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Probability Theory and Related Fields

Splitting Intervals II: Limit Laws for Lengths

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Summary. In the processes under consideration, a particle of size L splits with exponential rate L^{α}, $0<\alpha<\infty$, and when it splits, it splits into two particles of size LV and $L(1-V)$ where V is independent of the past with d.f. F on (0, 1). Let Z_t be the number of particles at time t and L_t the size of a randomly chosen particle. If $\alpha=0$, it is well known how the system evolves: $e^{-t}Z_t$ converges a.s. to an exponential r.v. and $-\log L_t \approx t + Ct^{1/2}X$ where X is a standard normal t.v. Our results for $\alpha>0$ are in sharp contrast. In "Splitting Intervals" we showed that $t^{-1/\alpha}Z_t$ converges a.s. to a constant $K>0$, and in this paper we show $-\log L_t = \frac{1}{\alpha}\log t + 0(1)$. We show that the empirical d.f. of the rescaled lengths, $Z_t^{-1}\sum I\{t^{1/\alpha}L_i \leqq \cdot\}$, converges a.s. to a non-degenerate limit depending on F that we explicitly describe. Our results with $\alpha=2/3$ are relevant to polymer degradation.

1. Introduction

We consider a class of Markov processes in which particles undergo binary splitting at a rate determined by their size. A particle of size L waits a mean $L^{-\alpha}$ exponential time and then splits into two particles of size LV and size $L(1-V)$. V is independent of the past with a fixed distribution F on (0, 1) and $0<\alpha<\infty$. This process with $\alpha=2/3$ has been used as a model for polymer degradation (Basedow, Ebert and Ederer, 1978), so for this and future applications it is of interest to find the limiting behavior of the particle sizes. Let Z_t be the number of particles at time t and L_t be the size of a particle picked at random. In "Splitting Intervals", we showed that $t^{-1/\alpha}Z_t$ converges almost surely to a constant $K>0$, and in this paper, we will show that $t^{1/\alpha}L_t$ con-

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verges weakly to a nondegenerate limit depending on F which we will explicitly describe.

This result is in sharp contrast to the situation when $\alpha=0$. In this case the particles always split at rate one so the number of particles is a binary Yule process and the logarithms of the particle sizes form a branching random walk. It is well known how the system evolves; $e^{-t}Z_t$ converges almost surely to a mean one exponential, and if $E(\log(V(1-V)))^2<\infty$, there is a constant C so that $(t+\log L_t)/Ct^{1/2}$ converges weakly to a standard normal distribution, i.e. $-\log L_t \approx t+Ct^{1/2}X$ where X is a standard normal random variable.

The difference between $\alpha=0$ and $\alpha>0$ is easy to explain. For $\alpha>0$, the logarithms of the particle sizes form a generalized branching random walk in which a particle at $x=-\log L$ splits at rate $e^{-\alpha x}=L^\alpha$. The exponential decay of the splitting rate slows down the growth of $-\log L_t$ and since particles that get ahead divide much more slowly, the particles stay close together. We will show that they stay so close in fact that $-\log L_t = \frac{1}{\alpha}\log t + 0(1)$!

To see that this is true at least in one case, consider what happens starting with one particle of size one with $\alpha=1$ and F uniform on (0, 1). The total splitting rate is always one so splits occur at the times of a Poisson process and a little thought reveals that when $Z_t=n+1$, the sizes have the same distribution as the spacings between the points in an iid sample of size n that is uniform on (0, 1). According to Blum (1955), if $L(n)$ is the length of a randomly chosen spacing from the $n+1$ spaces determined by an iid uniform sample, then $(n+1)L(n)$ converges weakly to the mean one exponential distribution. Since $Z_t/t\to 1$ a.s., we see that tL_t converges weakly to the mean one exponential distribution.

Blum's result is easy to prove. If $0=T_0<T_1<T_2\ldots$ are the arrival times in a rate one Poisson process, then it is well known that $\{T_1/T_{n+1}, T_2/T_{n+1}, \ldots, T_n/T_{n+1}\}$ has the same distribution as the order statistics from an iid sample of size n from the uniform distribution. By the strong law of large numbers, $T_{n+1}/(n+1)\to 1$ a.s., so it follows that

$$\frac{1}{n+1}\sum_{k=1}^{n+1} I_{\left\{(n+1)\left(\frac{T_k}{T_{n+1}}-\frac{T_{k-1}}{T_{n+1}}\right)>x\right\}} \to e^{-x} \quad \text{a.s.}$$

The above proof is quick, but other cases, even with $\alpha=1$, require drastically different methods. The first steps in developing our proof were taken in "Splitting Intervals", hereafter called SI, where we studied processes in which the unit interval undergoes subdivision according to the rules given above. From hereon we will work in the framework of SI. The initial particle is the interval [0, 1] which splits into subintervals. Z_t is the number of subintervals at time t and $\{L_i: 1\leqq i\leqq Z_t\}$ are the subinterval lengths at time t. When an interval of length L splits, it splits into a left interval of length LV and a right interval of length $L(1-V)$. A moment's reflection shows that the distribution of the length of a randomly chosen interval L_t is not changed if a coin is tossed to decide if, for each split, LV becomes the left subinterval or the right subinterval; i.e. the distribution of L_t depends on F only through $\bar{F}(x)=(F(x)+1-F(1-x-))/2$, the symmetrization of F about 1/2. For simplicity and with no real loss of generality, we will assume F is initially symmetric about 1/2. We also assume that F has a non-zero Lebesgue component. We will also consider the interval splitting process splitting with a different distribution $\hat{F}$. Sometimes $\hat{F}$ is computed from F and sometimes not. $\hat{F}$ is not assumed symmetric. The only assumption on $\hat{F}$ is that it does not concentrate on a set of the form $\{e^{-\lambda n}: n \geqq 1, \lambda > 0\}$. Another distribution function we will use frequently is $\hat{G}(x)=1-\hat{F}(e^{-x}-)$.

We are interested in the length of a randomly chosen interval, but first we will answer an easier question. What does a "tagged" interval look like? Specifically, what does the left interval look like? A little reflection will convince the reader that the left interval has a different distribution than a randomly chosen interval, so it is not clear we are moving in the right direction. However, we will find there is a relationship between the moments of the left interval and the moments of a randomly chosen interval.

