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SPLITTING CRITERIA FOR $\mathfrak{g}$-MODULES INDUCED FROM A PARABOLIC AND THE BERNSTEIN-GELFAND-GELFAND RESOLUTION OF A FINITE DIMENSIONAL, IRREDUCIBLE $\mathfrak{g}$-MODULE

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ABSTRACT. Let $\mathfrak{g}$ be a finite dimensional, complex, semisimple Lie algebra and let V be a finite dimensional, irreducible $\mathfrak{g}$-module. By computing a certain Lie algebra cohomology we show that the generalized versions of the weak and the strong Bernstein-Gelfand-Gelfand resolutions of V obtained by H. Garland and J. Lepowsky are identical.

Let G be a real, connected, semisimple Lie group with finite center. As an application of the equivalence of the generalized Bernstein-Gelfand-Gelfand resolutions we obtain a complex in terms of the degenerate principal series of G, which has the same cohomology as the de Rham complex.

Introduction. Let $\mathfrak{g}$ be a finite dimensional, complex, semisimple Lie algebra and let V be a finite dimensional, irreducible $\mathfrak{g}$-module. In [**2**, Theorem 9.9], I. N. Bernstein, I. M. Gelfand and S. I. Gelfand constructed a resolution of V by certain $\mathfrak{g}$-modules with Verma composition series. In the same paper another resolution of V is constructed which resolves V by direct sums of Verma modules, improving Theorem 9.9. (Cf. [**2**, Theorem 10.1′].) In their work, Bernstein, Gelfand and Gelfand study systematically a certain category of $\mathfrak{g}$-modules known as the category $\mathcal{O}$. Both the weak and the strong resolutions were generalized by H. Garland and J. Lepowsky in the papers [**9**] and [**14**] where generalized Verma modules play the same role as Verma modules in [**2**]. We will refer to these resolutions as the generalized weak BGG resolution and the generalized strong BGG resolution. In this paper we prove that the two generalized BGG resolutions are identical.

Our main theorem, Theorem 9.3 shows that each $\mathfrak{g}$-module in the generalized weak BGG resolution splits into a direct sum of generalized Verma modules. Consequently each such $\mathfrak{g}$-module is isomorphic to the $\mathfrak{g}$-module at the corresponding level in the generalized strong BGG resolution.

One of our methods consists in studying a certain category of $\mathfrak{g}$-modules and later using such category as a framework in order to obtain a key lemma, Lemma 9.1. We would like to point out that, in the light of Yoneda's interpretation of cohomology, Lemma 9.1 implies vanishing theorems on Lie algebra cohomology. (See the remark following the proof of Lemma 9.1.)

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The first section contains some standard notation used throughout the text.

In §2 we recall the basic material on the category $\mathcal{O}$ and on Verma modules, and in the third section we define the category $\mathcal{O}_S$, derive its elementary properties and review the necessary prerequisites on generalized Verma modules.

The results on projective modules in the category $\mathcal{O}_S$ obtained in §4 and the material on generalized Verma composition series developed in the subsequent section are the basic ingredients in the proof of our generalization of the duality theorem of [**1**] (Theorem 6.1). The proofs of the results on the category $\mathcal{O}_S$ use the basic ideas in [**1**] but some technical changes are necessary.

§§7–9 are devoted to the proof of the splitting theorem. In §7 we recall Garland and Lepowsky's construction of the generalized weak BGG resolution. In §8 we prove a preliminary lemma to the proof of Theorem 9.3 using the additivity of the Ext bifunctor. §9 is devoted to the proofs of Lemma 9.1 and Theorem 9.3. In the Appendix we give an alternate proof of Theorem 9.3 which does not use the framework of the category $\mathcal{O}_S$. The proof of the particular case of Lemma 9.1, where $n = 1$ and $S = \varnothing$, is due to James Humphreys. For the case where $n = 1$ and S is arbitrary we follow the general lines of his proof. The proof of Lemma 9.1 for Ext^n with $n \geqslant 2$ was an outgrowth of conversations with Nolan Wallach.

In §§10 and 11 we prove our second main theorem, Theorem 11.4 which shows that the isomorphisms of the $\mathfrak{g}$-modules in the generalized weak BGG resolution and the $\mathfrak{g}$-modules in the generalized strong BGG resolution can be chosen so that they commute with the differential maps, implying that the resolutions are identical. In §10 we give an almost entirely self-contained construction of the strong BGG resolution, using as prerequisites only the results on Verma modules exposed in Chapter 7 of [**6**]. The methods used here are based on the study of certain constants associated to pairs of elements of the Weyl group which are gotten from Theorem 9.3. The same technique involved in this new construction of the strong BGG resolution is used to prove our uniqueness theorem in the Verma module case (Lemma 10.5). Lemma 10.5 implies the exactness of the strong BGG resolution (Corollary 10.6) and the exactness of any complex of direct sums of Verma modules that satisfies some nonvanishing hypotheses (Corollary 10.7). In §11 we extend the results obtained in §10 to the generalized Verma modules situation.

In §12 we consider G a real, connected, linear, semisimple Lie group. As an application of our main result we obtain, using Zuckerman's interpretation of the de Rham complex, a complex in terms of the degenerate principal series of G that has the same cohomology as the de Rham complex. In Corollary 2.4 and Theorem 2.5 of the appendix to M. W. Silva's thesis (Rutgers University, 1977) a special case of the weak form of this complex was used to analyze the imbeddings of the discrete series in the principal series.

We call the reader's attention to the fact that some of the hypotheses made on $\mathfrak{g}$ and on G could be weakened. We note that all the results proved here are still true if we assume that $\mathfrak{g}$ is a finite dimensional, split semisimple Lie algebra over a field of characteristic zero and that G is a real, connected, semisimple Lie group with finite center. We also note that we have made no attempt to change standard terminology. The reader should observe that the word "split" has three meanings in this paper. The appropriate meaning should be clear from the context.

This paper contains parts of the author's thesis at Rutgers University (1978). The author takes the opportunity to thank Professor Nolan Wallach, her thesis advisor, for his advice and constant encouragement. Thanks are also due to James Humphreys who read the manuscript and made many suggestions.

1. Preliminaries. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ (i.e., a maximal abelian subalgebra of $\mathfrak{g}$, such that if $X \in \mathfrak{h}$, ad X: $\mathfrak{g} \to \mathfrak{g}$ is diagonalizable). Let $\mathfrak{g} = \mathfrak{h} \oplus \amalg_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the corresponding root space decomposition of $\mathfrak{g}$; here Δ is the root system of $(\mathfrak{g}, \mathfrak{h})$. Fix Δ^+ a system of positive roots in Δ and let π be the system of simple roots of Δ^+. Set $\rho = (1/2) \sum_{\alpha \in \Delta^+} \alpha$.

Let $\mathfrak{n} = \amalg_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\mathfrak{n}^- = \amalg_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. $\mathfrak{b}$ is called the Borel subalgebra defined by $\mathfrak{h}$ and π.

$\langle\, ,\, \rangle$ denotes the Killing form of $\mathfrak{g}$, and for $\lambda, \mu \in \mathfrak{h}^*$ we set $\langle \lambda, \mu \rangle = \langle h_\lambda, h_\mu \rangle = \mu(h_\lambda)$, where h_λ is the unique element in $\mathfrak{h}$ such that $\lambda(H) = \langle h_\lambda, H \rangle$ for all $H \in \mathfrak{h}$, $\lambda \in \mathfrak{h}^*$.

If $\alpha \in \Delta$, we denote by σ_α the endomorphism of $\mathfrak{h}^*$ defined by

$$\sigma_\alpha(\lambda) = \lambda - 2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, \qquad \lambda \in \mathfrak{h}^*.$$

σ_α is called the reflection relative to α. The *Weyl group* W of $(\mathfrak{g}, \mathfrak{h})$ is the group of automorphisms of $\mathfrak{h}^*$ generated by the σ_α, $\alpha \in \Delta$. In fact, the σ_α, $\alpha \in \pi$, suffice to generate W (cf. **[18]**).

Let $\pi = \{\alpha_1, \ldots, \alpha_n\}$. If $w \in W$, any expression of w as a product (composition) $\sigma_1 \cdots \sigma_q$ of reflections $\sigma_i = \sigma_{\alpha_i}$, $i \in \{1, \ldots, n\}$, with q minimal, is called a *reduced expression* of w; the integer q is called the *length* of w, denoted by $l(w)$. Let $W^{(k)}$ be the subset of W consisting of all elements of length k, $k \in \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$.

For $w \in W$ and $\lambda \in \mathfrak{h}^*$ we will use the abbreviation due to R. Moody: $w \cdot \lambda = w(\lambda + \rho) - \rho$.

If V is a $\mathfrak{g}$-module and $\lambda \in \mathfrak{h}^*$, set $V(\lambda) = \{v \in V$ such that $Hv = \lambda(H)v$, all $H \in \mathfrak{h}\}$. λ is said to be a *weight* of V if $V(\lambda) \neq (0)$ and every nonzero vector in $V(\lambda)$ is called a *weight vector* of weight λ. We denote by $P(V)$ the set of weights of V.

Let Q^+ denote the set of all linear combinations of the elements of π with nonnegative integral coefficients. We define a partial ordering on $\mathfrak{h}^*$ as follows.

$$\lambda \leqslant \mu \quad \text{if and only if} \quad \mu - \lambda \in Q^+.$$

If V is a $\mathfrak{g}$-module, and $\lambda \in P(V)$ is such that for no $\mu \in P(V)$ with $\mu \neq \lambda$ we can have $\mu - \lambda \in Q^+$, then λ is said to be a *highest weight* of V.

We denote by P^+ the set of dominant integral weights of Δ, that is,

$$P^+ = \left\{\lambda \in \mathfrak{h}^* \,\middle|\, \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z}^+, \text{all } \alpha \in \pi\right\}.$$

If $\mathfrak{a} \subset \mathfrak{g}$ is a subalgebra we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$ and by $Z(\mathfrak{a})$ the center of $U(\mathfrak{a})$. We will identify Lie algebra modules with the corresponding universal enveloping algebra modules.

2. The category $\mathcal{O}$ and Verma modules.

DEFINITION 2.1 [2]. We denote by $\mathcal{O}$ the full subcategory of the category of left $U(\mathfrak{g})$-modules consisting of the modules V such that

(1) V is $U(\mathfrak{g})$-finitely generated,

(2) V is $\mathfrak{h}$-diagonalizable,

(3) V is $U(\mathfrak{n})$-finite (i.e. $\dim U(\mathfrak{n})v < \infty$ for $v \in V$).

DEFINITION 2.2 [6]. If $\lambda \in \mathfrak{h}^*$, $\mathbf{C}$ can be looked upon as a $\mathfrak{b}$-module $\mathbf{C}_\lambda$ via the action

$$(H + X)z = \lambda(H)z, \qquad H \in \mathfrak{h}, \qquad X \in \mathfrak{n}, \qquad z \in \mathbf{C}.$$

(This action actually describes all the finite dimensional irreducible $U(\mathfrak{b})$-modules, as λ varies in $\mathfrak{h}^*$.) The left $U(\mathfrak{g})$-module induced by $\mathbf{C}_\lambda$, $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\lambda$, is called the *Verma module* associated with $\mathfrak{g}$, $\mathfrak{h}$, π and λ and denoted by V_λ.

For the convenience of the reader we list here some basic facts about the category $\mathcal{O}$ and Verma modules as well as recent results of Bernstein-Gelfand-Gelfand. Basic references for the material 2.3–2.12 below are [2] and [6]. The reader should note that [2] and [6] use a different formalism, namely, twisted induction vs. induction.

PROPOSITION 2.3. (1) *The category $\mathcal{O}$ is stable under the operations of taking submodules, quotients and finite direct sums.*

(2) *If $V \in \mathcal{O}$, all the subspaces $V(\mu)$, $\mu \in \mathfrak{h}^*$, are finite dimensional and $V = \coprod_{\mu \in \mathfrak{h}^*} V(\mu)$. Furthermore $P(V) \subset \bigcup_{i=1}^{q} (\lambda_i - Q^+)$, $\lambda_i \in \mathfrak{h}^*$, $q \in \mathbf{N} = \{1, 2, 3, \dots\}$.*

(3) *Every $V \in \mathcal{O}$ has a Jordan-Hölder (J-H) series.*

(4) *V_λ is $U(\mathfrak{n}^-)$-free.*

(5) *V_λ has a unique irreducible quotient which we denote by L_λ.*

(6) *Every irreducible module in $\mathcal{O}$ is an L_λ, $\lambda \in \mathfrak{h}^*$.*

Let Θ denote the set of all homomorphisms from $Z(\mathfrak{g})$ to $\mathbf{C}$.

DEFINITION 2.4. Let $\Theta(V)$ be the set of elements $\theta \in \Theta$ such that there exists a nonzero vector $v \in V$ satisfying the condition

$$Zv = \theta(Z)v, \quad \text{for all } Z \in Z(\mathfrak{g}).$$

PROPOSITION 2.5. (1) *$\Theta(V_\lambda)$ has only one element denoted by θ_λ.*

(2) *$\theta_\lambda = \theta_\mu$ if and only if there exists $w \in W$ such that $w \cdot \lambda = \mu$.*

PROPOSITION 2.6. *Let $V \in \mathcal{O}$.*

(1) *$\Theta(V)$ is finite.*

(2) *If ${}_\theta V = \{v \in V \mid$ there exists $k \in \mathbf{N}$ such that $(Z - \theta(Z))^k v = 0$, all $Z \in Z(\mathfrak{g})\}$, then $V = \coprod_{\theta \in \Theta(V)} {}_\theta V$.*

(3) *The functor $V \to {}_\theta V$ preserves exact sequences.*

THEOREM 2.7 (VERMA). *Let* $\lambda, \mu \in \mathfrak{h}^*$. *The vector space* $\mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_\lambda)$ *is either trivial or one-dimensional and every nonzero element in* $\mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_\lambda)$ *is injective.*

REMARK. The above says that if there is a nonzero $\mathfrak{g}$-homomorphism of V_μ into V_λ it is injective and unique up to scalar multiple. If $\mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_\lambda) \neq 0$ we will simply write $V_\mu \subset V_\lambda$.

THEOREM 2.8 (BERNSTEIN-GELFAND-GELFAND). *Let* $\lambda, \lambda' \in \mathfrak{h}^*$. *The following are equivalent.*

(1) $V_\lambda \subset V_{\lambda'}$.

(2) L_λ *is an irreducible subquotient of* $V_{\lambda'}$.

(3) *There exist* $\gamma_1, \ldots, \gamma_r \in \Delta^+$ *such that*

$$\lambda' = \lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_r = \lambda$$

where λ_i *is recursively defined as* $\sigma_{\gamma_i} \cdot \lambda_{i-1}$, $i = 1, \ldots, r$.

NOTE. Verma also proved the implication (3) $\Rightarrow$ (1) of Theorem 2.8 and conjectured its converse as well (cf. [**17**]).

DEFINITION 2.9. Let $w, w' \in W$ and $\gamma \in \Delta^+$. We write $w \overset{\gamma}{\leftarrow} w'$ if $w = \sigma_\gamma w'$ and $l(w) = l(w') + 1$, and $w \leftarrow w'$ if there exists $\gamma \in \Delta^+$ such that $w \overset{\gamma}{\leftarrow} w'$. We write $w \leqslant w'$ if $w = w'$ or if there exist $w_1 = w, w_2, \ldots, w_r = w'$ in W such that $w_1 \leftarrow w_2 \leftarrow \cdots \leftarrow w_r$. This gives a partial ordering in W.

LEMMA 2.10. *Let* $w_1, w_2 \in W$, $\gamma \in \Delta^+$, $\alpha \in \pi$, *with* $\alpha \neq \gamma$. *The following are equivalent.*

(1) $\sigma_\alpha w_1 \overset{\alpha}{\leftarrow} w_1$ *and* $\sigma_\alpha w_1 \overset{\gamma}{\leftarrow} w_2$.

(2) $w_2 \overset{\alpha}{\leftarrow} \sigma_\alpha w_2$ *and* $w_1 \overset{\sigma_\alpha \gamma}{\leftarrow} \sigma_\alpha w_2$.

LEMMA 2.11. *Let* $w_1, w_2 \in W$. *The number of elements* $w \in W$ *such that* $w_1 \leftarrow w \leftarrow w_2$ *is* 0 *or* 2.

THEOREM 2.12 (BERNSTEIN-GELFAND-GELFAND). *Let* $\lambda \in P^+$.

(1) $V_{w \cdot \lambda} \subset V_{w' \cdot \lambda}$ *if and only if* $w \leqslant w'$, $w, w' \in W$.

(2) *If* $l(w) = l(w') + 2$, $w, w' \in W$, *the number of* $w'' \in W$ *such that* $V_{w \cdot \lambda} \subsetneqq V_{w'' \cdot \lambda} \subsetneqq V_{w' \cdot \lambda}$ *is* 0 *or* 2.

3. The category $\mathcal{O}_S$ and generalized Verma modules. Denote by X_α, Y_α and H_α elements in $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ and $\mathfrak{h}$, respectively, satisfying the conditions

$$[X_\alpha, Y_\alpha] = H_\alpha, \qquad \alpha(H_\alpha) = 2, \qquad \alpha \in \Delta.$$

If $\alpha = \alpha_i$, set $X_i = X_{\alpha_i}$, $Y_i = Y_{\alpha_i}$, $H_i = H_{\alpha_i}$, $i \in \{1, \ldots, n\}$. (Cf. [**6**].)

