Czechoslovak Mathematical Journal, 28 (103) 1978, Praha

SPLITTING IN ABELIAN GROUPS

LADISLAV BICAN, Praha

(Received May 23, 1975)

In my previous papers I studied the splitting problem for two classes of mixed groups. I have shown that a mixed group from any of these classes splits if and only if it satisfies Conditions (α), (β). In the present paper I characterize the class $\mathscr{A}$ of finite rank torsion free groups such that if G is a mixed abelian group and G/T is in $\mathscr{A}$ then G splits if and only if it satisfies Conditions (α), (β).

By the word "group" we shall always mean an additively written abelian group. As in [1] we use the notions "characteristic" and "type" in the wide sense, i.e. we deal with these notions in mixed groups. However, it is clear that some of their properties do not hold in general. The symbols $h_p^G(g)$, $\tau^G(g)$, $\hat{\tau}^G(g)$ denote respectively the p-height, the characteristic, the type of the element g in the group G. If T is a torsion group, then T_p will denote the p-primary component of T and similarly if π is a set of primes, then T_π is defined by $T_\pi = \sum_{p \in \pi}{}_d T_p$. Next, we shall deal with mixed groups G with torsion part T and $\bar{G}$ will denote the factor-group G/T. The bar over the elements will denote the elements from $\bar{G}$. If H is a torsionfree group then the set of all elements g of H with $h_p^H(g) = \infty$ is a subgroup of H which will be denoted by $H[p^\infty]$. Any maximal linearly independent set of elements of a torsionfree group H is called a *basis*. It is well-known (see [4]) that if H is a torsionfree group of finite rank and K its free subgroup of the same rank then the number $r_p(H)$ of summands $C(p^\infty)$ in H/K is independent of the particular choice of K and this number is called *the p-rank* of H. If r is an integer of the form $r = p^k r'$, $(r', p) = 1$ then we shall write $h_p(r) = k$. A subgroup K of a torsionfree group H is called *regular* if $\hat{\tau}^K(g) = \hat{\tau}^H(g)$ for every $g \in K$ and it is called *generalized regular* if for every $g \in K$, $\tau^K(g)$ and $\tau^H(g)$ differ in finitely many places. The other notation will be essentially that from [3] and we shall frequently use the results of [1].

Now we shall formulate Conditions (α), (β) and the main result.

Condition (α). We say that a mixed group G with torsion part T satisfies Condition (α) if to any $g \in G \dot{-} T$ there exists an integer m such that $\hat{\tau}^G(mg) = \hat{\tau}^{\bar{G}}(\bar{g})$.

Condition (β). We say that a mixed group G with torsion part T satisfies Condition (β) if to any $g \in G \dotdiv T$ there exists an integer $m \neq 0$ such that for any prime p with $h_p^{\bar{G}}(\bar{g}) = \infty$, mg has a p-sequence (i.e. there exist elements $h_0^{(p)} = mg, h_1^{(p)}, h_2^{(p)}, \ldots$ such that $ph_{n+1}^{(p)} = h_n^{(p)}$, $n = 0, 1, \ldots$).

Theorem. *The following are equivalent for a torsionfree group H of finite rank:*

(i) $r_p(H) = r(H[p^\infty])$ *for every prime p and, for every generalized regular subgroup K of H of the same rank, the factor-group H/K has only a finite number of non-zero primary components;*

(ii) *if G is a mixed group with $\bar{G} \cong H$ then G splits if and only if G satisfies Conditions* (α), (β).

We shall call a torsionfree group H *purely finitely generated* if H contains elements $g_1, \ldots, g_m$ such that $H = \sum_{i=1}^{m} \{g_i\}_*^H$.

Proposition 1. *A torsionfree group H is purely finitely generated if and only if it is of finite rank and satisfies the statement* (i) *of Theorem.*

Proof. Suppose that H is purely finitely generated, $H = \sum_{i=1}^{m} \{g_i\}_*^H$. Then, obviously, $r(H) < \infty$ and to prove the equality $r_p(H) = r(H[p^\infty])$ for every prime p it clearly suffices to consider the case $H[p^\infty] = 0$. But then the Z_p-module $H \otimes Z_p$ is finitely generated, hence free, and the assertion follows. Let K be a generalized regular subgroup of H of the same rank as H. Then for every $i = 1, 2, \ldots, m$, $\{g_i\}_*^H/K \cap \{g_i\}_*^H$ is a torsion group with a finite number of non-zero primary components. Hence $\sum_{d\, i=1}^{m} \{g_i\}_*^H/K \cap \{g_i\}_*^H$ has the same property and the sequence of natural epimorphisms $\sum_{d\, i=1}^{m} \{g_i\}_*^H/K \cap \{g_i\}_*^H \to \sum_{i=1}^{m} \{g_i\}_*^H/\sum_{i=1}^{m} K \cap \{g_i\}_*^H \to H/K$ completes the "necessary" part of the proof.