Let $\hat{L}_t$ be the length of the left-most interval at time t when the splitting is generated by $\hat{F}$. The range of values $\{-\log \hat{L}_t: t \geqq 0\} = \{S_0, S_1, S_2, \ldots\}$ where $0 = S_0 < S_1 < S_2 \ldots$ are the arrival times in a renewal process with $P(S_{n+1} - S_n \leqq x) = \hat{G}(x)$. Given $\{S_n\}_{n=0}^{\infty}$, the holding times of $-\log \hat{L}_t$ at each S_n are mean $\exp(\alpha S_n)$ exponential and independent. If $\{\xi'_n\}_{n=0}^{\infty}$ is an independent sequence of mean one exponentials and $M(y) = \sup\{n \geqq 0: S_n \leqq -\log y\}$, then

$$P(\hat{L}_t < y) = P\left(\sum_{n=0}^{M(y)} \exp(\alpha S_n)\, \xi'_n \leqq t\right).$$

Turning the sum around so the big terms come first, we obtain

$$P(\hat{L}_t < y) = P\left(\sum_{m=0}^{M(y)} \exp(-\alpha T_m(y))\, \xi_m \leqq y^{\alpha} t\right)$$

where $T_m(y) = -\log y - S_{M(y)-m}$ and $\{\xi_m\}$ is a new iid sequence of mean one exponentials. Renewal theory tells us if $\hat{G}$ has finite mean $\hat{\mu}$, then $\{T_0(y), T_1(y), T_2(y), \ldots\}$ converges weakly to $\{T_0, T_1, T_2, \ldots\}$ as $y \to 0+$ where $0 < T_0 < T_1 < T_2 \ldots$ is the stationary renewal process generated by $\hat{G}$, i.e. $\{T_m\}_{m=0}^{\infty}$ has positive independent increments, T_0 has density function $\hat{\mu}^{-1}(1-\hat{G})$ and the remaining increments have distribution function $\hat{G}$. Replacing y by $xt^{-1/\alpha}$ and letting $t \to \infty$, we obtain

$$\lim_{t \to \infty} P(t^{1/\alpha} \hat{L}_t < x) = P(Y_\alpha < x^\alpha) \tag{1}$$

for

$$Y_\alpha = \sum_{m=0}^{\infty} \exp(-\alpha T_m)\, \xi_m.$$

What does this result for $t^{1/\alpha}\hat{L}_t$ say about $t^{1/\alpha}L_t$? Letting $\hat{m}(t,\beta)=E\hat{L}_t^\beta$, a simple computation shows that

$$\hat{m}'(t,\beta)=-\hat{m}(t,\beta)+\int_0^1 x^\beta \hat{m}(x^\alpha t,\beta)\,d\hat{F}(x)$$
$$\hat{m}(0,\beta)=1$$

and if the splitting distribution is F, $L(t,\beta)=\sum_{i=1}^{Z_t} L_i^\beta$ and $m(t,\beta)=EL(t,\beta)$ then

$$m'(t,\beta)=-m(t,\beta)+\int_0^1 x^{\beta-1}\,m(x^\alpha t,\beta)\,2x\,dF(x)$$
$$m(0,\beta)=1$$

Since F is symmetric, $x\to\int_0^x 2y\,dF(y)$ defines a distribution function. If we let $\hat{F}$ be this distribution function and observe that

$$g'(t)=-g(t)+\int_0^1 x^\beta g(x^\alpha t)\,d\hat{F}(x)$$
$$g(0)=1$$

has a unique solution, then we see

(2) When $\hat{F}(x)=\int_0^x 2y\,dF(y)$,

$$\hat{m}(t,\beta-1)=m(t,\beta).$$

This equality, while extremely fortuitous, is not completely unprecedented. In the study of the voter model (Sudbury (1976), Kelly (1977), Sawyer (1979) and Bramson and Griffeath (1980)), a similar relationship exists between the size of the cluster at $0\in Z^d$ and the size of a randomly chosen cluster. In the voter model, one can deduce the distribution of a random cluster from the distribution of the cluster at 0, and in this paper we will use (2) to compute the limiting distribution of L_t from the limiting distribution of $\hat{L}_t$.

There are a lot of details left to verify, but (2) makes the path to our result fairly clear. We will need to identify the limit of $t^{\beta/\alpha}\hat{m}(t,\beta)$, so in Sect. 3 we will prove

(3) If $\int_0^1 x^\gamma\,d\hat{F}(x)<\infty$ and $\gamma<\beta$, then

$$\lim_{t\to\infty} t^{\gamma/\alpha}\hat{m}(t,\gamma)\quad\text{exists and }>0 \qquad \text{and}$$
$$\lim_{t\to\infty} t^{\beta/\alpha}\hat{m}(t,\beta)=EY_\alpha^{\beta/\alpha}.$$

To obtain an a.s. limit for $L(t,\beta)$, we will need an estimate for $\sigma(t,\beta)=\operatorname{Var}L(t,\beta)$. In Sect. 4 we will show

(4) $$\sigma(t,\beta)\leqq Ct^{2(1-\beta)/\alpha-\theta}$$

as $t\to\infty$ where $C>0$ and $\theta>0$ are constants depending on α and F.

From (2), (3), (4) and the Borel-Cantelli lemma, we can conclude that for $\beta>0$

(5) $$\lim_{t\to\infty} t^{(\beta-1)/\alpha} L(t, \beta) = E Y_\alpha^{(\beta-1)/\alpha} \quad \text{a.s.}$$

Using (5) and the method of moments, we will obtain our main result.

(6) **Theorem.** *For the α-splitting process with Z_t intervals of lengths $\{L_i\colon 1 \leqq i \leqq Z_t\}$ at time t, $\alpha>0$ and Y_α defined by* (1), *we have*

$$\lim_{t\to\infty} t^{-1/\alpha} Z_t = E Y_\alpha^{-1/\alpha} \quad \text{a.s.}$$

and

$$\lim_{t\to\infty} \frac{1}{Z_t} \sum_{i=1}^{Z_t} I_{\{t^{1/\alpha} L_i \leqq x\}} = \int_0^x \frac{1}{y E Y_\alpha^{-1/\alpha}} P(Y_\alpha^{1/\alpha} \in dy) \quad \text{a.s.}$$