Fix S, an arbitrary subset of $\{1, \ldots, n\}$. Let $\mathfrak{h}_S \subset \mathfrak{h}$ be the span of the H_i, $i \in S$, $\mathfrak{h}^S = \{H \in \mathfrak{h} | \alpha_i(H) = 0$, all $i \in S\}$, $\Delta_S = \Delta \cap \sum_{i \in S} \mathbf{Z}\alpha_i$, where $\mathbf{Z}$ is the ring of integers, $\Delta_S^+ = \Delta^+ \cap \Delta_S$, $\mathfrak{n}_S = \amalg_{\alpha \in \Delta_S^+} \mathfrak{g}_\alpha$, $\mathfrak{n}_S^- = \amalg_{\alpha \in \Delta_S^+} \mathfrak{g}_{-\alpha}$, $\mathfrak{g}_S = \mathfrak{n}_S \oplus \mathfrak{h}_S \oplus \mathfrak{n}_S^-$, $\mathfrak{m} = \mathfrak{n}_S \oplus \mathfrak{h}_S \oplus \mathfrak{h}^S \oplus \mathfrak{n}_S^-$. $\mathfrak{g}_S$ is semisimple with Cartan subalgebra $\mathfrak{h}_S$ and $\Delta_{S|_{\mathfrak{h}_S}} = \{\alpha_{|_{\mathfrak{h}_S}} | \alpha \in \Delta_S\}$ is a root system for $(\mathfrak{g}_S, \mathfrak{h}_S)$ with positive system $\Delta^+_{S|_{\mathfrak{h}_S}} = \{\alpha_{|_{\mathfrak{h}_S}} | \alpha \in \Delta_S^+\}$ and simple roots $\{\alpha_{i|_{\mathfrak{h}_S}} | i \in S\}$.

Let $\mathfrak{u} = \amalg_{\alpha\in\Delta^+\setminus\Delta_S^+} \mathfrak{g}_\alpha$, $\mathfrak{u}^- = \amalg_{\alpha\in\Delta^+\setminus\Delta_S^+} \mathfrak{g}_{-\alpha}$. Clearly $[\mathfrak{m}, \mathfrak{u}] \subset \mathfrak{u}$. Therefore $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{p} \supset \mathfrak{b}$ and $\mathfrak{p}$ is a *parabolic subalgebra* of $\mathfrak{g}$. In fact, if S varies in the set of all the subsets of $\{1, \ldots, n\}$ we obtain in the above described way all the possible parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{b}$. $\mathfrak{u}$ is called the *nilpotent part* of $\mathfrak{p}$.

Next we introduce a subcategory of the category $\mathcal{O}$.

DEFINITION 3.1. We denote by $\mathcal{O}_S$ the full subcategory of the category of $U(\mathfrak{g})$-modules consisting of the modules V such that

(1) V is $U(\mathfrak{g})$-finitely generated.

(2) Viewed as an $U(\mathfrak{m})$-module, V is a direct sum of finite dimensional irreducible $U(\mathfrak{m})$-modules.

(3) V is $U(\mathfrak{u})$-finite.[1]

Let $P_S^+ = \{\lambda \in \mathfrak{h}^* | \lambda(H_i) \in \mathbf{Z}^+, \text{ all } i \in S\}$. The theorem of the highest weight **[18]** implies that there exists a one-to-one correspondence between P_S^+ and the set of all equivalence classes of finite dimensional irreducible $\mathfrak{m}$-modules. We denote this correspondence by: $\lambda \mapsto [M_\lambda]$.

DEFINITION 3.2. *If* $\lambda \in P_S^+$, M_λ can be looked upon as a $\mathfrak{p}$-module, the action of $\mathfrak{u}$ being the trivial one. (It is not hard to see that this definition gives the most general finite dimensional irreducible $\mathfrak{p}$-module.) The left $U(\mathfrak{g})$-module induced by the $U(\mathfrak{p})$-module M_λ, $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M_\lambda$, is called the *Generalized Verma Module* associated with $\mathfrak{g}$, $\mathfrak{h}$, π, S and λ, and denoted by V^{M_λ}. (Cf. **[15]**.)

REMARK. In the extreme cases where $S = \varnothing$ or $\{1, \ldots, n\}$, V^{M_λ} corresponds to V_λ and L_λ, respectively. It is also clear that $V^{M_\lambda} \in \mathcal{O}_S$ for $\lambda \in P_S^+$.

Let $N \in \mathcal{O}_S$. The *isotypic component of type* M_λ in N is, by definition, the sum of all $\mathfrak{m}$-submodules of N isomorphic to M_λ, and we denote it by N^λ. Let $P_\mathfrak{m}(N) = \{\lambda \in P_S^+ | N^\lambda \neq (0)\}$ and m_λ be the number of summands in the expression of N^λ as the sum of M_λ's. Then $N = \amalg_{\lambda\in P_\mathfrak{m}(N)} m_\lambda M_\lambda$ as an $\mathfrak{m}$-module, by Definition 3.1 (2).

Denote by $\mathcal{P}$ *Kostant's partition function*, i.e., given $\nu \in \mathfrak{h}^*$, $\mathcal{P}(\nu)$ = the number of families $(n_\alpha)_{\alpha\in\Delta^+}$ with $n_\alpha \in \mathbf{Z}^+$ such that $\nu = \sum_{\alpha\in\Delta^+} n_\alpha \alpha$.

Let $S' = \{1, \ldots, n\} \setminus S$. We denote by Q, Q_S and Q^S the sets of linear combinations with coefficients in $\mathbf{Z}$ of the elements of π, $\{\alpha_i\}_{i\in S}$ and $\{\alpha_j\}_{j\in S'}$, respectively. We write $\mu >^S \nu$, $\mu, \nu \in \mathfrak{h}^*$, if $\mu - \nu \in Q$ and the Q^s part of $\mu - \nu$ is > 0 in the sense of §1.

We now state and prove the counterpart to Proposition 2.3 for the category $\mathcal{O}_S$.

PROPOSITION 3.3. (1) *The category* $\mathcal{O}_S$ *is stable under the operations of taking submodules, quotients and finite direct sums.*

(2) *Let* $N \in \mathcal{O}_S$. *Then* $m_\lambda < \infty$ *for all* $\lambda \in P_\mathfrak{m}(N)$ *and*

$$P_\mathfrak{m}(M) \subset \bigcup_{i=1}^{q} (\lambda_i - Q^+), \qquad \lambda_i \in P_S^+, \qquad q \in \mathbf{N}.$$

[1]The categories $\mathcal{O}$ and $\mathcal{O}_S$ are note closed under extensions. They are closed under extensions only in a certain subcategory of $\mathfrak{g}$-modules (see §8).

(3) *Every* $N \in \mathcal{O}_S$ *has a Jordan-Hölder series.*
(4) V^{M_λ} *is* $U(\mathfrak{u}^-)$*-free,* $\lambda \in P_S^+$.
(5) *Every* V^{M_λ} *has a unique irreducible quotient* L_{M_λ}, $\lambda \in P_S^+$.
(6) *Every irreducible module in* $\mathcal{O}_S$ *is an* $L_{M_\lambda} \simeq L_\lambda$ *for some* $\lambda \in P_S^+$.

PROOF. Let $v_1, \ldots, v_n$ be weight vectors of weights $\lambda_1, \ldots, \lambda_n$, respectively, $\lambda_i \in P_S^+$, such that $N = \sum_{i=1}^n U(\mathfrak{g})v_i$ and $U(\mathfrak{m})v_i \simeq M_{\lambda_i}$, $i = 1, \ldots, n$.

For each i, $U(\mathfrak{u})M_{\lambda_i}$ is a finite dimensional $\mathfrak{m}$-module, hence

$$U(\mathfrak{u})M_{\lambda_i} = \coprod_{j=1}^{m_i} M_{\lambda_{ij}}, \qquad i = 1, \ldots, n.$$

Relabel the λ_{ij}'s so that

$$\{\lambda_{11}, \ldots, \lambda_{1m_1}, \lambda_{21}, \ldots, \lambda_{2m_2}, \ldots, \lambda_{n_1}, \ldots, \lambda_{nm_n}\} = \{\lambda_1, \ldots, \lambda_q\}.$$

By the Poincaré-Birkhoff-Witt theorem,

$$N = \sum_{i=1}^{q} U(\mathfrak{u}^-)M_{\lambda_i}, \qquad \lambda_i \in P_S^+.$$

This implies that

$$P_{\mathfrak{m}}(N) \subset \bigcup_{i=1}^{q} (\lambda_i - Q^+), \qquad \lambda_i \in P_S^+,$$

and

$$m_\lambda \leqslant \sum_{i=1}^{q} \mathcal{P}(\lambda_i - \lambda) < \infty,$$

proving (2).

Now $V^{M_\lambda} = U(\mathfrak{u}^-) \otimes M_\lambda$ as an $\mathfrak{m}$-module and $V^{M_\lambda} = \coprod_{\mu \in P_{\mathfrak{m}}(V^{M_\lambda})} m_\mu M_\mu$ with $m_\lambda = 1$. Also $V^{M_\lambda} = U(\mathfrak{g})M_\lambda$. Let N be a proper $\mathfrak{g}$-submodule of V^{M_λ}. Then $N = \coprod_{\mu \in P_{\mathfrak{m}}(V^{M_\lambda})} N \cap M_\mu$ and $N \cap M_\lambda = (0)$. Therefore the sum of all proper $\mathfrak{g}$-submodules of V^{M_λ} is a proper $\mathfrak{g}$-submodule and consequently V^{M_λ} has a unique maximal $\mathfrak{g}$-submodule. The corresponding quotient gives the desired L_{M_λ}, proving (5).

Let $N \in \mathcal{O}_S$ be irreducible. Choose $\lambda \in P_{\mathfrak{m}}(N)$ maximal with respect to $>^S$ and let $v_\lambda \in N(\lambda)$ be a nonzero weight vector of weight λ such that $M_\lambda = U(\mathfrak{m})v_\lambda$ in N^λ. Then $\mathfrak{u}M_\lambda = 0$. Therefore there exists

$$\varphi: V^{M_\lambda} \to N,$$

a surjective homomorphism. By (5) $N \simeq L_{M_\lambda}$. But $\mathfrak{n}v_\lambda = 0$ and $Hv_\lambda = \lambda(H)v_\lambda$ all $H \in \mathfrak{h}$. Proposition 7.1.13 in [**6**] implies that $N \simeq L_\lambda$. This proves (6).

The other statements are either obvious or they follow from 2.3. Q.E.D.

Next we will review some results on generalized Verma modules that are established in [**15**].

PROPOSITION 3.4. *Let* $\lambda \in P_S^+$ *and* ξ_λ *be the canonical generator of* V_λ. *Then the sequence*

$$\coprod_{i \in S} V_{\sigma_i \cdot \lambda} \xrightarrow{\varphi} V_\lambda \xrightarrow{\psi} V^{M_\lambda} \to 0$$

is exact, where φ *sends the canonical generator of* $V_{\sigma_i \cdot \lambda}$ *to a nonzero multiple of the highest weight vector* $Y_i^{\lambda(H_i)+1}\xi_\lambda \in V_\lambda$ *and* ψ *sends* ξ_λ *to* $1 \otimes v_\lambda$, v_λ *being the canonical generator of* M_λ.

NOTE. The proof of 3.4 uses a theorem of Harish-Chandra (cf. [**6**, Lemme 7.2.5]) and the universal property of the tensor product.

The following is a corollary of Proposition 1.8.5 and Lemme 7.2.4 of [**6**].

PROPOSITION 3.5. *Suppose* V *satisfies condition* 3.1(2). *Let* v *be a weight vector of* V *having weight* $\lambda \in \mathfrak{h}^*$ *such that* $\mathfrak{n}_S v = 0$. *Then* $\lambda \in P_S^+$ *and* $U(\mathfrak{m})v \simeq M_\lambda$.

PROPOSITION 3.6. *Let* L_λ *be the finite dimensional irreducible* $\mathfrak{g}$-*module with highest weight* $\lambda \in P^+$, w_λ *a highest weight vector generating* V^{M_λ}, *and*

$$S' = \{1, \ldots, n\} \setminus S.$$

Then $\sigma_i \cdot \lambda \in P_S^+$ *for all* $i \in S'$ *and*

$$\coprod_{i \in S'} V^{M_{\sigma_i \cdot \lambda}} \xrightarrow{f} V^{M_\lambda} \xrightarrow{g} L_\lambda \to 0$$

is an exact sequence, where f *sends a highest weight vector generating* $V^{M_{\sigma_i \cdot \lambda}}$ *to a nonzero multiple of the highest weight vector* $Y_i^{\lambda(H_i)+1}w_\lambda \in V^{M_\lambda}$ *and* g *takes* w_λ *to a highest weight vector of* L_λ.

NOTE. The above is a consequence of Harish-Chandra's theorem (see Lemme 7.2.4 in [**6**]) and the last three propositions.

4. Projective modules of $\mathcal{O}_S$. Here we generalize the results of [**1**] for the category $\mathcal{O}$.

For each $\theta \in \Theta$ consider the subcategory ${}_\theta(\mathcal{O}_S)$ of $\mathcal{O}_S$ consisting of the modules V of $\mathcal{O}_S$ such that, for every $Z \in Z(\mathfrak{g})$, V is annihilated by some power of $Z - \theta(Z)$ (cf. [**2**]).

PROPOSITION 4.1. *Let* $\lambda \in P_S^+$, $\theta \in \Theta$. *There exists a* $\mathfrak{g}$-*module* $Q = Q(\theta, \lambda) \in {}_\theta(\mathcal{O}_S)$ *such that given* $N \in \mathcal{O}_S$ *the vector space* $\operatorname{Hom}_\mathfrak{g}(Q, N)$ *is naturally isomorphic with* $\operatorname{Hom}_\mathfrak{m}(M_\lambda, {}_\theta N)$.

PROOF. By Proposition 7.4.8 in [**6**], each $\theta \in \Theta$ has the form θ_μ for some $\mu \in \mathfrak{h}^*$. The irreducible subquotients of $V \in {}_\theta(\mathcal{O}_S)$ are the L_ν's such that $\nu = w \cdot \mu$ for some $w \in W$ by Propositions 3.3 and 2.5. If V_1, V_2 and V in ${}_\theta(\mathcal{O}_S)$ are such that

$$0 \to V_1 \to V \to V_2 \to 0$$

is exact, then $P(V) = P(V_1) \cup P(V_2)$. Therefore

$$P(V) \subset \bigcup_{w \in W} \{w \cdot \mu - Q_+\},$$

for all V in ${}_\theta(\mathcal{O}_S)$. Now, $({}_\theta N)^\lambda = \sum_i U(\mathfrak{n}_S^-)v_i$, $v_i \in ({}_\theta N)^\lambda(\lambda)$. This implies that there exists a number r, depending only on θ and λ, such that

$$(\mathfrak{u})^r v = 0, \quad \text{for all } v \text{ in } ({}_\theta N)^\lambda. \tag{1}$$

Consider the $\mathfrak{p}$-module

$$U(\mathfrak{u}) / (\mathfrak{u})^r U(\mathfrak{u}) \otimes M_\lambda,$$

where $\mathfrak{u}$ acts on the left and $\mathfrak{m}$ by the tensor product action. Let

$$\hat{Q} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \left(U(\mathfrak{u}) / (\mathfrak{u})^r U(\mathfrak{u}) \otimes M_\lambda \right).$$

Clearly $\hat{Q} \in \mathcal{O}_S$. Consider the map

$$\Phi\colon \mathrm{Hom}_{\mathfrak{g}}(\hat{Q}, {}_\theta N) \to \mathrm{Hom}_{\mathfrak{m}}(M_\lambda, {}_\theta N)$$

defined by

$$\Phi(A)(v) = A(1 \otimes \bar{1} \otimes v),$$

where $A \in \mathrm{Hom}_{\mathfrak{g}}(\hat{Q}, {}_\theta N)$, $v \in M_\lambda$ and $\bar{1} = 1 + U(\mathfrak{u})/(\mathfrak{u})^r U(\mathfrak{u})$.

If $B \in \mathrm{Hom}_{\mathfrak{m}}(M_\lambda, {}_\theta N)$, $X \in U(\mathfrak{g})$, $Y \in U(\mathfrak{u})$, $\bar{Y} = Y + U(\mathfrak{u})/(\mathfrak{u})^r U(\mathfrak{u})$ and $v \in M_\lambda$, define

$$\Psi(B)(X \otimes \bar{Y} \otimes v) = XYv.$$

(1) implies that $\Psi(B) \in \mathrm{Hom}_{\mathfrak{g}}(\hat{Q}, {}_\theta N)$. It is trivial to see that Ψ is the inverse of Φ. Now set $Q = {}_\theta\hat{Q}$ and observe that $\mathrm{Hom}_{\mathfrak{g}}(\hat{Q}, {}_\theta N)$ is naturally isomorphic with $\mathrm{Hom}_{\mathfrak{g}}(Q, N)$ (see 2.6). Q.E.D.

COROLLARY 4.2. *Every V in $\mathcal{O}_S$ is a quotient of a projective in $\mathcal{O}_S$.*

PROOF. It is clear that an object P in $\mathcal{O}_S$ is projective if and only if $\mathrm{Hom}_{\mathfrak{g}}(P, -)$ is an exact functor. Proposition 4.1 says that the functor $\mathrm{Hom}_{\mathfrak{g}}(Q, -)$ is equivalent to an exact functor. This implies that Q is projective.

Each V in $\mathcal{O}_S$ is generated by finitely many vectors v_{θ_i,λ_j} in $({}_{\theta_i}V)^{\lambda_j}(\lambda_j)$, where $\theta_i \in \Theta(V)$ and $\lambda_j \in P_{\mathfrak{m}}(V)$, by Propositions 2.6 and 3.3. Each v_{θ_i,λ_j} defines an element of $\mathrm{Hom}_{\mathfrak{m}}(M_{\lambda_j}, {}_{\theta_i}V)$. Applying Proposition 4.1 again we conclude that V is a quotient of the projective object $\amalg_{i,j}\, Q(\theta_i, \lambda_j)$. Q.E.D.

