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SPLITTING AN α-RECURSIVELY ENUMERABLE SET

BY

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ABSTRACT. We extend the priority method in α-recursion theory to certain arguments with no *a priori* bound on the required preservations by proving the splitting theorem for all admissible α. THEOREM: *Let C be a regular α-r.e. set and D be a nonrecursive α-r.e. set. Then there are regular α-r.e. sets A and B such that $A \cup B = C$, $A \cap B = \emptyset$, $A, B \leq_\alpha C$ and such that D is not α-recursive in A or B.* The result is also strengthened to apply to $\leq_{c\alpha}$, and various corollaries about the structure of the α and $c\alpha$ recursively enumerable degrees are proved.

In ordinary recursion theory one can distinguish various types of priority arguments. The major split is between finite and infinite injury constructions but finer distinctions can and should be drawn. Thus for example one should mark the difference between the arguments for the Friedburg-Muchnik solution of Post's problem [2] and Sacks' splitting theorem [3], [7]. In the former there is an *a priori* recursive bound on the preservations initiated for a given requirement and so on the injuries it inflicts on lesser requirements. There are, however, no such bounds available in the proof of the latter theorem.

As long as one remains within the confines of ordinary recursion theory these distinctions are relatively unimportant. All that one ever needs in the proofs is that the injuries to each initial segment of requirements are bounded. Since the union over an initial segment of ω of finite sets is finite, the proofs go through without regard to the more subtle questions of the existence of recursive bounds on the injury sets. The situation changes dramatically when one enters the realm of recursion theory generalized to all admissible ordinals α. The problem is that the union over an initial segment of α of sets each of which is α-finite need not be α-finite. In general the collection itself must be given as an α-

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[1]The results of this paper (except § 5) are contained in the author's Ph.D. thesis, M.I.T., 1972 which was written under the supervision of Professor G. E. Sacks. At that time the author was partially supported by an NSF Graduate Fellowship. While preparing this paper for publication he was partially supported by NSF Grant GP 28700.

finite union of α-finite sets for one to be sure that the union is α-finite. Even in the proof of the Friedburg-Muchnik theorem, however, one is not so lucky as to have everything presented on a silver platter. Indeed, a crucial point in the Sacks-Simpson proof of the theorem in α-recursion theory [6] is their Lemma 2.3 which enables them to handle unions which are only α-r.e. Unfortunately, their lemma only applies to collections with a uniform *a priori* bound on the size of the members. It is therefore admirably suited to priority arguments of the Friedburg-Muchnik type but does not seem to suffice for ones like the splitting theorem which lack appropriate bounds. Indeed to date the only priority arguments that have been carried out for all admissibles have been of the Friedburg-Muchnik type. In this paper we exhibit a construction that enables one to handle priority arguments of the second type.(2) In particular we prove a strong form of the splitting theorem for all admissible ordinals.

THEOREM. *Let C be a regular α-r.e. set and D be a nonrecursive α-r.e. set. Then there are regular α-r.e. sets A and B such that $A \cup B = C$, $A \cap B = \emptyset$, $A, B \leqslant_\alpha C$ and such that D is not α-recursive in A or B.*

We should also remark that the methods developed in this paper have become key elements in the generalization of an infinite injury priority argument to all admissible ordinals. Indeed, by extending these methods and introducing some other ideas as well we have used an infinite injury argument to show that the recursively enumerable α-degrees are dense for all admissible α [10].

As for background material for this paper, a familiarity with the splitting theorem in ordinary recursion theory would be helpful (particularly the account in [7]) but is not necessary as we will make our proof in §1–3 essentially self-contained. In §0 we give all the basic definitions for α-recursion theory and reference the few elementary but nonobvious facts that we need. After a heuristic description of our construction at the beginning of §1 we give the formal details of the procedure. In §2 we show that our goals for the construction are in fact achieved via a priority argument and so prove the theorem. §3 is devoted to deriving some corollaries about the structure of the α-r.e. degrees while in §4 we indicate how to strengthen our results to include α-calculability degrees as well. Finally in §5 we discuss the problems of making the construction uniform and of splitting nonregular sets.

(2)C. T. Chong has independently proven that there are incomparable α-r.e. degrees below every α-r.e. degree. Though an even stronger result is a simple corollary of our theorem (Corollary 3.3) a similar priority argument is needed even for the weaker result. His methods however are rather different from ours and do not seem to suffice for the splitting theorem.

