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SPLIT FACES AND IDEAL STRUCTURE OF OPERATOR ALGEBRAS

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1. Introduction and notation.

The study of facial vs. ideal structure in operator algebras was initiated in 1963 by the independent works of Effros [10] and Prosser [14]. They found a one-to-one correspondence between norm closed left ideals in a C*-algebra, norm closed faces in its positive cone, and weak*-closed faces of its state space. In this correspondence, two-sided ideals correspond to invariant faces.

However, Effros and Prosser failed to characterize the invariant faces in a purely geometric way. In [16; Thm. 3.2] Størmer proved that these were exactly the Archimedean faces, while Alfsen and Andersen introduced the concept of a split face and noted that invariant faces are split [2; Prop. 7.1].

In section 2 we generalized to JB-algebras the correspondence between two-sided ideals and split faces. (For the theory of JB-algebras and their ideals, see [7], [13], [15], [3; § 2], [8], and [9]). At the same time, and equally important, we get new and more direct proofs of known results for C*-algebras. The reader primarily interested in C*-algebras may substitute C*-algebras and two-sided ideals for JB-algebras and Jordan ideals in section 2. By trivial modifications in the proofs, she can then make them valid for the C*-algebra case.

It should be mentioned here that all the results of section 2 are due to E. M. Alfsen and F. W. Shultz (unpublished). We would like to thank Alfsen and Shultz for their kind permission to include this material.

Section 3 contains the main new result of this paper. We define the structure space Prim (K) for an arbitrary compact convex set K, and give necessary and sufficient conditions for the canonical surjection $\partial_e K \to \mathrm{Prim}(K)$ to be open.

In section 4 we apply Theorem 3.1 together with the results of section 2 to generalize to JB-algebras Glimm's result [12], that the canonical mapping $\partial_e K \to \mathrm{Prim}(\mathscr{A})$ is open when $\mathscr{A}$ is a C*-algebra with state space K. The proof is rather different from Glimm's original proof, because of the lack of inner automorphisms.

By a *Jordan ideal* in a JB-algebra A we shall mean a subspace J such that, whenever $a \in A$ and $b \in J$, then $a \circ b \in J$. Jordan ideals correspond to two-

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sided ideals in the following strict sense: A norm closed self-adjoint complex subspace $\mathscr{I}$ of a C*-algebra $\mathscr{A}$ is a two-sided ideal iff its self-adjoint part $\mathscr{I}_{sa}$ is a Jordan ideal of $\mathscr{A}_{sa}$. This can be seen either by considering the weak*-closure in $\mathscr{A}^{**}$ of $\mathscr{I}$ and using [8; Thm. 2.3], or by appealing to [11; Thm. 2].

If a is an element of a JB-algebra A and ϱ is a linear functional on A, we denote by $\langle a, \varrho \rangle$ the value of the functional ϱ at the element a. Note that any JBW-algebra is canonically order- and norm-isomorphic to the space $A^b(K)$ of bounded affine functions on its normal state space K.

We define the *annihilators* of a subset J of a JB-algebra A and a subset F of its state space K by

$$J^\perp = \{\varrho \in K : \langle a, \varrho \rangle = 0 \quad \text{for all } a \in J\}$$

$$F_\circ = \{a \in A : \langle a, \varrho \rangle = 0 \quad \text{for all } \varrho \in F\} .$$

Similarly, we define the annihilator $J_\perp$ of a subset J of the JBW-algebra M and the annihilator F° of a subset F of its *normal* state space K.

If a, b are elements of a JB-algebra A we define their *Jordan triple product* $\{aba\}$ by

$$\{aba\} = 2a \circ (a \circ b) - a^2 \circ b .$$

If ϱ is a functional on A, we define functionals $a \circ \varrho$ and $\{a \varrho a\}$ by the formulas,

$$\langle b, a \circ \varrho \rangle = \langle a \circ b, \varrho \rangle ,$$

$$\langle b, \{a \varrho a\} \rangle = \langle \{aba\}, \varrho \rangle .$$

Note that if ϱ is positive, then $\{a \varrho a\}$ is positive.

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2. Split faces.