For the proof of sufficiency suppose that H is not purely finitely generated and $r_p(H) = r(H[p^\infty])$ for all primes p. Let $h_1, h_2, \ldots, h_n$ be a basis of H, $F = \sum_{d\, i=1}^{n} \{h_i\}$. If we order all non-zero elements of H in a sequence $a_1, a_2, \ldots$, then $H/\{F, \{a_1\}_*^H, \ldots, \{a_m\}_*^H\} = \sum_{d\, p_i \in \pi_m} T_{p_i}$, where π_m is infinite, since H is not purely finitely generated and $r_p(H) = r(H[p^\infty])$ for all primes p. In every set π_m we select an element p_n such that all these primes are pairwise different. It is not too difficult to show that for each p_m there exists a subgroup K_m of H with $\{F, \{a_1\}_*^H, \ldots, \{a_m\}_*^H\} \subseteq K$ and $H/K_m = C(p_m^k)$ for some $k = 1, 2, \ldots$. If we put $K = \bigcap_{m=1}^{\infty} K_m$ then H/K is a torsion group since $F \subseteq K$ and it is easy to see that the p_m-primary part of H/K is non-zero for all m. Moreover, if $0 \neq g \in K$ then $g = a_m$ for some m and $\{g\}_*^K$ is p-pure in H for all primes $p \neq p_i$, $i = 1, 2, \ldots, m - 1$. Consequently, K is a generalized regular subgroup of H and the proof is complete.

Lemma 2. *Let p be a prime and H a torsionfree group of rank n with $r(H[p^\infty]) < r_p(H)$. Then there exists a mixed group G which satisfies Conditions (α), (β), $\bar{G} \cong H$ and G does not split.*

Proof. It is easy to see that H contains a subgroup K such that $H[p^\infty] \subseteq K$ and $H/K \cong C(p^\infty)$. Then H can be generated by K and the elements x_j, $j = 1, 2, \ldots$, satisfying

$$p^j x_j = \sum_{r=1}^{n} \lambda_r^{(j)} h_r \,, \tag{1}$$

where $h_1, \ldots, h_n$ is a basis of K. Moreover, $h_1, \ldots, h_n$ can be chosen in such a way that $(\lambda_r^{(j)})$ are p-adic integers, $r = 1, 2, \ldots, n$, and $u_j = \sum_{r=1}^{n} \lambda_r^{(j)} h_r$ is of zero p-height in K (see [4], [5]). Consequently

$$(\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_n^{(j)}, p) = 1 \,, \quad j = 1, 2, \ldots \,. \tag{2}$$

Put $p^i \mu_r^{(i)} = \lambda_r^{(i+1)} - \lambda_r^{(i)}$, $r = 1, 2, \ldots, n$, $v_i = \sum_{r=1}^{n} \mu_r^{(i)} h_r$ and define groups $U = K \dotplus \sum_{i=1}^{\infty}{}_d \{a_i\}$, $V = \{u_1 - pa_1, v_i - pa_{i+1} + a_i, i = 1, 2, \ldots\}$, $W = \{u_i - p^i a_i, i = 1, 2, \ldots\}$.

If $h + \sum_{i=1}^{s} \lambda_i x_i = 0$, $h \in K$, then the multiplication by p^s gives $p \mid \lambda_s$. Consequently, $G = U/W$ is a mixed group with torsion part $T = V/W$ and $\bar{G} = G/T \cong U/V \cong H$, where the last isomorphism is induced by $h + \sum_{i=1}^{s} \lambda_i a_i \overset{\varphi}{\longmapsto} h + \sum_{i=1}^{s} \lambda_i x_i$, $h \in K$ (the equality $\operatorname{Ker} \varphi = V$ follows easily by induction on s). Moreover, G obviously satisfies Conditions (α), (β). Suppose that G splits, $G = T \dotplus S$. Then S is naturally isomorphic to H and it is easily seen that x_j corresponds to an element $y_j \in S$ of the form $y_j = a_j + \lambda_0(u_1 - pa_1) + \sum_{i=1}^{l} \lambda_i(v_i - pa_{i+1} + a_i) + W$. Further, if we denote by g_r the elements corresponding to h_r, then $mg_r = mh_r + W$, $r = 1, 2, \ldots, n$, for a suitable non-zero integer m. Consider the equality