0. **Preliminaries.** Let L_α be the collection of sets constructed by level α in Gödel's constructible universe L. α is *admissible* if L_α satisfies the replacement axiom schema of ZF for Σ_1 formulas. A subset of L_α is *α-recursively enumerable* (α-r.e.) if it has a Σ_1 definition over L_α and a partial function f is called *partial α-recursive* if its graph is α-r.e. The function is *α-recursive* if its domain is all of α while subsets of α are recursive if their characteristic functions are. Note that there is a one-one α-recursive map of α onto L_α and so it suffices to consider functions on α and subsets of α. Thus we only required that a function be total on α rather than L_α to be α-recursive. We also remark that a nonempty subset A of α is α-r.e. if and only if it is the domain of a partial α-recursive function if and only if it is the range of an α-recursive function. Moreover, if A is not *α-finite* (i.e., not a member of L_α) this last function can be chosen to be one-one. For all such basic facts about α-recursion theory that do not seem obvious we refer the reader to [4] or [6].

The main fact about admissible ordinals α is that one can perform Δ_1 (= α-recursive) recursions in L_α. Thus for example we can Gödel number the α-r.e. subsets of α $\{R_\epsilon\}_{\epsilon<\alpha}$ and we can even define an α-recursive simultaneous enumeration of the α-r.e. sets. We indicate this process by writing R_ϵ^σ for the elements of R_ϵ enumerated by stage σ in this standard enumeration. We now use this enumeration to define relative computations and recursiveness. We begin with an approximation. $[\epsilon]_\sigma^C(\gamma) = \delta$ if and only if

$$(\exists\rho)(\exists\eta)[\langle\gamma, \delta, \rho, \eta\rangle \in R_\epsilon^\sigma \;\&\; K_\rho \subseteq C \cap \sigma \;\&\; K_\eta \subseteq (\alpha - C) \cap \sigma]$$

where we have coded all n-tuples $(\gamma, \delta, \rho, \eta)$ and α-finite sets K_ρ and K_η as the ordinals $\langle\gamma, \delta, \rho, \eta\rangle, \rho$ and η in an α-recursive manner. We then say that $[\epsilon]^C(\gamma) = \delta$ if $[\epsilon]_\sigma^C(\gamma) = \delta$ for some σ. We call $\langle\gamma, \delta, \rho, \eta\rangle$ a computation associated with $[\epsilon]_\sigma^C(\gamma) = \delta$ and say that it requires K_η to be outside of C. Note that this makes $[\epsilon]^C$ a possibly multivalued function. We will often use the α-recursive well ordering of L_α gotten by restricting the usual well ordering of L to L_α to choose a least computation associated with some $[\epsilon]_\sigma^C(\gamma) = \delta$. If C is α-recursive (e.g. an approximation to an α-r.e. set) this procedure will also be α-recursive. In general we say that $[\epsilon]_\sigma^A(\gamma)$ or $[\epsilon]^A(\gamma)$ is convergent if it equals δ for some δ.

Finally, we say that a partial function f is weakly α-recursive in C ($f \leqslant_{w\alpha} C$) if and only if $f = [\epsilon]^C$ for some ϵ (and so in particular $[\epsilon]^C$ is single valued). Of course, set B is weakly α-recursive in C if its characteristic function is.

Though it will be convenient to arrange our splitting of C into A and B so that our given set D is not weakly α-recursive in A or B, $\leqslant_{w\alpha}$ is not the reducibility in which we are really interested. The real notion of α-recursiveness that we want requires that one be able to recover all α-finite subsets of A and $\alpha - A$ (also written A^c) from B in order to say that A is α-recursive in B ($A \leqslant_\alpha B$). Technically, $\leqslant_\alpha$ has the advantage of being transitive while $\leqslant_{w\alpha}$ is not always so. Philosophically, it is really the α-finite sets, i.e. the members of L_α, which are the individuals of our universe of discourse and about which we should be inquiring of B. As for the formal definition, we say $A \leqslant_\alpha C$ if and only if there is an ϵ such that

$$K_\gamma \subseteq A \leftrightarrow (\exists \rho)(\exists \eta)(\langle \rho, \eta, \gamma, 0\rangle \in R_\epsilon \,\&\, K_\rho \subseteq C \,\&\, K_\eta \subseteq \alpha - C)$$

and

$$K_\gamma \subseteq \alpha - A \leftrightarrow (\exists \rho)(\exists \eta)(\langle \rho, \eta, \gamma, 1\rangle \in R_\epsilon \,\&\, K_\rho \subseteq C \,\&\, K_\eta \subseteq \alpha - C)$$

for all α-finite sets K_γ.