Let K be a convex set. A face F of K is called a *split face* [1; § II.6] if there exists a face F' such that K is a direct convex sum of F and F' in the following sense: Any $\varrho \in K$ can be written as

$$\varrho = \lambda \sigma + (1 - \lambda) \sigma' , \tag{2.1}$$

where $\lambda \in [0, 1]$ is unique, and $\sigma \in F$ (respectively $\sigma' \in F'$) is unique (except for the case $\lambda = 0$ (respectively $\lambda = 1$)).

Note that the face F' is uniquely determined by F. It is called the *complement* of F. Also, the mapping $\varrho \to \lambda$, where λ is determined by (2.1), is a bounded affine function in K which has F as its peak set. In our applications K will be the base in a base-norm space (E, K) [1; p. 77]. Then the above affine function on K extends to a bounded linear functional on E. Hence, *split faces are norm exposed* and, in particular, norm closed.

The following result is included in [4; Thm. 11.5], but the present proof makes no use of the machinery of [4].

THEOREM 2.1. *Let M be a* JBW-*algebra and K its normal state space. There is a one-to-one correspondence between split faces F of K and central projections e in M, given by*:

(i) $F=\{\varrho \in \mid \langle e,\varrho\rangle=0\}$

(ii) *e is the unique affine function in K which is identically* 0 *on F and* 1 *on F'.*

PROOF. Let F be a split face of K. Define e to be the affine function $\varrho \to 1-\lambda$, where λ is the scalar occuring in (2.1). It is easily seen that e is an extreme point in the positive unit ball of M, and hence a projection. We have to show that e is central.

To this end consider an arbitrary element $a \in A$. If $\varrho \in F$ we find, using the Cauchy–Schwarz inequality, that

$$|\langle e \circ a, \varrho\rangle|^2 \leqq \langle e,\varrho\rangle\langle a^2,\varrho\rangle = 0 .$$

Thus the affine function $e \circ a$ vanishes on F. Similarly, $(1-e)\circ a$ vanishes on F', so $e\circ a$ coincides with a on F'. Repeating the argument, we find that the same holds for $\{eae\}=2e\circ(e\circ a)-e\circ a$. Since an affine function on K is determined by its restrictions to F and F', we conclude that $e\circ a=\{eae\}$. Thus e is central by [7; Lemma 2.11].

The proof that, conversely, a central projection e determines a split face by (i) is left to the reader.

Combining Theorem 2.1 with [8; Thm. 2.3] we immediately obtain

COROLLARY 2.2. *There is a one-to-one correspondence between weak*-closed Jordan ideals J of M and split faces F of K, given by $F=J_\perp$ and $J=F^\circ$.*

Indeed, when the central projection e corresponds to the split face F, we have $J=\{eMe\}$.

Passing to the duality of a JB-algebra and its dual, we have:

THEOREM 2.3. *Let A be a* JB-*algebra and K its state space. There is a one-to-one correspondence between norm closed Jordan ideals J of A and weak*-closed split faces F of K, given by $F=J^\perp$ and $J=F_\circ$.*

PROOF. If J is a norm closed Jordan ideal of A, then $J^{\perp}$ is a split face and $J = (J^{\perp})_{\circ}$. This is a trivial consequence of Cor. 2.2 and the Hahn–Banach separation theorem. (Consider the weak*-closed ideal $\bar{J}$ in A^{**}).

Conversely, that $F_{\circ} = F^{\circ} \cap A$ is a Jordan ideal when F is a split face also follows trivially from Cor. 2.2. That $F = (F_{\circ})^{\perp}$ follows, for example, from [1; Thm. II.6.15]. A more elementary proof is the following: Note that the unit ball of $\operatorname{lin} F$ is $\operatorname{co}(F \cup -F)$. By the Krein–Smulian theorem it follows that $\operatorname{lin} F$ is weak*-closed. If $\varrho \in K - F$, we can then separate ϱ from $\operatorname{lin} F$ with some $a \in A$. Then $a \in F_{\circ}$, and so $\varrho \notin (F_{\circ})^{\perp}$. This completes the proof.

Our next is a generalization of [10; Cor. 6.2].

THEOREM 2.4. *Let F be a split face of the state space K of a* JB*-algebra A. Then its weak*-closure $\bar{F}$ is also a split face of K.*

PROOF. By Cor. 2.2, $F_{\circ} = F^{\circ} \cap A$ is a Jordan ideal of A, and hence $G = (F_{\circ})^{\perp}$ is a weak*-closed split face of K. We shall prove that $\bar{F} = G$.