The following result is standard.

LEMMA 4.3. *If P is an indecomposable projective in $\mathcal{O}_S$ then P has a unique maximal submodule.*

PROOF. Cf. for example **[12]**.

Corollary 4.2 and Lemma 4.3 imply

COROLLARY 4.4. *If $P \in \mathcal{O}_S$ is indecomposable and projective then P has a unique irreducible quotient. Therefore there is a one-to-one correspondence between the irreducible modules L_λ, $\lambda \in P_S^+$ of $\mathcal{O}_S$ and the indecomposable projective modules in $\mathcal{O}_S$.*

We denote by P_λ the unique indecomposable projective module in $\mathcal{O}_S$ with quotient L_λ.

PROPOSITION 4.5. *If $\lambda \in P_S^+$ and $V \in \mathcal{O}_S$ then*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(P_\lambda, V) = (V : L_\lambda), \tag{2}$$

where $(V : L_\lambda)$ denotes the number of occurrences of L_λ in a Jordan-Hölder series of V.

NOTE. The proof of 4.5 is similar to that of the category $\mathcal{O}$ case as it appears in [1].

5. Generalized Verma composition series.

DEFINITION 5.1. A $U(\mathfrak{g})$-module V is said to have a *generalized Verma composition series* (GVCS) if there exists a filtration

$$V = V_1 \supset V_2 \supset \cdots \supset V_r \supset V_{r+1} = (0)$$

such that $V_i / V_{i+1} \simeq V^{M_{\lambda_i}}$, $\lambda_i \in P_S^+$, $i = 1, \ldots, r$.

If $\nu \in \mathfrak{h}^*$, we denote by $\mathcal{P}_S(\nu)$ the number of families $(n_\alpha)_{\alpha \in \Delta^+ \setminus \Delta_S^+}$ with n_α a nonzero integer such that $\nu = \sum_{\alpha \in \Delta^+ \setminus \Delta_S^+} n_\alpha \alpha$.

Let $\mathbf{Z}^{\mathfrak{h}^*}$ be the additive group consisting of all maps from $\mathfrak{h}^*$ to $\mathbf{Z}$. If $\lambda \in \mathfrak{h}^*$, e^λ denotes the element of $\mathbf{Z}^{\mathfrak{h}^*}$ of support $\{\lambda\}$ such that $e^\lambda(\lambda) = 1$. We write $f = \sum_{\lambda \in \mathfrak{h}^*} f(\lambda) e^\lambda$ for every f in $\mathbf{Z}^{\mathfrak{h}^*}$.

Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. An $\mathfrak{a}$-module V is said to have a formal $\mathfrak{a}$-character, or simply, an $\mathfrak{a}$-character if $V = \amalg_{\lambda \in \mathfrak{h}^*} V(\lambda)$ and $\dim V(\lambda) < \infty$, for all λ in $\mathfrak{h}^*$. By the *$\mathfrak{a}$-character* of such V, $\operatorname{ch}_{\mathfrak{a}} V$, we mean the map: $\lambda \mapsto \dim V(\lambda)$ which is an element of $\mathbf{Z}^{\mathfrak{h}^*}$. If V is a $\mathfrak{g}$-module we set $\operatorname{ch} V = \operatorname{ch}_{\mathfrak{g}} V$. We will refer to the $\mathfrak{g}$-character of V simply as the *character* of V.

Every object in $\mathcal{O}$ has a character and, in particular, V^{M_λ} has a character. Furthermore, the character of V^{M_λ} is easily computed from its definition, namely

$$\operatorname{ch} V^{M_\lambda} = \operatorname{ch}_{\mathfrak{m}} M_\lambda \cdot \sum_{\gamma \in Q^+} \mathcal{P}_S(\gamma) e^{-\gamma},$$

where the multiplication is convolution (cf. [6]).

Combining this formula with Théorème 7.5.9 of [6] we have proved the following well-known

LEMMA 5.2. *Let $\lambda \in P_S^+$. If we denote by W_S the Weyl group of Δ_S then*

$$\sum_{w \in W_S} \varepsilon(w) e^{w\rho_S} \cdot \operatorname{ch} V^{M_\lambda} = \sum_{w \in W_S} \varepsilon(w) e^{w(\lambda + \rho_S)} \cdot \sum_{\gamma \in Q^+} \mathcal{P}_S(\gamma) e^{-\gamma},$$

where $\rho_S = (1/2) \sum_{\alpha \in \Delta_S^+} \alpha$ and $\varepsilon(w)$ is the determinant of w.

Let V be as in Definition 5.1. 7.5.3 in [6] implies that

$$\operatorname{ch} V = \sum_{i=1}^{r} \operatorname{ch} V^{M_{\lambda_i}}.$$

Therefore, by Lemma 5.2, we have

$$\frac{\sum_{w \in W_S} \varepsilon(w) e^{w\rho_S}}{\sum_{\gamma \in Q^+} \mathcal{P}_S(\gamma) e^{-\gamma}} \cdot \operatorname{ch} V = \sum_{i=1}^{r} \sum_{w \in W_S} \varepsilon(w) e^{w(\lambda_i + \rho_S)}.$$

Since $\lambda_i \in P_S^+$, $i = 1, \ldots, r$, the set

$$\{e^{w(\lambda_i + \rho_S)}\}_{w \in W_S}, \qquad i \in \{1, \ldots, r\},$$

is linearly independent.

The discussion above allows us to formulate:

DEFINITION 5.3. Let V be as in Definition 5.1. We denote by $(V : V^{M_\lambda})$ the number of i such that $\lambda = \lambda_i$, $\lambda_i \in P_S^+$, $i \in \{1, \ldots, r\}$.

The next two lemmas are immediate generalizations of results of [**1**].

LEMMA 5.4. *Suppose N in $\mathcal{O}_S$ has a GVCS. Let λ be an element of $P_{\mathfrak{m}}(N)$ which is maximal in $P(N)$, $v \in N^\lambda(\lambda) \setminus \{0\}$ and $N^1 = U(\mathfrak{g})v$. Then $N^1 \simeq V^{M_\lambda}$ and N/N^1 admits a GVCS.*

LEMMA 5.5. *If $N = N_1 \oplus N_2$ admits a GVCS then both N_1 and N_2 admit a GVCS.*

LEMMA 5.6. *If $\lambda \in P_S^+$ then the $U(g)$-module $\hat{Q}$ constructed in the proof of Proposition* 4.1 *has a GVCS.*

PROOF. Choose in $U(\mathfrak{u})$ a collection of weight vectors $u_1, \ldots, u_s$ of weights $\lambda_1, \ldots, \lambda_s$, respectively, such that their projections $\bar{u}_i$ in $U(\mathfrak{u})/(\mathfrak{u})^r U(\mathfrak{u})$ define a basis there, and $\lambda_i <^S \lambda_j$ implies $i > j$ (here the natural number r is as in the proof of Proposition 4.1). Let M_j be the $\mathfrak{p}$-submodule of $U(\mathfrak{u})/(\mathfrak{u})^r U(\mathfrak{u}) \otimes M_\lambda$ generated by $\bar{u}_i \otimes v_\lambda$, where v_λ is the canonical generator of M_λ, $i = 1, \ldots, j$. We therefore have

$$U(\mathfrak{u})/(\mathfrak{u})^r U(\mathfrak{u}) \otimes M_\lambda = M_s \supset M_{s-1} \supset \cdots \supset M_1 \supset M_0 = (0)$$

and M_j/M_{j-1} is a finite dimensional $\mathfrak{u}$-trivial $\mathfrak{m}$-module. Since $\mathfrak{m}$ is reductive, Corollaire 1.6.4 of [**6**] implies that for each j, M_j/M_{j-1} splits as a sum of M_λ's, $\lambda \in P_S^+$, with $\mathfrak{u} M_\lambda = 0$.

A natural refinement $\{N_i\}$ of $\{M_i\}$ gives a filtration

$$U(\mathfrak{u})/(\mathfrak{u})^r U(\mathfrak{u}) = N_p \supset N_{p-1} \supset \cdots \supset N_1 \supset N_0 = (0)$$

with $N_i/N_{i-1} \simeq M_{\mu_i}$, $\mu_i \in P_S^+$ and $\mathfrak{u} M_{\mu_i} = 0$, $i = 1, \ldots, p$.

The exactness of the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})}(\ )$ (cf. Proposition 5.1.4 of [**6**]) gives the desired GVCS of $\hat{Q}$. Q.E.D.

6. The duality theorem. In this section, we generalize the duality theorem of [**1**]. We will fix θ in Θ and restrict our attention to the objects of ${}_\theta(\mathcal{O}_S)$.

THEOREM 6.1. *Every projective module in ${}_\theta(\mathcal{O}_S)$ admits a GVCS. Furthermore if λ and μ are in P_S^+ the duality*

$$\left(P_\lambda : V^{M_\mu}\right) = \left(V^{M_\mu} : L_\lambda\right)$$

is true inside the category ${}_\theta(\mathcal{O}_S)$.

PROOF. For every $\lambda \in P_S^+$, $Q(\lambda, \theta)$ constructed in Proposition 4.1 admits a GVCS by the Lemmas 5.5 and 5.6. Thus Corollary 4.2 and Lemma 5.5 imply that each projective module in $\mathcal{O}_S$ admits a GVCS.

We observe that in order to prove the duality it suffices to show that

$$\left(P_\lambda : V^{M_\mu}\right) = \dim \operatorname{Hom}_{\mathfrak{g}}\left(P_\lambda, V^{M_\mu}\right) \tag{1}$$

holds for λ, μ such that $\theta_\mu = \theta_\lambda = \theta$, by Proposition 4.5.

Now,

$$Q(\lambda, \theta) = \coprod_{\substack{\nu \in P_S^+ \\ \theta_\nu = \theta}} n_\nu(\lambda) P_\nu$$

(cf. [1]). By Corollary 4.4 we have

$$n_\nu(\lambda) = \dim \operatorname{Hom}_{\mathfrak{g}}(Q(\lambda, \theta), L_\nu).$$

Hence, by Proposition 4.1,

$$n_\nu(\lambda) = \dim \operatorname{Hom}_{\mathfrak{m}}(M_\lambda, L_\nu).$$

This implies that $n_\nu(\lambda) = 0$ if $\lambda \nless \nu$ and $n_\lambda(\lambda) = 1$.

We order $(\nu_1, \ldots, \nu_s)$ the weights ν of P_S^+ such that $\theta_\nu = \theta$, in such a way that $\nu_i < \nu_j$ implies $i > j$. Then

$$\left(V^{M_{\nu_i}} : L_{\nu_j}\right) = 0 \quad \text{if } i > j.$$

Suppose we have proved that

$$\left(Q(\nu_i, \theta) : V^{M_{\nu_j}}\right) = \dim \operatorname{Hom}_{\mathfrak{g}}\left(Q(\nu_i, \theta), V^{M_{\nu_j}}\right)$$

for all i, j in $\{1, \ldots, s\}$. By the above we have that $Q(\nu_i, \theta) = \coprod_{j<i} n_{\nu_j}(\nu_i) P_{\nu_j} \oplus P_{\nu_i}$. It follows that $Q(\nu_1, \theta) = P_{\nu_1}$ and (1) is true for $P = P_{\nu_1}$. Proceeding by finite induction, we assume that (1) holds for $P_\lambda = P_{\nu_j}, j < i$. The linearity of (1) and the above expression of $Q(\nu_i, \theta)$ imply that (1) holds for P_{ν_i}, and, thus, for all $P = P_\nu$ with $\nu \in P_S^+, \theta_\nu = \theta$.

Therefore the second statement of the theorem will be proved when (1) is proved for $Q(\lambda, \theta)$, λ in P_S^+, with $\theta_\lambda = \theta$.

We may replace $Q(\lambda, \theta)$ by $\hat{Q}$ without changing (1). By the proof of Lemma 5.6, we have

$$(\hat{Q} : V^{M_\mu}) = \dim \operatorname{Hom}_{\mathfrak{m}}\left(M_\mu, U(\mathfrak{u})/ (\mathfrak{u})^r U(\mathfrak{u}) \otimes M_\lambda\right).$$

In order to conclude the proof of the theorem we need the following observations.

(a) It is a well-known fact that if $\mathfrak{a}$ is a complex Lie algebra and U, V and W are finite dimensional $\mathfrak{a}$-modules then $\operatorname{Hom}_{\mathfrak{a}}(U, V \otimes W)$ is isomorphic with $\operatorname{Hom}_{\mathfrak{a}}(V^* \otimes U, W)$.

(b) If $\alpha \in \Delta$ and $X \in \mathfrak{g}_\alpha$, then $\langle X, Y \rangle = 0$ for all Y in $\mathfrak{g}_{-\alpha}$ implies that $X = 0$. This allows us to identity the dual module of $U(\mathfrak{u})/(\mathfrak{u})^r U(\mathfrak{u})$ with $U(\mathfrak{u}^-)/(\mathfrak{u}^-)^r U(\mathfrak{u}^-)$.

(c) If $V \in \mathcal{O}_S$, 3.1(2) and Proposition 3.3 imply that $\dim \operatorname{Hom}_{\mathfrak{m}}(V, M_\lambda) = \dim \operatorname{Hom}_{\mathfrak{m}}(M_\lambda, V)$. (a) and (b) imply that

$$(\hat{Q} : V^{M_\mu}) = \dim \operatorname{Hom}_{\mathfrak{m}}\left(U(\mathfrak{u}^-)/ (\mathfrak{u}^-)^r U(\mathfrak{u}^-) \otimes M_\mu, M_\lambda\right).$$

Our choice of r in the proof of Proposition 4.1 implies that

$$\operatorname{Hom}_{\mathfrak{m}}\left(U(\mathfrak{u}^-) \otimes M_\mu, M_\lambda\right) \simeq \operatorname{Hom}_{\mathfrak{m}}\left(U(\mathfrak{u}^-)/ (\mathfrak{u}^-)^r U(\mathfrak{u}^-) \otimes M_\mu, M_\lambda\right).$$

As an $\mathfrak{m}$-module V^{M_μ} is isomorphic with $U(\mathfrak{u}^-) \otimes M_\mu$. Hence, $(\hat{Q} : V^{M_\mu}) = \dim \operatorname{Hom}_{\mathfrak{m}}(V^{M_\mu}, M_\lambda)$.

(1) now follows from (c) and Proposition 4.1. Q.E.D.

7. The generalized weak BGG resolution. We recall the construction of the weak generalized BGG resolution from [9]:

Let, for $k = 0, 1, 2, \ldots$, $D_k = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \bigwedge^k (\mathfrak{g}/\mathfrak{p})$. We define the operators $\tilde{\partial}_k$: $D_k \to D_{k-1}$ as follows: Let $X_1, \ldots, X_k$ be elements of $\mathfrak{g}/\mathfrak{p}$ represented by $Y_1, \ldots, Y_k$ in $\mathfrak{g}$, and X in $\mathfrak{g}$.

$$\tilde{\partial}_k(X \otimes X_1 \wedge \cdots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1}(XY_i) \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k$$
$$+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} X \otimes \overline{[Y_i, Y_j]} \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k,$$

where $\overline{Y}$ represents the canonical image of $Y \in \mathfrak{g}$ in $\mathfrak{g}/\mathfrak{p}$, and $\hat{X}_i$ means that X_i has been deleted.

It is easy to see that $\tilde{\partial}_k$ is a map.

Let $\tilde{\varepsilon}$: $D_0 \to \mathbf{C}$ be defined by

$$\tilde{\varepsilon}(X \otimes 1) = \text{the constant term of } X, X \in U(\mathfrak{g}).$$

Then it can be shown that

$$0 \to D_{\dim \mathfrak{u}} \xrightarrow{\tilde{\partial}_{\dim \mathfrak{u}}} \cdots \to D_1 \xrightarrow{\tilde{\partial}_1} D_0 \xrightarrow{\tilde{\varepsilon}} \mathbf{C} \to 0$$

is a complex which we denote by $V(\mathfrak{g}, \mathfrak{p})$.

The following is Theorem II.9.1 of [2] for $\mathfrak{a} = \mathfrak{g}$ and $\mathfrak{b} = \mathfrak{p}$.

PROPOSITION 7.1. *The complex* $V(\mathfrak{g}, \mathfrak{p})$ *is exact.*

NOTE. $V(\mathfrak{g}, \mathfrak{p})$ is isomorphic to $V(\mathfrak{u}^-, o)$ as a $\mathfrak{u}^-$-complex. In particular, $V(\mathfrak{g}, \mathfrak{p})$ is $\mathfrak{u}^-$-free (cf. [2]).

DEFINITION 7.2. If $\Phi = (\lambda_1, \ldots, \lambda_r)$ with λ_i in P_S^+ (some of the λ_i's may coincide), a $\mathfrak{g}$-module N in $\mathcal{O}_S$ is said to be of type Φ if N is as in Definition 5.1.