Note that if A is α-r.e. the first condition can always be satisfied, and so to show that $A \leqslant_\alpha C$ one only needs an R that indicates relative to C when $K \cap A = \emptyset$. On the other hand, to show in our construction that $D \nleqslant_\alpha A$ it clearly suffices to show that $D \nleqslant_{w\alpha} A$ i.e., $[\epsilon]^A \neq D$ for any ϵ. Finally as $\leqslant_\alpha$ is transitive we have the notion of α-degree: $\deg(A) = \{B \subseteq \alpha | B \leqslant_\alpha A \leqslant_\alpha B\}$. As usual the α-degrees form an upper semilattice under $\leqslant_\alpha$. The join of two degrees $\deg(A) \vee \deg(B)$ is given by $\deg(A) \vee \deg(B) = \deg(C)$ where $C = \{\lambda + 2n | \lim(\lambda) \,\&\, \lambda + n \in A\} \cup \{\lambda + 2n + 1 | \lim(\lambda) \,\&\, \lambda + n \in B\}$. Of course, among the α-r.e. degrees there is a greatest, $\deg(\{\langle x, y\rangle | x \in R_y\})$, called the complete α-r.e. degree.

Next we need the notion of projection. The Σ_1-projectum of α, called α^*, is the least ordinal β such that there is a one-one α-recursive map f (called a projection) of α into β. The key fact about α^* is that every Σ_1 subset of a proper initial segment of α^* is α-finite [4]. We can use such an α-recursive projection $f\colon \alpha \longrightarrow \alpha^*$ to push our Gödel numbering of reduction procedures below α^*. In particular, we now use $[f^{-1}\epsilon]^A$ as our reduction procedure by first defining $[f^{-1}\epsilon]^A_\sigma(\gamma) = \delta$ if and only if there is an $\eta < \sigma$ such that $f(\eta) = \epsilon$ and $[\eta]^A_\sigma(\gamma) = \delta$. We now continue as before to define $[f^{-1}\epsilon]^A$. By having the reduction procedures listed below α^* we will be able to take advantage of the fact that any α-r.e. subset of α bounded below α^* is α-finite.

Finally, we define an $A \subseteq \alpha$ to be *regular* if $A \cap \beta$ is α-finite for every $\beta < \alpha$. Thus, for example, if $\alpha^* = \alpha$ every α-r.e. subset of α is regular. The key fact here is that every α-r.e. degree contains a regular set, i.e., given any α-r.e. set D there is an α-r.e. D' such that $D \leqslant_\alpha D' \leqslant_\alpha D$ [5].

1. The construction.

1.1. *The intuitive picture*. We begin by noting that since every α-r.e. degree contains a regular set we can assume without loss of generality that D is regular. We let c and d be one-one α-recursive functions which enumerate C and D

respectively. As usual, we dentoe the associated approximations to C and D by $C^\sigma = \{c(i) | i < \sigma\}, D^\sigma = \{d(i) | i < \sigma\}$.

The general plan of the construction calls for putting $c(\sigma)$ into exactly one of A and B at stage σ. This will assure us that $A \cup B = C$, $A \cap B = \emptyset$ and that $A, B \leqslant_\alpha C$. We will also have various elements that we wish to keep out of A and B for the sake of requirements associated with the condition that D not be recursive in A or B. These requirements will be ordered by a priority system that will be developed along with the construction. At any stage σ we will have an approximation to the final priority listing which we will use to determine whether $c(\sigma)$ is put into A or B. Essentially we will choose whichever will preserve the requirements of the highest possible priority.

In line with Sacks' original approach to this theorem [6] we try to insure that $D \nleqslant_\alpha A$ by the roundabout method of preserving (for each ϵ) computations of $[\epsilon]^A$ on initial segments as long as they seem to agree with D. The idea is that if $[\epsilon]^A = D$ for some ϵ, then we would eventually be preserving the first available computation of $[\epsilon]^A(x)$ for each x. We would then be able to compute $[\epsilon]^A$ and so D α-recursively—a contradiction. On the basis of such considerations, however, we can only argue that there is some bound on the x's for which we preserve computations of $[\epsilon]^A(x)$. We cannot assign any uniform *a priori* value to this bound and so find ourselves heir to the problems discussed at the beginning of this paper.

Our strategy for handling these difficulties is first to arrange the requirements $[\epsilon]^A \neq D$ in blocks P_γ and then to consider $[\epsilon]^A \neq D$ for all ϵ in P_γ as a single requirement. Thus we will consider $D(x)$ as computed from block P_γ if we have any ϵ in P_γ for which we as yet have no counterexample to $[\epsilon]^A = D$ and for which $[\epsilon]^A(x) = D(x)$. Since we will prove that there is a bound on the appearance of counterexamples, this blocking will not prevent us from recovering D correctly should the preservations associated with P_γ be unbounded. Finally we will have the determination of the size and number of these blocks interwoven with the construction in such a way that the blocks progress through the list of reduction procedures cofinally with the progression of preservations and injuries through α.