Let e be the central projection in A^{**} such that $F^{\circ} = (1-e) \circ A^{**}$. Since $G_{\circ} = F^{\circ} \cap A$, the mapping $a \to e \circ a$ induces an injective, and hence isometric, homomorphism $A/G_{\circ} \to e \circ A^{**}$.

Let $a \in A$. As in the proof of Theorem 2.1, we note that $e \circ a$ is the unique affine function on K coinciding with a on F and vanishing on F'. Therefore,

$$\|e \circ a\| = \sup \{|\langle a, \varrho \rangle| : \varrho \in F\} .$$

On the other hand, the quotient norm of $a + G_{\circ}$ in $A/G_{\circ}$ satisfies

$$\|a + G_{\circ}\| \geqq \sup \{|\langle a, \varrho \rangle| : \varrho \in G\} .$$

Since $\|a + G_{\circ}\| = \|e \circ a\|$, an application of the Hahn–Banach separation theorem yields $G \subseteqq \bar{F}$.

Finally, we mention a geometric property of Jordan homomorphisms:

PROPOSITION 2.5. *Let M_1 and M_2 be* JBW*-algebras with normal state spaces K_1, K_2 respectively. If $\varphi \colon M_1 \to M_2$ is a weak*-continuous Jordan homomorphism, then the predual map φ_* maps split faces of K_2 onto split faces of K_1.*

PROOF. We only scetch the proof, since this result is not needed in the sequel. Let F be a split face of K_2. We claim

$$\varphi_*(F) = \varphi^{-1}(F^{\circ})_{\perp} ,$$

which will complete the proof, by Cor. 2.2.

The special case

$$\varphi_*(K_2) = \operatorname{Ker}(\varphi)_\perp$$

is, in fact, easily proved using the Hahn–Banach extension theorem. This special case then yields the general case when we consider the composition of φ with the canonical map $M_2 \to M_2/F^\circ$.

3. Structure space of an arbitrary compact convex set.

In this section K will be a compact convex set in a locally convex topological vector space. Given $\varrho \in \partial_e K$ there exists a smallest closed split face $\bar{F}_\varrho$ containing ϱ. (See [1; p. 146]. Note that our notation differs from that in [1]. We write $\bar{F}_\varrho$ although in this generality we attach no meaning to the symbol F_ϱ. This is for consistency with the notation of section 4). We call the split face $\bar{F}_\varrho$ *primitive*, and denote by Prim (K) the set of all primitive split faces. We endow Prim (K) with the *structure topology*, whose closed sets are those of the form

$$\{G \in \operatorname{Prim}(K) : G \subseteqq F\},$$

where F is a closed split face of K. This topology exists by virtue of [1; Prop. II. 6.20]; we remark that Størmer's axiom, as imposed in [1; Lemma 6.25] is not necessary for this definition.

We consider the map $\varrho \to \bar{F}_\varrho$ of $\partial_e K$ onto Prim (K). This mapping is continuous, with $\partial_e K$ given the relative topology. We will characterize those K for which this map is also open. First, however, we need a definition.

Following [1; p. 146] we say that K satisfies *Størmer's axiom* if, whenever (F_α) is a collection of closed split faces of K, the closed convex hull $\overline{\text{co}}\,(\bigcup_\alpha F_\alpha)$ is a split face.

The following Theorem is an improvement of [1; Lemma II.6.29]. Note that we do not use the concept of sufficiently many inner automorphisms, which was used in [1] and is also buried in Glimm's original proof of the corresponding C*-algebra result [12].

THEOREM 3.1. *Let K be a compact convex set in a locally convex topological vector space. The mapping $\varrho \to \bar{F}_\varrho$ is open from the relative topology of $\partial_e K$ to the structure topology of* Prim (K) *iff K satisfies Størmer's axiom and the following condition*:

(*) *For any $G \in$* Prim (K), *the set $\{\varrho \in \partial_e G : \bar{F}_\varrho = G\}$ is dense in $\partial_e G$.*