In Sect. 2 we will compute the moments of Y_α. When the splitting is uniform, $F(x)=x$ and $\hat{F}(x)=x^2$, we can recognize the moments of Y_α as those of a gamma distribution with parameter $2/\alpha$,

$$P(Y_\alpha \in dy) = \frac{1}{\Gamma(2/\alpha)} y^{2/\alpha-1} e^{-y} dy.$$

Substituting this result into (6), we obtain

(7) **Corollary.** *When the splitting is uniform,*

$$\lim_{t\to\infty} t^{-1/\alpha} Z_t = \frac{\Gamma(1/\alpha)}{\Gamma(2/\alpha)} \quad \text{a.s.}$$

and

$$\lim_{t\to\infty} \frac{1}{Z_t} \sum_{i=1}^{Z_t} I_{\{t^{1/\alpha} L_i \leqq x\}} = \int_0^x \frac{\alpha}{\Gamma(1/\alpha)} e^{-y^\alpha} dy \quad \text{a.s.}$$

It is interesting to note that this family of densities has been proposed, both on theoretical and empirical grounds, as the density function for the size distribution of crushed coal (Bennett 1936), and in that context is called Rosin's law of size distribution. α is called the Rosin number. Coal with a low Rosin number tends to crumble into powder and many small pieces when crushed, and coal with a high Rosin number crushes into pieces of roughly uniform size. (Note that the distribution in (7) approaches the uniform distribution on $(0,1)$ as $\alpha\to\infty$ and see Pyke (1980) and Brennan (1986) for a related result.)

2. Some Properties of Y_α

The following notation holds throughout this section: $\hat{G}$ is a distribution function on $(0,\infty)$ with finite mean $\hat{\mu}$ and with Laplace transform $\hat{g}(\lambda)$

$=\int_0^\infty e^{-\lambda y}\,d\hat{G}(y)$. $\{X_i\}_{i=1}^\infty$ are iid with distribution function $\hat{G}$, $S_0=0$ and $S_n=X_1+X_2+\ldots+X_n$. $0<T_0<T_1<T_2<\ldots$ is a stationary renewal process generated by $\hat{G}$, meaning $\{T_m\}_{m=0}^\infty$ has independent increments, T_0 has density function $\hat{\mu}^{-1}(1-\hat{G})$ and the remaining increments have distribution function $\hat{G}$. We will now look in some detail at the properties of Y_α defined by

$$Y_\alpha=\sum_{m=0}^\infty \exp(-\alpha T_m)\,\xi_m$$

where $\{\xi_m\}_{m=0}^\infty$ is an independent sequence of mean one exponential random variables.

At first glance, this definition seems formidable for computation, but we can decompose

$$Y_\alpha \overset{d}{=} \exp(-\alpha T_0)\,Z_\alpha \tag{1}$$

where T_0 and Z_α are independent and $Z_\alpha \overset{d}{=} \sum_{n=0}^\infty \exp(-\alpha S_n)\,\xi_n$. In turn

$$Z_\alpha \overset{d}{=} \exp(-\alpha X_1)\,Z_\alpha+\xi_0 \tag{2}$$

where the random variables on the right side are independent. Random variables satisfying equations like (2) have been extensively studied. (See Vervaat (1979) for a survey, our problem is Example 3.8.)

We can compute with (1) and (2)

$$EY_\alpha=\frac{1}{\alpha\hat{\mu}} \tag{3}$$

and for $k\geqq 2$,

$$EY_\alpha^k=\frac{(k-1)!}{\alpha\hat{\mu}}\prod_{j=1}^{k-1}\frac{1}{1-\hat{g}(j\alpha)}.$$

Proof. From (2),

$$EZ_\alpha=\hat{g}(\alpha)\,EZ_\alpha+1$$

so

$$EZ_\alpha=1/(1-\hat{g}(\alpha)).$$

This procedure can clearly be used to compute all the moments of Z_α. Computing for $k=2$, 3 and 4, we find the pattern,

$$EZ_\alpha^k=k!\prod_{j=1}^{k}\frac{1}{1-\hat{g}(j\alpha)}$$

which we will verify by induction. From (2),

$$EZ_\alpha^k=E((\exp(-\alpha X_1)\,Z_\alpha+\xi_0)^k)=\sum_{j=0}^{k}\frac{k!}{j!}\,\hat{g}(j\alpha)\,EZ_\alpha^j.$$

Solving for EZ_α^k and using the induction hypothesis, we obtain

$$EZ_\alpha^k = \frac{k!}{1-\hat{g}(k\alpha)}\left(1+\sum_{j=1}^{k-1}\hat{g}(j\alpha)\prod_{i=1}^{j}\frac{1}{1-\hat{g}(i\alpha)}\right)$$
$$=\frac{k!}{1-\hat{g}(k\alpha)}\left(\frac{1}{1-\hat{g}(\alpha)}+\sum_{j=2}^{k-1}\hat{g}(j\alpha)\prod_{i=1}^{j}\frac{1}{1-\hat{g}(i\alpha)}\right)$$
$$=\frac{k!}{1-\hat{g}(k\alpha)}\frac{1}{1-\hat{g}(\alpha)}\left(1+\sum_{j=2}^{k-1}\hat{g}(j\alpha)\prod_{i=2}^{j}\frac{1}{1-\hat{g}(i\alpha)}\right).$$

Repeating the last two manipulations $k-2$ more times, we get

$$EZ_\alpha^k = k!\prod_{j=1}^{k}\frac{1}{1-\hat{g}(j\alpha)}.$$

Since

$$E(\exp(-k\alpha T_0)) = \frac{1}{\hat{\mu}}\int_0^\infty e^{-k\alpha y}(1-\hat{G}(y))\,dy = \frac{1-\hat{g}(k\alpha)}{k\alpha\hat{\mu}}$$

we obtain from (1),

$$EY_\alpha^k = E(\exp(-k\alpha T_0))\,EZ_\alpha^k = \frac{(k-1)!}{\alpha\hat{\mu}}\prod_{j=1}^{k-1}\frac{1}{1-\hat{g}(j\alpha)}.$$

As a corollary of (3), we have

$$EY_\alpha^k \leqq \frac{k!}{\alpha\hat{\mu}(1-\hat{g}(\alpha))^k} \tag{4}$$

so it follows from Carleman's condition (Chung (1974), p. 98) that Y_α is determined by its moments.