1.2. *The actual construction.* Since A and B play entirely analogous roles in the construction we will describe explicitly only the A part. It is of course understood that similar steps are to be taken on behalf of B. Before beginning the construction we let $f\colon \alpha \xrightarrow{1\text{-}1} \alpha^*$ be an α-recursive projection. The rest of our terminology will be defined simultaneously with the construction.

At stage σ we will have blocks P_γ^σ each of which will be an initial segment of α^*. For each γ we find the least x for which there is no $\gamma - A$ requirement with argument x associated with a $\gamma - A$ active reduction procedure. If there is a $\gamma - A$ active reduction procedure ϵ in P^σ_γ for which $[f^{-1}\epsilon]^{A^\sigma}_\sigma(x) = D^\sigma(x)$, we *create a* $\gamma - A$ *requirement with argument* x *associated with* ϵ. This requirement consists of the elements required to be out of A^σ by the least computation associated with $[f^{-1}\epsilon]^{A^\sigma}_\sigma(x) = D^\sigma(x)$. ($A^\sigma$, of course, is the set of elements enumerated in A by stage σ.)

If at any stage we put an element of a $\gamma - A$ requirement into A we *destroy* the requirement. A reduction procedure $\epsilon < \alpha^*$ is $\gamma - A$ *active* at stage σ unless there is a $\gamma - A$ requirement (as yet undestroyed) with argument x associated with ϵ such that $[f^{-1}\epsilon]^{A^\sigma}_\sigma(x) = 0 \neq D^\sigma(x)$, i.e., x has been enumerated in D since the requirement was created. The idea is that as long as we seem to have a computation showing that $[f^{-1}\epsilon]^A \neq D$ we need pay no further attention to ϵ.

Now for the definition of the blocks P^σ_γ:

$$P^\sigma_0 = \{0\},\ \forall\sigma, \quad P^\sigma_\gamma = \bigcup\{P^\sigma_\delta | \delta < \gamma\} \cup f(y^\sigma_\gamma) + 1,$$

where $y^\sigma_\gamma = \bigcup\{\tau < \sigma | (\exists\delta < \gamma)$ a δ-requirement is created at stage τ or an element of a δ-requirement is enumerated in C at stage $\tau\}$.

The idea is that (for some λ) each block P^σ_γ, $\gamma < \lambda$, will eventually reach a constant value, P_γ, which will reflect (via f) a bound on all injuries caused by our having to preserve δ-requirements for $\delta < \gamma$. Once we have such a bound we can show that $P_{\gamma+1}$ is bounded via the argument about recovering D alluded to in §1.1. Moreover, if λ is the least ordinal such that $\bigcup\{P_\gamma | \gamma < \lambda\}$, then every P_γ will be so handled and we will succeed in getting $D \not\leq_\alpha A, B$.

Finally we take $c(\sigma)$ and put it into A or B so as to preserve as much as possible. More precisely, we consider the sets I_A (I_B) of $A(B)$ requirements which would be destroyed by putting $c(\sigma)$ into $A(B)$. Let δ_A (δ_B) be the least ordinal γ such that I_A (I_B) contains a γ-requirement. If $\delta_A \leq \delta_B$ we put $c(\sigma)$ into B; otherwise it goes into A. Thus we have given a $\gamma - A$ requirement priority over a $\delta - B$ requirement iff $\gamma \leq \delta$. This then completes our description of the construction.

2. **The priority argument.** Our primary concern is to show that enough blocks eventually settle down so that we can argue that $[f^{-1}\epsilon]^A \neq D$ for every $\epsilon < \alpha^*$. We proceed inductively.

LEMMA 2.1. *If, by some stage* τ, P^σ_δ *and* y^σ_δ *have stabilized* (*i.e. become constant*) *at values* P_δ *and* y_δ *respectively for* $\delta < \gamma$ *and* $\bigcup\{P_\delta | \delta < \gamma\} < \alpha^*$, *then* P^σ_γ *and* y^σ_γ *eventually stabilize* (*of course at values less than* α^* *and* α *respectively*).

PROOF. Note that $\bigcup\{P_\delta \mid \delta < \gamma\} < \alpha^*$ implies that $\bigcup\{y_\delta \mid \delta < \gamma\} < \alpha$ since $f(y_\delta) \leq \bigcup\{P_\delta \mid \delta < \gamma\}$ for each $\delta < \gamma$ by definition and no sequence unbounded in α can project down to one bounded in α^*. (f^{-1} is Σ_1, as f is, and so therefore is its domain. Thus restricting the domain of f^{-1} to any proper initial segment of α^* produces an α-finite set. The admissibility of α then says precisely that the range of f^{-1} on this α-finite set (which includes all the y_δ's) is bounded in α.)