$$p^j y_j = \sum_{r=1}^{n} \lambda_r^{(j)} g_r \,. \tag{1'}$$

Multiplying by m and comparing the coefficients we get

$$m\lambda_r^{(j)} = mp^j \lambda_0 \lambda_r^{(1)} + mp^j \sum_{i=1}^{l} \lambda_i \mu_r^{(i)} + \sum_{i=1}^{l} \eta_i p^i \lambda_r^{(i)} \,, \quad r = 1, 2, \ldots, n \,, \tag{3}$$

$$(4) \qquad 0 = mp^{j+1}\lambda_{i-1} - mp^j\lambda_i + \eta_i p^i, \quad i = 1, 2, \ldots, l, \quad i \neq j .$$

By (4), $p^j \mid \eta_i p^i$ for all $i = 1, 2, \ldots, l$ and consequently (3) yields $p^j \mid m\lambda_r^{(j)}$, $r = 1, 2, \ldots, n$. However, this contradicts (2), m being non-zero.

Lemma 3. *Let H be a torsionfree group of rank n containing a regular subgroup K such that $H/K = \sum_{d, p_i \in \pi'} C(p_i)$, π' infinite. Then there exists a mixed group G satisfying Conditions (α), (β), $\bar{G} \cong H$ and G does not split.*

Proof. Let $h_1, \ldots, h_n$ be a basis of K. For every $p_i \in \pi'$ select an element $x_i \in H \dot{-} K$ in the p_i-pure closure of $\sum_{d, i=1}^{n}\{h_i\}$ in H. Then $H = \{K, x_1, x_2, \ldots\}$, $p_i x_i = u_i \in K$ and

$$(5) \qquad p_i^{s_i} u_i = \sum_{r=1}^{n} \lambda_r^{(i)} h_r$$

for some integers s_i, $\lambda_r^{(i)}$, $r = 1, 2, \ldots, n$ satisfying

$$(6) \qquad (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_n^{(i)}, p_i) = 1 .$$

Define groups $U = K \dot{+} \sum_{d, i=1}^{\infty}\{a_i\}$, $V = \{u_i - p_i a_i,\ i = 1, 2, \ldots\}$, $W = \{p_i \sum_{r=1}^{n} \lambda_r^{(i)} h_r - p_i^{s_i+2} a_i,\ i = 1, 2, \ldots\}$. Then $G = U/W$ is a mixed group with torsion part $T = V/W$ and $\bar{G} = G/T \cong U/V \cong H$, where the last isomorphism is induced by $h + \sum_{i=1}^{m} \lambda_i a_i \mapsto h + \sum_{i=1}^{m} \lambda_i x_i$, $h \in K$. G satisfies Conditions (α), (β), since K is regular in H. Suppose that G splits, $G = T \dot{+} S$. Then S is naturally isomorphic to H and it is easily seen that x_i corresponds to an element $y_i \in S$ of the form $y_i = a_i + \sum_{j=1}^{l_i} \lambda_j(u_j - p_j a_j) + W$. Further, if we denote by g_r the elements of S corresponding to h_r then for a suitable non-zero integer m it is $m g_r = \sum_{k=1}^{n} \varrho_k^{(r)} h_k + W$, $r = 1, 2, \ldots, n$. Now let $p_i \in \pi'$ be arbitrary and consider the equality

$$(5') \qquad p_i^{s_i+1} y_i = \sum_{r=1}^{n} \lambda_r^{(i)} g_r .$$

Multiplying by m we get $\sum_{r=1}^{n} \lambda_r^{(i)} \sum_{k=1}^{n} \varrho_k^{(r)} h_k = m p_i^{s_i+1}(a_i + \sum_{j=1}^{l} \lambda_j(u_j - p_j a_j)) + \sum_{j=1}^{l} \eta_j (p_j \sum_{k=1}^{n} \lambda_k^{(j)} h_k - p_j^{s_j+2} a_j)$ (both the sums on the right hand side are suitably enlarged by zeros, if necessary). If we put $\varrho = \prod_{j=1}^{l} p_j^{s_j}$ and $\bar{\varrho}_j = \varrho/p_j^{s_j}$, then