To deduce (1.7) from (1.6) in Sect. 1, we needed to identify the distribution of Y_α when $\hat{F}(x)=x^2$ and $\hat{G}(x)=1-\hat{F}(e^{-x})=1-e^{-2x}$. $\hat{G}$ is exponential with parameter 2 so $\hat{\mu}=1/2$ and $\hat{g}(\lambda)=2/(2+\lambda)$. Substituting into (3), we have

$$EY_\alpha^k = \frac{2}{\alpha}\prod_{j=1}^{k-1}\frac{j}{1-\dfrac{2}{2+j\alpha}} = \prod_{j=0}^{k-1} 2/\alpha + j$$

which are the moments of a gamma distribution with parameter $2/\alpha$, i.e.

$$P(Y_\alpha \in dy) = \frac{1}{\Gamma(2/\alpha)}\,y^{2/\alpha-1}\,e^{-y}\,dy.$$

Our last result of this section is to obtain Y_α as the weak limit of a related collection of random variables. We let $N(z)=\sup\{n\geqq 0: S_n\leqq z\}$ and $T_m(z)=z-S_{N(z)-m}$. ($T_m(\cdot)$ is defined slightly differently in this section than in Sect. 1.) We define

$$Y_\alpha(z) = \sum_{m=0}^{N(z)}\exp(-\alpha T_m(z))\,\xi_m. \tag{5}$$

Denoting convergence in distribution as $\Rightarrow$, we have

$$Y_\alpha(z) \Rightarrow Y_\alpha \quad \text{as} \quad z \to \infty \tag{6}$$

and for $\beta \geqq 0$,

$$\lim_{z\to\infty} E Y_\alpha^\beta(z) = E Y_\alpha^\beta.$$

Proof. By (4), it suffices to show that the integer moments of $Y_\alpha(z)$ converge to those of Y_α. We let $R_k(z) = E Y_\alpha^k(z)$ for $k \geqq 0$ and $\Phi_k(\lambda) = \int_0^\infty e^{-\lambda z} R_k(z)\, dz$ and observe that

$$Y_\alpha(z) \overset{d}{=} e^{-\alpha z}\xi + Y_\alpha(z - X_1)\, I_{\{X_1 \leqq z\}} \tag{7}$$

where the three random variables on the right side are mutually independent, ξ is a mean one exponential and X_1 has distribution function $\hat{G}$.

Using induction, we will show for $k \geqq 1$

$$R_k(z) = k e^{-\alpha z} R_{k-1}(z) + \int_0^z R_k(z-y)\, d\hat{G}(y), \tag{8a}$$

$$\Phi_k(\lambda) = \frac{k!}{\lambda + k\alpha} \prod_{j=0}^{k-1} \frac{1}{1 - \hat{g}(\lambda + j\alpha)}, \tag{8b}$$

$$\lim_{z\to\infty} R_k(z) = E Y_\alpha^k. \tag{8c}$$

Taking expectations on (7), we obtain the renewal equation

$$R_1(z) = e^{-\alpha z} + \int_0^z R_1(z-y)\, d\hat{G}(y).$$

Taking the Laplace transform of this equation gives

$$\Phi_1(\lambda) = \frac{1}{\lambda + \alpha} + \Phi_1(\lambda)\, \hat{g}(\lambda)$$

so

$$\Phi_1(\lambda) = \frac{1}{\lambda + \alpha} \frac{1}{1 - \hat{g}(\lambda)}.$$

Applying the renewal theorem (Feller (1971), p. 363), we obtain

$$\lim_{z\to\infty} R_1(z) = \frac{1}{\hat{\mu}} \int_0^\infty e^{-\alpha z}\, dz = \frac{1}{\alpha\hat{\mu}} = E Y_\alpha.$$

Using (7) again,

$$\begin{aligned} R_k(z) &= E((e^{-\alpha z}\xi + Y_\alpha(z - X_1)\, I_{\{X_1 \leqq z\}})^k) \\ &= e^{-\alpha k z} E\xi^k + \sum_{j=1}^k \binom{k}{j} e^{-\alpha(k-j)} E\xi^{k-j} \int_0^z R_j(z-y)\, d\hat{G}(y). \end{aligned}$$

Applying the induction hypothesis, we have

$$R_k(z)=\sum_{j=0}^{k-1}\frac{k!}{j!}e^{-\alpha(k-j)}(R_j(z)-je^{-\alpha z}R_{j-1}(z))+\int_0^z R_k(z-y)\,d\hat{G}(y)$$
$$=ke^{-\alpha z}R_{k-1}(z)+\int_0^z R_k(z-y)\,d\hat{G}(y).$$

Taking the Laplace transform gives

$$\Phi_k(\lambda)=\frac{k\Phi_{k-1}(\lambda+\alpha)}{1-\hat{g}(\lambda)}=\frac{k!}{\lambda+k\alpha}\prod_{j=0}^{k-1}\frac{1}{1-\hat{g}(\lambda+j\alpha)}$$

and applying the renewal theorem,

$$\lim_{z\to\infty}R_k(z)=\frac{k}{\hat{\mu}}\int_0^\infty e^{-\alpha z}R_{k-1}(z)\,dz=\frac{k}{\hat{\mu}}\Phi_{k-1}(\alpha)$$
$$=\frac{(k-1)!}{\alpha\hat{\mu}}\prod_{j=1}^{k-1}\frac{1}{1-\hat{g}(j\alpha)}=EY_\alpha^k.$$

3. The Left-most Interval $\hat{L}_t$

In this section we will analyze the left-most interval $\hat{L}_t$ in the splitting process generated by $\hat{F}$. We will show that $t^{1/\alpha}\hat{L}_t\Rightarrow Y_\alpha^{1/\alpha}$ and find conditions on β so that $t^{\beta/\alpha}E\hat{L}_t^\beta\to EY_\alpha^{\beta/\alpha}$ as $t\to\infty$. Throughout this section, we assume $\int_0^1 -\log x\,d\hat{F}(x)<\infty$ so that $\hat{G}(x)=1-\hat{F}(e^{-x}-)$ has finite mean $\hat{\mu}$.

