Splitting criterion for reflexive sheaves

TAKURO ABE MASAHIKO YOSHINAGA

April 6, 2005

Abstract

The purpose of this paper is to study the structure of reflexive sheaves over projective spaces through hyperplane sections. We give a criterion for a reflexive sheaf to split into a direct sum of line bundles. An application to the theory of free hyperplane arrangements is also given.

0 Main Theorem

Vector bundles over the projective space $\mathbf{P}^n_{\mathbb{K}}$ are one of the main subjects in both (algebraic) geometry and commutative algebra. The most fundamental result in this area is the theorem due to Grothendieck which asserts that any holomorphic vector bundle over $\mathbf{P}^1_{\mathbb{K}}$ splits into a direct sum of line bundles. When $n \geq 2$, vector bundles over $\mathbf{P}^n_{\mathbb{K}}$ do not necessarily split. Indeed, the tangent bundle is indecomposable. In these cases, some sufficient conditions for vector bundles to split have been established. The following is one of such criterions, which we call "Restriction criterion".

Theorem 0.1 (Horrocks)
Let $\mathbb{K}$ be an algebraically closed field, n be an integer greater than or equal to 3, and let E be a locally free sheaf on $\mathbf{P}^n_{\mathbb{K}}$ of rank $r\ (\geq 1)$. *Then E splits into a direct sum of line bundles if and only if there exists a hyperplane $H \subset \mathbf{P}^n_{\mathbb{K}}$ such that $E|_H$ splits into a direct sum of line bundles.*

In other words, the splitting of a vector bundle can be characterized by using a hyperplane section. However, vector bundles, or equivalently locally free sheaves, form a small class among all coherent sheaves. There are some important wider classes of coherent sheaves, e.g., reflexive sheaves or torsion free sheaves. The purpose of this article is to generalize the "Restriction criterion" to one for reflexive sheaves, and we also show that it fails in the class of torsion free sheaves. Our main theorem is as follows.

Theorem 0.2
Let $\mathbb{K}$ be an algebraically closed field, n be an integer greater than or equal to 3, and let E be a reflexive sheaf on $\mathbf{P}^n_{\mathbb{K}}$ of rank r (≥ 1). *Then E splits into a direct sum of line bundles if and only if there exists a hyperplane $H \subset \mathbf{P}^n_{\mathbb{K}}$ such that $E|_H$ splits into a direct sum of line bundles.*

We give two proofs for Theorem 0.2. The first proof is basically parallel to that of Theorem 0.1, in which we also establish a general principle that the structure of a reflexive sheaf can be recovered from its hyperplane section (Theorem 2.2).

The second proof is based on a cohomological characterization for a coherent sheaf to be locally free. By using it, the proof is reduced to Theorem 0.1.

The organization of this paper is as follows. In §1, we recall some basic results on reflexive sheaves from [H2]. In §2, we give the first proof of the main theorem. In §3, we give the second proof by using a cohomological characterization for a coherent sheaf to be locally free.

To each hyperplane arrangement in a vector space, we can associate a reflexive sheaf over the projective space. The splitting of this reflexive sheaf defines an important class of arrangements, namely, free arrangements. As an application of our main theorem, we give a criterion for an arrangement to be free in §4, which has been also obtained in [Y].

Acknowledgement. The authors learned results of §3 from Professor F.-O. Schreyer. They are grateful to him. The authors also thank to Takeshi Abe and Florin Ambro for many helpful comments and pointing out mistakes in our draft. The second author was supported by the JSPS Research Fellowship for Young Scientists.

1 Preliminaries

In this section, we fix the notation and prepare some results for the proof of Theorem 0.2. We use the terms "vector bundle" and "locally free sheaf" interchangeably. The term "variety" means a integral scheme of finite type over a field. Let X be a smooth variety of dimension n over a field $\mathbb{K}$, where $n \geq 1$ and $\mathbb{K}$ is an algebraically closed field. For a coherent sheaf E on X we denote by $\mathrm{Sing}(E)$ the non-free locus of E, i.e., $\mathrm{Sing}(E) := \{x \in X | E_x \text{ is not a free } \mathcal{O}_{x,X}\text{-module}\}$. The dual of a coherent sheaf E (on X) is denoted by E^*.

