spline* if it can be written as

$$f = \sum_j x_j N^{(r)}_j, \quad x_j \in X.$$

The space of all such functions is denoted by $S^r_\Pi((A, B); X)$ and if $I = \langle a, b\rangle \subset (A, B)$ we denote by $S^r_\Pi(I; X)$ the set of all restrictions to I of $f \in S^r_\Pi((A, B); X)$. In the real-valued case we simply drop the X in these symbols.

The result below is a trivial extension of the de Boor's result (cf. [8], pp. 272–273) from the real-valued to X-valued splines.

Theorem 7.8. *To each $r \geqslant 1$ there is a constant $D_r > 0$ independent of Π and p, $1 \leqslant p \leqslant \infty$, such that*

$$D_r^{-1}\|x\|_{l_p(X)} \leqslant \Big\|\sum_j x_j N^{(r)}_{j,p}\Big\|_{L_p(A,B)} \leqslant \|x\|_{l_p(X)},$$

where $x = (x_j)$, $x_j \in X$, and

$$N^{(r)}_{j,p} = N^{(r)}_j \|N^{(r)}_j\|^{-1/p}_{L_1(A,B)}.$$

Imposing on Π additional conditions

(7.9) $$t_{-r+1} = \ldots = t_0 = a, \quad t_N = \ldots = t_{N+r-1} = b$$

we obtain

Corollary 7.10. *If $r \geqslant 1$, $1 \leqslant p \leqslant \infty$, $I = \langle a, b\rangle$, $x = (x_{-r+1}, \ldots \ldots, x_{N-1})$, $x_j \in X$, then for some $D_r > 0$ independent of Π and p we have*

$$D_r^{-1}\|x\|_{l_p^{N+r-1}(X)} \leqslant \Big\|\sum_{j=-r+1}^{N-1} x_j N^{(r)}_{j,p}\Big\|_{L_p(I;X)} \leqslant \|x\|_{l_p^{N+r-1}(X)},$$

i.e. $S^r_\Pi(I; X)$ is, uniformly in N, p and $\Pi \cap I$, linearly isomorphic to $l_p^{N+r-1}(X)$.

For later use we introduce additional integer parameter r', $1 \leqslant r' \leqslant r$. We impose on Π an additional condition

(7.11) $$t_i < t_{i+r'}, \quad \frac{t_{i+r'} - t_i}{t_{j+r'} - t_j} \leqslant \mu_{r'} < \infty, \quad i, j = -r'+1, \ldots, N-1.$$

Note that (7.9) and (7.11) imply

$$1 \leqslant \frac{N+r'-1}{r'\,|I|}\,|\Pi|_{r'} \leqslant \mu_{r'}, \tag{7.12}$$

where $|\Pi|_{r'} = \max\{t_{i+r'}-t_i;\ -r'+1 \leqslant i \leqslant N-1\}$.

Proposition 7.13 (Bernstein type inequality). *Let $\Pi, I = \langle a, b\rangle$, r' and r satisfy conditions (7.9) and (7.11). Then there is $C = C(r, I)$ such that for $f \in S^r_\Pi(I; X)$, $k = 0, \ldots, r-r'$, $0 \leqslant kh \leqslant |I|$ and $1 \leqslant p \leqslant \infty$ one has*

$$\|\Delta^k_h f\|_p\big(I(kh); X\big) \leqslant C\cdot (h|\Pi|^{-1}_{r'})^k \|f\|_p(I; X). \tag{7.14}$$

Inequality (7.14) can be proved in exactly the same way as Lemma 9.2 in Ciesielski [12]. The intermediate step in the proof is an application of Bernstein's inequality for the X-valued splines. It can be obtained from Corollary 7.10 and formula (7.5).

Beside the Bernstein type inequality the orders of approximation by X-valued splines are important in our considerations.

In what follows the space of X-valued splines of order r on I, corresponding to a Π satisfying (7.9) and (7.11), is denoted by $S^{r',r}_\Pi(I, X)$. The best approximation of $f \in W^0_p(I; X)$ by $S^{r',r}_\Pi(I; X)$ is

$$E^{r',r}_{\Pi,p}(f; X)_I = \inf\{\|f-g\|_p(I, X) \colon g \in S^{r',r}_\Pi(I; X)\}.$$

For the definition of the norm we refer to (2.2).

Proposition 7.15. *Let Π satisfy (7.9) and (7.11). Then there is a constant $C = C(I, r)$ such that*

$$E^{r',r}_{\Pi,p}(f; X) \leqslant C\,\omega_{r,p}(f; X; |\Pi|_{r'}/r)_I$$

holds for $f \in W^0_p(I; X)$, $1 \leqslant p \leqslant \infty$.

Proof. This result for real-valued functions is well-known and the proof as presented by De Vore ([20], Theorem 4.1, p. 136) can be easily adapted to the case of X-valued splines.

Let us now consider a sequence of partitions $\Pi_1, \Pi_2, \ldots$, of the same type as Π satisfying (7.11) and (7.9), and such that $\Pi_n \subset \Pi_{n+1}$ and $\Pi_{n+1}\setminus\Pi_n$ is a one-point set.

For simplicity we set

$$E_n(f) = E^{r',r}_{\Pi_n,p}(f; X), \qquad \omega_k(f; \delta) = \omega_{k,p}(f; X; \delta).$$

Proposition 7.16. *There is a constant $C = C(r, I)$ such that for $1 \leqslant k \leqslant r-r'$*

$$\omega_k(f; 1/n) \leqslant Cn^{-k}\Big(\|f\|_p(I; X) + \sum_{j=1}^n j^{k-1}E_j(f)\Big)$$

holds for $f \in L_p(I; X)$, $1 \leqslant p \leqslant \infty$.

To prove this we apply standard argument and use (7.14) (cf. [10], Theorem 10).

The last problem we want to discuss in this section is the best approximation of X-valued functions with small support by X-valued splines. The theorem we are going to prove exhibits the local character of the best spline approximation.

For the proof of the theorem it is convenient to have the following simple abstract

Lemma 7.17. *Let Y be a Banach space and E its subspace, and let T be linear operator such that*

$$T^2 = T, \qquad T(E) \subseteq E,$$
$$\mathring{Y} = \operatorname{Ker} T, \qquad \mathring{E} = E \cap \mathring{Y}.$$

For given $y_0 \in \mathring{Y}$ let z, z_0 denote the best approximations to y_0 in $E, \mathring{E}$, respectively. Then

$$\|y_0-z\| \leqslant \|y_0-z_0\| \leqslant (1+\|T\|)\|y_0-z\|.$$

Proof. Put $x_0 = z-Tz$. Clearly, $x_0 \in \mathring{E}$. Thus,

$$\|y_0-z_0\| \leqslant \|y_0-x_0\| = \|y_0-z+T(z-y_0)\| \leqslant (1+\|T\|)\|z-y_0\|.$$

In what follows we use I for $\langle 0,1\rangle$ and J for one of the intervals $\langle -1,1\rangle$ or $\langle -1,2\rangle$. The uniform mesh corresponding to the step $2^{-\mu}$, $\mu \geqslant 0$, with multiplicities r, a, β, γ, r at the points $-1, 0, 1/2, 1, 2$, respectively, is denoted by $\Pi(a,\beta,\gamma)$. It is assumed that $r \geqslant 1$ is fixed and $1 \leqslant a, \beta, \gamma \leqslant r$. The corresponding spline space is

$$S_{a,\beta,\gamma}(J) = S^r_{\Pi(a,\beta,\gamma)}(J; X).$$

Moreover, let

$$\mathring{S}_{a,\beta,\gamma}(I) = \{u \in S_{a,\beta,\gamma}(J) \colon \operatorname{supp} u \subseteq I\},$$
$$S_{a,\gamma}(J) = S_{a,1,\gamma}(J),$$
$$\mathring{S}_{a,\gamma}(I) = \mathring{S}_{a,1,\gamma}(I).$$

Theorem 7.18. *For given $r \geqslant 2$ there is a constant C_r such that for $2^\mu \geqslant 2(r-1)$ we have*

$$C_r^{-1}\operatorname{dist}\big(h, S_{a,\gamma}(J)\big) \leqslant \operatorname{dist}\big(h, \mathring{S}_{a,\gamma}(I)\big) \leqslant C_r \operatorname{dist}\big(h, S_{a,\gamma}(J)\big)$$

for $h \in W^0_p(J; X)$, $1 \leqslant p \leqslant \infty$, with $\operatorname{supp} h \subseteq I$. The dist is taken with respect to the norm in $W^0_p(J; X)$.

Proof. *The first case:* $J = \langle -1, 1\rangle$. Let $Y = W^0_p(J; X)$, $Z = W^0_p(J\setminus I, X)$, and let the operator restricting functions from J to $J\setminus I$ be denoted by R, $R\colon Y\to Z$. Moreover, let the extension operator $S\colon Z\to Y$

be defined similarly as in (2.7). Then, $T = SR\colon Y \to Y$ is bounded and $T^2 = T$, $\mathring{Y} = \ker T = \{f \in Y\colon \operatorname{supp} f \subseteq I\}$. If now $E = S_{a,\gamma}(J)$, then one checks easily that $TE \subseteq E$ and $\mathring{E} = \mathring{Y} \cap E = \mathring{S}_{a,\gamma}(I)$. An application of Lemma 7.17 completes the proof in the first case.

The second case: $J = \langle -1, 2\rangle$. Let S_1 and S_2 be extension operators defined similarly as in (2.7) extending from $\langle -1, 0\rangle$ to $\langle -1, 1/2\rangle$, and from $\langle 1, 2\rangle$ to $\langle 1/2, 2\rangle$, respectively. Moreover, let

$$Sh(t) = \begin{cases} S_1 h(t) & \text{for} \quad -1 \leqslant t < 1/2, \\ S_2 h(t) & \text{for} \quad 1/2 \leqslant t \leqslant 2. \end{cases}$$

We find as in the first case that $T = SR$ has similar properties, i.e. T is bounded, $T^2 = T$, $\ker T = \mathring{Y}$. Let now $E = S_{a,r,\gamma}(J)$. It then follows that $TE \subseteq E$. Moreover, $\mathring{E} = \mathring{Y} \cap E = \mathring{S}_{a,r,\gamma}(J)$. Application of Lemma 7.17 gives for $h \in \mathring{Y}$

$$\operatorname{dist}(h, \mathring{E}) \leqslant C_r \operatorname{dist}(h, E). \tag{7.19}$$

Introducing $F = S_{a,\gamma}(J)$ and $\mathring{F} = \mathring{S}_{a,\gamma}(J)$ we find $\operatorname{dist}(h, F) \leqslant \operatorname{dist}(h, \mathring{F})$, and this is the trivial part of the inequalities in question. The opposite inequality is proved as follows. We note that $\mathring{F} = F \cap \mathring{E}$ and that for $2^\mu \geqslant 2(r-1)$ $E = F + \mathring{E}$ where the sum is the algebraic one. Now, for every $e \in E$ we have a unique representation in terms of the B-splines corresponding to $\Pi(a, r, \gamma)$

$$e = \sum_j a_j N_j^{(r)}, \qquad a_j \in X.$$

We now define

$$Pe = \sum_{j \in \sigma} a_j N_j^{(r)}, \qquad Qe = \sum_{j \notin \sigma} a_j N_j^{(r)},$$

with $\sigma = \{j;\ N_j^{(r)} \in F\}$. Since $2^\mu \geqslant 2(r-1)$, it follows that for $j \notin \sigma$, $N_j^{(r)} \in \mathring{E}$.

Thus, $P+Q = \mathrm{Id}$ and $PE \subseteq F$, $QE \subseteq \mathring{E}$. Clearly P and Q are projections and, by Corollary 7.10, they are bounded in Y, and the bounds for their norms depend on r only.

The $\operatorname{dist}(h, \mathring{F})$ can now be estimated as follows. Let $f \in F$, $\mathring{e} \in \mathring{E}$. Then $f - \mathring{e} \in E$ and consequently

$$f - \mathring{e} = P(f - \mathring{e}) + Q(f - \mathring{e}),$$

whence we infer

$$f - P(f - \mathring{e}) = \mathring{e} + Q(f - \mathring{e}).$$

Now, the left-hand side is in F, and the right-hand side is in $\mathring{E}$. Thus, $f - P(f - \mathring{e}) \in F \cap \mathring{E} = \mathring{F}$ and for $h \in \mathring{Y}$

$$\begin{aligned} \operatorname{dist}(h, \mathring{F}) &\leqslant \|h - (f - P(f - \mathring{e}))\| \leqslant \|h - f\| + \|P\|\,\|f - \mathring{e}\| \\ &\leqslant (1 + \|P\|)\|h - f\| + \|P\|\,\|h - \mathring{e}\|. \end{aligned}$$

Since $f \in F$ and $\mathring{e} \in \mathring{E}$ are arbitrary, we get

$$\operatorname{dist}(f, \mathring{F}) \leqslant C[\operatorname{dist}(h, F) + \operatorname{dist}(h, \mathring{E})].$$

Moreover, $E \supseteq F$ implies $\operatorname{dist}(h, E) \leqslant \operatorname{dist}(h, F)$. Combining the last two inequalities with (7.19) we get the desired result.

COROLLARY 7.20. *Let $r \geqslant 2$ be given, and let J be defined as I_Z in* (2.38). *Then there is C_r such that*

$$\operatorname{dist}(h, \mathring{S}_{a,\gamma}(I)) \leqslant C_r \omega_{r,p}(h; X; 1/2^\mu)_J$$

holds for $h \in W_p^0(J; X)$, $1 \leqslant p \leqslant \infty$, with $\operatorname{supp} h \subseteq I$, and for $2^\mu \geqslant r|J|^{-1}$.

This corollary follows immediately by Proposition 7.15 and Theorem 7.18.

8. Fundamental estimates for the spline systems. We should keep in mind that in the definition of $W_p^m(I; X)$ the space X in general depends on the exponent p. To avoid any confusion we shall use occasionally in what follows the symbol X_p for X. In the space $L_2(I; X_2)$ we have the natural scalar product

$$(f, g)_{L_2(I;X_2)} = \int_I (f(t), g(t))_{X_2} dt. \tag{8.1}$$

In what follows we assume that $I = \langle a, b\rangle$ and that the partition Π satisfies condition (7.9). Moreover, let $E = \{1-r, \ldots, N-1\}$. To each $e \subseteq E$ there corresponds the X-valued spline space

$$S_\Pi^r(I; X; e) = \Big\{u\colon\ u = \sum_{j \in e} x_j N_j^{(r)},\ x_j \in X\Big\}.$$

Clearly, $S_\Pi^r(I; X; e) \subseteq S_\Pi^r(I; X; E) = S_\Pi^r(I; X)$, and, by Corollary 7.10, for each $e \subseteq E$ it is a closed subspace in $L_p(I; X)$, $1 \leqslant p \leqslant \infty$. In particular there is a unique orthogonal with respect to (8.1) projection $P_\Pi^{(r)}(\cdot, X_2; e)$ of $L_2(I; X_2)$ onto $S_\Pi^r(I; X_2; e)$. As previously, in the case $X = \boldsymbol{R}$ we simply write $S_\Pi^r(I; e)$ for $S_\Pi^r(I; X; e)$. In the real-valued case there is a unique biorthogonal system $(\underline{N}_{j,e}^{(r)}, j \in e)$ in $S_\Pi^r(I, e)$, i.e. such that

$$(N_i^{(r)}, \underline{N}_{j,e}^{(r)}) = \delta_{i,j} \quad \text{for} \quad i, j \in e$$

and

$$S_\Pi^r(I; e) = \operatorname{span}[N_j^{(r)}, j \in e] = \operatorname{span}[\underline{N}_{j,e}^{(r)}, j \in e].$$

For the kernel of the orthogonal projection operator $P_\Pi^{(r)}(\cdot; e)$ from $L_2(I)$ onto $S_\Pi^r(I; e)$ we have the formulae

$$K_{\Pi,e}^{(r)} = \sum_{i\in e} N_i^{(r)} \otimes \underline{N}_{i,e}^{(r)} = \sum_{i,j\in e} a_{i,j}^e N_i^{(r)} \otimes N_j^{(r)}. \tag{8.2}$$

The matrix $A^e = (a_{i,j}^e)_{i,j\in e}$ is the inverse to the Gram matrix $G^e = \big((N_i^{(r)}, N_j^{(r)})_{L_2(I)}\big)_{i,j\in e}$. For $p = 2$ we check directly

$$P_\Pi^{(r)}(f; X_p; e)(t) = \int_I K_{\Pi,e}^{(r)}(t, s) f(s)\, ds, \tag{8.3}$$

where $f \in L_p(I; X_p)$. Since the kernel $K_{\Pi,e}^{(r)}$ is bounded, formula (8.3) makes sense for all p: $1 \leqslant p \leqslant \infty$. Thus (8.3) is the definition of a projection $P_\Pi^{(r)}(\cdot; X; e)$ from $L_p(I, X)$ onto $S_\Pi^r(I; X; e)$. Actually by interpolation (application of Hölder inequality) we get

$$\|P_\Pi^{(r)}(\cdot; X; e)\|_p(I; X) \leqslant \sup_{s\in I} \int_I |K_{\Pi,e}^{(r)}(s, t)|\, dt, \tag{8.4}$$

and it should be mentioned that the symmetry of the kernel $K_{\Pi,e}^{(r)}$ was used here. Now, properties (7.3), (7.4), formula (8.2) and inequality (8.4) give

$$\sup_{s\in I} \int_I |K_{\Pi,e}^{(r)}(s, t)|\, dt \leqslant r^{-1} \sup_{j\in e} \sum_{i\in e} |a_{ij}^e| (t_{i+r} - t_i). \tag{8.5}$$

PROPOSITION 8.6. *Let $r \geqslant 1$ and $I = \langle a, b\rangle$ be given, and let Π satisfy* (7.9). *Then there are constants $C_r < \infty$, $0 < q_r < 1$, such that*

$$|a_{ij}^e| \leqslant C_r (t_{i+r} - t_i)^{-1/2} (t_{j+r} - t_j)^{-1/2} q_r^{|i-j|}, \quad i, j \in e. \tag{8.7}$$

This result is due to de Boor [7], Corollary 2, p. 17. The version presented here follows from Corollary 7.10 with $p = 2$ and from a slightly modified result of Demko ([19], Theorem 2.2). The modification means simply that Demko's theorem can be extended by the same proof to the case of matrices with entries indexed by pairs of elements from a countable metric space. In our case the metric space is the set e with the natural *dist* induced from the real line.