To establish the lemma it clearly suffices to show that y_γ^σ eventually stabilizes. For γ a limit ordinal this is immediate from our assumptions and the definition of y_γ^σ. We therefore consider the case $\gamma = \nu + 1$ and show that there is a bound on the stages at which ν-requirements are created and at which elements of such requirements are enumerated in C.

Let $y = \bigcup\{y_\delta \mid \delta < \gamma\}$ and look at the construction from stage y onward. By definition of the y_δ everything connected with δ-requirements for $\delta < \nu$ has stopped acting up by stage y. In particular we never have to worry about preserving any δ-requirement for $\delta < \nu$ after stage y. Thus any $\nu - A$ requirement existing at stage y or created thereafter is never destroyed. (It could only be destroyed if some element which is also in a $\delta - B$ requirement for $\delta < \nu$ is enumerated in C. This, of course, cannot occur by the definition of y.) The computation associated with such a requirement is therefore correct, i.e. $[f^{-1}\epsilon]_\sigma^{A^\sigma}(x) = [f^{-1}\epsilon]^A(x)$. We must show that there are not too many of them.

Consider the set W of ϵ in P_ν such that ϵ is ν-inactive at some stage after y. As being ν-inactive at stage σ is α-recursive, this is a Σ_1 subset of $P_\nu < \alpha^*$ and so α-finite. Moreover, once any ϵ becomes ν-inactive after stage y it remains so forever since by the above remarks the associated requirement can never be destroyed. Of course, the stages at which each ϵ in W becomes ν-inactive are given by an α-recursive function g with domain W (just carry out the α-recursive construction from stage y onward until it occurs). Since W is α-finite, the admissibility of α guarantees a bound, say τ, on the range of g. After stage τ no $\gamma - A$ requirement which is associated with any ϵ in W can be created. Moreover, any $\nu - A$ requirement with argument x created after stage τ is associated with a computation giving the correct value of $D(x)$. The point is that the elements required to be outside of A by this computation are never put into A while the only change in D that can occur is that a new element of D is enumerated. This, however, would put the reduction procedure involved into W by the definition of ν-inactive contradicting our choice of τ.

Now at stage τ there are active $\nu - A$ requirements with associated arguments x for each x less than some ordinal β with the possible omission of some α-finite subset of β. After stage τ new $\nu - A$ requirements are created with arguments taken from the remaining elements in order. As the construction is α-recursive, the admissibility of α guarantees that the only way such requirements could be created α-infinitely often is eventually to have one with argument x for each $x < \alpha$. Were this to occur we could calculate D α-recursively as follows: to decide if $K \cap D = \emptyset$, begin at stage τ and proceed until a $\nu - A$ requirement with argument x associated with an ϵ not in W has been created for every $x \in K$. (Since such a stage exists for each $x \in K$ and the map from x to that stage is recursive, there is one stage by which it has all happened.) Now simply check the values of the computations associated with each argument to get the true value of $D(x)$. Since D is not α-recursive, there is a bound on the stages at which $\nu - A$ requirements are created. Moreover the collection of $\nu - A$ requirements is an α-finite set and since C is regular there is a bound on the stages at which elements of these requirements are enumerated in C. Thus the contribution to y_γ^σ from $\nu - A$ requirements eventually stabilizes. Beginning at such a stage the same argument now shows that the contributions from $\nu - B$ requirements also stabilize. Thus y_γ^σ and hence P_γ^σ attain constant values below α and α^* respectively. □

In view of this lemma we can let $\lambda \leqslant \alpha^*$ be the least ordinal such that $\alpha = \bigcup\{P_\delta \mid \delta < \lambda\}$ and be assured that λ is a limit ordinal and that P_δ does in fact exist for each $\delta < \lambda$. We are now in a position to prove that our construction has succeeded. As noted before $A \cup B = C$ and $A \cap B = \emptyset$ are immediate while $A, B \leqslant_\alpha C$ is only slightly less obvious. To check if $K \cap A = \emptyset$ just find a stage σ such that $C^\sigma \cap K = C \cap K$ and ask if $A^\sigma \cap K = \emptyset$. This clearly represents a reduction procedure for A from C as well as a proof that A and B are regular. As A and B are α-r.e. by construction we only have to prove the following:

LEMMA 2.2. *D is not α-recursive in A or B.*

PROOF. Assume not. For the sake of definiteness say $[f^{-1}\epsilon]^A = D$. Let ν be the least ordinal $<\lambda$ such that $\epsilon < P_\nu$. By Lemma 2.1 there is a least x which is not the argument of a $\nu - A$ requirement associated with a $\nu - A$ active ϵ' at any stage after y_ν. As $[f^{-1}\epsilon]^A = D$, ϵ cannot be $\nu - A$ inactive at any stage after y_ν. Moreover, since A is α-r.e., there is a stage $\sigma \geqslant y_\nu$ by which the $K_\rho \subseteq A$ appearing in a computation of $[f^{-1}\epsilon]^A(x)$ is already contained in A^σ and so $[f^{-1}\epsilon]_\sigma^{A^\sigma}(x) = [f^{-1}\epsilon]^A(x)$ (of course $K_\eta \subseteq \alpha - A$ implies that $K_\eta \subseteq \alpha - A^\sigma$ for every σ). We may of course also assume that $D^\sigma(x) = D(x)$. Since $D(x) = [f^{-1}\epsilon]^A(x)$ we create, at stage σ, a $\nu - A$ requirement with argument x associated with the $\nu - A$ active reduction procedure ϵ. This, of course, contradicts our choice of x. □

3. Some corollaries about α-degrees. In order to derive some interesting consequences about α-degrees from the splitting theorem we first note a further consequence of our construction.

LEMMA 3.1. *$A \vee B \equiv_\alpha C$.*

PROOF. We have shown that $A, B \leqslant_\alpha C$ and so $A \vee B \leqslant_\alpha C$. On the other hand $A \cup B = C$ clearly implies that $C \leqslant_\alpha A \vee B$. □

We can now prove the usual corollaries of the splitting theorem.

COROLLARY 3.2. *Let c and d be α-r.e. degrees such that d is not α-recursive. Then there are α-r.e. degrees a and b such that $c = a \vee b$, $d \nleqslant_\alpha a$ and $d \nleqslant_\alpha b$.* □

COROLLARY 3.3. *If c is a nonzero α-r.e. degree, then there are α-r.e. degrees a and b such that $c = a \vee b$, $0 <_\alpha a <_\alpha c$, $0 <_\alpha b <_\alpha c$ and a is incomparable with b.*

PROOF. Let C be a regular set of degree c and let $D = C$ in the theorem. Let a and b be the degrees of the sets A and B guaranteed by the theorem. Then if a and b are comparable (e.g. one is α-recursive) then $a \vee b$ is a or b. But by the lemma $a \vee b = c$ and the theorem assures us that $c \nleqslant a$ and $c \nleqslant b$. □

COROLLARY 3.4. *No α-r.e. degree is minimal.* □

COROLLARY 3.5. *If d is an incomplete, non-α-recursive α-r.e. degree then there is an α-r.e. degree incomparable with d.*

PROOF. Let the C of the theorem be a regular complete α-r.e. set and let D be a regular set of degree d. Let a and b be the degrees of the sets given by the theorem. If both a and b are comparable with d then both are recursive in it and so $a \vee b \leqslant_\alpha d$ but $a \vee b = c$ and $d \lneqq_\alpha c$—a contradiction. □

4. A strengthening for α-calculability.

4.1. In this section we will show how to modify our construction to guarantee that D is not α-calculable from A or B.

Roughly speaking one says that D is α-calculable from A ($D \leqslant_{c\alpha} A$) if using a standard equation calculus system one can deduce precisely the correct values of the characteristic function of D from some finite set of equations supplemented by the complete graph of the characteristic function of A. This gives a stronger reducibility than $\leqslant_\alpha$ because the deductions need not be α-finite nor even of length less than α. For a precise description of the equation calculus and more details about $\leqslant_{c\alpha}$ we refer the reader to [4], [5] or [6]. For our purposes it is enough to know that if A is regular and hyperregular then $D \leqslant_{c\alpha} A$ iff $D \leqslant_{\alpha} A$. (A is *hyperregular* if every function $f \leqslant_{w\alpha} A$ with domain any $\beta < \alpha$ has its range bounded in α.) The idea here is that hyperregularity insures that individual deductions have length $<\alpha$ while regularity then guarantees that each such deduction uses only α-finite amounts of information about A. Again we refer to [4] and [5] for precise proofs.

Our goal will thus be merely to indicate how the sets A and B constructed above can be made hyperregular. Roughly speaking, we will add on new requirements as in [6] that try to preserve computations of $[\epsilon]^A(x)$ on the largest necessary $\beta < \alpha$ on which it is total. Once these preservations have highest priority $[\epsilon]^A$ will become essentially α-recursive on β and so will have bounded range as required by the definition of hyperregularity.