The first result we prove is

(1) $$t^{1/\alpha}\hat{L}_t\Rightarrow Y_\alpha^{1/\alpha}\quad\text{as}\quad t\to\infty.$$

Proof. The range of values $\{-\log\hat{L}_t\colon t\geqq 0\}=\{0=S_0<S_1<S_2<\dots\}$ are the arrival times in a renewal process with $P(S_{n+1}-S_n\leqq x)=\hat{G}(x)=1-\hat{F}(e^{-x}-)$. Given $\{S_n\}_{n=0}^\infty$, the holding times of $-\log\hat{L}_t$ at each S_n are mean $\exp(\alpha S_n)$ exponential and independent. If $\{\xi_n'\}_{n=0}^\infty$ is an independent sequence of mean one exponentials and $M(y)=\sup\{n\geqq 0\colon S_n\leqq -\log y\}$, then

$$P(\hat{L}_t<y)=P\left(\sum_{n=0}^{M(y)}\exp(\alpha S_n)\,\xi_n'\leqq t\right).$$

Rearranging the sum so the big terms come first, we have

$$P(\hat{L}_t<y)=P\left(\sum_{m=0}^{M(y)}\exp(-\alpha T_m(-\log y))\,\xi_m\leqq y^\alpha t\right)$$
$$=P(Y_\alpha(-\log y)\leqq y^\alpha t)$$

where $\{\xi_m\}_{m=0}^\infty$ is a new iid sequence of mean one exponentials and $T_m(\cdot)$ and $Y_\alpha(\cdot)$ were defined in Sect. 2 (see (2.5)). From the equation above and (2.6), we have

$$\lim_{t\to\infty} P(t^{1/\alpha}\hat{L}_t < y) = \lim_{t\to\infty} P\left(Y_\alpha\left(-\log y + \frac{1}{\alpha}\log t\right) \leqq y^\alpha\right) = P(Y_\alpha \leqq y^\alpha).$$

We now turn our attention to finding conditions on β so that $t^{\beta/\alpha} E\hat{L}_t^\beta \to EY_\alpha^{\beta/\alpha}$ as $t\to\infty$. (2.6) almost says this is true for $\beta>0$ and a proof for $\beta>0$ can probably be obtained from (2.6). However, we shall use a different approach that will also work for $\beta<0$.

Let T_1 be the time of the first split, X the size of $\hat{L}_t$ after the first split and $\hat{L}_1(\cdot)$ a copy of $\hat{L}_{(\cdot)}$ that is independent of T_1 and X, then

$$\hat{L}_t^\beta \stackrel{d}{=} \begin{cases} 1 & \text{for } 0 \leqq t < T_1 \\ X^\beta \hat{L}_1^\beta(X^\alpha(t-T_1)) & \text{for } T_1 \leqq t. \end{cases}$$

If $\int_0^1 x^\beta \, d\hat{F}(x) < \infty$, we can take expectations on both sides of this equation to obtain an integral equation for $\hat{m}(t,\beta) = E\hat{L}_t^\beta$. Differentiating the result, we get

$$\hat{m}'(t,\beta) = -\hat{m}(t,\beta) + \int_0^1 x^\beta \hat{m}(x^\alpha t, \beta)\, d\hat{F}(x) \tag{2}$$
$$\hat{m}(0,\beta) = 1.$$

We multiply both sides of (2) by $e^{\beta u}$ and make the change of variables $t = e^{\alpha u}$ and $y = -\log x$. Letting $g(u) = e^{\beta u}\hat{m}(e^{\alpha u},\beta)$ and $h(u) = e^{\beta u}\hat{m}'(e^{\alpha u},\beta)$, we obtain

$$g(u) = \int_0^\infty g(u-y)\, d\hat{G}(y) - h(u) \tag{3}$$

which is equation (2.5) of *SI*. Following the argument given in that paper up to (2.9), we find for $-\infty < T < \infty$

$$\lim_{u\to\infty} g(u) = \hat{\mu}^{-1}\int_0^\infty g(T-s)(1-\hat{G}(s))\, ds - \hat{\mu}^{-1}\int_T^\infty h(s)\, ds \tag{4}$$

provided the right side is not $\infty - \infty$. As $s\to\infty$, $g(T-s) \sim Ce^{-\beta s}$. (Here and below C denotes a positive constant whose value is unimportant and may change from line to line.) We have

$$\int_0^\infty e^{-\beta s}(1-\hat{G}(s))\, ds = \beta^{-1}\int_0^\infty (1-e^{-\beta y})\, d\hat{G}(y) = \beta^{-1}\left(1 - \int_0^1 x^\beta\, d\hat{F}(x)\right) < \infty$$

so the first term on the right of (4) is finite. To take care of the second term, we observe

$$\hat{m}'(t,\beta) = \hat{m}(t, \alpha+\beta)\int_0^1 (x^\beta - 1)\, d\hat{F}(x). \tag{5}$$

Proof. Differentiating (2), we find

$$\hat{m}''(t,\beta) = -\hat{m}'(t,\beta) + \int_0^1 x^{\alpha+\beta}\,\hat{m}'(x^\alpha t,\beta)\,d\hat{F}(x)$$
$$\hat{m}'(0,\beta) = \int_0^1 (x^\beta - 1)\,d\hat{F}(x)$$

so $\left(\int_0^1 (x^\beta-1)\,d\hat{F}(x)\right)^{-1}\hat{m}'(t,\beta)$ and $m(t,\alpha+\beta)$ both satisfy

$$h'(t) = -h(t) + \int_0^1 x^{\alpha+\beta}\,h(x^\alpha t)\,d\hat{F}(x)$$
$$h(0)=1.$$

Since this equation has a unique solution, (5) holds.

If $\beta<0$, then $\hat{m}'(t,\beta)>0$ by (5). Since $g(u)>0$ in (4), $\int_T^\infty h(s)\,ds=+\infty$ is not possible so $\lim_{t\to\infty} t^{\beta/\alpha}\hat{m}(t,\beta)$ exists and $\geqq E Y_\alpha^{\beta/\alpha}>0$ by Fatou's lemma.