Particular cases of Proposition 8.6 were known earlier (cf. [10] and [21]).

The inequalities (8.4), (8.5) and (8.7) give now

PROPOSITON 8.8. *Let $I = \langle a, b\rangle$, $r \geqslant 1$, and let Π satisfy* (7.9) *and* (7.11). *Then there is a constant C_r such that*

$$\|P_\Pi^{(r)}(\cdot; X; e)\|_p(I; X) \leqslant C_r < \infty$$

holds uniformly in p, $1 \leqslant p \leqslant \infty$.

In what follows we are going to specialize the partitions Π and the sets e. Thus we introduce the dyadic partitions Π_n, $n \geqslant 1$ of $I = \langle 0, 1\rangle$: $\Pi_n = \{s_{n,j}, j = 1-r, \ldots, n+r-1\}$ and $s_{n,1-r} = \ldots = s_{n,0} = 0$, $s_{n,n} = \ldots = s_{n,n+r-1} = 1$, and if $n = 2^\mu + \nu$, $\mu \geqslant 0$, $1 \leqslant \nu \leqslant 2^\mu$, then

$$s_{n,j} = \begin{cases} j/2^{\mu+1} & \text{for} \quad j = 0, \ldots, 2\nu, \\ (j-\nu)/2^\mu & \text{for} \quad j = 2\nu+1, \ldots, n. \end{cases}$$

Clearly each Π_n satisfies (7.9) with $N = n$, $a = 0$, $b = 1$, and (7.11) for $r = r'$.

The subsets $e \subseteq E = \{1-r, \ldots, n+r-1\}$ of particular interest to us are the following: (i) $e = E$ and (ii) $e = \{0, \ldots, n-1\}$. The two cases will be treated separately.

Case (i). Let us introduce the following notation for $n \geqslant 1$:

$$P_n^{(r)}(\cdot; X) = P_{\Pi_n}(\cdot; X; E), \qquad P_n^{(r)} = P_n^{(r)}(\cdot; \boldsymbol{R}),$$
$$S_n^r(I; X) = S_{\Pi_n}(I; X; E), \qquad S_n^r(I) = S_n^r(I; \boldsymbol{R}).$$

These definitions are extended to the indices $2-r \leqslant n \leqslant 0$ as follows: $P_n^{(r)}(\cdot; X)$ is the orthogonal projection of $L_2(I; X)$ onto

$$S_n^r(I; X) = \mathscr{P}_{n+r-1}(I; X),$$

where $\mathscr{P}_k(I; X)$ is the space of all polynomials $x_0 + tx_1 + \ldots + t^{k-1}x_{k-1}$ with $t \in I$ and $x_j \in X$. Thus, we have an increasing sequence of spaces

$$S_n^r(I; X) \subseteq S_{n+1}^r(I; X), \qquad n \geqslant 2-r$$

and the corresponding family of orthogonal projections $\{P_n^{(r)}(\cdot; X), n \geqslant 2-r\}$. Moreover, $\dim S_n^r(I) = n+r-1$. Let us now define an orthonormal system $(f_n^{(r)}, n \geqslant 2-r)$ in $L_2(I)$ as follows: $f_{2-r}^{(r)} = 1$, $f_n^{(r)} \in S_n^{(r)}(I)$ and $f_n^{(r)}$ is orthogonal in $L_2(I)$ to $S_{n-1}^{(r)}(I)$. If in addition to this we assume that $\|f_n^{(r)}\|_2(I) = 1$, then $f_n^{(r)}$ is unique up to a sign. It is clear that

$$P_n^{(r)}(f; X)(t) = \int_I K_n^{(r)}(s, t) f(s)\, ds,$$

$$K_n^{(r)}(s, t) = \sum_{j=2-r}^{n} f_j^{(r)}(s) f_j^{(r)}(t).$$

To construct suitable bases in Sobolev spaces we need to consider the differentiated and integrated system $(f_n^{(r)}, n \geqslant 2-r)$. For this purpose let

$$Hf(t) = \int_t^1 f(s)\, ds, \qquad Df(t) = \frac{d}{dt} f(t).$$

We now define for $-r \leqslant k < r$, $n \geqslant 2-r+|k|$

$$f_n^{(r,k)} = \begin{cases} D^k f_n^{(r)} & \text{for} \quad 0 \leqslant k < r, \\ H^{-k} f_n^{(r)} & \text{for} \quad -r \leqslant k < 0. \end{cases}$$

Integration by parts gives immediately for $|k| < r$

(8.9) $$(f_i^{(r,k)}, f_j^{(r,-k)})_{L_2(I)} = \delta_{i,j}, \quad i, j \geqslant 2-r+|k|.$$

Defining for $|k| < r$, $n \geqslant 2-r+|k|$

$$K_n^{(r,k)}(s,t) = \sum_{j=2-r+|k|}^{n} f_j^{(r,-k)}(s) f_j^{(r,k)}(t),$$

we check using (8.9) that

$$P_n^{(r,k)}(f; X)(t) = \int_I K_n^{(r,k)}(s,t) f(s)\, ds$$

is a projection in $L_p(I; X)$.

Proposition 8.10. *Let $r \geqslant 1$ be given. Then there are constants $C_r < \infty$ and q_r, $0 < q_r < 1$, such that*

(8.11) $$|K_n^{(r,k)}(s,t)| \leqslant C_r n q_r^{n|s-t|} \quad s, t \in I,$$

holds for $n \geqslant 2-r+|k|$ with $|k| < r$.

Moreover, for $t \in I$, $-r \leqslant k < r$ and $n \geqslant 2-r+|k|$ we have

(8.12) $$|f_n^{(r,k)}(t)| \leqslant C_r n^{k+1/2} q_r^{n|t-t_n|},$$

where $t_n = s_{n,2\nu-1} = (2\nu-1)/2^{\mu+1}$ if $n = 2^\mu+\nu$, $1 \leqslant \nu \leqslant 2^\mu$.

The proof of (8.11) is given in [14], and for the proof of (8.12) we refer to [15] and for a much simpler proof to [12].

Case (ii). Now we do not have ready results. The basis in question has to be constructed. We apply the same technique starting with orthogonal projections. Referring to the notation introduced earlier we recall that now $E = \{1-r, \ldots, n-1\}$, $e = \{0, \ldots, n-1\}$, $N = n$ and $\Pi = \Pi_n$. In analogy to the case (i) we introduce for $n \geqslant 1$ the notation

$$Q_n^{(r)}(\cdot; X) = P_{\Pi_n}(\cdot; X; e), \qquad Q_n^{(r)} = Q_n^{(r)}(\cdot; \boldsymbol{R}),$$

$$S_n^r(I; X; e) = S_{\Pi_n}^r(I; X; e), \qquad S_n^r(I; e) = S_n^r(I; \boldsymbol{R}; e).$$

Now $\dim S_n^r(I; e) = n$ and, clearly,

$$S_n^r(I; X; e) \subset S_{n+1}^r(I; X; e) \quad \text{for} \quad n \geqslant 1$$

and $\{Q_n^{(r)}(\cdot; X), n \geqslant 1\}$ is the corresponding family of orthogonal projections. We define the new orthonormal system as follows: $g_1^{(r)} = N_{1,0}^{(r)}/\|N_{1,0}^{(r)}\|_2(I)$, $g_n^{(r)} \in S_n^r(I; e)$ and $g_n^{(r)}$ is orthogonal in $L^2(I)$ to

$S_{n-1}^r(I; e)$, $\|g_n^{(r)}\|_2(I) = 1$. Again,

$$Q_n^{(r)}(f; X)(t) = \int_I L_n^{(r)}(s,t) f(s)\, ds,$$

where

$$L_n^{(r)}(s,t) = \sum_{j=1}^{n} g_j^{(r)}(s) g_j^{(r)}(t).$$

Moreover, we introduce for $n \geqslant 1$, $-r \leqslant k < r$,

$$g_n^{(r,k)} = \begin{cases} D^k g_n^{(r)} & \text{for} \quad 0 \leqslant k < r \\ H^{-k} g_n^{(r)} & \text{for} \quad -r \leqslant k < 0 \end{cases}$$

It then follows that for $|k| < r$

$$(g_i^{(r,k)}, g_j^{(r,-k)}) = \delta_{i,j}, \quad i, j \geqslant 1.$$

Thus the operators

$$Q_n^{(r,k)}(f; X)(t) = \int_I L_n^{(r,k)}(s,t) f(s)\, ds,$$

with

$$L_n^{(r,k)}(s,t) = \sum_{j=1}^{n} g_j^{(r,-k)}(s) g_j^{(r,k)}(t),$$

are projections for $|k| < r$.

Lemma 8.13. *There are constants $C_r < \infty$ and q_r, $0 < q_r < 1$, such that for $|k| < r$ and $n \geqslant 1$*

(8.14) $$|L_n^{(r,k)}(s,t)| \leqslant C_r n q_r^{n|s-t|}, \quad s, t \in I.$$

In particular the operators $Q_n^{(r,k)}$: $L_p(I) \to L_p(I)$ are bounded uniformly in n and p, $1 \leqslant p \leqslant \infty$.

Proof. It is sufficient to prove (8.14) for $0 \leqslant k < r$. The proof goes by induction in k. For $k = 0$ (8.14) follows by Proposition 8.6. Suppose now that (8.14) holds for some k, $0 \leqslant k < r-1$. Now, for $f \in S_n^r(I; e)$ by the Markov inequality for algebraic polynomials we obtain

(8.15) $$\left|\frac{f(t)-f(s)}{t-s}\right| \leqslant C_r n \frac{1}{|I_{n,j}|} \int_{I_{n,j}} |f|, \quad s, t \in I_{n,j},$$

where $I_{n,j} = (s_{n,j-1}, s_{n,j})$ and C_r is a constant depending on r only. In particular (8.15) implies

(8.16) $$|Df(s)| \leqslant C_r n \frac{1}{|I_{n,j}|} \int_{I_{n,j}} |f|, \quad s \in I_{n,j}.$$

Applying (8.16) to

$$f(s) = L_n^{(r,k)}(s, t),$$

and then using (8.14) we obtain

$$(8.17)\qquad |D_s L_n^{(r,k)}(s, t)| \leqslant C_r n^2 q_r^{n|t-s|}.$$

If now $t > s$, then (8.17) gives

$$|L_n^{(r,k+1)}(s, t)| = \Big|\int_t^1 D_s L_n^{(r,k)}(s, u)\,du\Big| \leqslant C_r n q_r^{n|t-s|}.$$

However for $t < s$ we have by (8.17)

$$|L_n^{(r,k+1)}(s, t)| = \Big|\int_0^t D_s L_n^{(r,k)}(s, u)\,du - \int_0^1 D_s L_n^{(r,k)}(s, u)\,du\Big|$$
$$\leqslant \int_0^t |D_s L_r^{(r,k)}(s, u)|\,du + |D_s(L_n^{(r,k)}1)(s)|$$
$$\leqslant C_r n q_r^{n|t-s|} + |D_s(L_n^{(r,k)}1)(s)|.$$

Since for $0 \leqslant t \leqslant s$ we have $q_r^{ns} \leqslant q_r^{n(s-t)}$, it is therefore sufficient to prove

$$(8.18)\qquad |D(L_n^{(r,k)}1)(s)| \leqslant C_r n q_r^{ns}.$$

In what follows we denote by $N_{n,j}^{(r)}$ the jth B-spline corresponding to the dyadic partition Π_n. The biorthogonal functions in $S_n^r(I; e)$ are denoted as $\underline{N}_{n,i,e}^{(r)}$, $i \in e = \{0, \ldots, n-1\}$. It is convenient to introduce also the operator

$$Gf(t) = \int_0^t f(s)\,ds.$$

Now,

$$(8.19)\qquad L_n^{(r,k)}1 = D^k \sum_{j=1}^{n} (1, H^k g_j^{(r)})_{L^2(I)} g_j^{(r)}$$
$$= D^k \sum_{j=1}^{n} (G^k 1, g_j^{(r)})_{L^2(I)} g_j^{(r)}$$
$$= D^k \sum_{j=0}^{n-1} (G^k 1, \underline{N}_{n,j,e}^{(r)}) N_{n,j}^{(r)}.$$

Moreover, let

$$(8.20)\qquad G^k 1 = \sum_{j=1-r}^{n-1} b_j N_{n,j}^{(r)}.$$

Since $(G^k 1)(s) = s^k/k!$ and $G^k 1 \in S_n^r(I)$, it follows that the b_j's are uniquely determined. For later convenience we introduce

$$g = \sum_{j=1-r}^{-1} b_j N_{n,j}^{(r)}.$$

According to (8.20) we have

$$(8.21)\qquad G^k 1 - g = \sum_{j=0}^{n-1} b_j N_{n,j}^{(r)} \in S_n^r(I; e),$$

and therefore for $j \in e$

$$(8.22)\qquad b_j = (G^k 1, \underline{N}_{n,j,e}^{(r)}) - (g, \underline{N}_{n,j,e}^{(r)}).$$

Using (8.22) and (8.19) we get

$$(8.23)\qquad DL_n^{(r,k)}1 = D^{k+1}\Big(\sum_{j=0}^{n-1} b_j N_{n,j}^{(r)} + \sum_{j=0}^{n-1} (g, \underline{N}_{n,j,e}^{(r)}) N_{n,j}^{(r)}\Big)$$
$$= D^{k+1}\Big(\sum_{j=0}^{n-1} (g, \underline{N}_{n,j,e}^{(r)}) N_{n,j}^{(r)} - g\Big).$$

Notice that $\operatorname{supp} g \subseteq \langle 0, s_{n,r-1}\rangle$.

Moreover, we are going to show that

$$(8.24)\qquad \|g\|_\infty(0, s_{n,r-1}) = \|g\|_\infty(I) \leqslant C_r n^{-k}, \qquad n \geqslant 1.$$

In order to see this we define $\Pi' = \{s'_{n,j}, j = 0, \pm 1, \pm 2, \ldots\}$ as follows

$$s'_{n,j} = \begin{cases} s_{n,j} & \text{for } j \leqslant r-1, \\ s_{n,r-1} & \text{for } j = r, \ldots, 2r-2, \\ \text{arbitrary increasing} & \text{for } j \geqslant 2r-1. \end{cases}$$

The corresponding B-splines are denoted by N'_j. Clearly, $N_{n,j}^{(r)} = N'_j$ for $j \leqslant -1$, and for $0 < s < s_{n,r-1}$ we have

$$(G^k 1)(s) = \sum_{j=1-r}^{r-2} b'_j N'_j.$$

An application of Corollary 7.10 gives

$$n^{-k} \sim \|G^k 1\|_\infty(0, s_{n,r-1}) \sim \max_{1-r \leqslant j < r-1} |b'_j| \geqslant \max_{1-r \leqslant j < 0} |b'_j| \geqslant \Big\|\sum_{j=1-r}^{-1} b'_j N_{n,j}^{(r)}\Big\|_\infty (0, s_{n,r-1}).$$

However, on $(0, s_{n,r-1})$ we have, by (8.20),

$$G^k 1 = \sum_{j=1-r}^{r-2} b_j N_{n,j}^{(r)}$$

and, moreover,

$$G^k 1 = \sum_{j=1-r}^{-1} b'_j N'_j + \sum_{j=0}^{r-2} b'_j N'_j.$$

The properties of the B-splines imply now $b_j = b'_j$ for $j = 1-r, \ldots$ $\ldots, -1$, whence we infer (8.24). Using (8.24) we shall estimate the right-hand side of (8.23). It follows by the definition that

$$\underline{N}^{(r)}_{n,j,e} = \sum_{i\in e} a^e_{n,i,j} N^{(r)}_{n,i}. \tag{8.25}$$

Proposition 8.6 implies inequality

$$|a^e_{n,i,j}| \leqslant C_r n q_r^{|i-j|}, \quad i, j \in e, \tag{8.26}$$

which together with (8.24) gives

$$|(g, \underline{N}^{(r)}_{n,j,e})| \leqslant C_r n^{-k} q_r^j, \quad j \in e.$$

This and the properties of the B-splines imply

$$\Big| D^{k+1} \sum_{j=0}^{n-1} (g, \underline{N}^{(r)}_{n,j,e}) N^{(r)}_{n,j}(s) \Big| \leqslant C_r n q_r^{ns}.$$

Finally, g by definition is a spline with support in $\langle 0, s_{n,r-1}\rangle$, whence, by (7.14),

$$|D^{k+1} g(s)| \leqslant \begin{cases} 0 & \text{for} \quad s_{n,r-1} < s \leqslant 1, \\ C_r n & \text{for} \quad 0 \leqslant s \leqslant s_{n,r-1} \end{cases}$$

and this gives

$$|D^{k+1} g(s)| \leqslant C_r n q_r^{ns}, \quad s \in I.$$

The combination of these inequalities and (8.23) give (8.18), and the proof is complete.