4.2. *The construction*. We begin by noting that we can set an *a priori* bound on the ordinals β that we must consider for a fixed $[\epsilon]^A$. First our indexing as usual is such that each α-r.e. set has α many indexes. Thus for any ϵ there are unboundedly many ϵ' such that $[\epsilon]^A = [\epsilon']^A$. We may thus assume that $\beta \leqslant \epsilon$. Moreover if $\alpha^* < \alpha$ it suffices to consider $\beta \leqslant \alpha^*$. (Any $\beta > \alpha^*$ can be mapped one-one into α^* by an α-finite map—just restrict the α-recursive projection of α into α^* to β.) Finally if α^* is singular in L_α, i.e. for some $\gamma < \alpha^*$ there is an α-finite map $f: \gamma \longrightarrow \alpha^*$ whose range is unbounded in α^*, we can restrict ourselves to $\beta < \alpha^*$. The only new point here is that if $h \leqslant_{w\alpha} A$ is bounded on every $\beta < \alpha^*$ but unbounded on α^* then the map $g: \gamma \longrightarrow \alpha$ given by $g(x) = \bigcup_{\delta < f(x)} h(x)$ is weakly recursive in A and maps γ unboundedly into α (f is a cofinality map $\gamma \longrightarrow \alpha^*$). Thus in our construction we need only worry about $\beta \leqslant \epsilon$ when $\alpha^* = \alpha$ or $\alpha^* < \alpha$ is singular in L_α (here $\epsilon < \alpha^*$ as usual) while if α^* is regular we consider $\beta \leqslant \alpha^*$. Keeping this in mind we make the following additions to our construction:

At stage σ we find, for each ν and each $\epsilon < P_\nu^\sigma$, the least $x < \epsilon(\alpha^*)$ such that there is no $\nu' - A$ requirement with argument x associated with ϵ. If there is such an x and $[f^{-1}\epsilon]_\sigma^{A^\sigma}(x)$ is convergent we create a $\nu' - A$ requirement with argument x associated with ϵ. The requirement consists of the negative facts about A^σ used in the least computation showing that it is convergent. We adjust the definition of P_ν^σ to bound the creation of η'-requirements and the enumeration in C of elements contained in η'-requirements for $\eta < \nu$. Of course an $\eta' - A$ requirement is destroyed if anyone of its elements is put into A. Finally we insert these requirements into the priority listing by giving ν'-requirements (for A and B) the same priority as the corresponding ν-requirements. (Since $\nu - A$ and $\nu' - A$ requirements do not conflict this presents no problems.) We now determine whether to put $c(\sigma)$ into A or B as before.

4.3. *The priority argument.* Our main goal is again to show that the P_ν^σ and y_ν^σ stabilize. Once that has been done we can argue for the hyperregularity of A as follows: Consider any $\epsilon < \alpha^*$ and suppose that $[f^{-1}\epsilon]^A$ is a function with domain $\beta \leq \epsilon(\alpha^*)$. Let ν be the least ordinal such that $\epsilon < P_\nu$. Once we are beyond stage y_ν any computations of $[f^{-1}\epsilon]_\sigma^{A^\sigma}(x)$ for $x < \epsilon(\alpha^*)$ covering an initial segment of $\epsilon(\alpha^*)$ are made into requirements which are never destroyed and so give the correct value of $[f^{-1}\epsilon]^A(x)$. Moreover if $[f^{-1}\epsilon]^A(x)$ is defined at all we eventually get such a computation (A is still α-r.e.). Thus we can compute $[f^{-1}\epsilon]^A$ on β by just going through the construction from stage y_ν onward. Since this is an α-recursive procedure $[f^{-1}\epsilon]^A$ is bounded on β and A is hyperregular. We therefore conclude our proof by establishing Lemma 2.1 in this context.

LEMMA 4.4. *If, by some stage τ, P_δ^σ and y_δ^σ have stabilized at values P_δ and y_δ respectively for every $\delta < \gamma$ and $\bigcup\{P_\delta \mid \delta < \gamma\} < \alpha^*$ then P_γ^σ and y_γ^σ eventually stabilize.*

PROOF. As before, we need only consider the case $\gamma = \nu + 1$ and we know that the contributions for ν-requirements are bounded. We have only to show that there is a bound on the stages at which ν'-requirements are created. (The regularity of C takes care of the other component of y_γ.) If $\alpha^* = \alpha$ or $\alpha^* < \alpha$ is singular we can argue directly for this bound. As they are never destroyed the ν'-requirements created after stage y_ν correspond in a one-one way with an α-r.e. subset of $P_\nu \times P_\nu$ (⟨the associated requirement, the argument of the requirement⟩) which is strictly less than α^* by assumption. Thus the set is α-finite and so enumerated in bounded time.

If $\alpha^* < \alpha$ is regular we first consider the set W of $\epsilon < P_\nu$ such that for every $x < \alpha^*$ there are undestroyed ν'-requirements at stage y_ν which are associated with ϵ or have such requirements created after stage y_ν. This subset of P_ν is α-r.e. and so α-finite. Similarly all the requirements associated with ϵ's in W form an α-finite set and so are bounded. Consider now the ϵ not in W.