$$\bar{\varrho}_i \sum_{r=1}^{n} \lambda_r^{(i)} \sum_{k=1}^{n} \varrho_k^{(r)} h_k = m p_i \varrho a_i + m p_i \sum_{j=1}^{l} \lambda_j \Big(\bar{\varrho}_j \sum_{k=1}^{n} \lambda_k^{(j)} h_k - \varrho p_j a_j\Big) +$$

$$+ \bar{\varrho}_i \sum_{j=1}^{l} \eta_j \Big(p_j \sum_{k=1}^{n} \lambda_k^{(j)} h_k - p_j^{s_j+2} a_j\Big).$$

Consequently,

$$\bar{\varrho}_i \sum_{r=1}^{n} \lambda_r^{(i)} \varrho_k^{(r)} = m p_i \sum_{j=1}^{l} \lambda_j \bar{\varrho}_j \lambda_k^{(j)} + \bar{\varrho}_i \sum_{j=1}^{l} \eta_j p_j \lambda_k^{(j)}, \quad k = 1, 2, \ldots, n, \tag{7}$$

$$0 = m \varrho p_i p_j \lambda_j + \bar{\varrho}_i \eta_j p_j^{s_j+2}, \quad j \neq i, \quad j = 1, 2, \ldots, l. \tag{8}$$

By (8), $p_i \mid \eta_j p_j$ for all $j = 1, 2, \ldots, l$ and consequently (7) yields $p_i \,\Big|\, \sum_{r=1}^{n} \lambda_r^{(i)} \varrho_k^{(r)}$, $k = 1, 2, \ldots, n$. From this one easily derives that $p_i \mid D\lambda_r^{(i)}$, $r = 1, 2, \ldots, n$, where $D = \det(\varrho_k^{(r)})$ and hence $p_i \mid D$ owing to (6). So $D = 0$ which contradicts the linear independence of $g_1, \ldots, g_n$.

Lemma 4. *Let H be a torsionfree group of rank n containing a generalized regular subgroup K such that $H/K = \sum_{d\, p_i \in \pi'} C(p_i^\infty)$, π' infinite. Suppose that $r_p(H) = r(H[p^\infty])$ for all primes p. Then there exists a mixed group G satisfying Conditions (α), (β), $\bar{G} \cong H$ and G does not split.*

Proof. Let $h_1, \ldots, h_n$ be a basis of K. For every $p_i \in \pi'$ select an element $x_i \in H \dotminus K$ in the p_i-pure closure of $\sum_{d\, i=1}^{n} \{h_i\}$ in H in such a way that $p_i x_i = u_i \in K$, $h_{p_i}^H(x_i) = \infty$ and

$$p_i^{s_i} u_i = \sum_{r=1}^{n} \lambda_r^{(i)} h_r \tag{9}$$

for some integers s_i, $\lambda_r^{(i)}$, $r = 1, 2, \ldots, n$, satisfying

$$(\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_n^{(i)}, p_i) = 1. \tag{10}$$

Then $H = \{K, x_{ij}, i, j = 1, 2, \ldots\}$, where $x_{i1} = x_i$, $p_i x_{i,j+1} = x_{ij}$, $i, j = 1, 2, \ldots$. Define groups $U = K \dotplus \sum_{d\, i,j=1}^{\infty} \{a_{ij}\}$, $V = \{u_i - p_i a_{i1}, a_{ij} - p_i a_{i,j+1}, i, j = 1, 2, \ldots\}$, $W = \{p \sum_{r=1}^{n} \lambda_r^{(i)} h_r - p_i^{s_i+2} a_{i1},\ a_{ij} - p_i a_{i,j+1},\ i, j = 1, 2, \ldots\}$. Then $G = U/W$ is a mixed group with torsion part $T = V/W$ and $\bar{G} = G/T \cong U/V \cong H$, where the last isomorphism is induced by $h + \sum_{i=1}^{k} \sum_{j=1}^{l_i} \lambda_{ij} a_{ij} \mapsto h + \sum_{i=1}^{k} \sum_{j=1}^{l_i} \lambda_{ij} x_{ij}$, $h \in K$. G satisfies Conditions (α), (β) since K is a generalized regular subgroup of H. Suppose that G splits, $G = T \dotplus S$. Then S is naturally isomorphic to H and it is easily seen that x_{ij} corresponds to an element $y_{ij} \in S$ of the form $y_{ij} = a_{ij} + \sum_{r=1}^{k_{ij}} \lambda_r(u_r - p_r a_{r1}) + \sum_{s=1}^{l_{ijr}} \mu_{rs}(a_{rs} - p_r a_{r,s+1}))$. Further, if we denote by g_r the elements of S corresponding to h_r, then $mg_r = \sum_{k=1}^{n} \varrho_k^{(r)} h_k + W$, $r = 1, 2, \ldots, n$, where m is a suitable non-zero integer. Now let $p_i \in \pi'$ be arbitrary and consider the equality