Lemma 8.27. *Let $r \geqslant 2$ be given and let $t_n = s_{n,2\nu-1}$ for $n = 2^\mu + \nu$, $1 \leqslant \nu \leqslant 2^\mu$. Then there exists $C_r < \infty$ such that*

$$|g_n^{(r,k)}(t)| \leqslant C_r n^{1/2+k} q_r^{n|t-t_n|} \tag{8.28}$$

holds for $-r \leqslant k < r$, $t \in I$, $n > 1$.

Proof. The function $g_n^{(r)}$ is orthogonal to $S^r_{n-1}(I; e)$ and therefore

$$g_n^{(r)} = \sum_{j\in e} (g_n^{(r)}, N^{(r)}_{n,j}) \underline{N}^{(r)}_{n,j,e} = \sum_{j=2\nu-1-r}^{2\nu-1} (g_n^{(r)}, N^{(r)}_{n,j}) \underline{N}^{(r)}_{n,j,e},$$

where $e = \{0, \ldots, n-1\}$. However, (7.3) and (7.6) give

$$|(g_n^{(r)}, N^{(r)}_{n,j})| \leqslant \|N^{(r)}_{n,j}\|_2 \leqslant C_r n^{-1/2}.$$

Thus,

$$|g_n^{(r)}(t)| \leqslant C_r n^{-1/2} \max_{2\nu-1-r \leqslant j \leqslant 2\nu-1} |\underline{N}^{(r)}_{n,j,e}(t)|.$$

Now, (8.25) and (8.26) imply

$$|\underline{N}^{(r)}_{n,j,e}(t)| \leqslant C_r n q_r^{n|t-t_n|}, \quad j \in e, \ t \in I, \tag{8.29}$$

and this proves

$$|g_n^{(r)}(t)| \leqslant C_r n^{1/2} q_r^{n|t-t_n|}. \tag{8.30}$$

Inequality (8.28) now follows for $0 \leqslant k < r$ from (8.30) by (8.16).

In the proof of the remaining case of (8.28) we use the representation $(0 < -k \leqslant r)$

$$(u-t)^{-k-1} = \sum_{j=1-r}^{n-2} b_j(t) N^{(r)}_{n-1,j}(u).$$

It now follows by the very definition of $g_n^{(r)}$ that

$$\int_I (u-t)^{-k-1} g_n^{(r)}(u)\,du = \sum_{j=1-r}^{-1} b_j(t) (N^{(r)}_{n-1,j}, g_n^{(r)})$$

whence, by (8.30), we obtain

$$\Big| \int_I (u-t)^{-k-1} g_n^{(r)}(u)\,du \Big| \leqslant C_r n^{-1/2} q_r^r \sum_{j=1-r}^{-1} |b_j(t)|.$$

To estimate the right-hand side we introduce a new partition $\Pi' = (s'_{n-1,j}, j = 0, \pm 1, \ldots)$ as follows

$$s'_{n-1,j} = \begin{cases} s_{n-1,j} & \text{for} \quad j < r \\ s_{n-1,r-1} & \text{for} \quad j = r, \ldots, 2r-2 \\ \text{arbitrary increasing} & \text{for} \quad j \geqslant 2r-1. \end{cases}$$

Moreover, for $0 < u < s_{n-1,r-1}$ let

$$(u-t)_+^{-k-1} = \sum_{j=1-r}^{r-2} b'_j(t) N^{(r)\prime}_{n-1,j}(u),$$

where $N^{(r)\prime}_{n-1,j}$ are the B-splines of order r corresponding to Π'.

It now follows by Corollary 7.10 that uniformly in $t \in I$ and $n \geqslant 1$

$$\int_0^{s_{n-1,r-1}} |(u-t)^{-k-1}|\,du \sim n^{-1} \sum_{j=1-r}^{r-2} |b'_j(t)|.$$

However, $N^{(r)\prime}_{n-1,j} = N^{(r)}_{n-1,j}$ for $j = 1-r, \ldots, -1$, and comparing for $0 < u < s_{n-1,r-1}$ the two representations for $(u-t)^{-k-1}$ we find that $b_j(t) = b'_j(t)$ for $j = 1-r, \ldots, -1$. Consequently,

$$\sum_{j=1-r}^{-1} |b_j(t)| \leqslant \sum_{j=1-r}^{r-2} |b'_j(t)| \leqslant C_r (t + n^{-1})^{-k-1},$$

and

$$(8.31)\qquad \Big|\int_I (u-t)^{-k-1} g_n^{(r)}(u)\,du\Big| \leqslant C_r n^{-1/2} q_s^{\nu}(t+n^{-1})^{-k-1}.$$

If now $t > t_n$, then (8.30) implies (8.28) for $-r \leqslant k < 0$. On the other hand, if $0 < t < t_n = s_{n,2\nu-1}$, then by (8.30) and (8.31)

$$|g_n^{(r,k)}(t)| = \Big|\frac{1}{(-k-1)!}\int_t^1 (u-t)^{-k-1} g_n^{(r)}(u)\,du\Big|$$
$$\leqslant C_r\Big|\int_0^t (u-t)^{-k-1} g_n^{(r)}(u)\,du\Big| + C_r\Big|\int_I (u-t)^{-k-1} g_n^{(r)}(u)\,du\Big|$$
$$\leqslant C_r\big(n^{1/2+k} q_r^{n|t-t_n|} + n^{-1/2}(t+n^{-1})^{-k-1} q_r^{\nu}\big)$$
$$\leqslant C_r n^{1/2+k} \tilde{q}_r^{n|t-t_n|}, \qquad 0 < q_r < \tilde{q}_r < 1$$

and this completes the proof.

LEMMA 8.32 (Jackson type inequality). *Let $-r < k < r-1$, $1 \leqslant p \leqslant \infty$. Then there is a constant C_r such that*

$$(8.33)\qquad \|f - Q_n^{(r,k)} f\|_p \leqslant C_r n^{-1} \|D(f - Q_n^{(r,k)} f)\|_p, \qquad n \geqslant 1,$$

holds for $f \in W_1^1(I)$ with $f(0) = 0$ if $k \geqslant 0$, and for $f \in W_1^1(I)$ with $f(1) = 0$ if $k < 0$.

Proof. The proof is based on the idea of G. Freud and V. Popov in [23]. For the proof let

$$S_n^{(r,k)} = \operatorname{span}[g_j^{(r,k)}, j = 1, \ldots, n].$$

It follows, by Lemma 8.13, that with some $C_r < \infty$

$$\|f - Q_n^{(r,k)} f\|_p \leqslant C_r \|f - h\|_p, \qquad n \geqslant 1,$$

holds for $f \in L_p(I)$ and $h \in S_n^{(r,k)}$.

In order to get (8.33) it is important to construct a suitable h. This is done below. We let

$$g = D(f - Q_n^{(r,k)} f), \qquad a_j = \int_{I_{n,j}} f, \qquad I_{n,j} = \langle s_{n,j}, s_{n,j+1}\rangle,$$
$$b_j = M_{n,j}^{(r-k-1)}, \qquad G = \int_0^t, \qquad H = \int_t^1,$$

where $M_{n,j}^{(r)}$ is the, corresponding to the nth partition, B-spline normalized in L_1 (cf. (7.5)).

Case: $k \geqslant 0$. We modify the nth dyadic partition assuming that it has multiplicity $r-k+1$ at 1. It should be clear that

$$S_n^{(r,k)} = \operatorname{span}[Gb_j, j = 0, \ldots, n-1].$$

We now define

$$h = Q_n^{(r,k)} f + \sum_{j=0}^{n-1} a_j G b_j.$$

Case: $k < 0$. It is assumed in this case that the nth dyadic partition has multiplicity r at 1. It then follows that

$$Hb_j \in S_n^{(r,k)}(I) \quad \text{for} \quad j = 0, \ldots, n+k+1,$$

and we define

$$h = Q_n^{(r,k)} f - \sum_{j=0}^{n+k} a_j H b_j - \Big(\sum_{j=n+k+1}^{n-1} a_j\Big) H b_{n+k+1}.$$

Having the proper h in both cases we complete the proof by applying an argument similar to that in the proof of Lemma 4.8 in [17].

In what follows we denote by $(h_n^{(r,k)}, n \geqslant n_k)$ either $(f_n^{(r,k)}, n \geqslant 2-r+|k|)$ or $(g_n^{(r,k)}, n \geqslant 1)$. Thus in the first case $n_k = 2-r+|k|$ and in the second case $n_k = 1$. For $|k| < r$ we introduce the operators

$$H_n^{(r,k)} f(t) = \int_I M_n^{(r,k)}(s,t) f(s)\,ds$$

with

$$M_n^{(r,k)}(s,t) = \sum_{j=n_k}^{n} h_j^{(r,-k)}(s)\, h_j^{(r,k)}(t).$$

To each system $(h_n^{(r,k)}, n \geqslant n_k)$ we assign an interval J and an integer m as follows (cf. (2.38)):

If $(h_n^{(r,k)}) = (f_n^{(r,k)})$, then $J = I = \langle 0, 1\rangle$ and $m = r-k$ for $k \geqslant 0$; $J = \langle -1, 2\rangle$ and $m = -k$ for $k < 0$.

If $(h_n^{(r,k)}) = (g_n^{(r,k)})$, then $J = \langle -1, 1\rangle$ and $m = r-k$ for $k \geqslant 0$; $J = \langle 0, 2\rangle$ and $m = -k$ for $k < 0$.

If f is a function on I we let f_J denote the extension of f to J which vanishes on $J \setminus I$.

We are now in a position to formulate the basic results on the $(h_n^{(r,k)})$ systems.

PROPOSITION 8.34. *Let $-r < k < r-1$. Suppose that $f \in W_1^1(I)$ and $f_J \in W_1^1(J)$. Then, for $n \geqslant \max(n_k, n_{k+1})$, one has*

$$DH_n^{(r,k)} f = H_n^{(r,k+1)} Df.$$

Moreover, for $|k| < r$ we have

$$(h_i^{(r,-k)}, h_j^{(r,k)}) = \delta_{i,j}, \quad i, j \geqslant n_k.$$

A direct proof of these formulas is omitted.

PROPOSITION 8.35. *Let $|k| < r$. Then there are constants $C_r < \infty$, $0 < q_r < 1$, depending on r only such that*

$$|M_n^{(r,k)}(s,t)| \leqslant C_r n q_r^{n|s-t|}, \quad n \geqslant 1,\ s, t \in I.$$

In particular, the projections

$$H_n^{(r,k)}: L_p(I) \to L_p(I), \quad 1 \leqslant p \leqslant \infty,$$

are bounded uniformly in n and p.

This proposition follows from (8.11) and (8.14).

PROPOSITION 8.36 (Bernstein and Jackson type inequalities). *Let $-r < k < r-1$, $1 \leqslant p \leqslant \infty$. Then, for $f \in L_1(I)$ we have*

$$\|DH_n^{(r,k)}f\|_p(I) \leqslant C_r n \|H_n^{(r,k)}f\|_p(I), \quad n \geqslant 1. \tag{8.37}$$

Moreover, for $f \in W_p^1(I)$ with $f_J \in W_p^1(J)$, we get

$$\|f - H_n^{(r,k)}f\|_p(I) \leqslant C_r n^{-1} \|D(f - H_n^{(r,k)}f)\|_p(I), \quad n \geqslant 1. \tag{8.38}$$

In both inequalities C_r depends on r only.

Inequality (8.37) follows from (7.14); (8.38) is a consequence of Lemma 8.32 and of Lemma 4.8 in [17].

PROPOSITION 8.39. *Let $-r \leqslant k < r$. There are constants C_r, $0 < q_r < 1$, such that*

$$|h_n^{(r,k)}(t)| \leqslant C_r n^{1/2+k} q_r^{n|t-t_n|}, \quad n \geqslant 1, \quad t \in I$$

holds.

This proposition follows by (8.28) and (8.12).

Applying similar arguments as in the proof of Theorem 7.1 in [12] and using the last proposition we obtain:

PROPOSITION 8.40. *Let $r \geqslant 1$ be given. There is $C_r < \infty$ such that, if $1 \leqslant p \leqslant \infty$, $N = 2^\mu$, $\mu = 0, 1, \ldots, a \in l_p^N$, then*

$$\Big\| \sum_{N+1}^{2N} |a_n h_n^{(r,k)}| \Big\|_p (I) \leqslant C_r N^{1/2-1/p+k} \|a\|_{l_p^N}$$

holds for $-r \leqslant k < r$ and

$$N^{1/2-1/p+k} \|a\|_{l_p^N} \leqslant C_r \Big\| \sum_{N+1}^{2N} a_n h_n^{(r,k)} \Big\|_p (I)$$

holds for $|k| < r$.

LEMMA 8.41. *Let $|k| < r$ and let J, m be defined as before. Then for some $C_r < \infty$*

$$\|f - H_n^{(r,k)}f\|_p(I) \leqslant C_r \omega_{m,p}(f_J; 1/n)_J \tag{8.42}$$

holds for $f \in L^p(I)$, $1 \leqslant p \leqslant \infty$, $n \geqslant 1$.

Proof. It follows by Proposition 8.35 that it is sufficient to prove (8.42) for $n = 2^\mu$, $\mu \geqslant 0$. In the rest of the proof we assume therefore that n is of such form. We now distinguish four cases. At first let $(h_n^{(r,k)}) = (f_n^{(r,k)})$ and let $0 \leqslant k < r$. In this case inequality (8.42) follows by Theorem 4.1 of [12]. In case $-r < k < 0$ we define

$$\mathring{S}_{\alpha,\gamma}(I) = \operatorname{span}[(h_j^{(r,k)})_J, j = n_k, \ldots, n],$$

where the order of the splines is $r-k$ and the multiplicities at 0, 1 are $\alpha = r$, $\beta = r$, respectively. To obtain (8.42) it is now sufficient to apply Corollary 7.20 and Proposition 8.35. The last proposition is used to pass from the best approximation to $\|f - H_n^{(r,k)}f\|_p(I)$.

In the case $(h_n^{(r,k)}) = (g_n^{(r,k)})$ and $0 \leqslant k < r$ the argument is similar and it is omitted. It remains to prove (8.42) in the case of the system $(g_n^{(r,k)})$ for $-r < k < 0$. To this case there corresponds $J = \langle 0, 2\rangle$, $I = \langle 0, 1\rangle$. For the given $f \in L_p(I)$ we use $\tilde{f}$ to define the Steklov mean; here $\tilde{f}$ is the extension by zero of f to $\langle 0, \infty)$. The definition is as follows: for $h > 0$, $t \in J$

$$g(t) = -\sum_{j=1}^{m} (-1)^{m+j} \binom{m}{j} \int_I \ldots \int_I \tilde{f}\big(t + jh(s_1 + \ldots + s_m)\big) ds_1 \ldots ds_m.$$

It then follows that $g \in W_p^m(J)$, $g|_{J \setminus I} = 0$ and that for the restriction g_0 of g to I we have

$$\begin{aligned} &\|f - g_0\|_p(I) \leqslant C_r \omega_{m,p}(f_J; h)_J, \\ &h^m \|D^m g_0\|_p(I) \leqslant C_r \omega_{m,p}(f_J; h)_J. \end{aligned} \tag{8.43}$$

On the other hand Proposition 8.35 gives

$$\|f - H_n^{(r,k)}f\|_p(I) \leqslant C_r \big(\|f - g_0\|_p(I) + \|g_0 - H_n^{(r,k)}g_0\|_p(I)\big),$$

and Propositions 8.34 and 8.36 imply

$$\|g_0 - H_n^{(r,k)}g_0\|_p(I) \leqslant C_r n^{-m} \|g_1 - H_n^{(r,0)}g_1\|_p(I)$$

with $g_1 = D^m g_0$, $m = -k$. However, Proposition 8.35 implies

$$\|g_1 - H_n^{(r,0)}g_1\|_p(I) \leqslant C_r \|D^m g_0\|_p(I).$$

Thus, the combination of these inequalities with (8.43), $h = n^{-1}$, gives (8.42), and this completes the proof.