In this article, we employ homological algebra to investigate properties of a coherent sheaf on a smooth variety X. Let us review some definitions and results. For a coherent sheaf E on X over $\mathbb{K}$ and for a point $x \in X$

(denoted by $\mathrm{depth}_{\mathcal{O}_X}(E_x)$) as the length of a maximal E_x-regular sequence in $\mathcal{M}_x$, where $\mathcal{M}_x$ is the unique maximal ideal of a local ring $\mathcal{O}_{x,X}$. Moreover, we define the projective dimension of an $\mathcal{O}_{x,X}$-module E_x (denoted by $\mathrm{pd}_{\mathcal{O}_{x,X}}(E_x)$) as the length of a minimal free resolution of E_x as an $\mathcal{O}_{x,X}$-module. It is known that every module which is finitely generated over a regular local ring has finite projective dimension. These two quantities are related by the famous Auslander-Buchsbaum formula as follows.

$$\mathrm{depth}_{\mathcal{O}_{x,X}}(E_x) + \mathrm{pd}_{\mathcal{O}_{x,X}}(E_x) = \dim \mathcal{O}_{x,X}.$$

Hence it follows easily that a coherent sheaf E on X is locally free if and only if $\mathrm{depth}_{\mathcal{O}_{x,X}}(E_x) = \dim \mathcal{O}_{x,X}$ for all $x \in X$. For details and proofs, see [Ma]. The projective dimension can also be characterized as follows (for example, see [OSS] Chapter II).

Lemma 1.1
Let X be a smooth variety and E be a coherent sheaf on X. Then $\mathrm{pd}_{\mathcal{O}_{x,X}}(E_x) \leq q$ if and only if for all $i > q$ we have

$$\mathcal{E}xt^i_{\mathcal{O}_X}(E, \mathcal{O}_X)_x = 0.$$

In particular, E is locally free if and only if $\mathcal{E}xt^i_{\mathcal{O}_X}(E, \mathcal{O}_X) = 0$ for all $i > 0$.

Next, let us review definitions and results on reflexive sheaves on $\mathbf{P}^n_{\mathbb{K}}$. Reflexive sheaves form a category between torsion free sheaves and vector bundles.

Definition 1.1
We say a coherent sheaf E on $\mathbf{P}^n_{\mathbb{K}}$ is reflexive if the canonical morphism $E \to E^{**}$ is an isomorphism.

In this article, we use the following results on reflexive sheaves. For the proofs and details, see [H2].

Proposition 1.2 ([H2], Proposition 1.3)
A coherent sheaf E on $\mathbf{P}^n_{\mathbb{K}}$ is reflexive if and only if E is torsion free and $\mathrm{depth}_{\mathcal{O}_{x,\mathbf{P}^n_{\mathbb{K}}}}(E_x) \geq 2$ *for all points $x \in \mathbf{P}^n_{\mathbb{K}}$ such that* $\dim \mathcal{O}_{x,\mathbf{P}^n_{\mathbb{K}}} \geq 2$.

Corollary 1.3 ([H2], Corollary 1.4)
$\mathrm{codim}_{\mathbf{P}^n_{\mathbb{K}}} \mathrm{Sing}(E) \geq 3$ *for a reflexive sheaf E on $\mathbf{P}^n_{\mathbb{K}}$.*

Proposition 1.4 ([H2], Proposition 1.6)
For a coherent sheaf E on $\mathbf{P}^n_{\mathbb{K}}$, the following are equivalent.

1. *E is reflexive.*

2. *E is torsion free and normal.*

3. *E is torsion free and for each open set $U \subset \mathbf{P}^n_{\mathbb{K}}$ and each closed set Z in U satistying $\operatorname{codim}_U(Z) \geq 2$, we have $E|_U \simeq j_*(E|_{U \setminus Z})$, where $j : U \setminus Z \to Z$ is an open immersion.*

2 The first proof of Theorem 0.2

Let us prove Theorem 0.2. It suffices to show the "if" part of the statement. First, let us assume that $\dim(\operatorname{Sing}(E)) \geq 1$. Then any hyperplane $H \subset \mathbf{P}^n_{\mathbb{K}}$ intersects $\operatorname{Sing}(E)$. Take a point $x \in H \cap \operatorname{Sing}(E) \neq \emptyset$. Note that $\operatorname{depth}_{\mathcal{O}_{x,\mathbf{P}^n_{\mathbb{K}}}}(E_x) \leq \dim \mathcal{O}_{x,\mathbf{P}^n_{\mathbb{K}}} - 1$. Since the equation $h \in \mathcal{O}_{x,\mathbf{P}^n_{\mathbb{K}}}$ which defines H at x is a regular element for the reflexive $\mathcal{O}_{x,\mathbf{P}^n_{\mathbb{K}}}$-module E_x, it follows that $\operatorname{depth}_{\mathcal{O}_{x,H}}(E|_H)_x < \dim \mathcal{O}_{x,\mathbf{P}^n_{\mathbb{K}}} - 1 = \dim \mathcal{O}_{x,H}$. From Auslander-Buchsbaum formula, we conclude that $E|_H$ can not even be locally free. Hence we may assume that $\dim(\operatorname{Sing}(E)) = 0$.