If $\beta>0$, the situation is more difficult because (5) shows $h(u)<0$ and we need to show $-\int_T^\infty h(s)\,ds<\infty$. We observe if $\beta>0$, then $\hat{m}'(t,\beta)<0$ and increasing by (5) so

$$\hat{m}'(t,\beta)\,t/2 \geqq \hat{m}(t,\beta)-\hat{m}(t/2,\beta) \geqq -\hat{m}(t/2,\beta) \geqq -1$$
$$|\hat{m}'(t,\beta)| \leqq 2t^{-1}\,\hat{m}(t/2,\beta) \leqq 2t^{-1}. \tag{6}$$

If $I=\{\beta>0\colon \lim_{t\to\infty} t^{\beta/\alpha}\hat{m}(t,\beta)$ exists$\}$ is not empty, then I is an interval. I is not empty for if $0<\beta<\alpha$, then

$$-\int_T^\infty h(s)\,ds = -\int_T^\infty e^{\beta s}\,\hat{m}'(e^{\alpha s},\beta)\,ds \leqq 2\int_T^\infty e^{\beta s}\,e^{-\alpha s}\,ds<\infty$$

by (6). We next suppose $\nu=\sup I<\infty$ and let $\nu<\beta<\nu+\alpha$. Since $0<\beta-\alpha<\nu$, $\hat{m}(t,\beta-\alpha)\sim C t^{(\alpha-\beta)/\alpha}$. Applying (6), we see

$$|\hat{m}'(t,\beta-\alpha)| \leqq C t^{-\beta/\alpha}.$$

Replacing β by $\beta-\alpha$ in (5) gives

$$\hat{m}'(t,\beta-\alpha) = \hat{m}(t,\beta)\int_0^1 (x^{\beta-\alpha}-1)\,d\hat{F}(x)$$

so

$$\hat{m}(t,\beta) \leqq C t^{-\beta/\alpha}$$

and by (6)

$$|\hat{m}'(t,\beta)| \leqq C t^{-\beta/\alpha-1}$$

so

$$-\int_T^\infty h(s)\,ds \leqq C\int_T^\infty e^{\beta s}\,e^{-(\beta+\alpha)s}\,ds<\infty$$

which shows that $\nu<\infty$ is impossible.

If $\gamma<\beta<0$ with $\int_0^1 x^\gamma d\hat{F}(x)<\infty$ or $\beta\geqq 0$, then the set $\{t^{\beta/\alpha}\hat{L}_t^\beta: t\geqq 0\}$ is uniformly integrable so we have established

(7) If $\int_0^1 x^\gamma d\hat{F}(x)<\infty$ and $\gamma<\beta$, then

$$\lim_{t\to\infty} t^{\gamma/\alpha}\hat{m}(t,\gamma) \text{ exists and } \geqq EY_\alpha^{\gamma/\alpha}>0$$

and

$$\lim_{t\to\infty} t^{\beta/\alpha}\hat{m}(t,\beta)=EY_\alpha^{\beta/\alpha}.$$

4. The Length of a Randomly Chosen Interval

In this section we determine the asymptotic distribution of a randomly chosen interval from the α-splitting process generated by F. Z_t is the number of intervals at time t and $\{L_i: 1\leqq i\leqq Z_t\}$ are the interval lengths. For $\beta\geqq 0$, we set

$$L(t,\beta)=\sum_{i=1}^{Z_t} L_i^\beta.$$

If T is the time of the first split and X is the position of the first split point, then

$$L(t,\beta)\stackrel{d}{=}\begin{cases}1 & \text{for } 0\leqq t<T\\ X^\beta L_1(X^\alpha(t-T),\beta)+(1-X)^\beta L_2((1-X)^\alpha(t-T),\beta) & \text{for } T\leqq t\end{cases}$$

where $L_1(\cdot,\beta)$ and $L_2(\cdot,\beta)$ are independent, have the same distribution as $L(\cdot,\beta)$ and are independent of T and X.

Taking expectations on both sides of this equation, we obtain an integral equation for $m(t,\beta)=EL(t,\beta)$. Differentiating the result, we get

$$\begin{aligned} m'(t,\beta)&=-m(t,\beta)+\int_0^1 x^{\beta-1} m(x^\alpha t,\beta)\,2x\,dF(x)\\ m(0,\beta)&=1.\end{aligned}$$

Thoughout this section, we set $\hat{F}(x)=\int_0^x 2y\,dF(y)$. $\hat{F}$ is a distribution function since F is symmetric about 1/2. With this notation, we have

(1)
$$\begin{aligned} m'(t,\beta)&=-m(t,\beta)+\int_0^1 x^{\beta-1} m(x^\alpha t,\beta)\,d\hat{F}(x)\\ m(0,\beta)&=1.\end{aligned}$$

Comparing (1) with (3.2), we see

(2)
$$m(t,\beta)=\hat{m}(t,\beta-1).$$

$\int_0^1 x^{-1} d\hat{F}(x)=2$ so by (3.7), we can conclude

(3) $$\lim_{t\to\infty} t^{(\beta-1)/\alpha} m(t,\beta) = EY_\alpha^{(\beta-1)/\alpha} \quad \text{for} \quad \beta>0$$

and

$$\lim_{t\to\infty} t^{-1/\alpha} m(t,0) = K \geqq EY_\alpha^{-1/\alpha} > 0.$$

At the end of this section, we will show

(4) $$\operatorname{Var} L(t,\beta) = \sigma(t,\beta) \leqq C t^{2(\beta-1)/\alpha-\theta}$$

as $t\to\infty$ for $\theta>0$ depending on α and F. For each $\varepsilon>0$,

$$P\left(\left|\frac{L(t,\beta)}{m(t,\beta)}-1\right|>\varepsilon\right) \leqq \frac{\sigma(t,\beta)}{\varepsilon^2 m^2(t,\beta)} \leqq C t^{-\theta}.$$

If we let $\lambda\theta>1$, then by the Borel-Cantelli lemma

$$\lim_{n\to\infty} \frac{L(n^\lambda,\beta)}{m(n^\lambda,\beta)} = 1 \quad \text{a.s.}$$

If $\beta>1$ and $n^\lambda \leqq t \leqq (n+1)^\lambda$, then since $L(t,\beta)$ is decreasing,

$$\frac{L((n+1)^\lambda,\beta)}{m(n^\lambda,\beta)} \leqq \frac{L(t,\beta)}{m(t,\beta)} \leqq \frac{L(n^\lambda,\beta)}{m((n+1)^\lambda,\beta)}.$$

Since $m(t,\beta) \sim C t^{(1-\beta)/\alpha}$, $\lim_{n\to\infty} m((n+1)^\lambda,\beta)/m(n^\lambda,\beta)=1$, and we conclude

$$\lim_{t\to\infty} \frac{L(t,\beta)}{m(t,\beta)} = 1 \quad \text{a.s.}$$

If $0\leqq\beta<1$, $L(t,\beta)$ is increasing, and a similar argument leads to the same conclusion. We have shown for $\beta>0$

(5) $$\lim_{t\to\infty} t^{(\beta-1)/\alpha} L(t,\beta) = EY_\alpha^{(\beta-1)/\alpha} \quad \text{a.s.} \qquad \text{and}$$
$$\lim_{t\to\infty} t^{-1/\alpha} Z_t = K \geqq EY_\alpha^{-1/\alpha} > 0 \quad \text{a.s.}$$