$$(9') \qquad p_i^{s_i+1} y_{i1} = \sum_{r=1}^{n} \lambda_r^{(i)} g_r .$$

Multiplying by m we get

$$\sum_{r=1}^{r} \lambda_r^{(i)} \sum_{k=1}^{n} \varrho_k^{(r)} h_k = m p_i^{s_i+1}\big(a_{i1} + \sum_{j=1}^{s} (\lambda_j(u_j - p_j a_{j1}) + \sum_{k=1}^{l} \mu_{jk}(a_{jk} - p_j a_{j,k+1})) +$$
$$+ \sum_{j=1}^{s} (\eta_j(p_j \sum_{k=1}^{n} \lambda_k^{(j)} h_k - p_j^{s_j+2} a_{j1}) + \sum_{k=1}^{l} \varrho_{jk}(a_{jk} - p_j a_{j,k+1})) ,$$

where the sums on the right hand side are suitably enlarged by zeros, if necessary. If we put $\varrho = \prod_{j=1}^{s} p_j^{s_j}$ and $\bar{\varrho}_j = \varrho / p_j^{s_j}$ then

$$\bar{\varrho}_i \sum_{r=1}^{n} \lambda_r^{(i)} \sum_{k=1}^{n} \varrho_k^{(r)} h_k = m \varrho p_i a_{i1} + m p_i \sum_{j=1}^{s} \lambda_j \bar{\varrho}_j \sum_{k=1}^{n} \lambda_k^{(j)} h_k -$$
$$- m p_i \varrho \sum_{j=1}^{s} \lambda_j p_j a_{j1} + m p_i \varrho \sum_{j=1}^{s} \sum_{k=1}^{l} \mu_{jk}(a_{jk} - p_j a_{j,k+1}) +$$
$$+ \bar{\varrho}_i \sum_{j=1}^{s} (\eta_j (p_j \sum_{k=1}^{n} \lambda_k^{(j)} h_k - p_j^{s_j+2} a_{j1}) + \sum_{k=1}^{l} \varrho_{jk}(a_{jk} - p_j a_{j,k+1})) .$$

Thus

$$(11) \qquad \bar{\varrho}_i \sum_{r=1}^{n} \lambda_r^{(i)} \varrho_k^{(r)} = m p_i \sum_{j=1}^{s} \lambda_j \bar{\varrho}_j \lambda_k^{(j)} + \bar{\varrho}_i \sum_{j=1}^{s} \eta_j p_j \lambda_k^{(j)} , \quad k = 1\ 2\ \ldots, n ,$$

$$(12) \qquad 0 = -m p_i \varrho \lambda_j p_j + m p_i \varrho \mu_{j1} - \bar{\varrho}_i \eta_j p_j^{s_j+2} + \bar{\varrho}_i \varrho_{j1} , \quad j = 1, 2, \ldots, s , \quad j \neq i ,$$

$$(13) \qquad 0 = m p_i \varrho \mu_{jk} - m p_i \varrho p_j \mu_{j,k-1} + \bar{\varrho}_i \varrho_{jk} - \bar{\varrho}_i p_j \varrho_{j,k-1} ,$$

$j = 1, 2, \ldots, s$, $j \neq i$ $\quad k = 2\ \ldots, l + 1$, where we put

$$\mu_{j,l+1} = \varrho_{j,l+1} = 0 , \quad j = 1, 2, \ldots, s , \quad j \neq i .$$

By (13) we have $p_i \mid \varrho_{jl}$, then $p_i \mid \varrho_{j,l-1}, \ldots, p_i \mid \varrho_{j1}, j \neq i$. Now (12) yields $p_i \mid \eta_j p_j$,

$j = 1, 2, \ldots, s$ and then (11) gives $p_i \mid \sum_{r=1}^{n} \lambda_r^{(i)} \varrho_k^{(r)}$, $k = 1, 2, \ldots, n$. From this one easily derives that $p_i \mid D\lambda_r^{(i)}$, $r = 1, 2, \ldots, n$, where $D = \det(\varrho_k^{(r)})$, and hence $p_i \mid D$ in view of (10). Thus $D = 0$ which contradicts the linear independence of $g_1, \ldots, g_n$.