The next lemma is a generalization of Theorem 2.5 in [H2].

Lemma 2.1
Let E be a reflexive sheaf on $\mathbf{P}^n_{\mathbb{K}}$ $(n \geq 3)$ with $\dim(\operatorname{Sing}(E)) = 0$. Suppose the restriction $E|_H$ to a hyperplane H splits into a direct sum of line bundles. Then

$$H^1(\mathbf{P}^n_{\mathbb{K}}, E(k)) = 0, \text{ for all } k \in \mathbb{Z}.$$

Proof of Lemma 2.1. We use the long exact sequence associated with the short exact sequence

$$0 \to E(k-1) \to E(k) \to E(k)|_H \to 0.$$

Because $E(k)|_H$ is a direct sum of line bundles, it follows that $H^1(H, E(k)|_H) = 0$. So we have surjections

$$H^1(\mathbf{P}^n_{\mathbb{K}}, E(k-1)) \twoheadrightarrow H^1(\mathbf{P}^n_{\mathbb{K}}, E(k)), \ \forall k \in \mathbb{Z}. \tag{1}$$

To see that these cohomology groups are equal to zero, let us consider the spectral sequence of local and global Ext functors:

$$E_2^{p,q} = H^p(\mathbf{P}^n_{\mathbb{K}}, \mathcal{E}xt^q_{\mathbf{P}^n_{\mathbb{K}}}(E, \omega)) \Rightarrow E^{p+q} = \operatorname{Ext}^{p+q}_{\mathbf{P}^n_{\mathbb{K}}}(E, \omega)$$

where ω is the dualizing sheaf of $\mathbf{P}^n_{\mathbb{K}}$. The assumption $\dim(\operatorname{Sing}(E)) = 0$ implies $\dim(\operatorname{Supp}(\mathcal{E}xt^q_{\mathbf{P}^n_{\mathbb{K}}}(E, \omega))) = 0$ for all $q > 0$. Thus it follows that $E_2^{p,q} = 0$ unless $p = 0$ or $q = 0$. Moreover, Proposition 1.2 implies $\operatorname{depth}_{\mathcal{O}_{x,\mathbf{P}^n_{\mathbb{K}}}}(E_x) \geq$

2. From Auslander-Buchsbaum formula, we have $\mathrm{pd}_{\mathcal{O}_{x,\mathbf{P}^n_{\mathbb{K}}}} E_x < n-1$ for all $x \in \mathbf{P}^n_{\mathbb{K}}$. It follows that $\mathcal{E}xt^q_{\mathbf{P}^n_{\mathbb{K}}}(E,\omega) = 0$ for $\forall q \geq n-1$. Hence we have $E_2^{p,q} = 0$ for $q \geq n-1$. Considering the convergence of this spectral sequence, we obtain the surjection

$$H^{n-1}(\mathbf{P}^n_{\mathbb{K}}, \mathcal{H}om_{\mathbf{P}^n_{\mathbb{K}}}(E,\omega)) \simeq H^{n-1}(\mathbf{P}^n_{\mathbb{K}}, E^* \otimes \omega) \twoheadrightarrow \mathrm{Ext}^{n-1}_{\mathbf{P}^n_{\mathbb{K}}}(E,\omega). \quad (2)$$

Since $\mathrm{Ext}^{n-1}_{\mathbf{P}^n_{\mathbb{K}}}(E(k),\omega)$ is the Serre dual to $H^1(\mathbf{P}^n_{\mathbb{K}}, E(k))$, they have the same dimension. From (2), we have

$$\dim H^1(\mathbf{P}^n_{\mathbb{K}}, E(k)) \leq \dim H^{n-1}(\mathbf{P}^n_{\mathbb{K}}, E^*(-k) \otimes \omega) \quad (3)$$

for all $k \in \mathbb{Z}$. The right hand side of (3) vanishes for $k \ll 0$. Then together with the surjectivity (1), we conclude that $H^1(\mathbf{P}^n_{\mathbb{K}}, E(k)) = 0$, for all $k \in \mathbb{Z}$. $\square$