Remark 8.44. It should be clear that all the results on the operators $H_n^{(r,k)}$, $|k| < r$, can be extended directly to the X_p-valued case.

The last lemma of this section is needed in the next section in the proof of the unconditionality of the systems $(h_n^{(r,k)})$, $|k|<r$. The next lemma is preliminary to the last one. We need some more notation. For a linear operator $T\colon L_2\to L_2$ we denote by T^* its Hilbert space adjoint. For $-r\leqslant k<r$ and $f\in C(I)$ we define

$$\Delta_\mu^{(r,k)}f=\sum_{N+1}^{2N}\Big(\int_I f\,dh_n^{(r,-k-1)}\Big)h_n^{(r,k)},\qquad N=2^\mu.$$

In particular, for $|k|<r$ we have the relation

$$\Delta_\mu^{(r,k)}=H_{2N}^{(r,k)}-H_N^{(r,k)}.$$

LEMMA 8.45. *Let* $-r\leqslant k<r$, $N=2^\mu$, $\mu=0,1,\ldots$ *Then for some* $C_r<\infty$,

$$\|\Delta_\mu^{(r,k)}1\|_2(I)\leqslant C_rN^{-1/2}.$$

Proof. Notice that $-r\leqslant k<r$ implies $-r\leqslant -k-1<r$. Thus by Proposition 8.40

$$\|\Delta_\mu^{(r,k)}1\|_2=\Big\|\sum_{N+1}^{2N}\big(h_n^{(r,-k-1)}(1_-)-h_n^{(r,-k-1)}(0_+)\big)h_n^{(r,k)}\Big\|_2$$
$$\leqslant 2\big(\sup_{N<n\leqslant 2N}\|h_n^{(r,k)}\|_2\big)\Big\|\sum_{N+1}^{2N}|h_n^{(r,-k-1)}|\Big\|_\infty$$
$$\leqslant C_rN^kN^{1/2-k-1}=C_rN^{-1/2}.$$

LEMMA 8.46. *Let* $|k|<r$. *Then* $(\Delta_\mu^{(r,k)})^*=\Delta_\mu^{(r,-k)}$ *and for some* $C_r<\infty$ *we have for* $f\in L_2(I)$, $\mu\geqslant 0$, $\nu\geqslant 0$

$$\|\Delta_\mu^{(r,k)}\Delta_\nu^{(r,-k)}f\|_2(I)\leqslant C_r2^{-|\mu-\nu|/2}\|f\|_2(I).$$

Proof. The case $k=0$ is trivial since $(\Delta_\mu^{(r,0)},\ \mu=0,1,\ldots)$ is an orthogonal family of orthogonal projections in $L_2(I)$. For later use let $N=2^\nu$, $M=2^\mu$ and $g=\Delta_\nu^{(r,-k)}f$. By duality we reduce the proof to the case where $\mu\leqslant\nu$ $(M\leqslant N)$, and this inequality is assumed whenever $-r+1<k<r$. Each of the following three cases will be treated separately: $0<k<r$, $-r+1<k<0$ and $k=-r+1$.

Case: $0<k<r$. Using Bernstein's inequality (7.14) and Proposition 8.34 we obtain

$$\|\Delta_\mu^{(r,k)}\Delta_\nu^{(r,-k)}f\|_2=\|\Delta_\mu^{(r,k)}g\|_2$$
$$=\|D\Delta_\mu^{(r,k-1)}Gg\|_2\leqslant C_rM\|\Delta_\mu^{(r,k-1)}Gg\|_2$$
$$\leqslant C_rM\big\{\big\|\Delta_\mu^{(r,k-1)}\big(Gg-(g,1)\big)\big\|_2+|(g,1)|\,\|\Delta_\mu^{(r,k-1)}1\|_2\big\}.$$

Applying now Lemma 8.45 we get

$$|(g,1)|\,\|\Delta_\mu^{(r,k-1)}1\|_2=|(\Delta_\nu^{(r,-k)}f,1)|\,O(M^{-1/2})$$
$$=|(f,\Delta_\nu^{(r,k)}1)|\,O(M^{-1/2})=\|f\|_2O\big((MN)^{-1/2}\big).$$

Moreover, Proposition 8.35 implies

$$\big\|\Delta_\mu^{(r,k-1)}\big(Gg-(g,1)\big)\big\|_2\leqslant C_r\|Gg-(g,1)\|_2.$$

However,

$$(g,1)-Gg=\sum_{N+1}^{2N}(f,h_n^{(r,k)})h_n^{(r,-k-1)},$$

and therefore by Propositions 8.40 and 8.35

$$\|Gg-(g,1)\|_2\leqslant C_rN^{-1}\|\Delta_\nu^{(r,-k)}f\|_2=O(N^{-1})\|f\|_2.$$

Combination of all these inequalities completes the proof.

Case: $-r+1<k<0$. The Bernstein's inequality implies

$$\|\Delta_\mu^{(r,k)}\Delta_\nu^{(r,-k)}f\|_2=\|DH\Delta_\mu^{(r,k)}g\|_2\leqslant C_rM\|H\Delta_\mu^{(r,k)}g\|_2. \tag{8.47}$$

For the latter function we write the following identity

$$H\Delta_\mu^{(r,k)}g=\Delta_\mu^{(r,k-1)}Hg+(g,1)F_\mu \tag{8.48}$$
$$=\Delta_\mu^{(r,k-1)}(Hg-\lambda)+(g,1)F_\mu+\lambda\Delta_\mu^{(r,k-1)}1,$$

where

$$F_\mu=\sum_{M+1}^{2M}h_n^{(r,-k)}(0)h_n^{(r,k-1)},$$
$$\lambda=\sum_{N+1}^{2N}(f,h_n^{(r,k)})h_n^{(r,-k-1)}(1).$$

Now,

$$\lambda-Hg=\sum_{N+1}^{2N}(f,h_n^{(r,k)})h_n^{(r,-k-1)}$$

and therefore by Propositions 8.35 and 8.40

$$\|\Delta_\mu^{(r,k-1)}(Hg-\lambda)\|_2\leqslant C_rN^{-1}\|f\|_2. \tag{8.49}$$

Next, Lemma 8.45 gives

$$|(g,1)|=|(f,\Delta_\nu^{(r,k)}1)|\leqslant C_rN^{-1/2}\|f\|_2, \tag{8.50}$$

and Proposition 8.40 implies

$$\|F_\mu\|_2\leqslant C_rM^{k-1}\sum_{M+1}^{2M}|h_n^{(r,-k)}(0)|\leqslant C_rM^{-1/2}, \tag{8.51}$$

and

(8.52) $$|\lambda| \leqslant C_r N^{-1/2}\|f\|_2.$$

Moreover, Lemma 8.45 gives

(8.53) $$\|\Delta_\mu^{(r,k-1)}1\|_2 \leqslant C_r M^{-1/2}.$$

Combining (8.47)–(8.53) we obtain the desired inequality.

Case: $k = -r+1$. Let $m = -k = r-1$, $h = \Delta_\nu^{(r,r-1)}f$ and let h_J be its extension by zero from $I = \langle 0, 1\rangle$ to $J = \langle 0, 2\rangle$. The functions h and h_J are splines of order 1 corresponding to the uniform dyadic partition with step $(2N)^{-1}$. By a duality argument it is sufficient to prove our inequality for $\mu \geqslant \nu$ $(M \geqslant N)$. Now, by Lemma 8.41,

$$\begin{aligned}\|\Delta_\mu^{(r,-r+1)}\Delta_\nu^{(r,r-1)}f\|_2 &= \|\Delta_\mu^{(r,-r+1)}h\|_2\\ &\leqslant C_r\omega_{r-1,2}(h_J; M^{-1})_J \leqslant C_r\omega_{1,2}(h_J; M^{-1})_J,\end{aligned}$$

and the L_p improved version of (7.14) (cf. [12], Lemma 9.3) gives

$$\omega_{1,2}(h_J; M^{-1})_J \leqslant C_r N^{1/2}M^{-1/2}\|h\|_2 \leqslant C_r N^{1/2}M^{-1/2}\|f\|_2.$$

This completes the proof in the last case.

9. Spline bases in Sobolev spaces with boundary conditions on cubes. For fixed integer $d \geqslant 1$ let $Q = I^d$, $I = \langle 0, 1\rangle$, and let Z, Z' be two complementary boundary sets as defined in Section 2 (cf. (2.37) and (2.47)).

Our aim in this section is to construct Schauder bases in the Sobolev spaces $W_p^m(Q)_Z$, $m \geqslant 0$, $1 \leqslant p \leqslant \infty$. The bases we are constructing are tensor products of one-dimensional bases. Therefore we start with the definition of the one-dimensional bases.

DEFINITION 9.1. Let the integers $m = r-2$, $r \geqslant 2$, and the set $Z_0 \subseteq \partial I$ be given. Moreover, let $n(Z_0) = 1$ for $Z_0 = \{0\}, \{1\}$ and $n(Z_0) = 2-r$ for $Z_0 = \emptyset, \{0, 1\}$. The basic functions are now defined for $n \geqslant n(Z_0)$ as follows

$$F_n^{(m)}(t; Z_0) = \begin{cases} f_n^{(2r,r)}(t) & \text{if} \quad Z_0 = \emptyset,\\ f_n^{(2r,-r)}(t) & \text{if} \quad Z_0 = \{0, 1\},\\ g_n^{(2r,r)}(t) & \text{if} \quad Z_0 = \{0\},\\ g_n^{(2r,-r)}(t) & \text{if} \quad Z_0 = \{1\}.\end{cases}$$

It should be now clear that for $n \geqslant n(Z_0)$

$$F_n^{(m)}(\cdot; Z_0) \in W_p^{m+1}(I)_{Z_0} \quad \text{for} \quad 1 \leqslant p < \infty$$

and

$$F_n^{(m)}(\cdot; Z_0) \in W_p^m(I)_{Z_0} \quad \text{for} \quad 1 \leqslant p \leqslant \infty.$$

The very definition of the functions $F_n^{(m)}$ implies for $i, j \geqslant n(Z_0) = n(Z_0')$.

(9.2) $$\big(F_i^{(m)}(\cdot; Z_0), F_j^{(m)}(\cdot; Z_0')\big)_{L_2(I)} = \delta_{i,j}.$$

For given $f \in L^1(I)$ we define the partial sum operators

(9.3) $$U_n^{(m)}(f; Z_0) = \sum_{j=n(Z_0)}^{n} \big(f, F_j^{(m)}(\cdot; Z_0')\big)F_j^{(m)}(\cdot; Z_0).$$

To treat $\big(F_n^{(m)}(\cdot; Z_0), n \geqslant n(Z_0)\big)$ in Sobolev spaces we need

DEFINITION 9.4. Let k, $0 \leqslant k \leqslant m+1$, be given, and let $n(Z_0; k)$ be equal to 1 for $Z_0 = \{0\}, \{1\}$ and to $2-r+k$ for $Z_0 = \emptyset, \{0, 1\}$. Then

$$F_n^{(m,k)}(\cdot; Z_0) = D^k F_n^{(m)}(\cdot; Z_0),$$

$$F_n^{(m,-k)}(\cdot; Z_0') = \begin{cases} H^k F_n^{(m)}(\cdot; Z_0') & \text{for} \quad Z_0 = \emptyset, \{0\},\\ (-1)^k G^k F_n^{(m)}(\cdot; Z_0') & \text{for} \quad Z_0 = \{1\}, \{0, 1\}.\end{cases}$$

We have again the biorthogonality condition

(9.5) $$\big(F_i^{(m,k)}(\cdot; Z_0), F_j^{(m,-k)}(\cdot; Z_0')\big) = \delta_{i,j}$$

satisfied for $i, j \geqslant n(Z_0; k)$, $0 \leqslant k \leqslant m+1$ and $Z_0 \subseteq \partial I$. The corresponding partial sum operators are given by the formula

(9.6) $$U_n^{(m,k)}(f; Z_0) = \sum_{n=n(Z_0;k)}^{n} \big(f, F^{(m,-k)}(\cdot; Z_0')\big)F_n^{(m,k)}(\cdot; Z_0),$$

where $0 \leqslant k \leqslant m+1$. It follows by (9.2) and (9.5) that $U_n^{(m)} = U_n^{(m,0)}$ and $U_n^{(m,k)}$ are projections. An application of Proposition 8.34 gives:

PROPOSITION 9.7. *Let $0 \leqslant k \leqslant m+1$, and let $f \in W_1^k(I)_{Z_0}$. Then*

$$D^k U_n^{(m)}(f; Z_0) = U_n^{(m,k)}(D^k f; Z_0), \qquad n \geqslant n(Z_0; k).$$

LEMMA 9.8. *Let $Z_0 \subseteq \partial I$, $0 \leqslant k \leqslant m+1$. Then*

$$\big(F_n^{(m,k)}(\cdot; Z_0),\ n \geqslant n(Z_0; k)\big)$$

is a Schauder basis in $L_p(I)$ if $1 \leqslant p < \infty$, and in $W_p^0(I)_{Z_0}$ if $k \leqslant m$, $1 \leqslant p \leqslant \infty$.

This lemma follows directly from Definitions 9.1 and 9.4 by Proposition 8.35.

LEMMA 9.9. *Let $Z_0 \subseteq \partial I$, $0 \leqslant k \leqslant m+1, 1 < p < \infty$. Then*

$$\big(F_n^{(m,k)}(\cdot; Z_0),\ n \geqslant n(Z_0; k)\big)$$

is an unconditional Schauder basis in $L_p(I)$.

Proof. According to Definitions 9.1 and 9.4 it is sufficient to prove that $(h_n^{(r,k)}, n \geqslant n_k)$, $|k| < r = m+2$, is an unconditional basis in $L_p(I)$. Duality arguments reduce the problem to $1 < p \leqslant 2$. Let us now define for given $\varepsilon = (\varepsilon_n)$, $\varepsilon_n = \pm 1$, the operator

$$T_\varepsilon f = \sum_{n=n_k}^{\infty} \varepsilon_n (f, h_n^{(r,-k)})h_n^{(r,k)}.$$

It is sufficient to check that T_s is of strong type $(2, 2)$ and of weak type $(1, 1)$ and then to apply Marcinkiewicz's interpolation theorem.

The strong type $(2, 2)$ of T_s can be proved as follows. Lemma 8.46 says tftha the hypotheses of Cotlar's lemma (cf. [22], pp. 102–103) are satisied. Thus in the notation of Section 8 we have

$$\|f\|_2^2(I) \sim \|H_1^{(r,k)}f\|_2^2(I) + \Big\|\sum_{\mu=0}^{\infty} \pm \Delta_\mu^{(r,k)}f\Big\|_2^2(I),$$

whence by Khinchin's inequalities

$$\|f\|_2^2(I) \sim \|H_1^{(r,k)}f\|_2^2(I) + \sum_{\mu=0}^{\infty} \|\Delta_\mu^{(r,k)}f\|_2^2(I).$$

Now, Proposition 8.40 gives

$$\|\Delta_\mu^{(r,k)}f\|_2^2(I) \sim \sum_{N+1}^{2N} \|(f, h^{(r,-k)})h^{(r,k)}\|_2^2(I).$$

Consequently,

$$\|f\|_2(I) \sim \|T_s f\|_2(I).$$

The weak type $(1, 1)$ of T_s can be proved with the help of Proposition 8.39 exactly in the same way as it was proved in [13] that this property holds in case $(h_n^{(r,k)}) = (f_n^{(r,k)})$. This completes the proof of the lemma.