If $N \in \mathcal{O}_S$ is of type $\Phi = (\lambda_1, \ldots, \lambda_r)$ let

$$N = N_1 \supset N_2 \supset \cdots \supset N_r \supset N_{r+1} = (0)$$

with

$$N_i/N_{i+1} \simeq V^{M_{\lambda_i}}, \qquad i = 1, \ldots, r,$$

be the corresponding GVCS. Let $\theta \in \Theta$. Then

$${}_\theta N = {}_\theta(N_1) \supset {}_\theta(N_2) \supset \cdots \supset {}_\theta(N_r) \supset {}_\theta(N_{r+1}) = 0$$

and Proposition 2.6(3) implies that

$${}_\theta(N_i)/{}_\theta(N_{i+1}) \simeq {}_\theta(V^{M_{\lambda_i}}).$$

Proposition 7.1.8 in [6] implies that

$${}_\theta(V^{M_{\lambda_i}}) = 0, \quad \text{unless } \theta = \theta_{\lambda_i}.$$

Let us denote by Φ_θ the set of all λ in Φ such that $\theta = \theta_\lambda$ with the subordering inherited from Φ. We have established:

LEMMA 7.3. *Let* $N \in \mathcal{O}_S$ *be of type* Φ *and* $\theta \in \Theta$. *Then* ${}_\theta N$ *is of type* Φ_θ.

The following is a well-known fact.

LEMMA 7.4. *Let* $\mathfrak{a} \subset \mathfrak{g}$ *be a subalgebra, M an* $\mathfrak{a}$*-module and N a* $\mathfrak{g}$*-module. Then the map*

$$\varphi: U(\mathfrak{g}) \otimes_{U(\mathfrak{a})} (M \otimes N) \to N \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{a})} M)$$

defined by

$$\varphi(X \otimes (v \otimes u)) = X(u \otimes (1 \otimes v)),$$

$X \in U(\mathfrak{g})$, $v \in M$, $u \in N$, *is a* $U(\mathfrak{g})$*-isomorphism.*

NOTE. For a proof of 7.4 see [9, pp. 44–45].

Next we state a particular case of Proposition 6.4 of [9].

PROPOSITION 7.5. *For each* $k = 0, \ldots, \dim \mathfrak{u}^-$, $\lambda \in P^+$, $\bigwedge^k(\mathfrak{g}/\mathfrak{p}) \otimes L_\lambda$ *has a* $\mathfrak{p}$*-filtration*

$$\bigwedge^k(\mathfrak{g}/\mathfrak{p}) \otimes L_\lambda = M_1 \supset M_2 \supset \cdots \supset M_r \supset M_{r+1} = (0)$$

such that each M_i/M_{i+1} *is* $\mathfrak{u}$*-trivial and* $\mathfrak{m}$*-isomorphic to* M_{λ_i}, $\lambda_i \in P_S^+$, *and* $\coprod_{i=1}^r M_{\lambda_i}$ *is* $\mathfrak{m}$*-isomorphic to* $\bigwedge^k(\mathfrak{g}/\mathfrak{p}) \otimes L_\lambda$.

Combining Lemma 7.4, Proposition 7.5 and the exactness of the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (\ )$, we obtain

PROPOSITION 7.6. *Let* $V(\mathfrak{g}, \mathfrak{p}; \lambda) = V(\mathfrak{g}, \mathfrak{p}) \otimes L_\lambda$, $D_k^\lambda = D_k \otimes L_\lambda$, $\lambda \in P^+$, $k = 0, 1, \ldots, \dim \mathfrak{u}^-$. *Then* $V(\mathfrak{g}, \mathfrak{p}; \lambda)$ *is a* $U(\mathfrak{u}^-)$*-free resolution of* L_λ *and* D_k^λ *is of type* $\Phi_k(\lambda)$ *where* $\bigwedge^k(\mathfrak{u}^-) \otimes L_\lambda \simeq \coprod M_{\lambda_i}$ *as an* $\mathfrak{m}$*-module and* $\Phi_k(\lambda)$ *consists of the* λ_i *in some order.*

DEFINITION 7.7. Let W_S be the subgroup of W generated by $\{\sigma_i | i \in S\}$. Then W_S is the Weyl group of Δ_S (cf. [13]).

Let $W^S = \{w \in W | w^{-1}\Delta_S^+ \subset \Delta^+\}$.

PROPOSITION 7.8. *Every element w in W can be uniquely written as* $w = w_1 w'$, *with* w_1 *in* W_S *and* w' *in* W^S. *Moreover,* $l(w) = l(w_1) + l(w')$.

PROOF. [13].

PROPOSITION 7.9. *Let* $\lambda \in P^+$ *and let* P^* *be the set of weights* μ *of* $\bigwedge^k(\mathfrak{u}^-) \otimes L_\lambda$ *such that* $\mathfrak{u} + \rho \in W(\lambda + \rho)$. *Then* $w \mapsto w \cdot \lambda$ *defines a bijection between* $(W^S)^{(k)} = \{w \in W^S | l(w) = k\}$ *and* P^* *and each weight of* P^* *occurs with multiplicity one.*

PROOF. [13].

LEMMA 7.10 [9, PROPOSITION 8.4]. *Let* λ, P^* *be as in Proposition* 7.9. *If* $\mu \in P^*$ *then* $\mu + \alpha_i$ *is not a weight of* $\bigwedge^k(\mathfrak{u}^-) \otimes L_\lambda$, *for* $i \in S$.

Let $\Psi_k(\lambda)$ consist of the $w \cdot \lambda$ with $w \in (W^S)^{(k)}$ in some order. Combining Lemmas 7.3 and 7.10 with Propositions 2.5(2) and 7.9 we obtain the following particular case of Theorem 8.7 of [9].

THEOREM 7.11. *If* $\lambda \in P^+$, ${}_{\theta_\lambda}(D_k^\lambda)$ *is of type* $\Psi_k(\lambda)$, $k = 0, 1, \ldots, \dim \mathfrak{u}^-$.

We now proceed to prove that the modules ${}_{\theta_\lambda}(D_k^\lambda)$ split as a sum of generalized Verma modules.

8. n-extensions of $\mathfrak{g}$-modules. We denote by $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$ the full subcategory of the category of $\mathfrak{g}$-modules satisfying Property 3.1(2). Here and in §9 we will extend our discussion to the category $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$. This will not play a significant role in this paper, but is of independent interest (cf. Remark before Corollary 9.2).

NOTES. (1) $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$ is invariant under the operations of taking submodules, quotients and direct sums.

(2) If M is in $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$ we can view M as an $\mathfrak{m}$-module and form $U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} M$. The latter is clearly a projective object in $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$. Hence $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$ has enough projectives.

Let M and N be in $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$. If

$$E: 0 \to N \to E_n \to \cdots \to E_1 \to M \to 0$$

is an n-extension of M by N in $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$, we write $[E]$ for the equivalence class of E. We denote by $\operatorname{Ext}^n_{(\mathfrak{g},\mathfrak{m})}(M, N)$ the group of all equivalence classes of n-extensions of M by N.

NOTE. $\operatorname{Ext}^n_{(\mathfrak{g},\mathfrak{m})}$ is the relative Ext bifunctor (cf. [**16**, Chapter IX]).

LEMMA 8.1. *Suppose* $M = M_0 \supset \cdots \supset M_d \supset M_{d+1} = (0)$ *is a filtration in* $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$ *with*

$$M_i/M_{i+1} \simeq W_i, \qquad i = 0, \ldots, d,$$

such that $\operatorname{Ext}^1_{(\mathfrak{g},\mathfrak{m})}(W_i, W_j)$ *is trivial*, $0 \leq i < j \leq d$. *Then*

$$M \simeq \coprod_{i=0}^{d} W_i.$$

PROOF. We proceed by induction on the length of the filtration. If $d = 1$, we have

$$M = M_0 \supset M_1 \supset M_2 = (0)$$

and

$$M/M_1 \simeq W_0, \qquad M_1 = W_1.$$

By hypothesis, $\operatorname{Ext}^1_{(\mathfrak{g},\mathfrak{m})}(W_0, W_1)$ is trivial. Therefore, $0 \to W_1 \to M \to W_0 \to 0$ splits and $M \simeq W_0 \oplus W_1$.

By the induction hypothesis,

$$M_1 \simeq \coprod_{i=1}^{d} W_i.$$

Since $M/M_1 \simeq W_0$, it follows that

$$0 \to \coprod_{i=1}^{d} W_i \to M \to W_0 \to 0 \qquad (*)$$

is exact. If $(*)$ does not split then $\operatorname{Ext}^1_{(\mathfrak{g},\mathfrak{m})}(W_0, \coprod_{i=1}^d W_i)$ is nontrivial. The additivity of the relative Ext bifunctor (cf. [**16**, Chapter XII]) implies that $\coprod_{i=1}^d \operatorname{Ext}^1_{(\mathfrak{g},\mathfrak{m})}(W_0, W_i)$ is nontrivial, a contradiction. Q.E.D.

9. The splitting theorem. We recall some facts about $\operatorname{Ext}^n_{(\mathfrak{g},\mathfrak{m})}$ following [**19**].

Let for $n = 0, 1, \ldots, \dim \mathfrak{u} = s$, $\lambda \in P_S^+$, $M_{n,\lambda} = U(\mathfrak{u}) \otimes \bigwedge^n \mathfrak{u} \otimes M_\lambda$. We define the operators β_n: $M_{n,\lambda} \to M_{n-1,\lambda}$ as follows: Let $x \in U(\mathfrak{u})$, $X_i \in \mathfrak{u}$, $i = 1, \ldots, n$, and $v \in M_\lambda$.

$$\beta_n(X \otimes X_1 \wedge \cdots \wedge X_n \otimes v)$$
$$= \sum_{i=1}^{n} (-1)^{i+1} XX_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n \otimes v$$
$$+ \sum_{1 \le i < j \le n} (-1)^{i+j} X \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n \otimes v.$$

Let $\sigma: M_{0,\lambda} \to M_\lambda$ be the trivial module action. Then the sequence of $U(\mathfrak{p})$-modules

$$0 \to M_{s,\lambda} \xrightarrow{\beta_s} \cdots \xrightarrow{\beta_1} M_{0,\lambda} \xrightarrow{\sigma} M_\lambda \to 0$$

is exact.

If V is in $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$ then the nth cohomology space of $\mathfrak{u}$ on V, $H^n(\mathfrak{u}, V)$, is an $\mathfrak{m}$-semisimple module. Let f be a representative of a class ω in $\mathrm{Hom}_{\mathfrak{m}}(M_\lambda, H^n(\mathfrak{u}, V))$. Then f can be looked upon as an element of $\mathrm{Hom}_{U(\mathfrak{p})}(M_{n,\lambda}, V)$, so that $f \circ \beta_{n+1} = 0$. Applying a standard argument in homological algebra, we obtain a commutative diagram with exact rows,

$$\begin{array}{ccccccccccccc} M_{n+1,\lambda} & \xrightarrow{\beta_{n+1}} & M_{n,\lambda} & \xrightarrow{\beta_n} & M_{n-1,\lambda} & \xrightarrow{\beta_{n-1}} \cdots \xrightarrow{\beta_1} & M_{0,\lambda} & \xrightarrow{\sigma} & M_\lambda & \to & 0 \\ \downarrow & & \downarrow f & & \downarrow \mu_{n-1} & & \downarrow \mu_0 & & \| & & \\ 0 & \to & E_{n,\lambda} & \xrightarrow{\alpha_n} & E_{n-1,\lambda} & \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_1} & E_{0,\lambda} & \xrightarrow{\alpha_0} & M_\lambda & \to & 0 \end{array}$$

where $E_{n,\lambda} = V$, (μ_{n-1}, α_n) is the push-out of (β_n, f), $E_{j,\lambda} = M_{j,\lambda}$ and $\mu_j = \mathrm{id}$, $j = 0, \ldots, n-2$. We now apply the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (\ )$ to the bottom row, and take the quotient of the resulting sequence by an appropriate sequence to obtain an exact sequence in $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$,

$$\bar{E}: 0 \to V \xrightarrow{\bar{\alpha}_n} \bar{E}_{n-1,\lambda} \xrightarrow{\bar{\alpha}_{n-1}} \cdots \xrightarrow{\bar{\alpha}_1} \bar{E}_{0,\lambda} \xrightarrow{\bar{\alpha}_0} V^{M_\lambda} \to 0,$$

where $\bar{E}_{j,\lambda} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E_{j,\lambda}$, for $j < n-1$. $[\bar{E}]$ depends only on ω and we set $\hat{\omega} = [\bar{E}]$.

Conversely, let $\bar{E}$ be an exact sequence in $\mathcal{C}_{(\mathfrak{g},\mathfrak{m})}$. Using a standard homological algebraic argument we obtain an element ξ in $\mathrm{Hom}_{\mathfrak{m}}(M_\lambda, H^n(\mathfrak{u}, V))$ which depends only on $[\bar{E}]$. We set $[\bar{E}]^\vee = \xi$. Then $\hat{\xi} = [\bar{E}]$ and $\check{\hat{\omega}} = \omega$.

The following is a key result in the proof of our splitting theorem.

Lemma 9.1. *Let λ, μ be in P_S^+. If $\mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{m})}(V^{M_\lambda}, V^{M_\mu})$ is nontrivial then there exist distinct $\nu_1, \ldots, \nu_n$ in P_S^+ such that*

$$V_\lambda \subsetneq V_{\nu_1} \subset \cdots \subset V_{\nu_n} \subset V_\mu.$$

Proof. Let $[E]$ be in $\mathrm{Ext}^n_{\mathfrak{g},\mathfrak{m}}(V^{M_\lambda}, V^{M_\mu})$, where E is the nonsplit exact sequence

$$0 \to V^{M_\mu} \xrightarrow{\alpha_n} E_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_1} E_0 \xrightarrow{\alpha_0} V^{M_\lambda} \to 0.$$

By the discussion preceding the lemma we may assume that E is obtained from an element of $\mathrm{Hom}_{\mathfrak{m}}(M_\lambda, H^n(\mathfrak{u}, V^{M_\mu}))$ as described there. We now set $M_{n,\lambda}^{j+1} = \mathfrak{u}^j U(\mathfrak{u}) \otimes \bigwedge^n \mathfrak{u} \otimes M_\lambda$, $j = 1, 2, \ldots$. Then $M_{n,\lambda} = M_{n,\lambda}^1 \supset M_{n,\lambda}^2 \supset \ldots$ and $\bigcap_{j=1}^\infty M_{n,\lambda}^j = (0)$. Since $Z(\mathfrak{g})$ acts on V^{M_μ} by θ_μ this implies that the exact sequence E may be assumed to be in ${}_{\theta_\mu}(\mathcal{O}_S)$. In particular θ_λ must equal θ_μ.

We assume $n = 1$ and consider

$$E: 0 \to V^{M_\mu} \xrightarrow{\alpha} E \xrightarrow{\beta} V^{M_\lambda} \to 0,$$

a representative of a nonzero element of $\mathrm{Ext}^1_{(\mathfrak{g},\mathfrak{m})}(V^{M_\lambda}, V^{M_\mu})$. By Theorem 6.1 there exists an indecomposable projective module P_λ in ${}_{\theta_\lambda}(\mathcal{O}_S)$ such that $P_\lambda = W_1 \supset W_2 \supset \cdots \supset W_r \supset W_{r+1} = (0)$ with $W_i/W_{i+1} \simeq V^{M_{\nu_i}}$. Furthermore $\nu_1 = \lambda$ and $V_\lambda \subsetneqq V_{\nu_i}$ for $i \geqslant 2$.

Let $\pi: P_\lambda \to V^{M_\lambda}$ be the projection from W_1 to W_1/W_2. By the projectivity of P_λ we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & V^{M_\mu} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & V^{M_\lambda} & \to & 0. \\ & & & & \eta \nwarrow & & \nearrow \pi & & \\ & & & & & P_\lambda & & & \end{array}$$

We claim that $\eta(W_2) = \eta(P_\lambda) \cap \alpha(V^{M_\mu}) \neq (0)$. Indeed, if v is in W_2 then $\beta\eta(v) = \pi(v) = 0$. Hence $\eta(v)$ is in $\eta(P_\lambda) \cap \alpha(V^{M_\mu})$. Conversely, if v is in P_λ with $\eta(v)$ in $\alpha(V^{M_\mu})$ then $\beta\eta(v) = 0$, or $\pi(v) = 0$ and, therefore, v is in $\mathrm{Ker}\, \pi = W_2$. If $\eta(W_2) = 0$ then η induces

$$\tilde{\eta}: P_\lambda/W_2 \to E$$

where $\tilde{\eta}(v + W_2) = \eta(v)$. Then $\beta\tilde{\eta}(v + W_2) = \beta\eta(v) = \pi(v) = v + W_2$, i.e., $\beta\tilde{\eta} = \mathrm{id}$ and, therefore, E splits, which is contrary to our assumption.

Now, $\eta(P_\lambda) \supset \eta(W_2) \supset \cdots \supset \eta(W_i) \supset \cdots$ and for all $i \geqslant 2$, $\eta(W_i)/\eta(W_{i+1})$ is generated by a highest weight vector of weight ν_i such that $V_\lambda \subset V_{\nu_i}$, or $\eta(W_i)/\eta(W_{i+1}) = (0)$. Since $\eta(W_2) \neq (0)$ and $\eta(W_{r+1}) = (0)$, there exists i in $\{2, \ldots, r\}$ such that $V_\lambda \subset V_{\nu_i}$ and L_{ν_i} is a subquotient of $\eta(W_2) \subset \alpha(V^{M_\mu})$. Theorem 2.8 implies that $V_{\nu_i} \subset V_\mu$. Thus $V_\lambda \subsetneqq V_{\nu_i} \subset V_\mu$ and the assertion of the lemma holds for $n = 1$.