Lemma 5. *Let H be a purely finitely generated torsionfree group of rank n, $H = \sum_{i=1}^{m} \{g_i\}_*^H$. Then to every prime p there exists a linearly independent subset $g_1^{(p)}, \ldots, g_{l_p}^{(p)}$ of the set $\{g_1, \ldots, g_m\}$ such that $l_p \leqq n$ and the p-pure closures of the elements $g_1^{(p)}, \ldots, g_{l_p}^{(p)}$ together with $\{g_1, \ldots, g_m\}$ generate the p-pure closure of $\{g_1, \ldots, g_m\}$ in H.*

Proof. Let l_p be the smallest integer such that there are elements $g_1^{(p)}, \ldots, g_{l_p}^{(p)}$ in the set $\{g_1, \ldots, g_m\}$ having the property stated in Lemma 5. Suppose that $g_i^{(p)}$ are not independent, $\sum_{i=1}^{l_p} \lambda_i g_i^{(p)} = 0$, $\sum_{i=1}^{l_p} \lambda_i^2 > 0$. Without loss of generality we can assume that $h_p^H(\lambda_1 g_1^{(p)}) \leqq h_p^H(\lambda_2 g_2^{(p)}) \leqq \ldots \leqq h_p^H(\lambda_{l_p} g_{l_p}^{(p)})$. Let $p^l u_1 = g_1^{(p)}$, $\lambda_1 = p^k \lambda_1'$, $(\lambda_1', p) = 1$. Then there are u_i in H such that $p^{k+l} u_i = \lambda_i g_i^{(p)}$, $i = 2, \ldots, l_p$ and hence $\lambda_1' u_1 = -\sum_{i=2}^{l_p} u_i$. Finally, $p^l \alpha + \lambda_1' \beta = 1$ for some integers α, β and $u_1 = -\beta \sum_{i=2}^{l_p} u_i + \alpha g_1^{(p)}$ which contradicts the choice of l_p.

Proof of Theorem. (i) implies (ii): Let G be a mixed group with $\bar{G} \cong H$. If G splits then it satisfies Conditions (α), (β) by [1], Lemma 4. Conversely, let G satisfy Conditions (α), (β). By Proposition 1, $\bar{G}$ is purely finitely generated so that G contains elements $g_1, \ldots, g_m$ such that $\bar{G} = \sum_{i=1}^{m} \{\bar{g}_i\}_*^G$. In view of [1], Lemmas 1 and 3 we can assume that $\tau^G(g_i) = \tau^{\bar{G}}(\bar{g}_i)$, $i = 1, 2, \ldots, m$ and that the integers m_i corresponding to g_i under Condition (β) are equal to 1. Moreover, taking suitable multiples of g_i's, we can suppose that $g_1, \ldots, g_n$ are linearly independent and

$$g_i = \sum_{j=1}^{n} \lambda_j^{(i)} g_j\,, \quad (\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) = 1\,, \quad i = 1, 2, \ldots, m\,. \tag{14}$$

Consequently, as is easily seen, $F \cap T = 0$, where $F = \{g_1, \ldots, g_m\}$.

Let p be any prime. By Lemma 5 there is a linearly independent subset $g_1^{(p)}, \ldots \ldots, g_k^{(p)}, g_{k+1}^{(p)}, \ldots, g_{l_p}^{(p)}$ of the set $\{g_1, \ldots, g_m\}$ such that $g_i^{(p)}$, $i = 1, \ldots, k_p$ are of infinite p-height, $g_i^{(p)}$, $i = k_p + 1, \ldots, l_p$ are of finite p-height and the p-pure closures of the elements $\bar{g}_i^{(p)}$, $i = 1, \ldots, l_p$ together with $\bar{F}$ generate the p-pure closure of $\bar{F}$ in $\bar{G}$. For every $i = 1, 2, \ldots, k_p$ let $x_{ij}^{(p)}$ be a p-sequence of $g_i^{(p)}$ and for $i = k_p + 1, \ldots \ldots, l_p$ let $x_i^{(p)}$ be a solution of $p^{h_p{}^G(g_i{}^{(p)})} x = g_i^{(p)}$ in G. Now let A be the subgroup of G generated by F and all $x_{ij}^{(p)}, x_k^{(p)}$, $i = 1, 2, \ldots, k_p$, $k = k_p + 1, \ldots, l_p$, $j = 1, 2, \ldots, p$ a prime.