We are now ready to pass to the d-dimensional case. Let Z be given as in (2.37). For given multi-indices $\boldsymbol{k} = (k_1, \ldots, k_d)$, $0 \leqslant k_j \leqslant m+1$, $\boldsymbol{n} = (n_1, \ldots, n_d)$, $n_j \geqslant n(Z_j; k_j)$,

$$F_{\boldsymbol{n}}^{(m,\pm\boldsymbol{k})}(\cdot; Z) = F_{n_1}^{(m,\pm k_1)}(\cdot; Z_1)\otimes \ldots \otimes F_{n_d}^{(m,\pm k_d)}(\cdot; Z_d),$$
$$F_{\boldsymbol{n}}^{(m)} = F_{\boldsymbol{n}}^{(m,\boldsymbol{0})}.$$

It now follows by (9.5) and (2.47) that

(9.10) $$\big(F_{\boldsymbol{i}}^{(m,-\boldsymbol{k})}(\cdot; Z'), F_{\boldsymbol{l}}^{(m,\boldsymbol{k})}(\cdot; Z)\big)_{L_2(Q)} = \delta_{\boldsymbol{i},\boldsymbol{l}},$$

where $\boldsymbol{i}, \boldsymbol{l} \geqslant \boldsymbol{n}(Z; \boldsymbol{k})$, i.e. $i_j, l_j \geqslant n(Z_j; k_j)$ for $j = 1, \ldots, d$. It follows from (9.10) that the following operation is a projection: for $f \in L_1(Q)$

(9.11) $$U_{\boldsymbol{n}}^{(m,\boldsymbol{k})}(f; Z) = \sum_{\boldsymbol{i}\leqslant\boldsymbol{n}} \big(f, F_{\boldsymbol{i}}^{(m,-\boldsymbol{k})}(\cdot; Z')\big) F_{\boldsymbol{i}}^{(m,\boldsymbol{k})}(\cdot; Z).$$

Clearly, for $f = f_1\otimes \ldots \otimes f_d$

(9.12) $$U_{\boldsymbol{n}}^{(m,\boldsymbol{k})}(f; Z) = U_{n_1}^{(m,k_1)}(f_1; Z_1)\otimes \ldots \otimes U_{n_d}^{(m,k_d)}(f_d; Z_d).$$

We use later on the identification: $U_{\boldsymbol{n}}^{(m,\boldsymbol{0})} = U_{\boldsymbol{n}}^{(m)}$. It is important that Proposition 9.7 in view of (9.11) and (9.12) extends to

PROPOSITION 9.13. *If* $0 \leqslant k_j \leqslant m+1$ *and* $f \in W_1^{\boldsymbol{k}}(Q)_Z$, *then*

$$D^{\boldsymbol{k}} U_{\boldsymbol{n}}^{(m)}(f; Z) = U_{\boldsymbol{n}}^{(m,\boldsymbol{k})}(D^{\boldsymbol{k}}f; Z).$$

The ordering of multi-indexed sequences of functions which was used in [14] is called *rectangular*.

LEMMA 9.14. *Let Z be given as in* (2.37) *and let* $\boldsymbol{0} \leqslant \boldsymbol{k} \leqslant (m+1, \ldots, m+1)$. *Then* $\big(F_{\boldsymbol{n}}^{(m,\boldsymbol{k})}(\cdot; Z), \boldsymbol{n} \geqslant \boldsymbol{n}(Z; \boldsymbol{k})\big)$ *in the rectangular ordering, is a Schauder basis in* $L_p(Q)$ *if* $1 \leqslant p < \infty$, *and in* $W_p^0(Q)_Z$ *if* $\boldsymbol{k} \leqslant (m, \ldots, m)$, $1 \leqslant p \leqslant \infty$.

This lemma follows from Lemma 9.8 by an argument similar to the one applied in [14].

LEMMA 9.15. *Let Z be given as in* (2.37) *and let* $\boldsymbol{0} \leqslant \boldsymbol{k} \leqslant (m+1, \ldots, m+1)$. *Then* $\big(F_{\boldsymbol{n}}^{(m,\boldsymbol{k})}(\cdot; Z), \boldsymbol{n} \geqslant \boldsymbol{n}(Z; \boldsymbol{k})\big)$ *is an unconditional Schauder basis in* $L_p(Q)$ *if* $1 < p < \infty$.

To obtain this result we use Lemma 9.9 and a result of McCarthy [31] on products of uniformly bounded commuting Boolean algebras of projections.

THEOREM 9.16. *Let* $m \geqslant 0$ *and let Z be given as in* (2.37). *Then* $\big(F_{\boldsymbol{n}}^{(m)}(\cdot; Z), \boldsymbol{n} \geqslant \boldsymbol{n}(Z)\big)$ *in its rectangular ordering is for each* k, $0 \leqslant k \leqslant m$, *a Schauder basis in* $W_p^k(Q)_Z$ *with* $1 \leqslant p \leqslant \infty$.

The proof of this theorem is a direct consequence of Lemma 9.14 and Proposition 9.13. From Lemma 9.15 and Proposition 9.13 follows

THEOREM 9.17. *Let* $m \geqslant 0$ *and let Z be given as in* (2.37). *Then* $\big(F_{\boldsymbol{n}}^{(m)}(\cdot; Z); \boldsymbol{n} \geqslant \boldsymbol{n}(Z)\big)$ *is for each* k, $0 \leqslant k \leqslant m+1$, *an unconditional Schauder basis in* $W_p^k(Q)_Z$ *with* $1 < p < \infty$.

Using the notation of Section 2 we now state the fundamental result on orders of approximation.

THEOREM 9.18. *Let* $m \geqslant 0$, $\mu \geqslant 0$, $N = 2^\mu$, $\boldsymbol{n} = (N, \ldots, N)$, *and let Z be given as in* (2.37). *Then for some constant* $C = C(m, d)$ *we have for* $f \in L_p(Q)$, $1 \leqslant p \leqslant \infty$

$$\|f - U_{\boldsymbol{n}}^{(m)}(f; Z)\|_p(Q) \leqslant C\omega_{m+2,p}(f_Z; N^{-1})_{Q_Z},$$

where f_Z is the extension of f by zero to Q_Z.

Proof. For given $i = 1, \ldots, d$ we define the following projection

$$U_{\boldsymbol{n},i}^{(m)}(\cdot; Z_i) = E_1\otimes \ldots \otimes E_{i-1}\otimes U_{n_i}^{(m)}(\cdot; Z_i)\otimes E_{i+1}\otimes \ldots \otimes E_d,$$

where the E_j's are copies of the identity operator acting in $L_p(I)$. It then follows by a telescoping argument (cf. [17], proof of Lemma 5.9) that

(9.19) $$\|f - U_{\boldsymbol{n}}^{(m)}(f; Z)\|_p(Q) \sim \sum_{j=1}^{d} \|f - U_{\boldsymbol{n},j}^{(m)}(f; Z_j)\|_p(Q).$$

We now apply the vector-valued version of Lemma 8.41 to obtain

$$\|f - U_{\boldsymbol{n},j}^{(m)}(f; Z_j)\|_p(Q) \leqslant C\omega_{m+2,p}^{(j)}(f_Z; N^{-1})_{Q_Z}.$$

Moreover, there is a trivial inequality, $\omega_{m+2,p}^{(j)} \leqslant \omega_{m+2,p}$, and this completes the proof.

THEOREM 9.20. *Let* $m \geq 0, \mu \geq 0, N = 2^\mu, \boldsymbol{n} = (N, \ldots, N)$, $0 \leq k \leq k+l \leq m$, *and let* Z *be given as in* (2.37). *Moreover, let*

$$U_N f = U_{\boldsymbol{n}}^{(m)}(f; Z).$$

Then there is a constant $C = C(m, d)$ *such that for* $1 \leq p \leq \infty$ *and* $f \in W_p^k(Q)_Z$ *we have Bernstein's and Jackson's inequalities*

$$\|U_N f\|_p^{(k+l)}(Q) \leq CN^l \|U_N f\|_p^{(k)}(Q), \tag{9.21}$$

$$\|f - U_N f\|_p^{(k)}(Q) \leq CN^{-l}\|f - U_N f\|_p^{(k+l)}(Q). \tag{9.22}$$

Proof. This result was established for $Z_j = \emptyset, j = 1, \ldots, d$ and for $Z_j = \{0, 1\}, j = 1, \ldots, d$, in [17], Theorem 5.16. Since we have (9.19), Propositions 8.36, 9.13 and Theorem 9.18, similar arguments can be used to establish the theorem in the remaining cases of Z.

Remark 9.23. It is important to realize that we have the following formula for the adjoint operator (in the sense of the Hilbert space $L_2(Q)$)

$$U_{\boldsymbol{n}}^{(m)}(\cdot; Z)^* = U_{\boldsymbol{n}}^{(m)}(\cdot; Z').$$

10. Spline bases in Besov spaces on cubes with boundary conditions. Let us start with some obvious conclusions which can be drawn from the results we have already established.

PROPOSITION 10.1. *Let* $Q = I^d$ *and* Z *be given as in* (2.37), *let* m *be an integer. Moreover, let* $0 < s < m$, $1 \leq p, q \leq \infty$. *Then* $\big(F_{\boldsymbol{n}}^{(m)}(\cdot; Z)\big)$, $\boldsymbol{n} \geq \boldsymbol{n}(Z)$, *in the rectangular ordering, is a basis in* $B_{p,q}^s(Q)_Z$.

Proof. Proposition 2.50 (cf. also Remark 2.51) characterizes $B_{p,q}^s(Q)_Z$ as an interpolation space between $W_p^0(Q)_Z$ and $W_p^m(Q)_Z$. Since, by Theorem 9.16, the sequence $\big(F_{\boldsymbol{n}}^{(m)}(\cdot; \boldsymbol{Z})\big)$ is a Schauder basis in the latter spaces, the proposition follows from the fundamental property of interpolation spaces (cf. e.g. [3], Theorem 3.1.2).

PROPOSITION 10.2. *Let the hypothesis of Proposition* 10.1 *be satisfied. Then for* $1 < p < \infty$ *the system* $\big(F_{\boldsymbol{n}}^{(m)}(\cdot; Z), \boldsymbol{n} \geq \boldsymbol{n}(Z)\big)$ *is an unconditional basis in* $B_{p,q}^s(Q)_Z$.

Proof. Apply similar argument as in the previous proof using Theorem 9.17 instead of 9.16.

The space $B_{p,q}^s(Q)_Z$, for arbitrary Z and for any choice of the parameters $s > 0$, $1 \leq p, q \leq \infty$, has an unconditional basis.

We are going to construct such a basis in two steps. The first step gives an unconditional Schauder decomposition into finite-dimensional subspaces, providing at the same time a new equivalent norm in $B_{p,q}^s(Q)_Z$. In the second step a recipe is given for choosing in each of the finite dimensional subspaces a suitable basis so that they will provide a linear isomorphism between $B_{p,q}^s(Q)_Z$ and certain sequence spaces.

DEFINITION 10.3. For given Z, $m \geq 0$, $N = 2^\mu$, $\mu \geq 0$, let

$$V_{\mu+1} f = U_{2\boldsymbol{n}}^{(m)}(f; Z) - U_{\boldsymbol{n}}^{(m)}(f; Z), \quad \boldsymbol{n} = (N, \ldots, N)$$

$$V_0 f = U_{\boldsymbol{1}}^{(m)}(f; Z), \quad \boldsymbol{1} = (1, \ldots, 1).$$

THEOREM 10.4. *Let* Z *be given as in* (2.37) *and let* $1 \leq p, q \leq \infty$, $0 < s < m$. *Then* $B_{p,q}^s(Q)_Z$ *has an equivalent norm*

$$\|f\|_{p,q}^{(s)}(Q)_Z = \Big(\sum_{\mu=0}^{\infty} \big(2^{s\mu}\|V_\mu f\|_p(Q)\big)^q\Big)^{1/q}. \tag{10.5}$$

Proof. Theorem 9.18 implies that for some constants $C = C(m, d)$ we have

$$\|V_\mu f\|_p(Q) \leq C\omega_{m,p}(f_Z; N^{-1})_{Q_Z},$$

whence we infer

$$\|f\|_{p,q}^{(s)}(Q)_Z \leq C(s, m, d)\|f_Z\|_{p,q}^{(s)}(Q_Z).$$

The opposite inequality can be proved as follows. In Section 2 we have defined

$$Q_Z = \prod_{i=1}^{d} I_i \quad \text{with} \quad I_i = I_{Z_j}.$$

Let now for fixed j, $1 \leq j \leq d$,

$$Q_0 = \prod_{j \neq i=1}^{d} I_i, \quad X = W_p^0(Q_0).$$

Our aim is to estimate from above

$$\omega_{m,p}^{(j)}(f_Z; N^{-1})_{Q_Z} = \omega_{m,p}(f_{Z_j}; X; N^{-1})_{I_j},$$

where f_{Z_j} has the obvious meaning. It follows by Proposition 7.16 that

$$\omega_{m,p}^{(j)}(f_Z; N^{-1})_{Q_Z} \leq CN^{-m}\Big(\|f_Z\|_p(Q_Z) + \sum_{k=1}^{N} k^{m-1}E_k(f)\Big),$$

where

$$E_k(f) = \inf\{\|f_{Z_j} - g\|_p(I_j; X) : g \in S_k(I_j; X)\}$$

and $S_k(I_j; X)$ is the space of X-valued splines on I_j corresponding to the kth dyadic partition such that it contains (see below)

$$\operatorname{span}\Big[\Big(\sum_{i=n(Z_j)}^{k} f_i F_i^{(m)}(\cdot; Z_j)\Big)_{Z_j}; f_i \in L_p(Q_0)\Big].$$

It then follows that $(\boldsymbol{n} = (N, \ldots, N))$

$$E_N(f) \leq C\|f - U_{\boldsymbol{n},j}^{(m)}(f; Z_j)\|_p(Q).$$

Applying (9.19) we find that

$$\sum_{j=1}^{d} \omega_{m,p}^{(j)}(f_Z;\ N^{-1})_{Q_Z} \leqslant CN^{-m} \sum_{\nu=0}^{\mu} 2^{\nu m} \|V_\nu f\|_p(Q).$$

This implies that

$$\sum_{j=1}^{d} \|f_Z\|_{p,q,j}^{(s)}(Q_Z) \leqslant C\|f\|_{p,q}^{(s)}(Q)_Z,$$

whence by Theorem 2.28 the proof is complete.

COROLLARY 10.6. *For arbitrary boundary conditions Z we have the following decomposition* (cf. Definition 10.3)

$$f = \sum_{\mu=0}^{\infty} V_\mu f$$

in $W_p^m(Q)_Z$ and $B_{p,q}^s(Q)_Z$ for $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $0 < s < m$. It is unconditional in the Sobolev space for $1 < p < \infty$, and in the Besov space for $1 \leqslant p, q \leqslant \infty$, $0 < s < m$.

This follows from Theorems 9.1 and 10.4.

We now pass to the construction of suitable bases in the finite dimensional subspaces. Let us start with a decomposition of the index set $N(Z) = N(Z_1) \times \ldots \times N(Z_d)$, where $N(Z_j) = \boldsymbol{N} + n(Z_j)$, $n(Z_j)$ is given as in Definition 9.1, and $\boldsymbol{N}$ is the set of all non-negative integers.

For given $\mu \geqslant 1$, $\mu \in \boldsymbol{N}$, $\emptyset \neq e \subseteq D = \{1, \ldots, d\}$ define

$$N_{\mu,e} = \{\boldsymbol{n}:\ 2^{\mu-1} < n_i \leqslant 2^\mu \text{ for } i \in e,\ n_i \leqslant 2^{\mu-1} \text{ for } i \in D \setminus e\},$$

$$N_\mu = \bigcup_{\emptyset \neq e \subseteq D} N_{\mu,e},$$

$$N_0 = \{\boldsymbol{n} \in N(Z):\ n_i \leqslant 1 \text{ for } i \in D\}.$$

Clearly, all the sets are disjoint and

$$N(Z) = \bigcup_{\mu=0}^{\infty} N_\mu = N_0 \cup \bigcup_{\mu=1}^{\infty} \bigcup_{\emptyset \neq e \subseteq D} N_{\mu,e}. \tag{10.7}$$

To each of the components in (10.7) there corresponds a finite dimensional multivariate spline space:

$$S_{\mu,e}(Z) = \operatorname{span}[F_{\boldsymbol{n}}^{(m)}(\cdot\,; Z):\ \boldsymbol{n} \in N_{\mu,e}],$$

$$S_0(Z) = \operatorname{span}[F_{\boldsymbol{n}}^{(m)}(\cdot\,; Z):\ \boldsymbol{n} \in N_0].$$

In $S_0(Z)$ the basis will be left as it is.

In $S_{\mu,e}(Z)$ there will be constructed a tensor product basis in such a way that its coefficient space will be, uniformly in μ and p, $1 \leqslant p \leqslant \infty$, linearly isomorphic to l_p^k, $k = \dim S_{\mu,e}(Z)$. Moreover, the dual basis will have the same property with respect to $S_{\mu,e}(Z')$.