We now assume that the lemma is true for $n - 1$ and consider

$$E: 0 \to V^{M_\mu} \xrightarrow{\alpha_n} E_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_1} E_0 \xrightarrow{\alpha_0} V^{M_\lambda} \to 0$$

a representative of a nonzero element of $\mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{m})}(V^{M_\lambda}, V^{M_\mu})$. Let P_λ be as above. Then there exist $E_0^1, \ldots, E_{n-1}^1$ in ${}_{\theta_\lambda}(\mathcal{O}_S)$, $\alpha_i^1: E_i^1 \to E_{i-1}^1$, $\alpha_0^1: E_0^1 \to V^{M_\lambda}$ and $\gamma_i: E_i^1 \to E_i$, $i = 0, \ldots, n - 1$, such that the diagram

$$\begin{array}{ccccccccccc} 0 & \to & V^{M_\mu} & \xrightarrow{\alpha_n^1} & E_{n-1}^1 & \xrightarrow{\alpha_{n-1}^1} \cdots \xrightarrow{\alpha_1^1} & E_0^1 & \xrightarrow{\alpha_0^1} & V^{M_\lambda} & \to & 0 \\ & & \| & & \downarrow\gamma_{n-1} & & \downarrow\gamma_0 & & \| & & \\ 0 & \to & V^{M_\mu} & \xrightarrow{\alpha_n} & E_{n-1} & \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_1} & E_0 & \xrightarrow{\alpha_0} & V^{M_\lambda} & \to & 0 \end{array}$$

is commutative and has exact rows and $E_1^1 = P_\lambda$, $\alpha_0^1 = \pi$. From the exactness of

$$0 \to W_r \to W_{r-1} \to W_{r-1}/W_r \to 0$$

we obtain a long exact $\mathrm{Ext}_{(\mathfrak{g},\mathfrak{m})}$-sequence,

$$\cdots \to \mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_{r-1}/W_r, V^{M_\mu})$$
$$\to \mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_{r-1}, V^{M_\mu}) \to \mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_r, V^{M_\mu}) \to \cdots .$$

If $\mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_{r-1}/W_r, V^{M_\mu})$ is nontrivial, or $\mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_r, V^{M_\mu})$ is nontrivial, then the lemma follows by the induction hypothesis, If both are trivial then $\mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_{r-1}, V^{M_\mu})$ is trivial.

Now, from the exactness of

$$0 \to W_{r-1} \to W_{r-2} \to W_{r-2}/W_{r-1} \to 0$$

we obtain a long exact $\mathrm{Ext}_{(\mathfrak{g},\mathfrak{m})}$-sequence:

$$\cdots \mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_{r-2}/W_{r-1}, V^{M_\mu}) \to \mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_{r-2}, V^{M_\mu}) \to \mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_{r-1}, V^{M_\mu}) \to \cdots.$$

If $\mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_{r-2}/W_{r-1}, V^{M_\mu})$ is nontrivial, then the lemma follows by the induction hypothesis. If not, then $\mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_{r-2}, V^{M_\mu})$ is trivial. We continue in this fashion and note that

$$\mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_2, V^{M_\mu}) \simeq \mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(\alpha_1^1(E_1^1), V^{M_\mu})$$

and hence that $\mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_2, V^{M_\mu})$ is nontrivial. We conclude that there exists i in $\{2, \ldots, r\}$ such that $\mathrm{Ext}^{n-1}_{(\mathfrak{g},\mathfrak{m})}(W_i/W_{i+1}, V^{M_\mu})$ is nontrivial and the result follows by the induction hypothesis. Q.E.D.

REMARK. It follows from Lemma 9.1 and the discussion that precedes it that $\mathrm{Hom}_{\mathfrak{m}}(M_\lambda, H^n(\mathfrak{u}, V^{M_\mu}))$ is trivial unless $V_\lambda \subsetneq V_{\nu_1} \subset \cdots \subset V_{\nu_n} \subset V_\mu$, for some distinct elements $\nu_1, \ldots, \nu_n$ in P_S^+.

COROLLARY 9.2. *Let λ be a dominant integral linear form on $\mathfrak{h}$ and and let w, w' be elements of W^S with $l(w) = l(w')$. Then*

$$\mathrm{Ext}^1_{(\mathfrak{g},\mathfrak{m})}(V^{M_{w\cdot\lambda}}, V^{M_{w'\cdot\lambda}})$$

is trivial.

PROOF. Suppose $\mathrm{Ext}^1_{(\mathfrak{g},\mathfrak{m})}(V^{M_{w\cdot\lambda}}, V^{M_{w'\cdot\lambda}})$ is not trivial. Then $V_{w\cdot\lambda} \subsetneq V_{w'\cdot\lambda}$, by Lemma 9.1. Therefore $w < w'$, by Theorem 2.12. But $w < w'$ contradicts the assumption that $l(w) = l(w')$. Q.E.D.

THEOREM 9.3. *If λ is in P^+, then*

$$_{\theta_\lambda}(D_k^\lambda) = \coprod_{w \in (W^S)^{(k)}} V^{M_{w\cdot\lambda}},$$

for $k = 0, 1, \ldots, \dim \mathfrak{u}^-$.

PROOF. The statement follows immediately from Theorem 7.11, Lemma 8.1 and Corollary 9.2. Q.E.D.

10. The strong BGG resolution. Fix $\lambda \in P^+$. For each $w \in W$ we choose an injection, $V_{w\cdot\lambda} \to V_\lambda$. This fixes, for each pair (w, w') of elements of W such that $w \leqslant w'$, an injection $i_{w,w'}$: $V_{w\cdot\lambda} \to V_{w'\cdot\lambda}$ (cf. Theorem 2.12). Let $C_k = \coprod_{w \in W^{(k)}} V_{w\cdot\lambda}$, $k = 0, 1, \ldots, \dim \mathfrak{n}^-$. Then every $\mathfrak{g}$-homomorphism of C_k into C_{k-1} is given by $\sum c^k_{w_1,w_2} i_{w_1,w_2}$, $c^k_{w_1,w_2} \in \mathbf{C}$, the sum over all $w_1 \in W^{(k)}$, $w_2 \in W^{(k-1)}$, $k = 1, \ldots, \dim \mathfrak{n}^-$.

Let D_k, ${}_{\theta_\lambda}(D_k^\lambda)$, $\tilde{\partial}_k$, $k = 1, \ldots, \dim \mathfrak{u}^-$, and $\tilde{\varepsilon}$ be as in §7. We will use the same notation $\tilde{\partial}_k$ for the restriction of $\tilde{\partial}_k \otimes \mathrm{id}$ to ${}_{\theta_\lambda}(D_k^\lambda)$ and write $\tilde{\partial}_k$: ${}_{\theta_\lambda}(D_k^\lambda) \to {}_{\theta_\lambda}(D_{k-1}^\lambda)$, $k = 1, \ldots, \dim \mathfrak{u}^-$; $\tilde{\varepsilon}$ will also denote

$$\tilde{\varepsilon} \otimes \mathrm{id}|_{\theta_\lambda(D_0^\lambda)}\colon {}_{\theta_\lambda}(D_0^\lambda) \to L_\lambda.$$

If $S = \varnothing$ then $\mathfrak{p} = \mathfrak{b}$. In this case we set ${}_{\theta_\lambda}(B_k^\lambda) = {}_{\theta_\lambda}(D_k^\lambda)$, $\partial_k = \tilde{\partial}_k$, $k = 1, \ldots, \dim \mathfrak{u}^- = \dim \mathfrak{n}^-$ and $\varepsilon = \tilde{\varepsilon}$. That is,

$${}_{\theta_\lambda}(B_k^\lambda) = {}_{\theta_\lambda}\big(\big(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \textstyle\bigwedge^k (\mathfrak{g}/\mathfrak{b})\big) \otimes L_\lambda\big), \qquad k = 0, 1, \ldots, \dim \mathfrak{n}^-.$$

Theorem 9.3 implies that

$${}_{\theta_\lambda}(B_k^\lambda) = \coprod_{w \in W^{(k)}} V_{w\cdot\lambda} = C_k, \qquad k = 0, 1, \ldots, \dim \mathfrak{n}^-. \qquad (*)$$

We abuse notation and write

$$\partial_k\colon C_k \to C_{k-1}, \qquad k = 1, \ldots, \dim \mathfrak{n}^-.$$

LEMMA 10.1. *Let*

$$0 \to C_q \xrightarrow{\alpha_q} \cdots \xrightarrow{\alpha_2} C_1 \xrightarrow{\alpha_1} C_0 \xrightarrow{\alpha_0} L_\lambda \to 0$$

be an exact complex, with the C_i as above and $q = \dim \mathfrak{n}^-$. Then $\alpha_{i|V_{w\cdot\lambda}}$ is injective for all $w \in W^{(i)}$, $i = 1, \ldots, q$.

PROOF. If $w' \in W^{(i+1)}$ and $w \in W^{(i)}$, then the image of any map $V_{w'\cdot\lambda} \to V_{w\cdot\lambda}$ is a proper submodule. So $V_{w\cdot\lambda}$ cannot lie in the image of α_{i+1} and therefore it cannot lie in the kernel of α_i. Q.E.D.

LEMMA 10.2. *Let*

$$0 \to C_q \xrightarrow{\alpha_q} \cdots \xrightarrow{\alpha_2} C_1 \xrightarrow{\alpha_1} C_0 \xrightarrow{\alpha_0} L_\lambda \to 0$$

be a complex. Suppose that $\alpha_{i|V_{w\cdot\lambda}} \neq 0$, for all $w \in W^{(i)}$, $i = 1, \ldots, q$, where $q = \dim \mathfrak{n}^-$. Denote by $(a^i_{w_1,w_2})$, $w_1 \in W^{(i)}$, $w_2 \in W^{(i-1)}$ the complex matrix associated with α_i, $i = 1, \ldots, q$. Then $a^i_{w_1,w_2} \neq 0$ for all $w_1 \in W^{(i)}$, $w_2 \in W^{(i-1)}$ such that $w_1 \leftarrow w_2$, all $i = 1, \ldots, q$.

PROOF. The result is clear for $i = 1$. Suppose that the lemma holds for $i = 1, \ldots, j-1$ and let $w_0 \in W^{(j)}$. We assert that there exists $\alpha \in \pi$ such that $w_0 \overset{\alpha}{\leftarrow} \sigma_\alpha w_0$ and $a^j_{w_0,\sigma_\alpha w_0} \neq 0$. Indeed, since α_j restricted to $V_{w_0\cdot\lambda}$ is nonzero, there exists $w_1 \in w^{(j-1)}$ such that $a^j_{w_0,w_1} \neq 0$. Theorem 2.12 implies that $w_0 \overset{\gamma}{\leftarrow} w_1$ for some $\gamma \in \Delta^+$. If $\gamma \in \pi$ then our assertion is true. If $\gamma \notin \pi$ let $\alpha \in \pi$ be such that $w_0 \overset{\alpha}{\leftarrow} \sigma_\alpha w_0$. Set $t_0 = \sigma_\alpha w_0$. By Lemma 2.10 we have

$$\begin{array}{ccccc} & & w_1 & & \\ & \gamma \swarrow & & \nwarrow \alpha & \\ w_0 & & & & \sigma_\alpha w_1 \\ & \nwarrow \alpha & & \sigma_\alpha\gamma \swarrow & \\ & & t_0 & & \end{array}$$

Let ξ_w denote the canonical generator of $V_{w\cdot\lambda}$. If $w \leqslant w' \leqslant w''$ let $\xi_{w,w'}$ denote the image of ξ_w in $V_{w'\cdot\lambda}$ under $i_{w,w'}$, and $\xi_{w,w',w''}$ denote the image of $\xi_{w,w'}$ under $i_{w',w''}$.

Clearly $\xi_{w,w''} = \xi_{w,w',w''}$.

Now

$$\alpha_{j-1}\alpha_j(\xi_{w_0}) = \alpha_{j-1}\left(\coprod_{\substack{t \\ w_0 \leftarrow t}} a^j_{w_0,t}\xi_{w_0,t}\right) = \coprod_{\substack{r \\ t \leftarrow r}} \coprod_{\substack{t \\ w_0 \leftarrow t}} a^j_{w_0,t} a^{j-1}_{t,r}\xi_{w_0,r} = 0.$$

Using Lemma 2.11 we obtain

$$a^j_{w_0,w_1} a^{j-1}_{w_1,\sigma_\alpha w_1} + a^j_{w_0,t_0} a^{j-1}_{t_0,\sigma_\alpha w_1} = 0. \tag{1}$$

Since $a^j_{w_0,w_1} \neq 0$, and $a^{j-1}_{w_1,\sigma_\alpha w_1} \neq 0$ by the induction hypothesis, (1) implies that $a^j_{w_0,t_0} \neq 0$, proving our assertion.

Now, let w be an arbitrary element of $W^{(j-1)}$ such that $w_0 \overset{\beta}{\leftarrow} w$, $\beta \in \Delta^+$. Applying Lemma 2.10 we get

$$\begin{array}{ccccc} & & w & & \\ & \beta \swarrow & & \nwarrow \alpha & \\ w_0 & & & & \sigma_\alpha w \\ & \nwarrow \alpha & & \sigma_\alpha\beta \swarrow & \\ & & \sigma_\alpha w_0 & & \end{array}$$

Using the argument we have just used to prove the assertion above, we obtain

$$a^j_{w_0,\sigma_\alpha w_0} a^{j-1}_{\sigma_\alpha w_0,\sigma_\alpha w} + a^j_{w_0,w} a^{j-1}_{w,\sigma_\alpha w} = 0.$$

Therefore $a^j_{w_0,w} \neq 0$. Q.E.D.

DEFINITION 10.3 **[2]**. A *square* is a quadruple (w_1, w_2, w_3, w_4) of elements of W such that $w_1 \leftarrow w_2 \leftarrow w_4$ and $w_1 \leftarrow w_3 \leftarrow w_4$, $w_2 \neq w_3$ (cf. Definition 2.9).

The following lemma plays an important role in the construction of the strong BGG resolution. Here we give a simple proof of the lemma using (*) above.

First, we recall some concepts that will be needed in the proof.

(i) A real subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is called a *real form* of $\mathfrak{g}$ if $\mathfrak{a} \oplus (-1)^{1/2}\mathfrak{a} = \mathfrak{g}$.

(ii) If $\mathfrak{a}$ is a finite dimensional semisimple Lie algebra over a field F of characteristic zero, a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{a}$ (see **[6]** for the definition of a Cartan subalgebra of $\mathfrak{a}$) is said to be a *splitting Cartan subalgebra* of $\mathfrak{a}$ if the characteristic roots of every ad H: $\mathfrak{a} \to \mathfrak{a}$, $H \in \mathfrak{c}$, are in F. We say that the pair $(\mathfrak{a}, \mathfrak{c})$ is a *split semisimple Lie algebra* if $\mathfrak{c}$ is a splitting Cartan subalgebra of $\mathfrak{a}$.

LEMMA 10.4 **[2**, LEMMA 10.4**]**. *To every arrow $w_1 \leftarrow w_2$ it is possible to associate a number $c(w_1, w_2) = \pm 1$, such that the product of all numbers associated to the four arrows of any square (w_1, w_2, w_3, w_4) is equal to -1.*

PROOF. For each $\alpha \in \Delta$, let X_α, Y_α, H_α be as in §3. Observe that X_α and Y_α were chosen so that $\langle X_\alpha, Y_\alpha \rangle = 1$. If α, $\beta \in \Delta$ and $\alpha + \beta \neq 0$ define $N_{\alpha,\beta} \in \mathbf{C}$ by $[X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta}$ if $\alpha + \beta$ is a root, and set $N_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root. It can be shown that the numbers $N_{\alpha,\beta}$ are real numbers (see e.g. **[18**, Theorem 3.7.5**]**). Set $\mathfrak{h}_{\mathbf{R}} = \Sigma_{\alpha \in \Delta} \mathbf{R}H_\alpha$ and $\mathfrak{g}_{\mathbf{R}} = \mathfrak{h}_{\mathbf{R}} + \Sigma_{\alpha \in \Delta} \mathbf{R}X_\alpha$. The discussion above implies that $\mathfrak{g}_{\mathbf{R}}$ is a real form of $\mathfrak{g}$. The integrality properties of the roots imply that $\mathfrak{h}_{\mathbf{R}}$ is a splitting Cartan subalgebra of $\mathfrak{g}_{\mathbf{R}}$.

We now note that what has been said in the previous sections about $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{g}$ complex semisimple and $\mathfrak{h}$ Cartan, is also true for $(\mathfrak{g}_{\mathbf{R}}, \mathfrak{h}_{\mathbf{R}})$. If μ is a real valued linear form on $\mathfrak{h}_{\mathbf{R}}$, denote by $(V_\mu)_{\mathbf{R}}$ and $({}_{\theta_\mu}(B_k)^\mu)_{\mathbf{R}}$ the $U(\mathfrak{g}_{\mathbf{R}})$-modules corresponding to V_μ and ${}_{\theta_\mu}(B_k^\mu)$, respectively, when we substitute $(\mathfrak{g}_{\mathbf{R}}, \mathfrak{h}_{\mathbf{R}})$ for $(\mathfrak{g}, \mathfrak{h})$.