We next let

$$H_t(\cdot) = \frac{1}{Z_t} \sum_{i=1}^{Z_t} I_{\{t^{1/\alpha} L_i \leqq \cdot\}}$$

denote the empirical distribution function of the rescaled lengths. We have for $\beta>0$

$$\lim_{t\to\infty} \int_0^\infty x^\beta \, dH_t(x) = \lim_{t\to\infty} \frac{1}{Z_t} \sum_{i=1}^{Z_t} (t^{1/\alpha} L_i)^\beta$$
$$= \lim_{t\to\infty} \frac{t^{(\beta-1)/\alpha} L(t,\beta)}{t^{-1/\alpha} Z_t} = K^{-1} EY_\alpha^{(\beta-1)/\alpha} \quad \text{a.s.}$$

Since the moments of $H_t(\cdot)$ converge almost surely, the set $\{H_t(\cdot)\}$ is almost surely tight, and if $H(\cdot)$ is any weak sequential limit point of $\{H_t(\cdot)\}$ and $\beta>0$, then

$$\int_0^\infty x^\beta\, dH(x) = K^{-1} E Y_\alpha^{(\beta-1)/\alpha} = \int_0^\infty x^\beta (xK)^{-1} P(Y_\alpha^{1/\alpha} \in dx). \tag{6}$$

Letting $\beta \to 0+$ in (6), we see that $K = \lim_{t\to\infty} t^{-1/\alpha} Z_t = E Y_\alpha^{-1/\alpha}$. The last step to establish

$$\lim_{t\to\infty} H_t(x) = \int_0^x \frac{1}{y E Y_\alpha^{-1/\alpha}} P(Y_\alpha^{1/\alpha} \in dy) \quad \text{a.s.}$$

is to show that the measure on the right side of (6) is determined by its moments.

(7) Suppose $\int_0^\infty x^\beta\, dH(x) = \int_0^\infty x^\beta (xK)^{-1} P(Y_\alpha^{1/\alpha} \in dx)$ for $\beta \geqq 0$, then

$$H(x) = \int_0^x (yK)^{-1} P(Y_\alpha^{1/\alpha} \in dy).$$

Proof. Taking $\beta = n\alpha$ where $n \geqq 0$ is an integer, we have

$$\int_0^\infty x^{n\alpha}\, dH(x) = \int_0^\infty x^{n\alpha} (xK)^{-1} P(Y_\alpha^{1/\alpha} \in dx)$$

$$\begin{aligned}\int_0^\infty z^n\, dH(z^{1/\alpha}) &= \int_0^\infty z^n z^{-1/\alpha} K^{-1} P(Y_\alpha \in dz)\\ &\leqq \int_0^1 z^{-1/\alpha} K^{-1} P(Y_\alpha \in dz) + K^{-1} \int_1^\infty z^n P(Y_\alpha \in dz)\\ &\leqq 1 + K^{-1} \int_0^\infty z^n P(Y_\alpha \in dz) \leqq A B^n n!\end{aligned}$$

by (2.4). By Carleman's condition (Chung (1974), p. 98), it follows that

$$H(z^{1/\alpha}) = \int_0^z y^{-1/\alpha} K^{-1} P(Y_\alpha \in dy)$$

or

$$H(x) = \int_0^{x^\alpha} y^{-1/\alpha} K^{-1} P(Y_\alpha \in dy) = \int_0^x (yK)^{-1} P(Y_\alpha^{1/\alpha} \in dy).$$

Having completed all the painless computations, our last remaining task is to obtain the upper bound (4) for $\sigma(t,\beta) = \operatorname{Var} L(t,\beta)$. A straightforward but somewhat tedious computation gives

$$\begin{aligned}\sigma'(t,\beta) = &-\sigma(t,\beta) + \int_0^1 x^{2\beta-1} \sigma(x^\alpha t,\beta)\, d\hat{F}(x) - 2m(t,\beta)\, m'(t,\beta)\\ &+ \int_0^1 f(x,t)\, h(x,t)\, dF(x)\end{aligned} \tag{8}$$

for $f(x,t)=x^\beta m(x^\alpha t,\beta)+(1-x)^\beta m((1-x)^\alpha t,\beta)+m(t,\beta)$ and $h(x,t)=x^\beta m(x^\alpha t,\beta)+(1-x)^\beta m((1-x)^\alpha t,\beta)-m(t,\beta)$.

We need bounds on the last two terms of (8). $|m(t,\beta)\,m'(t,\beta)|\sim Ct^{(1-2\beta)/\alpha}$ and $0<f(x,t)\leqq Ct^{(1-\beta)/\alpha}$ as $t\to\infty$. C in the last bound does not depend on x.

We next establish

$$\int_0^1 |h(x,t)|\,dF(x)\leqq Ct^{(1-\beta)/\alpha-\theta} \tag{9}$$

as $t\to\infty$ for a $\theta>0$ which depends on α and F.

Proof. The argument breaks into the cases: $m(t,\beta)$ is increasing, $0\leqq\beta<1$, and $m(t,\beta)$ is decreasing, $\beta>1$. When $\beta=1$, $m(t,\beta)\equiv 1$ and $h(x,t)=0$. The case $\beta=0$ is Lemma 2.5 of SI and the proof of that result applies to $0\leqq\beta<1$ with virtually no change so we will concentrate on the case $\beta>1$ and sketch the differences when $0\leqq\beta<1$.