Now, let us put

$$E|_H \simeq \oplus_{i=1}^r \mathcal{O}_H(a_i)$$

and $F := \oplus_{i=1}^r \mathcal{O}_{\mathbf{P}^n_{\mathbb{K}}}(a_i)$. Noting that $\mathrm{Ext}^1_{\mathbf{P}^n_{\mathbb{K}}}(F, E(-1)) \simeq H^1(\mathbf{P}^n_{\mathbb{K}}, E(-a_i - 1)) = 0$, Theorem 0.2 follows from the following theorem, which asserts that, roughly speaking, the structure of a reflexive sheaf can be recovered from its restriction to a hyperplane.

Theorem 2.2
Let E and F be reflexive sheaves on $\mathbf{P}^n_{\mathbb{K}}$ $(n \geq 2)$ and H be a hyperplane in $\mathbf{P}^n_{\mathbb{K}}$. Suppose $E|_H \cong F|_H$ and $\mathrm{Ext}^1_{\mathbf{P}^n_{\mathbb{K}}}(F, E(-1)) = 0$. Then $E \cong F$.

Proof of Theorem 2.2. We want to extend the isomorphism $\varphi : F|_H \to E|_H$ to one over $\mathbf{P}^n_{\mathbb{K}}$. That is possible since there is an exact sequence

$$\begin{aligned} 0 \;\to\; & \mathrm{Hom}_{\mathbf{P}^n_{\mathbb{K}}}(F, E(-1)) \to \mathrm{Hom}_{\mathbf{P}^n_{\mathbb{K}}}(F, E) \to \mathrm{Hom}_{\mathbf{P}^n_{\mathbb{K}}}(F, E|_H) \\ \to\; & \mathrm{Ext}^1_{\mathbf{P}^n_{\mathbb{K}}}(F, E(-1)) = 0, \end{aligned}$$

and every morphism $F|_H \to E|_H$ has a canonical extension to a morphism $F \to E|_H$. Let us fix an extended morphism $f : F \to E$ which satisfies $f|_H = \varphi$. Now, let us consider the morphism $\det f : \det F \to \det E$. This is a monomorphism because f is already a monomorphism. Since $E|_H \simeq F|_H$, ranks and first Chern classes of E and F are the same. Henceforth we can see that $\det f$ is a multiplication of some constant element in $\mathbb{K}$. Note that this constant is not zero. For $\det f$ is not zero on H. Thus at each point $x \in \mathbf{P}^n_{\mathbb{K}} \setminus (\mathrm{Sing}(E) \cup \mathrm{Sing}(F))$, the morphism f_x is an isomorphism because at these points f_x are the endomorphism of a direct sum of local rings of the same rank. Since $\mathrm{codim}_{\mathbf{P}^n_{\mathbb{K}}}(\mathrm{Sing}(E) \cup \mathrm{Sing}(F)) > 2$ and both of E and F are reflexive, the third condition of Proposition 1.4 implies that f is also an isomorphism on $\mathbf{P}^n_{\mathbb{K}}$. □

Remark 2.1
In Theorem 0.2, we can not omit the assumption that E is reflexive, i.e., "Restriction criterion" fails for torsion free sheaves. For example, consider the ideal sheaf I_p on $\mathbf{P}^3_{\mathbb{K}}$ which corresponds to a closed point $p \in \mathbf{P}^3_{\mathbb{K}}$. Note that I_p is not reflexive. Indeed, let us put $U = \mathbf{P}^3_{\mathbb{K}} \setminus \{p\}$ and $j : U \to \mathbf{P}^3_{\mathbb{K}}$ be an open immersion. It is easy to see that $I_p|_U \simeq \mathcal{O}_U$. If I_p is reflexive, then according to Proposition 1.4, $j_*(I_p|_U) \simeq I_p$ must hold. However, clearly this is not ture. Hence I_p is not reflexive. Now, if we cut I_p by a plane H which does not contain p, then it is easily seen that $I_p|_H \simeq \mathcal{O}_H$. However, of course, I_p is not a line bundle on $\mathbf{P}^3$.

3 The second proof

Instead of Theorem 2.2, we can use the following result, which is the generalization of the famous Horrocks' splitting criterion (For example, see [OSS]). Combining this criterion with usual cohomological arguments and Lemma 2.1, we can give the second proof of Theorem 0.2. However, it seems that this theorem is not so familiar. Hence let us show the result with a complete proof.