Let λ be a dominant integral real valued linear form on $\mathfrak{h}_{\mathbf{R}}$. (*) implies that

$$\left({}_{\theta_\lambda}(B_k^\lambda)\right)_{\mathbf{R}} = \coprod_{w \in W^{(k)}} (V_{w\cdot\lambda})_{\mathbf{R}}.$$

Let $\partial_k^{\mathbf{R}}$ and $\varepsilon^{\mathbf{R}}$ be the $U(\mathfrak{g}_{\mathbf{R}})$-module maps ∂_k and ε, respectively, in the real case. Suppose $w_1 \leftarrow w_2$, $w_1, w_2 \in W$. If $w_1 \in W^{(k)}$, for some $k \in \{1, \ldots, \dim \mathfrak{n}^-\}$, then $w_2 \in W^{(k-1)}$, and we let $s(w_1, w_2)$ be the real entry corresponding to the pair (w_1, w_2) in the real matrix $(s(w, w'))_{w \in W^{(k)}, w' \in W^{(k-1)}}$ associated with $\partial_k^{\mathbf{R}}$. An argument used in the proof of Lemma 10.2 implies that if (w_1, w_2, w_3, w_4) is any square then

$$s(w_1, w_2)s(w_2, w_4) = -s(w_1, w_3)s(w_3, w_4).$$

Proposition 7.1 combined with Lemmas 10.1 and 10.2 imply that $s(w_1, w_2) \neq 0$. Now set

$$c(w_1, w_2) = \frac{s(w_1, w_2)}{|s(w_1, w_2)|}. \quad \text{Q.E.D.}$$

For $k = 0, 1, \ldots, \dim \mathfrak{n}^-$, let $d_k\colon C_k \to C_{k-1}$ be the map defined by the matrix $(d^k_{w_1,w_2})$, $w_1 \in W^{(k)}$, $w_2 \in W^{(k-1)}$, where $d^k_{w_1,w_2} = c(w_1, w_2)$ (see Lemma 10.4), if $w_1 \leftarrow w_2$ and $d^k_{w_1,w_2} = 0$, otherwise.

Let $\eta\colon C_0 \to L_\lambda$ be the canonical surjection defined by $\eta(1 \otimes 1) = v_\lambda$, where v_λ is the canonical generator of L_λ.

The result that follows simplifies the work of BGG (see [**2**]). It shows that the weak BGG resolution [**2**, Theorem 9.9] and the strong BGG resolution [**2**, Theorem 10.1] coincide. In particular, it gives another proof of Theorem 10.1 of [**2**].

LEMMA 10.5. *There exist $U(\mathfrak{g})$-isomorphisms $\mu_k\colon C_k \to C_k$ such that all the diagrams*

$$\begin{array}{ccccccccccc}
\cdots & \to & C_k & \xrightarrow{d_k} & C_{k-1} & \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} & C_0 & \xrightarrow{\eta} & L_\lambda & \to & 0 \\
 & & \uparrow \mu_k & & \uparrow \mu_{k-1} & & \uparrow \mu_0 & & \| & & \\
\cdots & \to & C_k & \xrightarrow{\partial_k} & C_{k-1} & \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} & C_0 & \xrightarrow{\varepsilon} & L_\lambda & \to & 0
\end{array}$$

commute, $k = 0, 1, \ldots, \dim \mathfrak{n}^-$.

PROOF. Let μ_0 be the obvious nonzero multiple of the identity, γ_1^0 id, and suppose we have defined $\mu_0, \ldots, \mu_{k-1}$ $U(\mathfrak{g})$-isomorphisms, such that $\eta\mu_0 = \varepsilon$ and

$$d_{j-1}\mu_{j-1} = \mu_{j-2}\partial_{j-1}, \qquad j = 2, \ldots, k. \tag{2}$$

Let $\mu_j(\xi_w) = \gamma_w^j \xi_w$, $w \in W^{(j)}$, $j = 0, 1, \ldots, k-1$, where ξ_w is the canonical generator of $V_{w\cdot\lambda}$. If $w_0 \in W^{(k)}$, there exists α in π such that $\sigma_\alpha w_0 \in W^{(k-1)}$. Set $t_0 = \sigma_\alpha w_0$.

$$\partial_k(\xi_{w_0}) = \coprod_{\substack{t \\ w_0 \leftarrow t}} a^k_{w_0,t}\xi_{w_0,t},$$

with $a^k_{w_0,t} \in \mathbf{C} \setminus \{0\}$, by Proposition 7.1, and Lemmas 10.1 and 10.2. Here $\xi_{w_0,t}$ is as in the proof of Lemma 10.2.

$$d_{k-1}\mu_{k-1}\partial_k(\xi_{w_0}) = d_{k-1}\mu_{k-1}\Big(\coprod_{\substack{t\\ w_0\leftarrow t}} a^k_{w_0,t}\xi_{w_0,t}\Big)$$
$$= d_{k-1}\Big(\coprod_{\substack{t\\ w_0\leftarrow t}} a^k_{w_0,t}\gamma_t^{k-1}\xi_{w_0,t}\Big)$$
$$= \coprod_{\substack{r\\ t\leftarrow r}}\coprod_{\substack{t\\ w_0\leftarrow t}} a^k_{w_0,t}\gamma_t^{k-1}c(t,r)\xi_{w_0,t,r} = 0$$

by (2) above, where $\xi_{w_0,t,r}$ is as in the proof of Lemma 10.2.

Let $t \in W^{(k-1)}$ be such that $w_0 \overset{\beta}{\leftarrow} t$, $\beta \in \Delta^+$, $t \neq t_0$. By an argument used in the proof of Lemma 10.2, we obtain

$$\frac{a^k_{w_0,t_0}\gamma_{t_0}^{k-1}}{a^k_{w_0,t}\gamma_t^{k-1}} = -\frac{c(t,\sigma_\alpha t)}{c(t_0,\sigma_\alpha t)}.$$

But

$$-\frac{c(t,\sigma_\alpha t)}{c(t_0,\sigma_\alpha t)} = \frac{c(w_0,t_0)}{c(w_0,t)}.$$

Therefore

$$a^k_{w_0,t}\gamma_t^{k-1} = \frac{a^k_{w_0,t_0}\gamma_{t_0}^{k-1}}{c(w_0,t_0)}c(w_0,t)$$

for all t such that $w_0 \leftarrow t$. Let

$$\gamma^k_{w_0} = \frac{a^k_{w_0,t_0}\gamma_{t_0}^{k-1}}{c(w_0,t_0)},$$

and define $\mu_k(\xi_{w_0}) = \gamma^k_{w_0}\xi_{w_0}$. Then

$$\mu_{k-1}\partial_k(\xi_{w_0}) = \coprod_{\substack{t\\ w_0\leftarrow t}} a^k_{w_0,t}\gamma_t^{k-1}\xi_{w_0,t}$$
$$= \coprod_{\substack{t\\ w_0\leftarrow t}} \gamma^k_{w_0}c(w_0,t)\xi_{w_0,t} = d_k\mu_k(\xi_{w_0}),$$

and the result follows by recurrence. Q.E.D.

COROLLARY 10.6 (THEOREM 10.1, [2]). *The sequence*

$$0 \to C_q \xrightarrow{d_q} C_{q-1} \xrightarrow{d_{q-1}} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\eta} L_\lambda \to 0$$

is exact, where $q = \dim \mathfrak{n}^-$.

COROLLARY 10.7. *If* $\alpha_k\colon C_k \to C_{k-1}$ *is such that* $\alpha_{k|_{V_{w\cdot\lambda}}} \neq 0$, $w \in W^{(k)}$, $k = 1, \ldots, \dim \mathfrak{n}^-$, *and* $\alpha_{k-1}\alpha_k = 0$, $k = 1, \ldots, \dim \mathfrak{n}^-$, *then the complex*

$$0 \to C_q \xrightarrow{\alpha_q} C_{q-1} \xrightarrow{\alpha_{q-1}} \cdots \xrightarrow{\alpha_2} C_1 \xrightarrow{\alpha_1} C_0 \xrightarrow{\alpha_0} L_\lambda \to 0,$$

where $q = \dim \mathfrak{n}^-$, *is exact.*

PROOF. Note that the only property of the maps ∂_k needed in the proof of Lemma 10.5 was that the numbers $a^k_{w_1,w_2}$, $w_1 \in W^{(k)}$, $w_2 \in W^{(k-1)}$ be nonzero. Therefore, by Lemma 10.2, Lemma 10.5 applies with ∂_k replaced by α_k. The corollary now follows from 10.6 above. Q.E.D.

11. The generalized strong BGG resolution and a uniqueness theorem.

PROPOSITION 11.1 [**14**, PROPOSITION 3.1]. *Let $\lambda \in h^*$, $\mu \in P_S^+$ and f: $V_\lambda \to V_\mu$ a nonzero map. Denote by π_μ: $V_\mu \to V^{M_\mu}$ the natural projection. If $\pi_\mu \circ f$: $V_\lambda \to V^{M_\mu}$ is nonzero, then $\lambda \in P_S^+$ and $(\pi_\mu \circ f)(K_\lambda) = 0$, where $K_\lambda \subset V_\lambda$ is the kernel of π_λ.*

REMARK. Proposition 11.1 implies that if $\lambda \in P_S^+$ then $f(K_\lambda) \subset K_\mu$ and, therefore, f induces $\tilde{f}$: $V^{M_\lambda} \to V^{M_\mu}$ defined by

$$\tilde{f}(v + K_\lambda) = f(v) + K_\mu.$$

DEFINITION 11.2 [**14**]. Let λ, $\mu \in P_S^+$, f: $V_\lambda \to V_\mu$ a nonzero $\mathfrak{g}$-homomorphism. The map $\tilde{f}$: $V^{M_\lambda} \to V^{M_\mu}$ above is called the *standard map* associated with f.

PROPOSITION 11.3 [**14**, PROPOSITION 3.7]. *Let $\lambda \in P^+$, w, $w' \in W^S$ be such that $l(w) = l(w') + 1$. Then there exists a nonzero $\mathfrak{g}$-homomorphism*

$$V^{M_{w\cdot\lambda}} \to V^{M_{w'\cdot\lambda}},$$

if and only if $w \leftarrow w'$. If $w \leftarrow w'$ then the standard map from $V^{M_{w\cdot\lambda}}$ to $V^{M_{w'\cdot\lambda}}$ (which is unique up to nonzero scalar multiple) is nonzero.

Fix $\lambda \in P^+$. For each $k = 0, 1, \ldots, \dim \mathfrak{u}^-$, define

$$E_k = \coprod_{w \in (W^S)^{(k)}} V^{M_{w\cdot\lambda}}.$$

If w, $w' \in W^S$ with $l(w) = l(w') + 1$, let

$$\tilde{d}^k_{w,w'}: V^{M_{w\cdot\lambda}} \to V^{M_{w'\cdot\lambda}}$$

be the map defined by

$$\tilde{d}^k_{w,w'} = c(w, w')\tilde{i}_{w,w'}, \quad w \leftarrow w',$$

and

$$\tilde{d}^k_{w,w'} = 0, \quad \text{otherwise}.$$

Here $\tilde{i}_{w,w'}$ is the standard map associated with $i_{w,w'}$ and $c(w, w')$ is as in Lemma 10.4. We have therefore maps

$$\tilde{d}_k: E_k \to E_{k-1}, \qquad k = 1, \ldots, \dim \mathfrak{u}^-.$$

Let $\tilde{\eta}$: $V^{M_\lambda} \to L_\lambda$ be the map g of Proposition 3.6.

Consider the natural projections

$$\begin{array}{ccccccc} \mathfrak{g} & \to & \mathfrak{g}/\mathfrak{b} & \text{and} & \mathfrak{g} & \to & \mathfrak{g}/\mathfrak{p} \\ X & \mapsto & \bar{X} & & X & \mapsto & \bar{\bar{X}}. \end{array}$$

Then π: $\mathfrak{g}/\mathfrak{b} \to \mathfrak{g}/\mathfrak{p}$ defined by $\pi(\bar{X}) = \bar{\bar{X}}$ is a linear map. Let π_k: $\bigwedge^k(\mathfrak{g}/\mathfrak{b}) \to \bigwedge^k(\mathfrak{g}/\mathfrak{p})$ be the map defined by

$$\pi_k(\bar{X}_1 \wedge \cdots \wedge \bar{X}_k) = \bar{\bar{X}}_1 \wedge \cdots \wedge \bar{\bar{X}}_k, \qquad k = 0, 1, \ldots, \dim \mathfrak{u}^-,$$

and $\pi_k = 0$, $k = (\dim \mathfrak{u}^-) + 1, \ldots, \dim \mathfrak{n}^-$.

The adjoint action of $\mathfrak{p}$ (resp. $\mathfrak{b}$) on $\mathfrak{g}$ gives a $\mathfrak{p}$-module (resp. $\mathfrak{b}$-module) structure to $\mathfrak{g}/\mathfrak{p}$ (resp. $\mathfrak{g}/\mathfrak{b}$). We consider now the morphisms of vector spaces

$$\hat{\pi}_k: U(\mathfrak{g}) \times \bigwedge^k(\mathfrak{g}/\mathfrak{b}) \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \bigwedge^k(\mathfrak{g}/\mathfrak{p}),$$

$$\hat{\pi}_k(X, \bar{X}_1 \wedge \cdots \wedge \bar{X}_k) = X \otimes \bar{\bar{X}}_1 \wedge \cdots \wedge \bar{\bar{X}}_k,$$

$k = 0, 1, \ldots, \dim \mathfrak{u}^-$, $\hat{\pi}_k = 0$, $k > \dim \mathfrak{u}^-$. Since $\mathfrak{b} \subset \mathfrak{p}$, there exists a $\mathfrak{g}$-module map

$$\tilde{\pi}_k: U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^k(\mathfrak{g}/\mathfrak{b}) \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \bigwedge^k(\mathfrak{g}/\mathfrak{p}),$$

defined by

$$\tilde{\pi}_k(X \otimes \bar{X}_1 \wedge \cdots \wedge \bar{X}_k) = X \otimes \bar{\bar{X}}_1 \wedge \cdots \wedge \bar{\bar{X}}_k,$$

$k = 0, 1, \ldots, \dim \mathfrak{u}^-$, $\tilde{\pi}_k = 0$, $k > \dim \mathfrak{u}^-$. Therefore we obtain maps from ${}_{\theta_\lambda}(B_k^\lambda)$ to ${}_{\theta_\lambda}(D_k^\lambda)$ which we also denote by $\tilde{\pi}_k$. It is clear from the definitions of $\tilde{\partial}_k$ and ∂_k that

$$\tilde{\partial}_k \circ \tilde{\pi}_{k+1} = \tilde{\pi}_k \circ \partial_{k+1}, \qquad k = 0, 1, \ldots, (\dim \mathfrak{u}^-) - 1.$$

Let $K^k = \ker \tilde{\pi}_k$ in ${}_{\theta_\lambda}(B_k^\lambda)$. Then

$$\tilde{\partial}_k: {}_{\theta_\lambda}(B_k^\lambda)/K^k \to {}_{\theta_\lambda}(B_{k-1}^\lambda)/K^{k-1}$$

is given by

$$\tilde{\partial}_k(v + K^k) = \partial_k(v) + K^{k-1}.$$

By Theorem 9.3 we have

$${}_{\theta_\lambda}(D_k^\lambda) = \coprod_{w \in (W^S)^{(k)}} V^{M_{w\cdot\lambda}} = E_k$$

and

$${}_{\theta_\lambda}(B_k^\lambda) = \coprod_{w \in W^{(k)}} V_{w\cdot\lambda} = C_k.$$

Now $V^{M_{w\cdot\lambda}} = V_{w\cdot\lambda}/K_{w\cdot\lambda}$, $w \in W^S$ (see Proposition 11.1). Hence

$$K^k = \coprod_{w \in (W^S)^{(k)}} K_{w\cdot\lambda} \oplus \coprod_{\substack{w \in W^{(k)} \\ w \notin W^S}} V_{w\cdot\lambda}.$$

If $k = 0, 1, \ldots, \dim \mathfrak{u}^-$ and $w \in (W^S)^{(k)}$, let γ_w^k id: $V_{w\cdot\lambda} \to V_{w\cdot\lambda}$ be as in the proof of Lemma 10.5, and let $\widetilde{\gamma_w^k \text{ id}}$: $V^{M_{w\cdot\lambda}} \to V^{M_{w\cdot\lambda}}$ be the standard map associated with γ_w^k id. Let

$$\tilde{\mu}_k = \coprod_{w \in (W^S)^{(k)}} \widetilde{\gamma_w^k \text{ id}}: E_k \to E_k.$$

According to Lemma 10.5 we obtain

$$\tilde{d}_k \circ \tilde{\mu}_k = \widetilde{d_k \circ \mu_k} = \widetilde{\mu_{k-1} \circ \partial_k} = \tilde{\mu}_{k-1} \circ \tilde{\partial}_k.$$

Summarizing, we have

THEOREM 11.4. *If* $k = 0, 1, \ldots, \dim \mathfrak{u}^- = r$, *there are* $U(\mathfrak{g})$*-isomorphisms* $\tilde{\mu}_k$: $E_k \to E_k$ *such that all the diagrams*

$$\begin{array}{ccccccccccc} 0 & \to & E_r & \overset{\tilde{d}_r}{\to} \cdots \overset{\tilde{d}_{k+1}}{\to} & E_k & \overset{\tilde{d}_k}{\to} \cdots \overset{\tilde{d}_1}{\to} & E_0 & \overset{\tilde{\varepsilon}}{\to} & L_\lambda & \to & 0 \\ & & \uparrow \tilde{\mu}_r & & \uparrow \tilde{\mu}_k & & \uparrow \tilde{\mu}_0 & & \| & & \\ 0 & \to & E_r & \overset{\tilde{\partial}_r}{\to} \cdots \overset{\tilde{\partial}_{k+1}}{\to} & E_k & \overset{\tilde{\partial}_k}{\to} \cdots \overset{\tilde{\partial}_1}{\to} & E_0 & \overset{\tilde{\eta}}{\to} & L_\lambda & \to & 0 \end{array}$$

commute, where $\tilde{\mu}_0$ *is a nonzero multiple of the identity.*

REMARK. Theorem 11.4 shows that the generalized weak BGG resolution and the generalized strong BGG resolution coincide, simplifying the work of Lepowsky [**14**]. The Corollary 11.5 below gives another proof of the exactness of the generalized strong BGG resolution.

COROLLARY 11.5 [**14**, THEOREM 4.3]. *The sequence*

$$0 \to E_{\dim \mathfrak{u}^-} \overset{\tilde{d}_{\dim \mathfrak{u}^-}}{\to} \cdots \overset{\tilde{d}_1}{\to} E_0 \overset{\tilde{\eta}}{\to} L_\lambda \to 0$$

is exact.