For $g \in G$ we have a finite expression

$$\bar{g} = \sum_{p \in \pi} \left(\sum_{i=1}^{k_p} \alpha_i^{(p)} \bar{x}_{ir_i(p)}^{(p)} + \sum_{k=k_p+1}^{l_p} \beta_k^{(p)} \bar{x}_k^{(p)} \right) + \sum_{i=1}^{n} \gamma_i \bar{g}_i$$

from which it follows immediately that

$$G = \{A, T\} . \tag{15}$$

If π' is any set of primes then an integer ϱ will be called *a π'-integer* if all prime divisors of ϱ lie in π'. Let π_1, π_2 be two disjoint sets of primes and let $g \in \{A, T_{\pi_2}\} \cap \cap T_{\pi_1}$, $g = \sum_{p \in \bar{\pi}} (\sum_{i=1}^{k_p} \alpha_i^{(p)} x_{ir_i(p)}^{(p)} + \sum_{k=k_p+1}^{l_p} \beta_k^{(p)} x_k^{(p)}) + \sum_{i=1}^{n} \gamma_i g_i + t$, $t \in T_{\pi_2}$, $\bar{\pi}$ finite. First, we shall show that we can assume $\bar{\pi} \subseteq \pi_1$ and $t = 0$. Suppose that $\bar{\pi} \cap \pi_2 \neq \emptyset$ and write g in the form $h_1 + h_2 + h_3 + t$, where h_1 is the above sum taken over $\bar{\pi} \cap \pi_2$, h_2 is the same sum taken over $\bar{\pi} \dot{-} \bar{\pi} \cap \pi_2$, $h_3 = \sum_{i=1}^{n} \gamma_i g_i \in F$. With respect to the choice of $x_{ij}^{(p)}, x_i^{(p)}$ and $g \in T_{\pi_1}$ there is a π_1-integer $\varrho \neq 0$ such that $\varrho g = 0$ and $\varrho h_2 \in F$ and hence $\varrho(h_1 + t) \in F$. On the other hand, for some π_2-integer $\sigma \neq 0$, $\sigma(h_1 + t) \in F$. Hence $(h_1 + t) \in F$, since $(\varrho, \sigma) = 1$, and g can be written in the desired form. If $\bar{\pi} \subseteq \pi_1$ then $g = h_1 + h_2 + t$, $h_2 \in F$. Then $\varrho h_1 \in F$, $\varrho g = 0$ for a non-zero π_1-integer ϱ. However, $\varrho t = -\varrho(h_1 + h_2) \in T \cap F = 0$ and hence $t = 0$, since $t \in T_{\pi_2}$.

So we have $g = \sum_{i=1}^{k_p} \alpha_i^{(p)} x_{ir_i(p)}^{(p)} + \sum_{k=k_p+1}^{l_p} \beta_k^{(p)} x_k^{(p)} + \sum_{q \in \bar{\pi}} (\sum_{i=1}^{k_q} \alpha_i^{(q)} x_{ir_i(q)}^{(q)} + \sum_{k=k_q+1}^{l_q} \beta_k^{(q)} x_k^{(q)}) + \sum_{i=1}^{n} \gamma_i g_i = h_1 + h_2 + f \in \{A, T_{\pi_2}\} \cap T_{\pi_1}$ where $\bar{\pi} \subset \{p, \bar{\pi}\} \subseteq \pi_1$ and $\bar{\pi} \neq \{p, \bar{\pi}\}$. Let $h_p^G(g_i^{(p)}) = s_i^{(p)}$, $i = k_p+1, \ldots, l_p$, $s = \max\{r_1^{(p)}, \ldots, r_{k_p}^{(p)}, s_{k_p+1}^{(p)}, \ldots, s_{l_p}^{(p)}\}$ and let ϱ be an $\bar{\pi}$-integer such that $\varrho h_2 \in F$. Then $p^s \varrho g \in F \cap T = 0$. Suppose that $g_i^{(p)} = g_{j_i}$, $i = 1, 2, \ldots, l_p$. Now $0 = p^s \varrho g = \varrho(\sum_{i=1}^{k_p} \alpha_i^{(p)} p^{s - r_i(p)} g_{j_i} + \sum_{i=k_p+1}^{l_p} \beta_i^{(p)} p^{s - s_i(p)} g_{j_i}) + p^s \sum_{i=1}^{n} \gamma_i g_i$ for suitable integers γ_i, $i = 1, 2, \ldots, n$. Using (14) and the linear independence of $g_1, \ldots, g_n$ we get