Theorem 3.1
Let $\mathbb{K}$ be an algebraically closed field, n be a integer greater than or equal to 2, and let E be a coherent sheaf on $\mathbf{P}^n_{\mathbb{K}}$. Then E splits into a direct sum of line bundles if and only if $H^i(\mathbf{P}^n_{\mathbb{K}}, E(k)) = 0$ for all $k \in \mathbb{Z}$, $i = 1, \cdots, n-1$ and $H^0(\mathbf{P}^n_{\mathbb{K}}, E(k)) = 0$ for all $k \ll 0$.

Remark 3.1
Note that when E is torsion free, then $H^0(\mathbf{P}^n_{\mathbb{K}}, E(k)) = 0$ for all $k \ll 0$. This follows from the fact that all torsion free sheaves can be embedded into a direct sum of line bundles on $\mathbf{P}^n_{\mathbb{K}}$. So in the theorem, the condition $H^0(\mathbf{P}^n_{\mathbb{K}}, E(k)) = 0$ is automatically satisfied for torsion free sheaves.

When E is a vector bundle, Theorem 3.1 is just the splitting criterion of Horrocks. Thus for the proof of this theorem, it suffices to show the following lemma.

Lemma 3.2
Let X be a nonsingular projective variety over an algebraically closed field $\mathbb{K}$ of dimension $n > 1$, L be an ample line bundle on X, and let E be a

coherent sheaf on X.Then E is locally free if and only if $H^i(X, E(k)) = 0$ for all $k \ll 0$ and $i = 0, 1, \cdots, n-1$, where $E(k) = E \otimes L^k$.

Proof of Lemma 3.2. From Serre duality, the "only if" part follows immediately. Let us show the "if" part of the statement. Recall that E is locally free on X if and only if $\mathcal{E}xt^i_X(E, \mathcal{O}_X) = 0$ for all $i > 0$, see §1. Consider the spectral sequence

$$E_2^{p,q}(k) = H^p(X, \mathcal{E}xt^q_X(E(k), \omega)) \Rightarrow E^{p+q}(k) = \mathrm{Ext}^{p+q}_X(E(k), \omega),$$

where $k \in \mathbb{Z}$ and ω is the dualizing sheaf on X. By Serre duality, $H^i(X, E(k))^* \simeq \mathrm{Ext}^{n-i}_X(E(k), \omega)$ for $i = 0, 1, \cdots, n$. So for each $i > 0$, $E^i(k) = \mathrm{Ext}^i_X(E(k), \omega) = 0$ for sufficiently small $k \in \mathbb{Z}$. Now let us assume that there exists an integer $i > 0$ such that $\mathcal{E}xt^i_X(E, \mathcal{O}_X) \neq 0$, and we show that this leads to a contradiction. It is easy to see that

$$E_2^{0,i}(k) = H^0(X, \mathcal{E}xt^i_X(E, \omega) \otimes \mathcal{O}_X(-k)) \neq 0, \text{ for } \forall k \ll 0.$$

On the other hand, for $p > 0$,

$$E_2^{p,q}(k) = H^p(X, \mathcal{E}xt^q_X(E, \omega) \otimes \mathcal{O}_X(-k)) = 0, \text{ for } \forall k \ll 0.$$

From the definition of spectral sequence,

$$\mathrm{Ext}^i_X(E(k), \omega) = E_2^{0,i}(k) \neq 0,$$

for $\forall k \ll 0$. This contradicts the assumption that for each $i > 0$, $E^i(k) = 0$ for sufficiently small $k \in \mathbb{Z}$. Hence we can see that $\mathcal{E}xt^i_X(E, \mathcal{O}_X) = 0$ for all $i > 0$, so E is a locally free sheaf. □

4 Application to hyperplane arrangements

In this section, we describe an application of our main theorem to the theory of hyperplane arrangements. As mentioned in §0, each hyperplane arrangement determines a reflexive sheaf. We start with a more general setting. To every divisor D in a complex manifold M we can associate a reflexive sheaf as follows.

Definition 4.1
A vector field δ on an open set $U \subset M$ is said to be logarithmic tangent to D if for a local defining equation h of $D \cap U$ on U, $\delta h \in (h)$. The sheaf associated with logarithmic vector fields is denoted by $\mathrm{Der}_M(-\log D)$.