12. The de Rham complex. Let G be a connected, linear, semisimple Lie group, that is G is a connected, semisimple Lie subgroup of $GL(n, \mathbf{R})$ for some n. If H is any Lie subgroup of G we denote by $\mathfrak{h}_0$ and $\mathfrak{h}$ the Lie algebra of H and its complexification, respectively. If $\mathfrak{a} \subset \mathfrak{g}$ is a subalgebra $U(\mathfrak{a})$ denotes the universal enveloping algebra of $\mathfrak{a}$, as in §1.

Let $G = KA_1N_1$ be an Iwasawa decomposition of G. Let M_1 denote the centralizer of A_1 in K. The closed subgroup $P_1 = M_1A_1N_1$ is called a *minimal parabolic subgroup* of G.

By definition, a *parabolic subgroup* of G is a subgroup that equals its own normalizer and such that the complexification of its Lie algebra contains a Borel subalgebra of $\mathfrak{g}$.

Let P be a parabolic subgroup of G containing P_1. We write $P = MAN$ for the Langlands decomposition of P (see [**22**]). Let M^0 be the identity component of M, so that $P^0 = M^0AN$ is the identity component of P. Then $G = KP^0$. Set $X = G/P^0$. X is isomorphic with $K/K \cap P^0$ as a K-homogeneous space.

Let $\mathfrak{D}^i(X)$ denote the space of all complex valued i-forms on X, that is, $\mathfrak{D}^i(X)$ is the space of all C^∞ cross sections of the complexification of the C^∞ $\mathbf{R}$-vector bundle $(\bigwedge^i T(X))^*$, where $T(X)$ is the tangent bundle of X. For instance, if $\omega \in \mathfrak{D}^i(X)$ and $x \in X$ then $\omega_x = \omega(x)$ is an element of $(\bigwedge^i T(X)_x \otimes \mathbf{C})^*$, $i = 0, 1, \ldots, \dim X$.

Let $\Gamma^\infty((\bigwedge^i T(X))^*)$ denote the space of all C^∞ cross sections of the C^∞ $\mathbf{R}$-vector bundle $(\bigwedge^i T(X))^*$ and define

$$d_i\colon \Gamma^\infty((\bigwedge^i T(X))^*) \to \Gamma^\infty((\bigwedge^{i+1} T(X))^*)$$

by

$$d_i(\omega)(X_1, \ldots, X_{i+1}) = \sum_{j=1}^{i+1} (-1)^{j+1} X_j \omega(X_1, \ldots, \hat{X}_j, \ldots, X_{i+1})$$
$$+ \sum_{1 < j < k < i+1} (-1)^{j+k} \omega([X_j, X_k], X_1, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_{i+1}),$$

where $\omega \in \Gamma^\infty((\bigwedge^i T(X))^*)$, X_j is a vector field over X, and $\hat{X}_j$ means that X_j has been deleted, $j = 1, \ldots, i+1$, $i = 1, 2, \ldots, \dim X - 1$. Let $d_0\colon \Gamma^\infty((\bigwedge^0 T(X))^*) \to \Gamma^\infty((\bigwedge^1 T(X))^*)$ be defined by $d_0(\omega) = \omega_*$, the differential of ω, $\omega \in \Gamma^\infty((\bigwedge^0 T(X))^*)$. We also denote by $d_i\colon \mathfrak{D}^i(X) \to \mathfrak{D}^{i+1}(X)$ the corresponding extensions ot the complexified vector bundles. Using Stokes' theorem we obtain the complex:

$$0 \to \mathbf{C} \xrightarrow{\eta} \mathfrak{D}^0(X) \xrightarrow{d_0} \mathfrak{D}^1(X) \xrightarrow{d_1} \cdots \xrightarrow{d_{r-1}} \mathfrak{D}^r(X) \xrightarrow{\varepsilon} \mathbf{C} \to 0, \qquad (1)$$

where η is the inclusion map and $\varepsilon(\omega) = \int_X \omega$, $\omega \in \mathfrak{D}^r(X)$, $r = \dim X$. (1) is known as the de Rham complex of X.

De Rham's theorem implies that the ith cohomology of equation (1) is given by $H^i(K/K \cap P^0, \mathbf{C})$, if $0 < i < r$, and is equal to zero if $i = 0$ or $i = r$.

G acts on X by $l(g)x = gx$, $g \in G$, $x \in X$. If M and N are C^∞ manifolds and $\varphi\colon M \to N$ is a C^∞ map, we denote by φ_* and φ^* the differential and codifferential of φ, respectively. If $g \in G$, $x \in X$ and $v \in (\bigwedge^i T(X)_x)^*$, define

$$(g \cdot v)(X_{1_{gx}}, \ldots, X_{i_{gx}}) = v\left(l(g^{-1})_{*_{gx}} X_{1_{gx}}, \ldots, l(g^{-1})_{*_{gx}} X_{i_{gx}}\right),$$

where the X_j are vector fields over X, $j = 1, \ldots, i$, $i = 1, \ldots, r$. This action extends to an action on $(\bigwedge^i T(X)_x \otimes \mathbf{C})^*$, $x \in X$, $i = 1, \ldots, r$, and

$$g^{-1}((\bigwedge^i T(X)_{gx} \otimes \mathbf{C})^*) \subset (\bigwedge^i T(X)_x \otimes \mathbf{C})^*,$$

for all $g \in G$, $x \in X$, $i = 1, \ldots, r$.

If $\omega \in \Gamma^\infty((\bigwedge^i T(X))^*)$ and $g \in G$, define

$$g\omega = \bigwedge^i l(g^{-1})^* \omega.$$

We denote by π the extension of this action of G to $\mathfrak{D}^i(X)$. Clearly, if $\omega \in \mathfrak{D}^i(X)$, $g \in G$, and $x \in X$, then $(\pi(g)\omega)_x = g(\omega_{g^{-1}x})$.

Let H be a Lie subgroup of G and let V be a finite dimensional representation of $H \cap P^0$. We denote by $C^\infty(H; \tau)$ the space of all C^∞ functions $f\colon H \to V$ such that $f(gp) = \tau(p^{-1})f(g)$, for all $p \in H \cap P^0$, $g \in H$. Here τ is the action of $H \cap P^0$ on V.

Let τ_i be the restriction to P^0 of the action of G on $(\bigwedge^i T(X)_{eP^0} \otimes \mathbf{C})^*$, $i = 0, \ldots, r$. If $g_0, g \in G$ and $f \in C^\infty(G; \tau_i)$, let $\tilde{\pi}(g_0)(f)(g) = f(g_0^{-1}g)$.

LEMMA 12.1. *If* $\omega \in \mathfrak{D}^i(X)$, *define* $f_\omega\colon G \to (\bigwedge^i T(X)_{eP^0} \otimes \mathbf{C})^*$ *by*

$$f_\omega(g) = \pi(g^{-1})\omega(gP^0).$$

The map $A\colon \omega \mapsto f_\omega$ *is an isomorphism of* $\mathfrak{D}^i(X)$ *and* $C^\infty(G;\tau_i)$ *such that* $A \circ \pi(g) = \tilde{\pi}(g) \circ A$, *for all* $g \in G$.

PROOF. This is a consequence of the Lemmas 5.2.3 and 5.3.4 of **[18]**. Q.E.D.

Let $\psi\colon G \to G/P^0$ be the coset map $\psi(g) = gP^0$, $g \in G$, and let $\psi_{*_e}\colon T(G)_e \otimes \mathbf{C} \to T(G/P^0)_{eP^0} \otimes \mathbf{C}$ denote the extension of the differential map of ψ at the identity. Using the natural identification of $T(G)_e$ with $\mathfrak{g}_0$ we may write ψ_{*_e}: $\mathfrak{g} \to T(G/P^0)_{eP^0} \otimes \mathbf{C}$. Now, ker $\psi_{*_e} = \mathfrak{p}$, the complexification of the Lie algebra of P. Therefore, ψ_{*_e} induces $\tilde{\psi}\colon \mathfrak{g}/\mathfrak{p} \to T(G/P^0)_{eP^0} \otimes \mathbf{C}$.

Let Ad denote the adjoint representation of G, as usual. If $p \in P^0$, then Ad(p) extends to a map from $\mathfrak{g}$ to $\mathfrak{g}$. We abuse notation and write Ad(p) for the induced map from $\mathfrak{g}/\mathfrak{p}$ to $\mathfrak{g}/\mathfrak{p}$.

A trivial computation shows that

$$\tilde{\psi} \circ \mathrm{Ad}(p) = l(p)_{*_{eP^0}} \circ \tilde{\psi}, \quad \text{for all } p \in P^0. \tag{$*$}$$

Let $g \in G$, and let X_j be an element in $\mathfrak{g}_0/\mathfrak{p}_0$ represented by $Y_j \in \mathfrak{g}_0$, for $j = 1, \ldots, i+1$. If $f\colon G \to (\bigwedge^i(\mathfrak{g}_0/\mathfrak{p}_0))^*$ is C^∞, define

$$(\alpha_{i+1} f)(g)(X_1, \ldots, X_{i+1}) = \sum_{j=1}^{i+1} (-1)^{j+1} Y_{j_g} f(X_1, \ldots, \hat{X}_j, \ldots, X_{i+1})$$
$$+ \sum_{1<j<k<i+1} (-1)^{j+k} f(g)\left(\overline{[Y_j, Y_k]}, X_1, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_{i+1}\right),$$

where $\overline{Y}$ represents the canonical image of $Y \in \mathfrak{g}_0$ in $\mathfrak{g}_0/\mathfrak{p}_0$ and $\hat{X}_j$ means that X_j has been deleted, $j = 1, \ldots, i+1$, $i = 0, \ldots, r-1$. We also denote by α_i the extension of α_i to the space of all C^∞ functions $f\colon G \to (\bigwedge^i(\mathfrak{g}/\mathfrak{p}))^*$, $i = 0, \ldots, r-1$.

LEMMA 12.2. *Set* $\sigma_i = \bigwedge^i \mathrm{Ad}^*|_{P^0}$ *and let* $\hat{\pi}(g_0)(f)(g) = f(g_0^{-1}g)$, *for* g_0, $g \in G$, $f \in C^\infty(G;\sigma_i)$. *Then there are isomorphisms* $\varphi\colon \mathfrak{D}^i(X) \to C^\infty(G;\sigma_i)$ *such that*:

(1) $\varphi \circ \pi(g) = \hat{\pi}(g) \circ \varphi$, *all* $g \in G$.

(2) *If* $\eta' = \varphi \circ \eta$ *and* $\varepsilon' = \varepsilon \circ \varphi^{-1}$ *then all the diagrams*

$$\begin{array}{ccccccccccccc}
0 & \to & \mathbf{C} & \xrightarrow{\eta} & \mathfrak{D}^0(X) & \xrightarrow{d_0} \cdots \xrightarrow{d_{i-1}} & \mathfrak{D}^i(X) & \xrightarrow{d_i} \cdots \xrightarrow{d_{r-1}} & \mathfrak{D}^r(X) & \xrightarrow{\varepsilon} & \mathbf{C} & \to & 0 \\
 & & \| & & \downarrow\varphi & & \downarrow\varphi & & \downarrow\varphi & & \| & & \\
0 & \to & \mathbf{C} & \xrightarrow{\eta'} & C^\infty(G;\sigma_0) & \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_{i-1}} & C^\infty(G;\sigma_i) & \xrightarrow{\alpha_i} \cdots \xrightarrow{\alpha_{r-1}} & C^\infty(G;\sigma_r) & \xrightarrow{\varepsilon'} & \mathbf{C} & \to & 0
\end{array}$$

commute.

PROOF. Let A be the isomorphism of Lemma 12.1, and let $\hat{\psi}$ be the map from $C^\infty(G;\tau_i)$ to $C^\infty(G;\sigma_i)$ induced by $\tilde{\psi}$. Now, set $\varphi = \hat{\psi} \circ A$ and use $(*)$. Q.E.D.

Let $\mathfrak{D}^i(X)^K$ be the space of all K-invariant elements of $\mathfrak{D}^i(X)$. If we denote by d_i the restrictions of d_i to $\mathfrak{D}^i(X)^K$, $i = 0, \ldots, r-1$, and by ε the restriction of ε to $\mathfrak{D}^r(X)^K$, we obtain the complex

$$0 \to \mathbf{C} \xrightarrow{\eta} \mathfrak{D}^0(X)^K \xrightarrow{d_0} \mathfrak{D}^1(X)^K \xrightarrow{d_1} \cdots \xrightarrow{d_{r-1}} \mathfrak{D}^r(X)^K \xrightarrow{\varepsilon} \mathbf{C} \to 0. \tag{2}$$

Since K is connected, $l(k)$ is homotopic to the identity, for all $k \in K$. Therefore, the homotopy invariance theorem and de Rham's theorem imply that the cohomology of the complex (2) is the cohomology of the complex (1) (cf. [**18**, Theorem 3.8.2]). This fact together with Lemma 12.2 implies the following.

COROLLARY 12.3. *Let $C_F^\infty(G; \sigma_i)$ denote the space of all K-finite vectors of $C^\infty(G; \sigma_i)$, where σ_i is as in Lemma* 12.2. *Then the cohomology of the complex*

$$0 \to \mathbf{C} \xrightarrow{\eta'} C_F^\infty(G; \sigma_0) \xrightarrow{\alpha_0} C_F^\infty(G; \sigma_1) \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{r-1}} C_F^\infty(G; \sigma_r) \xrightarrow{\varepsilon'} \mathbf{C} \to 0 \qquad (3)$$

is the cohomology of (1). *Here α_i and ε' are the restrictions of α_i and ε' to $C_F^\infty(G; \sigma_i)$ and $C_F^\infty(G; \sigma_r)$, respectively, $i = 0, 1, \ldots, r-1$ (see Lemma* 12.2 *for the definitions of η' and ε').*

If $f \in C_F^\infty(G; \sigma_i)$ ($C_F^\infty(G; \sigma_i)$ as in Corollary 12.3) and $Y \in \mathfrak{g}_0$, define

$$(\tilde{\pi}(Y)f)(g) = \frac{d}{dt}\bigg|_{t=0} f(\exp(-tY)g), \qquad g \in G.$$

This action extends to an action of $U(\mathfrak{g})$ on $C_F^\infty(G; \sigma_i)$, which we also denote by $\tilde{\pi}$.

Define μ: $C_F^\infty(G; \sigma_i) \to \operatorname{Hom}(U(\mathfrak{g}), \bigwedge^i(\mathfrak{g}/\mathfrak{p})^*)$ by $\mu(f)(Y) = (\tilde{\pi}(Y)f)(1)$, $Y \in U(\mathfrak{g}), f \in C_F^\infty(G; \sigma_i)$.

We denote by ad the representation of $\mathfrak{p}_0$ on $\mathfrak{g}_0/\mathfrak{p}_0$ induced by the restriction to $\mathfrak{p}_0$ of the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_0$. ad extends to an action of $U(\mathfrak{p})$ on $\mathfrak{g}/\mathfrak{p}$ which we also denote by ad. It is easy to see that, if $Y \in U(\mathfrak{g})$, $Z \in U(\mathfrak{p})$ and $f \in C_F^\infty(G; \sigma_i)$, then

$$\tilde{\pi}(Z)\tilde{\pi}(Y)f(1) = \textstyle\bigwedge^i \operatorname{ad}^*(Z)(\tilde{\pi}(Y)f(1)).$$

This implies that for $f \in C_F^\infty(G; \sigma_i)$, $\mu(f) \in \operatorname{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \bigwedge^i(\mathfrak{g}/\mathfrak{p})^*)$, where $\bigwedge^i(\mathfrak{g}/\mathfrak{p})^*$ is equipped with the $U(\mathfrak{p})$-module structure given by $\bigwedge^i \operatorname{ad}^*$. If Y, $Z \in U(\mathfrak{g})$, $B \in \operatorname{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \bigwedge^i(\mathfrak{g}/\mathfrak{p})^*)$, define $\tilde{\pi}(Y)B(Z) = B(ZY)$. It can be easily seen that μ is a $U(\mathfrak{g})$-module map from $C_F^\infty(G; \sigma_i)$ into

$$\operatorname{Hom}_{U(\mathfrak{p})}\big(U(\mathfrak{g}), \textstyle\bigwedge^i(\mathfrak{g}/\mathfrak{p})^*\big)_F,$$

the space of all $U(\mathfrak{k})$-finite vectors of $\operatorname{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \bigwedge^i(\mathfrak{g}/\mathfrak{p})^*)$.

PROPOSITION 12.4. *The map*

$$\mu: C_F^\infty(G; \sigma_i) \to \operatorname{Hom}_{U(\mathfrak{p})}\big(U(\mathfrak{g}), \textstyle\bigwedge^i(\mathfrak{g}/\mathfrak{p})^*\big)_F$$

is a $U(\mathfrak{g})$-module isomorphism.

PROOF. Let $f \in C_F^\infty(G; \sigma_i)$ and suppose that $\mu(f) = 0$. Then $\mu(f)(Y) = 0$ for all $Y \in U(\mathfrak{g})$, that is, $\tilde{\pi}(Y)f(1) = 0$ for all $Y \in U(\mathfrak{g})$. If $g \in G$, say $g = kp$, $k \in K$, $p \in P^0$, then $f(g) = f(kp) = \bigwedge^i \operatorname{Ad}^*(p^{-1})f(k) = \bigwedge^i \operatorname{Ad}^*(p^{-1})\tilde{\pi}(k^{-1})f(1)$. The K-finiteness of f implies that f is analytic. Since G is connected, Taylor's theorem implies that $f \equiv 0$, and we have proved that μ is injective.