In what follows we consider a uniform partition $\Pi(\mu)$ with step $2^{-\mu}$ and with multiplicities one at all knots but 0 and 1 where the multiplicities are assumed to be $3r$ and $2r$, respectively, i.e. $\Pi(\mu) = (t_{\mu,i})$,

$$t_{\mu,i} = \begin{cases} (i+3r-1)2^{-\mu} & \text{for } i \leqslant -3r, \\ 0 & \text{for } -3r < i \leqslant 0, \\ i2^{-\mu} & \text{for } 0 < i < 2^\mu, \\ 1 & \text{for } 2^\mu \leqslant i < 2^\mu + 2r, \\ (i-2r+1)2^{-\mu} & \text{for } i \geqslant 2^\mu + 2r. \end{cases}$$

The B-spline corresponding to the support $\langle t_{\mu,i}, t_{\mu,i+r'}\rangle$ is denoted by $N_{\mu,i}^{(r')}(t) = N^{(r')}(t_{\mu,i}, \ldots, t_{\mu,i+r'};\ t)$. The function $N_{\mu,i}^{(r')}$ is well defined for r', i satisfying the condition $t_{\mu,i} < t_{\mu,i+r'}$. With such understanding of the B-splines we can state and prove easily:

PROPOSITION 10.8. *Let $H = \int_t^1$, $\mu \geqslant 1$, $r \geqslant 2$. Then we have*

$$\operatorname{span}[f_j^{(2r,r)},\ j = 2-r, \ldots, 2^\mu] = \operatorname{span}[N_{\mu,j}^{(r)},\ j = 1-r, \ldots, 2^\mu - 1],$$

$$\operatorname{span}[f_j^{(2r,-r)},\ j = 2-r, \ldots, 2^\mu] = \operatorname{span}[N_{\mu,j}^{(3r)},\ j = -2r+1, \ldots, 2^\mu - r - 1],$$

(10.9)

$$\operatorname{span}[g_j^{(2r,r)},\ j = 1, \ldots, 2^\mu] = \operatorname{span}[N_{\mu,j}^{(r)},\ j = 0, \ldots, 2^\mu - 1],$$

$$\operatorname{span}[g_j^{(2r,-r)},\ j = 1, \ldots, 2^\mu] = \operatorname{span}[\tilde{N}_{\mu,j}^{(3r)},\ j = 0, \ldots, 2^\mu - 1],$$

where

$$\tilde{N}_{\mu,j}^{(3r)} = \begin{cases} 2^{\mu r} H^r N_{\mu,j}^{(2r)} & \text{for } j = 0, \ldots, r-1, \\ N_{\mu,j-r}^{(3r)} & \text{for } j = r, \ldots, 2^\mu - 1. \end{cases}$$

PROPOSITION 10.10. *There is a constant C_r such that for $\mu \geqslant 1$ and $1 \leqslant p \leqslant \infty$*

$$C_r^{-1} 2^{-\mu/p} \leqslant \|\tilde{N}_{\mu,j}^{(3r)}\|_p(I) \leqslant C_r 2^{-\mu/p}, \quad j = 0, \ldots, 2^\mu - 1.$$

Moreover,

$$\|a\|_p C_r^{-1} \leqslant \Big\|\sum_{j=0}^{2^\mu - 1} a_j \tilde{N}_{\mu,j,p}^{(3r)}\Big\|_p(I) \leqslant C_r \|a\|_p, \tag{10.11}$$

where $\|a\|_p$ is the norm in $l_p^{2^\mu}$ and

$$\tilde{N}_{\mu,j,p}^{(3r)} = \tilde{N}_{\mu,j}^{(3r)} \big(\|\tilde{N}_{\mu,j}^{(3r)}\|_1(I)\big)^{-1/p}.$$

Proof. The first two inequalities for $r \leqslant j < 2^\mu$ follow by (7.3) and (7.6). Now, $\operatorname{supp} H^r N_{\mu,j}^{(2r)} \subseteq \operatorname{sup} N_{\mu,j}^{(2r)} \subseteq \langle 0, (3r-1)2^{-\mu}\rangle$ for $j = 0, \ldots, r-1$, and

$$\|H^r N_{\mu,j}^{(2r)}\|_\infty \leqslant C_r 2^{-\mu r}.$$

Thus,

$$\|\tilde{N}^{(3r)}_{\mu,j}\|_p(I) \leqslant C_r 2^{-\mu/p}, \qquad j = 0, \ldots, r-1.$$

The opposite inequality follows from Bernstein's inequality

$$2^{\mu r}\|H^r N^{(2r)}_{\mu,j}\|_p(I) C_r \geqslant \|D^r H^r N^{(2r)}_{\mu,j}\|_p(I) = \|N^{(2r)}_{\mu,j}\|_p(I) \geqslant C_r^{-1} 2^{-\mu/p}.$$

The right-hand side of (10.11) can be proved in a similar way as in the case of the B-splines (as in Corollary 7.10). The left-hand side can be proved as follows. We notice that there are coefficients $b_{1-3r}, \ldots, b_{-1}$ such that on $I = \langle 0, 1\rangle$

$$\sum_{j=1-3r}^{-1} b_j N^{(3r)}_{\mu,j,p} = \sum_{j=0}^{r-1} a_j \tilde{N}^{(3r)}_{\mu,j,p},$$

where $N^{(3r)}_{\mu,j,p}$ is defined as in Theorem 7.8. Thus by the theorem just quoted

$$\Big\|\sum_{j=0}^{2^\mu-1} a_j \tilde{N}^{(3r)}_{\mu,j,p}\Big\|_p(I) = \Big\|\sum_{j=1-3r}^{-1} b_j N^{(3r)}_{\mu,j,p} + \sum_{j=0}^{2^\mu-r-1} a_{j+r} N^{(3r)}_{\mu,j,p}\Big\|_p(I)$$
$$\geqslant C_r\Big(\sum_{j=1-3r}^{-1} |b_j|^p + \sum_{j=r}^{2^\mu-1} |a_j|^p\Big)^{1/p}$$
$$\geqslant C_r\Big(\Big\|\sum_{j=-3r}^{-1} b_j N^{(3r)}_{\mu,j,p}\Big\|_p^p(I) + \sum_{j=r}^{2^\mu-1} |a_j|^p\Big)^{1/p}.$$

However, by Bernstein's inequality we obtain

$$\Big\|\sum_{j=1-3r}^{-1} b_j N^{(3r)}_{\mu,j,p}\Big\|_p^p(I) = \Big\|\sum_{j=0}^{r-1} a_j \tilde{N}^{(3r)}_{\mu,j,p}\Big\|_p^p(I)$$
$$= \Big\|2^{\mu r} H^r \sum_{j=0}^{r-1} a_j N^{(2r)}_{\mu,j,p}\Big\|_p^p(I)\|N^{(2r)}_{\mu,j}\|_1(I)/\|\tilde{N}^{(3r)}_{\mu,j}\|_1(I)$$
$$\geqslant C_r^p\Big\|\sum_{j=0}^{r-1} a_j N^{(2r)}_{\mu,j,p}\Big\|_p^p(I) \geqslant C_r^p \sum_{j=0}^{r-1} |a_j|^p,$$

and this completes the proof of (10.11).

LEMMA 10.12. *Let* $B_k = (b^{(k)}_{i,j})$, $k = 0, 1$, *be the Gram matrices with*

$$b^{(0)}_{i,j} = (N^{(r)}_{\mu,i}, N^{(3r)}_{\mu,j-r}), \qquad i, j = 1-r, \ldots, 2^\mu-1,$$
$$b^{(1)}_{i,j} = (N^{(r)}_{\mu,i}, \tilde{N}^{(3r)}_{\mu,j}), \qquad i, j = 0, \ldots, 2^\mu-1.$$

Let $A_k = B_k^{-1} = (a^{(k)}_{i,j})$, $k = 0, 1$. *Then there are constants* C_r *and* q_r, $0 < q_r < 1$, *such that*

$$|a^{(k)}_{i,j}| \leqslant C_r n q_r^{|i-j|} \quad \text{for} \quad i, j = (1-k)(1-r), \ldots, 2^\mu-1,$$

where $k = 0, 1$.

Proof. We are going to prove the case $k = 1$ only. The other case is quite similar and the proof is omitted. Since (cf. Section 8) for $n = 2^\mu$

$$L_n^{(2r,r)}(s, t) = \sum_{i,j=0}^{n-1} a^{(1)}_{i,j} \tilde{N}^{(3r)}_{\mu,i}(s) N^{(r)}_{\mu,j}(t),$$

it follows that

$$a^{(1)}_{i,j} = \int_I\int_I L_n^{(2r,r)}(s, t) \underline{\tilde{N}}^{(3r)}_{\mu,i}(s) \underline{N}^{(r)}_{\mu,j}(t)\,ds\,dt. \tag{10.13}$$

Here, $(\underline{\tilde{N}}^{(3r)}_{\mu,i})$ and $(\underline{N}^{(r)}_{\mu,j})$ are the dual bases to $(\tilde{N}^{(3r)}_{\mu,i})$ and $(N^{(r)}_{\mu,j})$, respectively. Using Corollary 7.10, (10.11) and Proposition 8.6 (suitably adapted in case of $(\tilde{N}^{(3r)}_{\mu,i})$) we obtain

$$|\underline{N}^{(r)}_{\mu,i}(t)| \leqslant C_r n q_r^{|nt-i|}, \qquad 0 < q_r < 1,$$
$$|\underline{\tilde{N}}^{(3r)}_{\mu,i}(t)| \leqslant C_r n q_r^{|nt-i|}.$$

These inequalities, Lemma 8.13 and (10.13) give the desired result.

DEFINITION 10.14.

$$\overset{0}{N}{}^{(3r)}_{\mu,j} = \sum_{i=1-r}^{2^\mu-1} a^{(0)}_{ij} N^{(3r)}_{\mu,i-r}, \qquad j = 1-r, \ldots, 2^\mu-1,$$
$$\overset{1}{N}{}^{(3r)}_{\mu,j} = \sum_{i=0}^{2^\mu-1} a^{(1)}_{ij} \tilde{N}^{(3r)}_{\mu,i}, \qquad j = 0, \ldots, 2^\mu-1.$$

LEMMA 10.15. *The following relations hold*

$$(N^{(r)}_{\mu,i}, \overset{0}{N}{}^{(3r)}_{\mu,j}) = \delta_{i,j}, \qquad i, j = 1-r, \ldots, 2^\mu-1,$$
$$(N^{(r)}_{\mu,i}, \overset{1}{N}{}^{(3r)}_{\mu,j}) = \delta_{i,j}, \qquad i, j = 0, \ldots, 2^\mu-1.$$

Moreover, if $(f_{\mu,i})$ *is one of the systems* $(N^{(r)}_{\mu,i}, i = 1-r, \ldots, 2^\mu-1)$ $(\overset{0}{N}{}^{(3r)}_{\mu,i}, i = 1-r, \ldots, 2^\mu-1)$, $(N^{(r)}_{\mu,i}, i = 0, \ldots, 2^\mu-1)$ *or* $(\overset{1}{N}{}^{(3r)}_{\mu,i}, i = 0, \ldots, 2^\mu-1)$, *then for some constant* C_r *we have*

$$\|a\|_{l_p^\lambda} C_r^{-1} \leqslant \Big\|\sum_i a_i \frac{f_{\mu,i}}{\|f_{\mu,i}\|_p}\Big\|^p \leqslant C_r \|a\|_{l_p^\lambda},$$

where $\lambda = \dim \operatorname{span}[f_{\mu,i}]$. *In addition to this we have*

$$\|\overset{0}{N}{}^{(3r)}_{\mu,i}\|_p \sim \|\overset{1}{N}{}^{(3r)}_{\mu,i}\|_p \sim 2^{\mu/q}, \qquad 1/p + 1/q = 1.$$

Proof. The duality relations follow by Definition 10.14, Lemma 10.12 and Definition 10.14 imply for $k = 0, 1$; $n = 2^\mu$,

$$|\overset{k}{N}{}^{(3r)}_{\mu,i}(t)| \leqslant C_r n q_r^{|nt-i|}, \qquad 0 < q_r < 1.$$

This, the duality relations and the properties of the B-splines imply the remaining statements of the lemma.

DEFINITION 10.16. Let

$$f_{\mu,j}^{(2r,r)} = N_{\mu,j-1}^{(r)}/\|N_{\mu,j-1}^{(r)}\|_2 \quad \text{for} \quad j = 2-r, \ldots, 2^\mu;$$

$$f_{\mu,j}^{(2r,-r)} = \overset{0}{N}{}_{\mu,j-1}^{(3r)}/\|\overset{0}{N}{}_{\mu,j-1}^{(3r)}\|_2 \quad \text{for} \quad j = 2-r, \ldots, 2^\mu;$$

$$g_{\mu,j}^{(2r,r)} = N_{\mu,j-1}^{(r)}/\|N_{\mu,j-1}^{(r)}\|_2 \quad \text{for} \quad j = 1, \ldots, 2^\mu;$$

$$g_{\mu,j}^{(2r,-r)} = \overset{1}{N}{}_{\mu,j-1}^{(3r)}/\|\overset{1}{N}{}_{\mu,j-1}^{(3r)}\|_2 \quad \text{for} \quad j = 1, \ldots, 2^\mu.$$

DEFINITION 10.17. The function $F_{\mu,n}^{(m)}(t; Z_0)$ for $\mu \geqslant 1$, $n(Z_0) \leqslant n \leqslant 2^\mu$ is defined by the formula as given in Definition 9.1 if we replace formally F_n, f_n, g_n by $F_{\mu,n}, f_{\mu,n}, g_{\mu,n}$, respectively. Moreover, let $F_{0,n}^{(m)} = F_n^{(m)}$ for $n(Z_0) \leqslant n \leqslant 1$.

We are now ready to define our new basis.

DEFINITION 10.18. The new system for given Z is defined according to the decomposition (10.7) as follows

$$G_{\boldsymbol{n}}^{(m)}(\cdot; Z) = F_{\boldsymbol{n}}^{(m)}(\cdot; Z) \quad \text{for} \quad \boldsymbol{n} \in N_0,$$

$$G_{\boldsymbol{n}}^{(m)}(\cdot; Z) = \bigotimes_{i \in e} F_{n_i}^{(m)}(\cdot; Z_i) \otimes \bigotimes_{i \in D \setminus e} F_{\mu-1,n_i}^{(m)}(\cdot; Z_i)$$

for $\boldsymbol{n} \in N_{\mu,e}$, $\mu \geqslant 1$, $\emptyset \neq e \subseteq D$.

THEOREM 10.19. *For given $m \geqslant 0$ and Z we have*

$$(10.20) \qquad \big(G_{\boldsymbol{k}}^{(m)}(\cdot; Z'), G_{\boldsymbol{n}}^{(m)}(\cdot; Z)\big)_{L_2(Q)} = \delta_{\boldsymbol{k},\boldsymbol{n}},$$

where $\boldsymbol{n}, \boldsymbol{k} \geqslant \boldsymbol{n}(Z)$. Moreover, for $\mu \geqslant 0$,

$$(10.21) \qquad \operatorname{span}[G_{\boldsymbol{n}}^{(m)}(\cdot; Z): \boldsymbol{n} \in N_\mu] = \operatorname{span}[F_{\boldsymbol{n}}^{(m)}(\cdot; Z): \boldsymbol{n} \in N_\mu]$$

and for $\mu \geqslant 1$, $f \in L_1(Q)$

$$(10.22) \qquad V_\mu f = \sum_{\boldsymbol{n} \in N_\mu} \big(f, G_{\boldsymbol{n}}^{(m)}(\cdot; Z')\big) G_{\boldsymbol{n}}^{(m)}(\cdot; Z),$$

where V_μ is given as in Definition 10.3.

Finally, for some constant $C = C(m, d)$

$$(10.23) \qquad \|a\|_{l_p^\lambda} C^{-1} \leqslant 2^{\mu(1/p-1/2)d} \Big\| \sum_{\boldsymbol{n} \in N_\mu} a_{\boldsymbol{n}} G_{\boldsymbol{n}}^{(m)}(\cdot; Z) \Big\|_p (I^d) \leqslant C \|a\|_{l_p^\lambda},$$

where λ = cardinality of N_μ.

Proof. Property (10.20) follows immediately from Definitions 10.18, 10.17, 10.16 and Lemma 10.15. Let us now introduce

$$S_\mu(Z_0) = \operatorname{span}[F_n^{(m)}(\cdot; Z_0), n(Z_0) \leqslant n \leqslant 2^\mu],$$

$$\Delta_\mu(Z_0) = \operatorname{span}[F_n^{(m)}(\cdot; Z_0), 2^{\mu-1} < n \leqslant 2^\mu].$$

It then follows (see the definition following (10.7)) that

$$S_{\mu,e}(Z) = \bigotimes_{i \in e} \Delta_\mu(Z_i) \otimes \bigotimes_{i \in D \setminus e} S_{\mu-1}(Z_i).$$

Proposition 8.40 gives with $\lambda = 2^{\mu-1}$

$$\Big\| \sum_{2^{\mu-1}+1}^{2^\mu} a_j F_j^{(m)}(\cdot; Z_i) \Big\|_p (I) \sim 2^{\mu(1/2-1/p)} \|a\|_{l_p^\lambda},$$

and Lemma 10.15 implies with $\lambda_i = 2^{k-1} - n(Z_i) + 1$

$$\Big\| \sum_{j=n(Z_i)}^{2^{\mu-1}} a_j F_{\mu-1,j}^{(m)}(\cdot; Z_i) \Big\|_p \sim 2^{\mu(1/2-1/p)} \|a\|_{l_p^{\lambda_i}}.$$

Applying a standard lemma on products of operators in L_p spaces we find

$$\Big\| \sum_{\boldsymbol{n} \in N_{\mu,e}} a_{\boldsymbol{n}} G_{\boldsymbol{n}}^{(m)}(\cdot; Z) \Big\|_p (Q) \sim 2^{\mu(1/2-1/p)d} \|a\|_{l_p^\lambda},$$

where $\lambda = \dim S_{\mu,e}(Z)$. Since

$$N_\mu = \bigcup_{\emptyset \neq e \subseteq D} N_{\mu,e},$$

it follows that (10.23) is satisfied. Introducing

$$S_\mu(Z) = S_\mu(Z_1) \otimes \ldots \otimes S_\mu(Z_d),$$

we find that

$$S_\mu(Z) = S_{\mu-1}(Z) \oplus \bigotimes_{\emptyset \neq e \subseteq D} S_{\mu,e}(Z)$$

whence (10.21) follows, and this completes the proof.