In the definition above, a vector field δ is identified with a derivation $\delta : \mathcal{O}_M \longrightarrow \mathcal{O}_M$, and $\mathrm{Der}_M(-\log D)$ can be considered as a subsheaf of the tangent sheaf. The sheaf of logarithmic vector fields $\mathrm{Der}_M(-\log D)$ is not necessarily locally free, but in [S], K. Saito proved the following.

Theorem 4.1 ([S])
$\mathrm{Der}_M(-\log D)$ *is a reflexive sheaf.*

From now on, we restrict ourselves to the case where D is a hyperplane arrangement.

Let V be an ℓ-dimensional linear space over $\mathbb{K}$ and $S := \mathbb{K}[V^*]$ be the algebra of polynomial functions on V that is naturally isomorphic to $\mathbb{K}[z_1, z_2, \cdots, z_\ell]$ for any choice of basis $(z_1, \cdots, z_\ell)$ of V^*.

A (central) hyperplane arrangement $\mathcal{A}$ is a finite collection of codimension one linear subspaces in V. For each hyperplane H of $\mathcal{A}$, fix a nonzero linear form $\alpha_H \in V^*$ vanishing on H and put $Q := \prod_{H \in \mathcal{A}} \alpha_H$.

The characteristic polynomial of $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}, t) = \sum_{X \in L_{\mathcal{A}}} \mu(X) t^{\dim X},$$

where $L_{\mathcal{A}}$ is a lattice which consists of the intersections of elements of $\mathcal{A}$, ordered by reverse inclusion, $\hat{0} := V$ is the unique minimal element of $L_{\mathcal{A}}$ and $\mu : L_{\mathcal{A}} \longrightarrow \mathbb{Z}$ is the Möbius function defined as follows:

$$\begin{aligned} \mu(\hat{0}) &= 1, \\ \mu(X) &= -\sum_{Y < X} \mu(Y), \text{ if } \hat{0} < X. \end{aligned}$$

The characteristic polynomial is one of the most important concepts in the theory of hyperplane arrangements. Actually there are a lot of combinatorial or geometric interpretations of characteristic polynimial. For details, see [OT].

Denote by $\mathrm{Der}_V := \mathbb{K}[V^*] \otimes V$ the S-module of all polynomial vector fields on V. The following definition was given by G. Ziegler.

Definition 4.2 ([Z])
For a given arrangement $\mathcal{A}$ and a map $m : \mathcal{A} \longrightarrow \mathbb{Z}_{\geq 0}$, we define modules of logarithmic vector fields with multiplicity m by

$$D(\mathcal{A}, m) = \{\delta \in \mathrm{Der}_V \mid \delta\alpha_H \in S\alpha^{m(H)}, \ \forall H \in \mathcal{A}\}$$

When the multiplicity m is the constant map $\underline{1}(H) \equiv 1$ $(\forall H \in \mathcal{A})$, $D(\mathcal{A}, \underline{1})$ is simply denoted by $D(\mathcal{A})$.

It is known that the graded S-module $D(\mathcal{A}, m)$ is a reflexive module of rank $l = \dim V$.

Definition 4.3

(1) *An arrangement with a multiplicity* $(\mathcal{A}, m)$ *is called free with exponents* $(e_1, \cdots, e_\ell)$ *if* $D(\mathcal{A}, m)$ *is a free* S*-module, with a homogeneous basis* $\delta_1, \cdots, \delta_\ell$ *such that*

$$\deg \delta_i = e_i.$$

Note that a vector field

$$\delta = \sum_i f_i \frac{\partial}{\partial x_i}$$

is said to be homogeneous if coefficients $f_1, \cdots, f_\ell$ *are all homogeneous with the same degree and put* $\deg \delta := \deg f_i$.

(2) *An arrangement* $\mathcal{A}$ *is called free if* $(\mathcal{A}, \underline{1})$ *is free, i.e.,* $D(\mathcal{A})$ *is a free* S*-module.*

Since $D(\mathcal{A})$ contains the Euler vector field $\theta_E := \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i}$, the exponents $(e_1, \cdots, e_\ell)$ of a free arrangement $\mathcal{A}$ contains 1. H. Terao proved that the freeness of $\mathcal{A}$ implies a remarkable behavior of the characteristic polynomial.