It remains to prove that μ is surjective.

Since G/P^0 is isomorphic with $K/K \cap P^0$ as a K-homogeneous space, there exists an isomorphism from $C_F^\infty(G; \sigma_i)$ to $C_F^\infty(K; \sigma_{i|_{K \cap P^0}})$ commuting with the respective actions of K.

For any compact Lie group H, let $\hat{H}$ denote the set of all equivalence classes of finite dimensional, unitary, irreducible representations of H. Theorem 5.3.6 of **[18]** implies that

$$C_F^\infty(K; \sigma_{i|_{K\cap P^0}}) = \coprod_{\gamma\in\hat{K}} \dim \operatorname{Hom}_{K\cap P^0}\big(V_\gamma, \textstyle\bigwedge^i(\mathfrak{g}/\mathfrak{p})^*\big)V_\gamma,$$

algebraic direct sum, where (π_γ, V_γ) is a representative of $\gamma \in \hat{K}$.

On the other hand, Proposition 5.5.8 of **[6]** implies that, as a representation of $\mathfrak{k}$, $\operatorname{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \bigwedge^i(\mathfrak{g}/\mathfrak{p})^*)_F$ is isomorphic with $\operatorname{Hom}_{U(\mathfrak{k}\cap\mathfrak{p})}(U(\mathfrak{k}), \bigwedge^i(\mathfrak{g}/\mathfrak{p})^*)_F$, the space of $U(\mathfrak{k})$-finite vectors of $\operatorname{Hom}_{U(\mathfrak{k}\cap\mathfrak{p})}(U(\mathfrak{k}), \bigwedge^i(\mathfrak{g}/\mathfrak{p})^*)$.

Now, $\mathfrak{k}$ is reductive and its center is contained in $\mathfrak{p}$. This implies that the action of the center of $\mathfrak{k}$ on $\operatorname{Hom}_{U(\mathfrak{k}\cap\mathfrak{p})}(U(\mathfrak{k}), \bigwedge^i(\mathfrak{g}/\mathfrak{p})^*)$ is semisimple. Therefore, by Lemme 1.6.4 and Proposition 5.5.3 of **[6]** we get

$$\operatorname{Hom}_{U(\mathfrak{k}\cap\mathfrak{p})}\big(U(\mathfrak{k}), \textstyle\bigwedge^i(\mathfrak{g}/\mathfrak{p})^*\big) = \coprod_{\gamma\in\hat{k}} \dim \operatorname{Hom}_{U(\mathfrak{k})}\big(V_\gamma, \textstyle\bigwedge^i(\mathfrak{g}/\mathfrak{p})^*\big)V_\gamma,$$

where $\hat{k}$ is the set of all equivalence classes of finite dimensional, irreducible representations of $\mathfrak{k}$, and (π_γ, V_γ) is a representative of $\gamma \in \hat{k}$.

The surjectivity of μ now follows. Q.E.D.

If $Y \in U(\mathfrak{g})$, we write $Y \mapsto {}^tY$ for the principal anti-automorphism of $U(\mathfrak{g})$.

Let $Y \in U(\mathfrak{g})$, and let X_j be an element of $\mathfrak{g}/\mathfrak{p}$ represented by $Y_j \in \mathfrak{g}$, for $j = 1, \dots, i+1$. If $B \in \operatorname{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \bigwedge^i(\mathfrak{g}/\mathfrak{p})^*)_F$, define

$$(\beta_i B)(Y)(X_1, \dots, X_{i+1}) = \sum_{j=1}^{i+1} (-1)^{j+1} B({}^tY_j Y)(X_1, \dots, \hat{X}_j, \dots, X_{i+1})$$
$$+ \sum_{1 \le j < k \le i+1} (-1)^{j+k} B(Y)\big(\overline{[Y_j, Y_k]}, X_1, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_{i+1}\big).$$

Here $\bar{Z}$ represents the canonical image of $Z \in \mathfrak{g}$ in $\mathfrak{g}/\mathfrak{p}$, and $\hat{X}_j$ means that X_j has been deleted.

The following lemma is straightforward.

LEMMA 12.5. *Let η', ε' and α_j, $j = 0, 1, \dots, r-1$ be as in Corollary* 12.3. *Then the diagrams*

$$\begin{array}{ccccccccc}
0 \to \mathbf{C} & \xrightarrow{\eta'} & C_F^\infty(G;\sigma_0) & \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_{r-1}} & C_F^\infty(G;\sigma_r) & \xrightarrow{\varepsilon'} & \mathbf{C} \to 0 \\
\| & & \downarrow\mu & & \downarrow\mu & & \| \\
0 \to \mathbf{C} & \xrightarrow{\xi} & \operatorname{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \mathbf{C})_F & \xrightarrow{\beta_0} \cdots \xrightarrow{\beta_{r-1}} & \operatorname{Hom}_{U(\mathfrak{p})}\big(U(\mathfrak{g}), \bigwedge^r(\mathfrak{g}/\mathfrak{p})\big) & \xrightarrow{\nu} & \mathbf{C} \to 0
\end{array}$$

are commutative. Here $\xi = \mu \circ \eta'$ and $\nu = \varepsilon' \circ \mu^{-1}$.

By Proposition 5.5.4 of **[6]** the space $\operatorname{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \bigwedge^i(\mathfrak{g}/\mathfrak{p})^*)_F$ is $U(\mathfrak{g})$-isomorphic with $(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \bigwedge^i(\mathfrak{g}/\mathfrak{p}))_F^*$, the $U(\mathfrak{k})$-finite dual of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \bigwedge^i(\mathfrak{g}/\mathfrak{p})$.

Let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$ and Θ the set of all homomorphisms θ: $Z(\mathfrak{g}) \to \mathbf{C}$. If $\theta \in \Theta$ we define ${}^t\theta(Z) = \theta({}^tZ)$, for all $Z \in Z(\mathfrak{g})$. Then

$${}_\theta\big(\big(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \textstyle\bigwedge^i(\mathfrak{g}/\mathfrak{p})\big)^*\big) = \big({}_{{}^t\theta}\big(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \textstyle\bigwedge^i(\mathfrak{g}/\mathfrak{p})\big)\big)^*,$$

for all θ in Θ (here, if V is a $\mathfrak{g}$-module, ${}_\theta V$ is defined as in §2).

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$, Δ the root system of $(\mathfrak{g}, \mathfrak{h})$. Fix Δ^+ a system of positive roots in Δ and let $\pi = \{\alpha_1, \ldots, \alpha_n\}$ be the system of simple roots of Δ^+. Set $\rho = (1/2) \sum_{\alpha \in \Delta^+} \alpha$.

Let S be a subset of $\{1, \ldots, n\}$ giving $\mathfrak{p}$ and let Δ_S^+ and P_S^+ be as in §3. If $\lambda \in P_S^+$, let M_λ be defined as in §3, and denote by σ_λ the corresponding representation of P^0. Let V^{M_λ} be the generalized Verma module associated with $\mathfrak{g}$, $\mathfrak{h}$, π, S and λ, and let θ_λ be the infinitesimal character of V^{M_λ}.

Let W denote the Weyl group of Δ and let $(W^S)^{(i)}$ be defined as in Proposition 7.9.

Theorem 9.3 implies

$$_{\theta_0}\left(\mathrm{Hom}_{U(\mathfrak{p})}\left(U(\mathfrak{g}), \textstyle\bigwedge^i(\mathfrak{g}/\mathfrak{p})^*\right)_F\right) = \coprod_{w \in (W^S)^{(i)}} (V^{M_{w\cdot 0}})^*_F, \qquad (**)$$

where $(V^{M_{w\cdot 0}})^*_F$ is the $U(\mathfrak{k})$-finite dual of $V^{M_{w\cdot 0}}$.

If $H \in \mathfrak{a}_0$, set $\rho(H) = (1/2)\,\mathrm{tr}(\mathrm{ad}\, H_{|\mathfrak{n}_0})$. If $m \in M$, $a \in A$ and $n \in N$, define $\delta(man) = e^{2\rho(\log a)}$, where log: $A \to \mathfrak{a}_0$ is the inverse of exp: $\mathfrak{a}_0 \to A$.

If σ is a finite dimensional representation of P^0 set $H_{P^0,\sigma} = C_F^\infty(G; \delta^{1/2}\sigma)$. $H_{P^0,\sigma}$ is known as the degenerate principal series.

Let $\tilde{\partial}_i$ be as in §10 and denote by $\tilde{\partial}_i^*$ the dual map of $\tilde{\partial}_i$. We abuse notation and write

$$\tilde{\partial}_i^*: \coprod_{w \in (W^S)^{(i)}} H_{P^0,\delta^{-1/2}\sigma^*_{w\cdot 0}} \to \coprod_{w \in (W^S)^{(i+1)}} H_{P^0,\delta^{-1/2}\sigma^*_{w\cdot 0}}, \qquad i = 0, \ldots, r-1.$$

COROLLARY 12.6. *The ith cohomology of the complex*

$$0 \to \mathbf{C} \xrightarrow{\tilde{\xi}} H_{P^0,\delta^{-1/2}\sigma_0^*} \xrightarrow{\tilde{\partial}_0^*} \cdots \xrightarrow{\tilde{\partial}_{r-1}^*} \coprod_{w \in (W^S)^{(r)}} H_{P^0,\delta^{-1/2}\sigma^*_{w\cdot 0}} \xrightarrow{\tilde{\nu}} \mathbf{C} \to 0 \qquad (4)$$

is the cohomology of X if $1 < i < r$ and is zero if $i = 0$ or $i = r$. Here $\tilde{\xi}$ and $\tilde{\nu}$ are induced from ξ and ν of Lemma 12.5, *respectively.*

PROOF. We assert that $_{\theta_0}(C_F^\infty(G; \sigma_i)) = \coprod_{w \in (W^S)^{(i)}} C_F^\infty(G; \sigma^*_{w\cdot 0})$, where $C_F^\infty(G; \sigma^*_{w\cdot 0})$ is the space of all K-finite vectors of $C^\infty(G; \sigma^*_{w\cdot 0})$, $w \in (W^S)^{(i)}$, $i = 0, \ldots, r$. Indeed,

$$_{\theta_0}(C_F^\infty(G; \sigma_i)) = {}_{\theta_0}\left(\mathrm{Hom}_{U(\mathfrak{p})}\left(U(\mathfrak{g}), \textstyle\bigwedge^i(\mathfrak{g}/\mathfrak{p})^*\right)_F\right),$$

by Proposition 12.4. Therefore $_{\theta_0}(C_F^\infty(G; \sigma_i)) = \coprod_{w \in (W^S)^{(i)}}(V^{M_{w\cdot 0}})^*_F$, by (**). Applying Proposition 5.5.4 of **[6]** to $(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M_{w\cdot 0})^*_F$ and Proposition 12.4 to $\mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), M^*_{w\cdot 0})_F$, $w \in (W^S)^{(i)}$, $i = 0, \ldots, r$, we find that $(V^{M_{w\cdot 0}})_F = C_F^\infty(G; \sigma^*_{w\cdot 0})$, for all $w \in (W^S)^{(i)}$, all $i = 0, \ldots, r$. The result now follows from Lemma 12.2, Corollary 12.3 and Lemma 12.5. Q.E.D.

NOTES. (1) $K/K \cap P^0$ is a real homology sphere if and only if (4) is exact.

(2) It is well known that if $\mathrm{rk}_{\mathbf{R}}\, G = 1$ and P is a proper parabolic subgroup of G then $K/K \cap P^0$ is a sphere. In this case (4) is exact, by (1).

(3) Let $G = \mathrm{SL}(n, \mathbf{R})$. An Iwasawa decomposition of G is obtained by taking $K = \mathrm{SO}(n)$, A_1 the group of positive diagonal matrices in G and N_1 the group of $n \times n$ upper triangular matrices with ones on the main diagonal. Here P_1 is the

group of all upper triangular matrices in G. Let P be the group of all matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$, A $(n-1) \times (n-1)$, in G. Then $X = G/P^0 = K/K \cap P^0 = \mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1}$ and (4) is exact, by (1).

Appendix. Here we give an alternate proof of Theorem 9.3 which does not use the formalism of the category $\mathcal{O}_S$.

LEMMA A.1 [**14**, PROPOSITION 3.8]. *Let $w \in W$, $w \notin W^S$, $w' \in W^S$, and assume that $w \leftarrow w'$ (see Definition 2.9). Then $w \overset{\alpha_i}{\leftarrow} w'$ for some $i \in S$.*

We now give a proof of the analog of Corollary 9.2 in $\mathcal{O}$.

Let λ be dominant integral in $\mathfrak{h}^*$ and let $w, w' \in W^S$ be such that $l(w) = l(w')$. Suppose that $\mathrm{Ext}^1_{\mathcal{O}}(V^{M_{w\cdot\lambda}}, V^{M_{w'\cdot\lambda}})$ is nontrivial (here $\mathrm{Ext}^1_{\mathcal{O}}$ means the Ext^1 bifunctor in $\mathcal{O}$). Let

$$E\colon 0 \to V^{M_{w'\cdot\lambda}} \xrightarrow{\alpha} E \xrightarrow{\beta} V^{M_{w\cdot\lambda}} \to 0$$

be a representative of a nonzero element of $\mathrm{Ext}^1_{\mathcal{O}}(V^{M_{w\cdot\lambda}}, V^{M_{w'\cdot\lambda}})$.

By Corollary 4.1 and Proposition 6.2 of [**1**], there exists an indecomposable projective object $I_{w\cdot\lambda}$ in $\mathcal{O}$ having a Verma composition series by V_ν such that

$$(I_{w\cdot\lambda} : V_\nu) = (V_\nu : L_{w\cdot\lambda}).$$

That is, there exists a filtration

$$I_{w\cdot\lambda} = I_1 \supset I_2 \supset \cdots \supset I_r \supset I_{r+1} = (0),$$

such that $I_i/I_{i+1} \simeq V_{\nu_i}$, with $V_{w\cdot\lambda} \subset V_{\nu_i}$, $i = 1, \ldots, r$. Furthermore, $V_{\nu_1} = V_{w\cdot\lambda}$, and $V_{w\cdot\lambda} \subsetneqq V_{\nu_i}$ for all $i \geqslant 2$.

Denote by f the composite

$$f\colon I_{w\cdot\lambda} \xrightarrow{\pi} V_{w\cdot\lambda} \xrightarrow{\eta} V^{M_{w\cdot\lambda}},$$

where π is the projection, $I_1 \to I_1/I_2$, and η is the map ψ of Proposition 3.4. Then there exists M, $I_2 \subset M \subset I_{w\cdot\lambda}$ such that

$$M/I_2 \simeq \coprod_{i \in S} V_{\sigma_i w\cdot\lambda} \subset V_{w\cdot\lambda},$$

by Proposition 3.4.

By the projectivity of $I_{w\cdot\lambda}$ we have the commutative diagram in $\mathcal{O}$,

$$\begin{array}{ccccccccc} 0 & \to & V^{M_{w'\cdot\lambda}} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & V^{M_{w\cdot\lambda}} & \to & 0 \\ & & & & g \nwarrow & & \nearrow f & & \\ & & & & & I_{w\cdot\lambda} & & & \end{array}$$

Now, $g(M) = g(I_{w\cdot\lambda}) \cap \alpha(V^{M_{w'\cdot\lambda}}) \neq (0)$, since E is nonsplit. We have two cases to consider.

Case 1. $g(I_2) \neq (0)$. Then, for every $i \geqslant 2$, $g(I_i)/g(I_{i+1})$ is generated by a highest weight vector of weight ν with $V_{w\cdot\lambda} \subsetneqq V_\nu$, or else $g(I_i)/g(I_{i+1}) = (0)$. Since $g(I_2) \neq (0)$ and $g(I_{r+1}) = (0)$, there exists $i \geqslant 2$ such that $g(I_i)/g(I_{i+1}) \neq (0)$. Therefore there exists ν such that $V_{w\cdot\lambda} \subsetneqq V_\nu$ and L_ν is an irreducible subquotient of $g(I_2) \subset g(M) \subset \alpha(V^{M_{w'\cdot\lambda}})$. This implies that $V_\nu \subset V_{w'\cdot\lambda}$ and therefore $V_{w\cdot\lambda} \subsetneqq V_{w'\cdot\lambda}$, a contradiction.

Case 2. $g(I_2) = (0)$. In this case, write $M/I_2 \simeq \amalg_{i \in S} V_{\sigma_i w \cdot \lambda}$. Let $v_i \in M$ be such that $v_i + I_2$ is the canonical generator of $V_{\sigma_i w \cdot \lambda}$, $i \in S$. Then $g(M)$ is generated by $\{g(v_i)\}_{i \in S}$. Since $g(M) \neq (0)$, $g(M)$ contains the module generated by a highest weight vector of weight $\sigma_i w \cdot \lambda$ for some $i \in S$. Therefore, $L_{\sigma_i w \cdot \lambda}$ is an irreducible subquotient of $V_{w' \cdot \lambda}$. Hence $V_{\sigma_i w \cdot \lambda} \subset V_{w' \cdot \lambda}$ for some $i \in S$. Thus $\sigma_i w \leftarrow w'$ and $w' \in W^S$, $\sigma_i w \notin W^S$. Lemma A.1 now implies that $w = w'$, a contradiction. This proves that $\mathrm{Ext}^1_{\mathcal{O}}(V^{M_{w \cdot \lambda}}, V^{M_{w' \cdot \lambda}})$ is trivial. We combine this with the analog of Lemma 8.1 in the Category $\mathcal{C}_{(\mathfrak{g},\mathfrak{h})}$ of $\mathfrak{g}$-modules satisfying 2.1(2). Theorem 9.3 now follows. Q.E.D.

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