PROOF. 1. Assume that the map $\partial_e K \to \operatorname{Prim}(K)$ is open. Let (F_α) be a collection of closed split faces of K, and consider the following (relatively) open subset of $\partial_e K$:

$$V = \partial_e K - \overline{\bigcup_\alpha \partial_e F_\alpha} \tag{3.1}$$

By assumption the set $\{\bar{F}_\varrho : \varrho \in V\}$ is open in Prim (K). By definition of the structure topology, there exists a closed split face F of K such that, whenever $\varrho \in \partial_e K$:

$$\varrho \notin F \Leftrightarrow \bar{F}_\varrho = \bar{F}_\sigma \quad \text{for some } \sigma \in V. \tag{3.2}$$

If $\varrho \in \partial_e F_\alpha$ then $\bar{F}_\varrho \subseteqq F_\alpha$, and so, by (3.1), $\bar{F}_\varrho \neq \bar{F}_\sigma$ for all $\sigma \in V$. By (3.2), $\varrho \in F$, and therefore $F_\alpha \subseteqq F$. We claim that $F = \overline{\text{co}}\,(\bigcup_\alpha F_\alpha)$. If not, we find some $\varrho \in \partial_e F$ with $\varrho \notin \overline{\text{co}}\,(\bigcup_\alpha F_\alpha)$. By (3.1) $\varrho \in V$, so by (3.2) $\varrho \notin F$. This contradiction proves our claim, and the validity of Størmer's axiom is proved.

Next, assume that (*) does not hold and choose $G \in$ Prim (K) not satisfying (*). Then there exists an open set $V \subseteqq \partial_e K$ such that $V \cap G \neq \emptyset$ and $G \neq \bar{F}_\varrho$, whenever $\varrho \in V$. As above, there is a closed split face F of K such that (3.2) holds. If $G = \bar{F}_\varrho$ then, by (3.2), $\varrho \in F$ and so $G \subseteqq F$. By (3.2) this implies that $G \cap V = \emptyset$, which is a contradiction. Thus (*) is necessary.

2. Assume that K satisfies Størmer's axiom and the property (*). Let V be a (relatively) open subset of $\partial_e K$, and let

$$F = \overline{\text{co}}\,(\bigcup \{G \in \text{Prim}\,(K) : G \cap V = \emptyset\}) . \tag{3.3}$$

By Størmer's axiom, F is a split face. We claim that

$$\{\bar{F}_\varrho : \varrho \in V\} = \{G \in \text{Prim}\,(K) : G \nsubseteq F\} , \tag{3.4}$$

which will complete the proof since the righthand side of (3.4) is an open subset of Prim (K).

Milman's theorem implies that the union of all $\partial_e G$, where $G \in$ Prim (K) and $G \cap V = \emptyset$, is dense in $\partial_e F$. In particular, since V is open, $V \cap \partial_e F = \emptyset$. Thus, if $\varrho \in V$ then $\bar{F}_\varrho \nsubseteq F$ and one inclusion in (3.4) is proved.

On the other hand, if $G \in$ Prim K and $G \nsubseteq F$ then (3.3) implies that $G \cap V \neq \emptyset$. By the property (*), $G = \bar{F}_\varrho$ for some $\varrho \in G \cap V$. Now the second inclusion in (3.4) follows, and the proof is complete.

REMARK. If K is a Choquet simplex, any extreme point of K is a split face, and so the property (*) is trivial. However, K does not satisfy Størmer's axiom unless $\partial_e K$ is closed [1; Thm. II.7.19]. Thus (*) does not imply Størmer's axiom.

To see that, conversely, Størmer's axiom is not sufficient in the above Theorem, we consider a compact convex set K which contains only one non-trivial closed split face F. Then Størmer's axiom is trivially satisfied. If F contains an extreme point ϱ which is isolated in $\partial_e K$, then $\{\varrho\}$ is an open subset of $\partial_e K$ whose image $\{F\}$ in Prim (K) is not open.

We briefly indicate how such a set can be constructed. Let K_1 be a compact convex set containing only one non-trivial closed split face $\{\sigma_1\}$, e.g., the state space of the algebra of compact operators on an infinite dimensional Hilbert space, with the unit adjoined. Let K_2 be a square. In the direct convex sum of K_1 and K_2, identify σ_1 with a corner σ_2 of K_2. More precisely, K is the state space of the order unit space

$$A = \{(a_1, a_2) \in A(K_1) \oplus A(K_2) : \langle a_1, \sigma_1 \rangle = \langle a_2, \sigma_2 \rangle\} .$$

Then the only non-trivial closed split face of K is (the image of) K_2, and any corner of K_2 other than σ_2 is isolated in $\partial_e K$.