Were there some bound strictly less than α^* on the arguments for which we create ν'-requirements after stage y_ν for these ϵ, we could argue as above. On the other hand, if there were no such bound, we could define an α-recursive map from a final segment of α^* into P_ν by sending x to the ϵ associated with the first ν'-requirement with argument x created after y_ν. Since this would contradict the regularity of α^* in L_α, we can indeed conclude as in the previous cases. □

Of course there is now no difficulty in deriving all the corollaries of §3 for α-calculability degrees as well. We can also draw additional ones about the

α-r.e. degrees by adding on the requirement that the constructed degrees be hyperregular. As an example we state

COROLLARY 4.5. *Every α-r.e. degree is the join of two hyperregular α-r.e. degrees.* □

5. **Open questions.** We consider two problems in this section. The first is whether the hypothesis that C is regular can be eliminated. On the basis of considerations of α-degree alone one can show that it is not, in general, possible to split a nonregular α-r.e. set. In particular, if $\alpha^* < \alpha$ and the cofinality of α^* in L_α is less than the Σ_2-projectum of α (the least β such that there is a Σ_2/L_α map of α into β). There is exactly one α-r.e. degree containing a nonregular α-r.e. set [9]. Under such conditions it is of course impossible to split a nonregular α-r.e. set since the constructed sets must be nonregular and of strictly lower α-degree. We therefore ask if these conditions do not hold (i.e., $cf(\alpha^*) \geqslant \sigma 2p(\alpha)$), can one then split an arbitrary nonregular α-r.e. set C. We can only show that there are nonregular α-r.e. sets A and B such that $\deg(A) \vee \deg(B) = \deg(C)$ [9].

For our second question we turn to the more technical problem of uniformity. Basically we are interested in ways that the constructions of §§1 and 4 can be made uniform, i.e., independent of any extraneous parameters or choices related to α. Essentially there were two such nonuniform choices in our basic construction. The first amounted to assuming that D was regular while the second consisted of choosing a projection $f\colon \alpha \longrightarrow \alpha^*$. Although we know of no uniform procedure for producing a regular α-r.e. set with the same degree as a given α-r.e. set, we can handle the first difficulty by a less direct method. As for the second problem we can eliminate the choice of a projection but only at the expense of a nonuniform splitting of the construction depending on α^* being less than α or not.

To be more specific, we solve the first problem by going uniformly from a given α-r.e. set D to a regular α-r.e. set E such that $E \leqslant_{w\alpha} D$. (E is the deficiency set of D, i.e., $\{\sigma \mid (\exists \tau > \sigma)(f(\tau) < f(\sigma))\}$ where f is a one-one recursive enumeration of D.) We then substitute E for D in our construction. Since we then show that in fact $E \not\leqslant_{w\alpha} A$ and $E \not\leqslant_{w\alpha} B$ it is clear that $D \not\leqslant_\alpha A$ and $D \not\leqslant_\alpha B$.

The second problem can be attacked with the methods of [8] to eliminate a choice of projection. The idea is that one keeps guessing at α^* and a projection using a parameterless Σ_1-skolem function. Although we omit the details all goes well if in fact $\alpha^* < \alpha$. One then eventually has correct guesses and the construction becomes essentially that of §1. The problem is that if $\alpha^* = \alpha$ we cannot use the same constuction and still guarantee that the blocks dealt with include every reduction procedure. Of course, for $\alpha^* = \alpha$ no choices or infinite parameters were needed in the original construction. Thus we can prove that there exist two integers k and l which are Gödel numbers for two placed functions such that given an index β for a regular α-r.e. set C and one γ for a nonrecursive α-r.e. set D, $k(\beta, \gamma)$ and $l(\beta, \gamma)$ each give indexes of two regular α-r.e. sets (A_0, B_0) and (A_1, B_1) such that $A_i \cup B_i = C$, $A_i \cap B_i = \emptyset$. Moreover $D \not\leq_\alpha A_i$ and $D \not\leq_\alpha B_i$, where $i = 0$ if $\alpha^* < \alpha$ and $i = 1$ if $\alpha^* = \alpha$. Although this seems to be sufficient for applications to recursion theory in higher type objects like that of [8] in [0], the result is not entirely satisfactory and we ask if a single integer can be found which does the job for all α. We feel confident that the answer to this question is yes. We are less sure of the answer to the more difficult question of whether our subterfuge to avoid the nonregularity of D can be eliminated. To be precise we repeat a question of Sacks [5]: Is there a uniform method of going from an index of an α-r.e. set to an index of a regular α-r.e. set of the same α-degree?

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