$$\varrho\left(\sum_{i=1}^{k_p} \alpha_i^{(p)} p^{s - r_i(p)} \lambda_k^{(j_i)} + \sum_{i=k_p+1}^{l_p} \beta_i^{(p)} p^{s - s_i(p)} \lambda_k^{(j_i)} \right) + p^s \gamma_k = 0 \,, \quad k = 1, 2, \ldots, n \,. \tag{16}$$

Consider the (n, m)-matrix $M = (\lambda_j^{(i)})$, $j = 1, \ldots, n$, $i = 1, \ldots, m$ and denote by $(D(i_1, \ldots, i_n)$ the determinant of the submatrix of M having the columns $i_1, \ldots, i_n$. Obviously, the linearly independent elements g_{j_i}, $i = 1, \ldots, l_p$ can be embedded into a linearly independent subset $\{g_{j_i}, i = 1, 2, \ldots, n\}$ of the set $\{g_1, \ldots, g_m\}$ and consequently $D = D(j_1, \ldots, j_n) \neq 0$.

Now let π_2 be the set of all primes dividing some non-zero $D(i_1, \ldots, i_n)$. Then π_2 is clearly finite and if we put $\pi_1 = \pi - \pi_2$ then $T = T_{\pi_1} \dotplus T_{\pi_2}$. Moreover,

$$(p, D) = 1 . \tag{17}$$

From (16), and (17) one easily derives that $p^s \mid \varrho D\alpha_i^{(p)} p^{s-r_i(p)}$, $i = 1, 2, \ldots, k_p$ and $p^s \mid \varrho D\beta_i^{(p)} p^{s-s_i(p)}$, $i = k_p + 1, \ldots, l_p$. Thus $p^{r_i(p)} \mid \alpha_i^{(p)}$, $i = 1, \ldots k_p$ and $p^{s_i(p)} \mid \beta_i^{(p)}$ and hence $h_1 \in F$. Repeating this argument we finally get $g \in F \cap T = 0$. Thus

$$G = \{A, T_{\pi_2}\} \dotplus T_{\pi_1} = G_1 \dotplus T_{\pi_1} . \tag{18}$$

However, if we take $\pi_1 = \{p\}$ in the above part, we can choose the elements $g_1, \ldots, g_n$ in such a way that $g_i^{(p)} = g_i$, $i = 1, 2, \ldots, l_p$ and consequently $D = D(1, 2, \ldots, n) = 1$. Then $G_1 = T_p \dotplus G_p$ for all $p \in \pi_2$ and so G_1 splits. Thus (15) and (18) complete the proof of this part.

(ii) implies (i): By Lemma 2, $r_p(H) = r(H[p^\infty])$ for every prime p. If H contains a generalized regular subgroup K such that H/K is a torsion with infinitely many non-zero primary components, then it obviously contains a generalized regular subgroup K such that either $H/K = \sum_{d,\, p_i \in \pi'} C(p_i)$ or $H/K = \sum_{d,\, p_i \in \pi'} C(p_i^\infty)$, π' infinite, and it suffices to use Lemmas 3 and 4.

References

[1] *L. Bican:* Mixed abelian groups of torsionfree rank one, Czech. Math. J. *20 (95)*, (1970), 232—242.

[2] *L. Bican:* A note on mixed abelian groups, Czech. Math. J. *21 (96)*, (1971), 413—417.

[3] *L. Fuchs:* Abelian groups, Budapest, 1958.

[4] *L. Procházka:* On p-rank of torsionfree abelian groups of finite rank, Czech. Math. J. *12 (87)*, (1962), 3—43, (Russian).

[5] *A. Malcev:* Torsionfree abelian groups of finite rank, Mat. Sb. *4 (46)*, (1938), 45—68, (Russian).

Author's address: 186 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).