4. The primitive ideal space of a JB-algebra.

In this section A will be a JB-algebra with a unit 1, and K its state space.

We define the primitive ideal space of A to be $\operatorname{Prim}(A) = \{\ker \varphi_\varrho : \varrho \in \partial_e K\}$, where $\varphi_\varrho : A \to A_\varrho$ is the dense representation associated with ϱ. (See [3; § 2]). Note that φ_ϱ^* maps the normal state space of A_ϱ bijectively onto the smallest split face F_ϱ of K containing ϱ. By Theorem 2.4, $\bar{F}_\varrho$ is a split face, and indeed by the proof of that theorem, $\bar{F}_\varrho = (\ker \varphi_\varrho)^\perp$. Thus $\ker \varphi_\varrho \to \bar{F}_\varrho$ is a bijection of $\operatorname{Prim}(A)$ and $\operatorname{Prim}(K)$. Defining the Jacobson topology on $\operatorname{Prim}(A)$ in analogy with the C*-algebra case, we see (using Theorem 2.3) that $\operatorname{Prim}(A)$ and $\operatorname{Prim}(K)$ are homeomorphic.

THEOREM 4.1. *Let A be a* JB-*algebra with state space K. The mapping $\varrho \to \ker \varphi_\varrho$ is a continuous and open map from $\partial_e K$ with weak*-topology onto* $\operatorname{Prim}(A)$.

PROOF. We shall prove that K satisfies the requirements of Theorem 3.1.

We start with Størmer's axiom. If F_α is a closed split face of K, the Krein–Milman theorem implies that F_α is the closed convex hull of the union of all F_ϱ, where $\varrho \in \partial_e F_\alpha$. Thus, we need only assume given a subset C of $\hat{K} = \{F_\varrho : \varrho \in \partial_e K\}$, and we have to prove that $\overline{\operatorname{co}} \bigcup_{F \in C} F$ is a split face.

In [6; Cor. 5.8] it is proved that the σ-convex hull of $\partial_e K$, defined as

$$\sigma - \operatorname{co}(\partial_e K) = \left\{ \sum_{j=1}^{\infty} \lambda_j \varrho_j : \lambda_j \geqq 0, \sum \lambda_j = 1, \varrho_j \in \partial_e K \right\}$$

is a split face of K. We claim that $\sigma - \operatorname{co}(\partial_e K)$ is a direct σ-convex sum of the split faces $G \in \hat{K}$. By this we mean that any $\varrho \in \sigma - \operatorname{co}(\partial_e K)$ is uniquely representable in the form

$$\varrho = \sum_{F \in \hat{K}} \lambda_F \varrho_F , \tag{4.1}$$

where $\lambda_F \geqq 0$, $\sum \lambda_F = 1$, and $\varrho_F \in F$. We omit the trivial proof. (At one stage one has to use that F' is norm closed, so that $(1-\lambda_F)^{-1} \sum_{G \neq F} \lambda_G \varrho_G \in F'$).

Returning to our subset C of $\hat{K}$, we find at once from the decomposition (4.1) that $\sigma-\mathrm{co}\,(\bigcup\{F : F \in C\})$ is a split face of $\sigma-\mathrm{co}\,(\partial_e K)$, and hence of K. By Theorem 2.4, its closure $\overline{\mathrm{co}}\,(\bigcup\{F : F \in C\})$ is also a split face, so the validity of Størmer's axiom is proved.

Next, if $\varrho \in \partial_e K$ then $F_\varrho = \sigma-\mathrm{co}\,(\partial_e F_\varrho)$, and so $\bar{F}_\varrho = \overline{\mathrm{co}}\,(\partial_e F_\varrho)$. By Milman's theorem, $\partial_e F_\varrho$ is dense in $\partial_e \bar{F}_\varrho$. However, if $\sigma \in \partial_e F_\varrho$ then $F_\sigma = F_\varrho$, so $\bar{F}_\sigma = \bar{F}_\varrho$. From this the condition (*) of Theorem 3.1 follows, and the proof is complete.

COROLLARY 4.2. *If A is a* JB-*algebra then* Prim (A) *is a Baire space in the Jacobson topology.*

PROOF. By [1; Cor. I.5.14] $\partial_e K$ is a Baire space in the weak*-topology. The Corollary now follows from Theorem 4.1.

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