Theorem 4.2 ([T])

Suppose $\mathcal{A}$ *is a free arrangement with the exponents* $(e_1, \cdots, e_\ell)$*, then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - e_i).$$

As we will see later, in Corollary 4.5, the freeness is equivalent to the splitting of a reflexive sheaf, and exponents are corresponding to the splitting type. On the other hand, the left hand side of the Theorem 4.2 is obtained from the intersection poset, thus determined by the combinatorial structure. This theorem connects two regions in mathematics: combinatorics of arrangements and geometry of reflexive sheaves. It enables us to study combinatorics of arrangements via a geometric method. For example, in [Y] characteristic polynomials for some arrangements are computed by using this interpretation.

In [Z], Ziegler studied the relation between the freeness and the freeness with a multiplicity. Fixing a hyperplane $H_0 \in \mathcal{A}$, let us define an arrangement

$$\mathcal{A}^{H_0} := \{H_0 \cap K \mid K \in \mathcal{A},\ K \neq H_0\},$$

over H and the natural multiplicity

$$\underline{m}(X) := \sharp\{K \in \mathcal{A} \mid K \cap H_0 = X\}$$

for $X \in \mathcal{A}^{H_0}$.

Theorem 4.3 ([Z])
If $\mathcal{A}$ is a free arrangement with exponents $(1, e_2, \cdots, e_\ell)$, then the restricted arrangement with natural multiplicity $(\mathcal{A}^{H_0}, \underline{\mathrm{m}})$ is also free with exponents $(e_2, \cdots, e_\ell)$.

More precisely, let $\alpha = \alpha_{H_0}$ be a defining equation of H_0 and define

$$D_0(\mathcal{A}) := \{\delta \in D(\mathcal{A}) \mid \delta\alpha = 0\}.$$

It is easily seen that $D(\mathcal{A})$ has a direct sum decomposition into graded S-modules

$$D(\mathcal{A}) = S \cdot \theta_E \oplus D_0(\mathcal{A}).$$

Ziegler proved that if $\delta_1 = \theta_E, \delta_2, \cdots, \delta_\ell$ is a basis of $D(\mathcal{A})$ with $\delta_2, \cdots, \delta_\ell \in D_0(\mathcal{A})$, then $\delta_2|_{H_0}, \cdots, \delta_\ell|_{H_0}$ form a basis of $D(\mathcal{A}^{H_0}, \underline{\mathrm{m}})$.

Recall that a graded S-module $M = \oplus_{k\in\mathbb{Z}} M_k$ determines a coherent sheaf $\tilde{M}$ over $\mathbf{P}^{\ell-1} = \operatorname{Proj} S$. Conversely for any coherent sheaf $\mathcal{F}$ over $\mathbf{P}^{\ell-1}$, $\Gamma_*(\mathcal{F}) := \bigoplus_{k\in\mathbb{Z}} \Gamma(\mathbf{P}^{\ell-1}, \mathcal{F}(k))$ defines the graded S-module associated with $\mathcal{F}$. We have the natural S-homomorphism $\alpha : M \to \Gamma_*(\tilde{M})$, which is neither injective nor surjective in general. In the case of $M = D(\mathcal{A})$, however, we have the following lemma.

Lemma 4.4
$\alpha : D(\mathcal{A}) \xrightarrow{\cong} \Gamma_*\left(\mathbf{P}^{\ell-1}, \widetilde{D(\mathcal{A})}\right)$ *is isomorphic.*

Proof of Lemma 4.4. We prove the surjectivity. Since $\bigcup_{i=1}^{\ell} D(z_i) = \mathbf{P}^{\ell-1}$, any element in $\Gamma(\mathbf{P}^{\ell-1}, \widetilde{D(\mathcal{A})}(k))$ can be expressed as

$$\delta = \frac{\delta_1}{z_1^{d_1}} = \frac{\delta_2}{z_2^{d_2}} = \cdots = \frac{\delta_\ell}{z_\ell^{d_\ell}},$$

where $\delta_i \in D(\mathcal{A})_{d_i+k}$. From the facts that δ_i is an element of a S-free module Der_V and S is UFD, it is easily seen that δ is also a polynomial vector field, so contained in Der_V. Let α_H be a defining linear form of $H \in \mathcal{A}$, and we may choose i such that α_H and z_i are linearly independent. Then the right hand side of

$$z_i^{d_i} \cdot \delta\alpha_H = \delta_i \alpha_H$$

is divisible by α_H, so is the left. Hence $\delta\alpha_H$ is also divisible by α_H, and we can conclude that $\delta \in D(\mathcal{A})$. □

The above lemma enable us to connect freeness and splitting.