COROLLARY 10.24. *The system $\big(G_{\boldsymbol{n}}^{(m)}(\cdot; Z), \boldsymbol{n} \geqslant \boldsymbol{n}(Z)\big)$ is an unconditional Schauder basis in $B_{p,q}^s(Q)_Z$ for $0 < s < m$, $1 \leqslant p, q \leqslant \infty$. Moreover, if*

$$f = \sum_{\boldsymbol{n} \geqslant \boldsymbol{n}(Z)} a_{\boldsymbol{n}} G_{\boldsymbol{n}}^{(m)}(\cdot; Z),$$

then letting $\sigma = s/d + 1/2 - 1/p$

$$\|f\|_{p,q}^s(Q)_Z \sim \Big\{ \sum_{\mu=0}^{\infty} \Big[2^{\mu\sigma d} \Big(\sum_{\boldsymbol{n} \in N_\mu} |a_{\boldsymbol{n}}|^p \Big)^{1/p} \Big]^q \Big\}^{1/q}.$$

Remark. The system $\big(G_{\boldsymbol{n}}^{(m)}(\cdot; Z), \boldsymbol{n} \geqslant \boldsymbol{n}(Z)\big)$ in suitable ordering is a Schauder basis in $W_p^0(Q)_Z$ for $1 \leqslant p \leqslant \infty$.

11. Final comments. Let us complete the proof of Theorems A and B formulated in the introduction to Part I. Recall that M is a compact d-dimensional C^∞ manifold and $m \geqslant 1$ is a fixed integer. The sequence $(f_n)_{n=1}^\infty \subset C^m(M)$ referred to in Theorem A can be obtained as follows.

Let $Q_1, \ldots, Q_N$ be the decomposition of M into non-overlapping d-cubes constructed in Section 3. Let T be the isomorphism constructed in Theorem 4.9, i.e.

$$T\colon \mathscr{F}(M) \to \sum_{i\leqslant N} \oplus \mathscr{F}(Q_i)_{Z_i} \tag{11.1}$$

for $\mathscr{F} = W_p^k$, $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$.

Recall that, if we fix diffeomorphisms $\Phi_i\colon I^d \to Q_i$, for $i = 1, \ldots, N$, then each summand $\mathscr{F}(Q_i)_{Z_i}$ is in a natural way identified with the space $\mathscr{F}(I^d)_{\Phi_i^{-1}(Z_i)}$. Consider, for $i = 1, \ldots, N$, the system

$$\mathfrak{F}_i \subseteq C^m(Q_i)_{Z_i} \subset T\big(C^m(M)\big)$$

which corresponds in this identification to the system

$$\big\{F_{\boldsymbol{n}}^{(m)}\big(\cdot, \Phi_i^{-1}(Z_i)\big)\colon \boldsymbol{n} \geqslant \boldsymbol{n}\big(\Phi_i^{-1}(Z_i)\big)\big\} \subseteq C^m(I^d)_{\Phi_i^{-1}(Z_i)}$$

constructed in Section 9 (cf. Theorems 9.16 and 9.17).

The union $\mathfrak{F}_1 \cup \ldots \cup \mathfrak{F}_N$ can be ordered as $(h_n)_{n=1}^{\infty}$ so that $(h_{(j-1)N+i})_{j=1}^{\infty}$ is the enumeration of $\mathfrak{F}_i$ which corresponds to the so-called rectangular ordering of $\mathfrak{F}_i$ (as defined in [14]). Put $f_n = T^{-1}(h_n) \in C^m(M)$ for $n = 1, 2, \ldots$ Then (A1) follows from Theorems 9.16 and 4.9 and (A2) follows from Theorems 9.17 and 4.9.

Recall that we have fixed a smooth measure μ on M (cf. Section 4). In particular, on each Q_i one has $d\mu = q_i dx$ for some positive function $q_i \in C^{\infty}(Q_i)$, where dx is the measure transported from the Lebesgue measure on I^d by means of Φ_i. Since $(f_n)_{n=1}^{\infty}$ is, in particular, a Schauder basis in $L_1(M, d\mu)$, there is a unique μ-biorthogonal sequence $(g_n)_{n=1}^{\infty}$ which consists of elements of $L_{\infty}(M, d\mu)$. A different description of this sequence will show that (A3), (A1′) and (A2′) are satisfied.

By Theorem 4.9 the operator V_0 (which is the inverse to T^*) induces linear isomorphisms

$$\mathring{V}\colon \mathring{\mathscr{F}}(M) \to \sum_{i\leqslant N} \oplus \mathscr{F}(Q_i)_{Z_i'},$$

for $\mathscr{F} = W_p^k$, $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$.

For $i = 1, \ldots, N$ the system

$$\mathfrak{F}_i' \subseteq C^m(Q_i)_{Z_i'} \subseteq \mathring{V}\big(\mathring{C}(M)\big)$$

which corresponds to the system

$$\big\{F_{\boldsymbol{n}}^{(m)}\big(\cdot, \Phi_i^{-1}(Z_i')\big)\colon \boldsymbol{n} \geqslant \boldsymbol{n}\big(\Phi_i^{-1}(Z_i')\big)\big\} \subseteq C^m(I^d)_{\Phi_i^{-1}(Z_i')}$$

is, by (9.10), dx-biorthogonal to $\mathfrak{F}_i$.

By Theorem 9.16, the system $\mathfrak{F}_i'$ forms (in the rectangular ordering) a basis in $\mathscr{F}(Q_i)_{Z_i'}$ which is unconditional if $1 < p < \infty$ by Theorem 9.17. It follows that the system

$$q_i^{-1} \cdot \mathfrak{F}_i' = \{q_i^{-1} \cdot f\colon f \in \mathfrak{F}_i'\}$$

is μ-biorthogonal to $\mathfrak{F}_i$. Since the multiplication by q_i^{-1} defines an automorphism of each space $\mathscr{F}(Q_i)_{Z_i'}$ this system remains a Schauder basis in $W_p^k(Q_i)_{\Phi_i^{-1}(Z_i')}$.

Finally, since $V_0^{-1} = T^*$, the set

$$V_0^{-1}(q_1^{-1} \cdot \mathfrak{F}_1' \cup \ldots \cup q_N^{-1} \cdot \mathfrak{F}_N') \subseteq \mathring{C}(M)$$

is μ-biorthogonal to $(f_n)_{n=1}^{\infty}$ and hence, suitably ordered, it coincides with $(g_n)_{n=1}^{\infty}$. Therefore, conditions (A3), (A1′), (A2′) are satisfied thanks to Theorems 4.9, 9.16 and 9.17. This completes the proof of Theorem A.

As we already mentioned in the introduction the sequence $(f_n)_{n=1}^{\infty}$ is actually a basis in $W_p^k(M)$ for $-m \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$ and is an unconditional basis if $1 < p < \infty$. Analogously $(g_n)_{n=1}^{\infty}$ is a basis in $\mathring{W}_p^k(M)$ for $-m \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$ and is an unconditional basis if $1 < p < \infty$.

The construction of the basis from Theorem B is similar. Instead of the $\mathfrak{F}_i$'s one uses the systems $\mathfrak{G}_i$, $i = 1, \ldots, N$, corresponding to

$$\big\{G_{\boldsymbol{n}}^{(m)}\big(\cdot, \Phi_i^{-1}(Z_i)\big)\colon \boldsymbol{n} \geqslant \boldsymbol{n}\big(\Phi_i^{-1}(Z_i)\big)\big\} \subseteq C^m(I^d)_{\Phi^{-1}(Z_i')}.$$

Also Theorems 9.16 and 9.17 are replaced by Corollary 10.24 and the subsequent remark.

One checks first that (B1) and (B2) are satisfied for $0 < s < m$. This is not difficult but rather tedious. It may be more convenient to order the union of the $\mathfrak{G}_i$'s a bit differently. Recall that each $\mathfrak{G}_i$ is split into blocks $N_0, N_1, \ldots$ with $\mathrm{Card}(N_j)$ of the order 2^{jd}. Now we require that the jth block of $\mathfrak{G}_i$ precede the jth block of $\mathfrak{G}_{i+1}$, for $i = 1, \ldots, N-1$ and the jth block of $\mathfrak{G}_N$ precede the $(j+1)$-st block of $\mathfrak{G}_1$. This new ordering will not spoil the properties of the sequence as a basis in $W_p^0(M)$.

Using Corollary 10.24 and Corollary 4.11 we obtain, for $0 < s < m$, formulas similar to (B1) and (B2) with $b_{p,q}^s$ replaced by another, explicitly defined, sequence space. It is not difficult to check that the latter space is equal as a set to $b_{p,q}^s$, the respective norms being equivalent. We omit this verification.

To complete the proof of Theorem B we need some facts concerning duality and interpolation between the Besov spaces on M (cf. [17]).

Dualizing (B2) for $0 < s < m$, we obtain that (B1) holds also for $0 > s > -m$. Moreover, since

$$B_{p,q}^0(M) = \big(B_{p,q}^{-1/2}(M), B_{p,q}^{1/2}(M)\big)_{1/2,q}$$

and (cf. [3], Theorems 5.6.1 and 6.4.2)

$$b_{p,q}^{1/2-1/p} = (b_{p,q}^{-1/p}, b_{p,q}^{1-1/p})_{1/2,q},$$

we obtain that (B1) is true for $s = 0$ as well.

The proof of (B2) for $-m < s < m$ is analogous.

It should be rather clear that the constant C in (B1) and (B2) can be taken uniformly bounded for $|s| \leqslant m-\varepsilon$, if $\varepsilon > 0$ is fixed.

The bases constructed in Theorems A and B satisfy inequalities of Bernstein and Jackson type, i.e. one has

COROLLARY 11.2. *There exists $C < \infty$ such that if S_N, $N = 2^\mu$, denotes the N^{-1} projection operator with respect to $(f_n)_{n=1}^\infty$ (or to $(g_n)_{n=1}^\infty$) corresponding to U_N in Theorem 9.20, then*

(11.3) $$\|S_N\colon W_p^k(M) \to W_p^{k+l}(M)\| \leqslant CN^l,$$

(11.4) $$\|I-S_N\colon W_p^{k+l}(M) \to W_p^k(M)\| \leqslant CN^{-l},$$

whenever $-m \leqslant k \leqslant k+l \leqslant m$.

Proof. The special case where $l = 0$ follows from Theorem A (resp. Theorem B).

Since T in (11.1) is an isomorphism for $\mathscr{F} = W_p^k(M)$, $-m \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$, and the basis in $\mathscr{F}(M)$ is composed from bases in the summands in the way described above, it will suffice to verify the analogous statement for the bases in $\mathscr{F}(Q_i)_{Z_i}$ for $i = 1, \ldots, N$.

This, however, has been done in Theorem 9.20 at least in the case $0 \leqslant k \leqslant k+l \leqslant m$. The verification in the case $-m \leqslant k \leqslant k+l \leqslant 0$ is reduced (by a duality argument) to the case $0 \leqslant k' \leqslant k'+l' \leqslant m$ (with $k' = -k-l$, $l' = l$) for the biorthogonal system. Finally, if $-m \leqslant k < 0 < k+l \leqslant m$, since the S_N's are idempotents, we can simply use the inequality

$$\|S_N\colon W_p^k \to W_p^{k+l}\| \leqslant \|S_N\colon W_p^k \to W_p^0\| \cdot \|S_n\colon W_p^0 \to W_p^{k+l}\|$$

and a similar one for $I - S_N$ (cf. [17]). This completes the proof.

Theorem B simplifies considerably the study of the embedding maps between Besov spaces on M by reducing them to diagonal maps between the sequence spaces $b_{p,q}^\varrho$. In the sequel we always put

$$\varrho = r/d - 1/p + 1/2, \qquad \sigma = s/d - 1/q + 1/2.$$

Observe first that the inclusion

(11.5) $$B_{p,u}^r(M) \subseteq B_{q,v}^s(M)$$

is equivalent to $r > s$, $\varrho > \sigma$ if $u > v$ and to $r \geqslant s$, $\varrho \geqslant \sigma$ if $v \geqslant u$. This follows easily from Theorem B. Recall also that

(11.6) $$B_{p,1}^k(M) \subseteq W_p^k(M) \subseteq B_{p,\infty}^k(M)$$

(cf. e.g. [17]). (If k is not an integer, for W_p^k one can take the complex interpolation space.)

Let $\mathfrak{A}$ be a quasi-normed operator ideal in the sense of [47]. One can ask whether the embedding map (11.5) belongs to $\mathfrak{A}$. By Theorem B this is equivalent to $j \in \mathfrak{A}$, where j is the embedding $b_{p,u}^\varrho \to b_{q,v}^\sigma$. Clearly, the latter is equivalent to the condition

(11.7) $$(D_\lambda\colon b_{p,u}^0 \to b_{q,v}^0) \in \mathfrak{A},$$

where $\lambda = \varrho - \sigma$ and D_λ is the diagonal operator $(D_\lambda)(x_n) = (n^{-\lambda} x_n)$.

It is easy to see that there is a $\lambda_0 \in \boldsymbol{R}$ such that (11.7) holds if $\lambda > \lambda_0$ and (11.7) does not hold if $\lambda < \lambda_0$. The number λ_0 depends on $\mathfrak{A}, p, q$ but does not depend on either u or v. This follows from the inclusions

$$b_{t,\infty}^{a+\varepsilon} \subseteq b_{t,1}^a \subseteq b_{t,w}^a \subseteq b_{t,\infty}^a$$

for $\varepsilon > 0$, $1 \leqslant t, w \leqslant \infty$. Taking $u = p$, $v = q$, we have $b_{p,p}^0 = l_p$, $b_{q,q}^0 = l_q$, hence λ_0 coincides with the number $\lambda(\mathfrak{A}, p, q)$ called the *limit order* of the ideal $\mathfrak{A}$ (cf. [47]). This generalizes the result of H. König [44] (formulated for Sobolev spaces and $r \geqslant s \geqslant 0$). (This follows from (11.6).)

Limit orders for many important ideals are computed in [47]. We shall discuss briefly the case where $\mathfrak{A} = \mathfrak{P}_t$ is the ideal of t-absolutely summing operators (see [47]).

A sufficient condition for (11.5) to belong to $\mathfrak{P}_t$ is that D_λ in (11.7) admits a factorization of the type

$$b_{p,u}^0 \to l_\infty \xrightarrow[D_\alpha]{} l_t \xrightarrow[D_\beta]{} b_{q,v}^0.$$

For this it suffices that $\alpha + \beta = \lambda$, $\alpha > 1/t$, $\beta > 1/q - 1/t$, $\beta > 0$. Numbers α, β with these properties exist iff $\lambda > 1/q$ and $t > 1/\lambda$. Hence we have obtained:

COROLLARY 11.8. *If $r - s > d/p$, then the embedding* (11.5) *is t-absolutely summing for $t > 1/\lambda$, where*

$$\lambda = \varrho - \sigma = (r-s)/d - 1/p + 1/q.$$

This corollary, the inclusions (11.6) and the well-known relationship between t-absolutely summing and t-Radonifying maps (cf. [47]) imply, e.g., the following result concerning random fields on the manifold M (abstract Wiener measures).

COROLLARY 11.9. *Let γ be a cylindrical Gaussian probability on the space $B_{p,u}^r(M)$ (or $W_p^r(M)$). If $s < r - d/p$, then γ can be uniquely extended to a Radon probability measure on $B_{\infty,1}^s(M)$.*

Let us pass to applications concerning the distribution of the eigenvalues of operators acting in Besov spaces. In the results we shall formulate $(\lambda_n(S))$ will denote the sequence of non-zero eigenvalues of the (bounded linear) operator $S\colon X \to X$. (They are counted according to their algebraic multiplicities and are ordered so that $|\lambda_n(S)|$ decreases as n tends to ∞.)

First we shall extend a result of B. Carl [37] who considered the case $d = 1$, $p \leqslant q$.

COROLLARY 11.10. *Let $1 \leqslant p, q \leqslant \infty$, $r > s$ and $(r-s) > d(1/p-1/q)$. Let M be a compact d-dimensional C^∞ manifold and let $S\colon B^s_{q,1}(M) \to B^s_{q,1}(M)$ be a bounded linear operator such that $S\big(B^s_{q,1}(M)\big) \subseteq B^r_{p,\infty}(M)$. Then*

$$|\lambda_n(S)| = O(n^{-(r-s)/d}).$$

Proof. Clearly, without loss of generality we may assume that $p \leqslant q$. Carl's proof for this case can be carried with some obvious changes. Indeed, now we know by Theorem B that the entropy numbers of the embedding $B^r_{p,\infty}(M) \to B^s_{q,1}(M)$ decay at the same rate as those of the embedding $B^{r/d}_{p,\infty}(I) \to B^{s/d}_{q,1}(I)$.

A stronger result was obtained earlier by A. Pietsch [48] in case S is a sort of "integral operator with kernel K". Also he had to consider only the case $d = 1$ because Theorem B was not available for $d > 1$ at that time. The following theorem, conjectured at the end of [48] can be obtained using Pietsch's method and Theorem B.

COROLLARY 11.11. *Let M be a compact d-dimensional C^∞ manifold. Suppose that $1 \leqslant p, q, u, v \leqslant \infty$, $r > s$, $r-s > d(1/p-1/q)$. If $S\colon B^s_{q,v}(M) \to B^s_{q,v}(M)$ is the integral operator defined by a kernel K of class $[B^r_{p,u}, B^{-s}_{q',v'}]$, then $(\lambda_n(S))$ belongs to the Lorentz space $l_{t,u}$, where*

$$1/t = (r-s)/d + \min(1/2, 1/q').$$

In particular, if $u = \infty$, then $|\lambda_n(S)| = O(n^{-1/t})$.

Theorem B enables us also to simplify the estimates of the s-numbers of the embedding operators by reducing to finite dimensional problems. For the definition of the general notion of the s-number function we again refer to [47]. Here we shall only describe a typical case in which almost all is known (cf. [43]), namely the Kolmogorov diameters $\big(d_n(T)\big)$.

Recall that for the operators $T\colon X \to Y$ and $n = 0, 1, 2, \ldots$ one puts

$$d_n(T) = \inf\{\|Q_E T\|\colon E \subseteq Y, \dim E \leqslant n\},$$

where $Q_E\colon Y \to Y/E$ denotes the quotient map. Clearly, $d_n(T) \to 0$ iff T is a compact operator.

If $X \subseteq Y$ and T is the embedding operator, one writes $d_n(X, Y)$ instead of $d_n(T\colon X \to Y)$. The embeddings (11.6) show that

$$C^{-1} d_n\big(B^r_{p,1}(M), B^s_{q,\infty}(M)\big) \leqslant d_n\big(W^r_p(M), W^s_q(M)\big) \leqslant C d_n\big(B^r_{p,\infty}(M), B^s_{q,1}(M)\big),$$

where C does not depend on n. In order to estimate $d_n(B^r_{p,\infty}, B^s_{q,1})$ from above and $d_n(B^r_{p,1}, B^s_{q,\infty})$ from below one can use Theorem B which reduces the problem to the sequence spaces or, alternatively, to the case where $d = 1$. In most cases settled so far the best upper and lower estimates coincide up to a constant factor.

The diameters $d_n(b^\varrho_{p,u}, b^\sigma_{q,v})$ are now almost completely determined. Let

$$\alpha = \min\big(\varrho - \sigma + a(p, q), (\varrho-\sigma)\max(1, q/2)\big) \geqslant 0,$$

where

$$a(p, q) = \begin{cases} 0 & \text{if } 1 \leqslant p \leqslant q \leqslant 2, \\ 1/2 - 1/q & \text{if } 1 \leqslant p \leqslant 2 < q, \\ 1/p - 1/q & \text{otherwise.} \end{cases}$$

Then (assuming $u \geqslant v$ if $\alpha = 0$) one has for $n \geqslant 2$

$$C_1 n^{-\alpha} \leqslant d_n(b^\varrho_{p,u}, b^\sigma_{q,v}) \leqslant C_2 n^{-\alpha} (\log n)^\beta \tag{11.12}$$

for some $\beta \leqslant 3$ and $0 < C_1 \leqslant C_2 < \infty$ depending only on $\varrho - \sigma$, p, q, u, v and β. Moreover, (11.12) obtains with $\beta = 0$ in each of the cases $q \leqslant p$, $q \leqslant 2$, $q = \infty$, $\varrho - \sigma > (1/p-1/q)/(q/2-1)$, $p = 1 \neq q(\varrho - \sigma)$.

Let us indicate how these results are deduced from the corresponding finite dimensional facts.

Observe that if $n, m \geqslant 0$, then

$$d_n(b^\varrho_{p,u}, b^\sigma_{q,v}) \geqslant 2^{-m(\varrho-\sigma)} d_n(l_p^{2^m}, l_q^{2^m}) \tag{11.13}$$

(recall that $d_n(l_p^k, l_q^k) = 0$ if $k \leqslant n$). On the other hand, if $k_m \geqslant 0$ are integers and $\sum_{m=0}^{\infty} k_m \leqslant n$, then

$$d_n(b^\varrho_{p,u}, b^\sigma_{q,v}) \leqslant \sum_{m=0}^{\infty} 2^{-m(\varrho-\sigma)} d_{k_m}(l_p^{2^m}, l_q^{2^m}). \tag{11.14}$$

(This is a version of the method of V. E. Maiorov [45].)

The lower estimates in (11.12) follow from (11.13) and

$$d_n(l_p^N, l_q^N) \geqslant C n^{-a(p,q)} \quad \text{if } N \geqslant 2n, \tag{11.15}$$

$$d_n(l_p^N, l_q^N) \geqslant 1/2 \quad \text{if } 1 \leqslant p \leqslant q > 2 \text{ and } N \geqslant C' n^{q/2}. \tag{11.16}$$

Inequality (11.15) can be found in [43] or [41], for a slightly weaker form of (11.16) we refer to E. D. Gluskin [41].

Upper estimates in (11.12) follow from (11.14) after a suitable choice of the k_m's. One may assume that $\alpha > 0$.

The case where $q \leqslant p$ or $q \leqslant 2$ is easy, because then $d_0(l_p^N, l_q^N) = N^{-a(p,q)}$. (Just put $k_m = 2^m$ if $2^m \leqslant n/2$, $k_m = 0$ otherwise.)

Hence we may assume that $2 \leqslant p < q \leqslant \infty$ (the case where $1 \leqslant p \leqslant 2 < q$ being an easy consequence). Let $j = [\log_2 n] - 1$, $l = [(1/2)qj]$. Given $r > 0$, let b satisfy $b(r^j + r^{j+1} + \ldots + r^l) = n - 2^j$. Put $k_m = 2^m$ if $0 \leqslant m < j$, $k_m = [b \cdot r^m]$ if $j \leqslant m \leqslant l$, $k_m = 0$ if $m > l$. Using these values in (11.14) and B. S. Kashin's estimate (valid for $2 \leqslant p < q \leqslant \infty$)

$$d_k(l_p^N, l_q^N) \leqslant C\big(k^{-1/2} N^{1/q} (\log(eN/k))^{3/2}\big)^\gamma, \tag{11.17}$$

where $\gamma = (1/p - 1/q)/(1/2 - 1/q)$ (cf. [43] and [41]), we obtain (11.12) if $r = 1$. In some cases, e.g. if $\varrho - \sigma > \gamma/q$, a better choice of r yields (11.12) with $\beta = 0$.

It is not known whether the logarithmic term in (11.17) can be replaced by a constant depending on $(\log N)/\log k$. If this can be done for some $2 < q < \infty$, then one obtains easily that (11.12) holds for this q with $\beta = 0$ except for the case $a(p, q) = (q/2 - 1)(\varrho - \sigma)$ (cf. Added in proof).

In some cases, however, the factor $(\log n)^\beta$ may be necessary. Namely, if $q > 2$ and $u > v \geqslant 1$, then

$$d_n(b^{1/q}_{1,u}, b^0_{q,v}) \geqslant Cn^{-1/2}(\log n)^{1/v-1/u}.$$

This can be deduced using (11.16).

We close the paper with bibliographical comments. The first spline basis in $C(I)$ and $L_p(I)$ were treated by Haar [42], Faber [40], Schauder [51] and Franklin [39]. Extensions of those results to spaces of smooth functions over cubes are given in the papers of Ciesielski [11], Schonefeld [52], Ciesielski and Domsta [14] and Ryll [50].

The first result on unconditionality of the spline bases concerns the Haar system and it is due to Marcinkiewicz [46]. In Bochkariev [36] we find the proof that the Franklin system is an unconditional basis in $L_p(I)$, $1 < p < \infty$. For extension of those results to higher order splines we refer to Ciesielski [13].

Special cases of Theorem B were proved in Ciesielski [38], [10] and [12] and by Ropela in [49].

Theorems A and B in a less general form are presented in Ciesielski and Figiel [16].

Added in proof. The correct order of the diameters $d_n(l_p^N, l_q^N)$ in all the remaining cases has been recently found by E. D. Gluskin [53]. In particular, he proved that for $2 < p < q < \infty$ estimate (11.17) holds with $C = C(p, q)$ and 3/2 replaced by 0. Using this in the argument sketched above (now with $r > 1$), we obtain that the logarithmic factor in (11.12) may be necessary only if $q > \max(2, p)$ and $q(\varrho - \sigma) = \min(1, \gamma)$.

References to Part II

[3] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, Berlin–Heidelberg–New York 1976.

[6] C. de Boor, *On calculating with B-splines*, J. Approximation Theory 6 (1972), 50–62.

[7] — *Splines as linear combinations of B-splines*, in: *Approximation Theory II*, Ed. G. G. Lorentz et al., Academic Press, New York 1976, 1–47.

[8] — *The quasi interpolant as a tool in elementary spline theory*, in: *Approximation Theory*, Ed. G. G. Lorentz, Academic Press, New York 1973, 269–276.

[9] Z. Ciesielski, *Properties of the orthonormal Franklin system*, Studia Math. 23 (1963), 141–157.

[10] — *Properties of the orthonormal Franklin system II*, ibid. 27 (1966), 289–323.

[11] — *A construction of basis in $C^{(1)}(I^2)$*, ibid. 33 (1969), 243–247.

[12] — *Constructive function theory and spline systems*, ibid. 53 (1975), 277–302.

[13] — *Equivalence, unconditionality and convergence a.e. of the spline bases in L_p spaces*, in: *Approximation Theory*, Banach Center Publications, vol. 4, PWN–Polish Scientific Publishers, Warsaw 1979, 55–68.

[14] Z. Ciesielski and J. Domsta, *Construction of orthonormal basis in $C^m(I^d)$ and $W_p^m(I^d)$*, Studia Math. 41 (1972), 211–224.

[15] —, — *Estimates for the spline orthonormal functions and for their derivatives*, ibid. 44 (1972), 315–320.

[16] Z. Ciesielski and T. Figiel, *Construction of Schauder bases in function spaces on smooth compact manifolds*, in: *Approximation and Function Spaces*, Ed. Z. Ciesielski, PWN, Warszawa–North Holland, Amsterdam–New York–Oxford 1981, 217–232.

[17] —, — *Spline approximation and Besov spaces on compact manifolds*, Studia Math. 75 (1982), 13–36.

[18] H. B. Curry and I. J. Schoenberg, *On Pólya frequence functions IV*, J. Analyse Math. 17 (1966), 71–107.

[19] S. Demko, *Inverses of band matrices and local convergence of spline projections*, SIAM J. Numerical Analysis 14.4 (1977), 616–619.

[20] R. A. De Vore, *Degree of Approximation*, in: *Approximation Theory II*, Ed. G. G. Lorentz et al., Academic Press, New York 1976, 117–161.

[21] J. Domsta, *A theorem on B-splines*, Studia Math. 41 (1972), 291–314.

[22] C. Fefferman, *Recent Progress in Classical Fourier Analysis*, in: Proc. Int. Congr. of Mathematicians, vol. 1, August 21–29, 1974, Vancouver, Canada, 95–118.

[23] G. Freud and V. Popov, *Some questions that are related to approximation by spline functions and polynomials*, Studia Sci. Math. Hungary 5 (1970), 161–171 [Russian].

[31] C. A. McCarthy, *Commuting Boolean Algebras of Projections II, Boundedness in L_p*, Proc. Amer. Math. Soc. 15 (1964), 781–787.

[36] S. V. Bočkariev, *Some inequalities for Franklin series*, Analysis Mathematica 1 (1975), 249–257.

[37] E. Carl, *Entropy numbers, s-numbers and eigenvalue problems*, J. Functional Analysis 41 (1981), 290–306.

[38] Z. Ciesielski, *On the isomorphisms of the spaces H_α and m*, Bull. Acad. Polon. Sci. 4 (1960), 217–222.

[39] Ph. Franklin, *A set of continuous orthogonal functions*, Math. Ann. 100 (1928), 522–529.

[40] G. Faber, *Über die orthogonal Funktionen des Herrn Haar*, Jahresbericht d. deutsch. Math. Vereinigung 19 (1910), 104–112.

[41] E. D. Gluskin, *On certain finite-dimensional problems of the diameter theory*, Vest. Leningrad Univ. 13 (1981), 5–10 [Russian].

[42] A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Math. Ann. 69 (1910), 331–371.

[43] B. S. Kashin, *Diameters of some-finite dimensional sets and classes of smooth functions*, Izv. Akad. Nauk SSSR 41 (1977), 334–351 [Russian].

[44] H. König, *Grenzordnungen von Operatorenidealen*, Math. Ann. 212 (1974), 51–77.

[45] V. E. Maiorov, *Discretization of the diameter problem*, Uspehi Mat. Nauk 30 (1975), 179–180 [Russian].

[46] J. Marcinkiewicz, *Quelques théorèmes sur les séries orthogonales*, Ann. Soc. Polon. Math. 16 (1937), 85–86.

[47] A. Pietsch, *Operator ideals*, Verlag der Wissenschaften and North Holland, Berlin 1978/1980.
[48] — *Eigenvalues of integral operators I*, Math. Ann. 247 (1980), 169–178.
[49] S. Ropela, *Spline bases in Besov spaces*, Bull. Acad. Polon. Sci. 24 (1976), 467–470.
[50] J. Ryll, *Interpolating bases for spaces of differentiable functions*, Studia Math. 63 (1978), 125–144.
[51] J. Schauder, *Eine Eigenschaft des Haarschen Orthogonalsystems*, Math. Z. 28 (1928), 317–320.
[52] S. Schonefeld, *Schauder bases in the Banach spaces $C^k(T^q)$*, Trans. Amer. Math. Soc. 165 (1971), 309–318.
[53] E. D. Gluskin, *The norms of random matrices and diameters of finite dimensional sets*, Mat. Sb. 120 (1983), 180–189.

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О строении безусловных базисов некоторых пространств Кёте

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Резюме. В пространствах Кёте числовых последовательностей

$$l_p[a_r(n)] = \Big\{\xi = (\xi_n): \Big\{\sum_n [|\xi_n| a_r(n)]^p\Big\}^{1/p} = |\xi|_r < +\infty,\ r = 1, 2, \ldots\Big\},$$

$1 < p < \infty$, показано, что для любой последовательности элементов (f_m), имеющей равностепенно непрерывную биортогональную систему функционалов, выполнено условие: существуют такие отображения в λ: $N \to R^+$, σ: $N \to N$, s: $N \to N$, c: $N \to R^+$, что

$$c_r^{-1} a_r(\sigma(m)) \leqslant \lambda_m |f_m|_{s(r)} \leqslant c_r a_{s(r+1)}(\sigma(m)), \quad r, m \in N.$$

Это даёт возможность получить простое доказательство квазиэквивалентности безусловных базисов в пространствах Кёте $l_p[a_r(n)]$, $p = 1, 2, \infty$, имеющих правильный безусловный базис, а также доказать квазиэквивалентность базисов и гипотезу Бессаги в некоторых других классах пространства.

Введение. Нашей целью является изучение в пространствах Кёте свойств базисов, основными из которых считаем квазиэквивалентность безусловных базисов (КББ) и характеризацию безусловных базисов дополняемых подпространств (проблема Бессаги).

Настоящая работа продолжает многочисленные исследования в этой области (см., напр., [1], [2], [4], [5], [10], [17], [19], [20], [23]). Подробнее с историей вопросов и библиографией можно познакомиться в обзорах [21], [22].

Важную роль в работе играет теорема 1, которая утверждает, что из матрицы преднорм стандартного базиса ортов всегда можно получить матрицу, эквивалентную матрице преднорм любой последовательности элементов, имеющей равностепенно непрерывную биортогональную систему функционалов, путём повторения одних столбцов и удаления других. Это обобщает известную теорему Драгилева–Бессаги [2], [5] для базисов дополняемых подпространств ядерных пространств и недавний результат В. П. Захарюты и автора для p-абсолютных базисов дополняемых подпространств пространств Кёте.

Рассуждения теоремы 1 в комбинации с усовершенствованием приёмов работ [17], [20], [7] даёт значительно упрощённое по сравнению