Corollary 4.5
$\mathcal{A}$ is free with exponents $(e_1, \cdots, e_\ell)$ if and only if

$$\widetilde{D(\mathcal{A})} = \mathcal{O}_{\mathbf{P}^{\ell-1}}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^{\ell-1}}(-e_\ell)$$

Now, the following theorem, which has been proved and played an important role in the proof of Edelman and Reiner conjecture in [Y], is naturally proved from Theorem 0.2.

Theorem 4.6 ([Y])
$\mathcal{A}$ is free if and only if there exists a hyperplane $H_0 \in \mathcal{A}$ such that

(a) $(\mathcal{A}^{H_0}, \underline{\mathrm{m}})$ *is free, and*

(b) $\mathcal{A}_x := \{H \in \mathcal{A} \mid H \ni x\}$ *is free for all* $x \in H_0 \setminus \{0\}$.

Proof of Theorem 4.6. Let us denote by $\mathbf{P}(V)$ the projective space of one-dimensional subspaces in a vector space V. Recall that $D_0(\mathcal{A})$ is a graded reflexive S-module. So it determines a reflexive sheaf $\widetilde{D_0(\mathcal{A})}$ over $\mathbf{P}(V)$. As is mentioned in [MS], the local structure of $\widetilde{D_0(\mathcal{A})}$ is determined by the local structure of $\mathcal{A}$, i.e.,

$$\widetilde{D_0(\mathcal{A})}_{\bar{x}} = \widetilde{D_0(\mathcal{A}_x)}_{\bar{x}},$$

for $\bar{x} \in \mathbf{P}(V)$. Using Theorem 4.3 locally, condition (b) in Theorem 4.6 implies that

$$\widetilde{D_0(\mathcal{A})}_{\bar{x}}|_{\mathbf{P}(H_0)} = D\widetilde{(\mathcal{A}^{H_0}, \underline{\mathrm{m}})}_{\bar{x}}.$$

Now condition (a) in Theorem 4.3 means that $\widetilde{D_0(\mathcal{A})}|_{\mathbf{P}(H_0)}$ splits into a direct sum of line bundles. From Theorem 0.2, we may conclude that $\widetilde{D_0(\mathcal{A})}$ is also splitting. Hence

$$\bigoplus_{k \in \mathbb{Z}} \Gamma\left(\mathbf{P}(V), \widetilde{D_0(\mathcal{A})}(k)\right) = D_0(\mathcal{A})$$

is a free module over S. Thus $\mathcal{A}$ is a free arrangement.

□

References

[EFS] D. Eisenbud, G. Floystad, F.-O. Schreyer, Sheaf cohomology and free resolutions over exterior algebra. Trans. Amer. Math. Soc, **355** (2003), 4397-4426.

[H1] R. Hartshorne, Algebraic Geometry. Graduated Texts in Mathematics, Springer-Verlag, 1977.

[H2] R. Hartshorne, Stable reflexive sheaves. Math. Ann. **254** (1980), 121-176.

[Ma] H. Matsumura, Commutative Algebra. W.A. Benjamin Co., New York (1970).

[MS] M. Mustaţă and H. Schenck, The module of logarithmic p-forms of a locally free arrangement. J. Algebra **241** (2001), no. 2, 699–719.

[OSS] C. Okonek, M. Schneider, H. Spindler, Vector Bundles on Complex Projective Spaces. **3** (1980), Birkhäuser.

[OT] P. Orlik, H. Terao, Arrangements of hyperplanes. Grundlehren der Mathematischen Wissenschaften, 300. Springer-Verlag, Berlin, 1992.

[S] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields. J. Fac. Sci. Univ. Tokyo Sect. IA Math.**27** (1980), no. 2, 265–291

[T] H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula. Invent. Math. **63** (1981), no. 1, 159–179.

[Y] M. Yoshinaga, Characterization of a free arrangement and conjecture of Edelman and Reiner. Invent. Math. **157**(2004), no.2, 449–454.

[Z] G. M. Ziegler, Multiarrangements of hyperplanes and their freeness. in *Singularities* (Iowa City, IA, 1986), pp. 345–359, Contemp. Math., **90**, Amer. Math. Soc., Providence, RI, 1989.

Takuro Abe
Department of Mathematics,
Kyoto University,
Kyoto 606-8502,
Japan,
abetaku@math.kyoto-u.ac.jp

Masahiko Yoshinaga
Research Institute for Mathematical Sciences,
Kyoto University,
Kyoto 606-8502,

Japan
yosinaga@kurims.kyoto-u.ac.jp