We let $0<z<x<1$ and $X_1, X_2, X_3\ldots$ be i.i.d. with distribution function $\hat F$. $X_0=1$, $W_i=X_0X_1\ldots X_i$ and $J=\inf\{i\geqq 1: W_i<z\}$. Starting with

$$m(t,\beta)=\int_0^1 y^{\beta-1} m(y^\alpha t,\beta)\,d\hat F(y)-m'(t,\beta) \tag{10}$$

we iterate this equation $J-1$ times to obtain

$$m(t,\beta)=E(W_J^{\beta-1} m(W_J^\alpha t,\beta))-E\left(\sum_{i=0}^{J-1} W_i^{\beta-1} m'(W_i^\alpha t,\beta)\right).$$

(For more details, see (2.13) and then the argument leading to (2.7) of SI). By (2) and (3.5), $m'(t,\beta)$ is negative and increasing so

$$-m'(W_i^\alpha t,\beta)\leqq C(W_i^\alpha t)^{(1-\beta)/\alpha-1}\leqq Cz^{1-\beta-\alpha}t^{(1-\beta)/\alpha-1} \tag{11}$$

and

$$E\left(\sum_{i=0}^{J-1} W_i^{\beta-1}\right)=z^{\beta-1}E\left(\sum_{i=0}^{J-1}\exp((\beta-1)(-\log z-S_i))\right)=z^{\beta-1}R(z)$$

where $S_i=-\log W_i$. Renewal theory tells us that $\sup_{0<z<1} R(z)<\infty$ so we have verified

$$0\leqq m(t,\beta)-E(W_J^{\beta-1} m(W_J^\alpha t,\beta))\leqq Cz^{-\alpha}t^{(1-\beta)/\alpha-1}. \tag{12a}$$

Next, we repeat the above procedure with a new i.i.d. sequence $\tilde X_1$, $\tilde X_2$, $\tilde X_3\ldots$. Everything is a s above except $\tilde X_0=x$, then (10) is replaced by

$$x^{\beta-1} m(x^\alpha t,\beta)=\int_0^1 (yx)^{\beta-1} m((yx)^\alpha t,\beta)\,d\hat F(y)-x^{\beta-1} m'(x^\alpha t,\beta).$$

The last term in (11) becomes $z^{\beta-1}R(z/x)$ and (12a) remains true with W_J replaced by $\tilde W_{\tilde J}$ and $m(t,\beta)$ replaced by $x^{\beta-1}m(x^\alpha t,\beta)$. C on the right side of (12a) does not depend on x in this case.

When $\beta<1$ and $\alpha+\beta\geqq 1$, $m'(t,\beta)$ is positive and non-increasing and we have

$$0\leqq E(W_J^{\beta-1} m(W_J^\alpha t,\beta))-m(t,\beta)\leqq C z^{\beta-1} \tag{12b}$$

and when $0<\alpha+\beta<1$, $m'(t,\beta)$ is positive and increasing and

$$0\leqq E(W_J^{\beta-1} m(W_J^\alpha t,\beta))-m(t,\beta)\leqq C z^{\beta-1} m'(t,\beta) \tag{12c}$$

(12b, c) remain true when W_J is replaced by $\tilde{W}_{\tilde{J}}$ and $m(t,\beta)$ is replaced by $x^{\beta-1} m(x^\alpha t,\beta)$.

In SI, using a coupling of Ney (1981), we showed that the sequences X_0, X_1, $X_2\dots$ and $\tilde{X}_0$, $\tilde{X}_1$, $\tilde{X}_2\dots$ can be constructed on the same probability space along with a random variable τ so that

$$W_J I_{\{\tau\leqq -\log z\}} \overset{d}{=} \tilde{W}_{\tilde{J}} I_{\{\tau\leqq -\log z\}} \tag{13}$$

and

$$P(\tau>-\log z)\leqq C x^{-\gamma} z^\gamma$$

for $0<\gamma<1$ depending on F. To apply Ney's coupling, we use the assumption that F has a non-zero Lebesgue component.

For $\beta>1$, we have

$$\begin{aligned}
&|x^{\beta-1} m(x^\alpha t,\beta)-m(t,\beta)|\\
&\quad\leqq E(m(t,\beta)-W_J^{\beta-1} m(W_J^\alpha t,\beta))+E(x^{\beta-1} m(x^\alpha t,\beta)-\tilde{W}_{\tilde{J}}^{\beta-1} m(\tilde{W}_{\tilde{J}}^\alpha t,\beta))\\
&\qquad+|E(W_J^{\beta-1} m(W_J^\alpha t,\beta)-\tilde{W}_{\tilde{J}}^{\beta-1} m(\tilde{W}_{\tilde{J}}^\alpha t,\beta))|\leqq C z^{-\alpha} t^{(1-\beta)/\alpha-1}\\
&\qquad+E(|W_J^{\beta-1} m(W_J^\alpha t,\beta)-\tilde{W}_{\tilde{J}}^{\beta-1} m(\tilde{W}_{\tilde{J}}^\alpha t,\beta)|\, I_{\{\tau>-\log z\}})\\
&\quad\leqq C z^{-\alpha} t^{(1-\beta)/\alpha-1}+E(W_J^{\beta-1} m(W_J^\alpha t,\beta)\, I_{\{\tau>-\log z\}})\\
&\qquad+E(\tilde{W}_{\tilde{J}}^{\beta-1} m(\tilde{W}_{\tilde{J}}^\alpha t,\beta)\, I_{\{\tau>-\log z\}})
\end{aligned}$$

by (12a) and (13). $m(t,\beta)\leqq C t^{(1-\beta)/\alpha}$ so $x^{\beta-1} m(x^\alpha t,\beta)\leqq C t^{(1-\beta)/\alpha}$ independent of x and

$$E(W_J^{\beta-1} m(W_J^\alpha t,\beta)\, I_{\{\tau>-\log z\}})\leqq C x^{-\gamma} z^\gamma t^{(1-\beta)/\alpha}$$

by (13). A similar calculation holds for the $\tilde{W}$ term. We have

$$|x^{\beta-1} m(x^\alpha t,\beta)-m(t,\beta)|\leqq C(z^{-\alpha} t^{(1-\beta)/\alpha-1}+x^{-\gamma} z^\gamma t^{(1-\beta)/\alpha}).$$

Replacing z by $t^{-\rho}$ gives

$$|x^{\beta-1} m(x^\alpha t,\beta)-m(t,\beta)|\leqq C(t^{\rho\alpha-1}+x^{-\gamma} t^{-\rho\gamma})\, t^{(1-\beta)/\alpha}$$

and

$$\begin{aligned}
\int_0^1 |h(x,t)|\, dF(x)&\leqq 2\int_0^1 x\,|x^{\beta-1} m(x^\alpha t,\beta)-m(t,\beta)|\, dF(x)\\
&\leqq C(t^{\rho\alpha-1}+t^{-\rho\gamma})\, t^{(1-\beta)/\alpha}
\end{aligned}$$

(9) is now established for $\beta>1$ by taking $0<\rho<1/\alpha$.

For $0 \leqq \beta < 1$ and $\alpha + \beta \geqq 1$, (12b) and a similar argument lead to

$$\int_0^1 |h(x,t)|\, dF(x) \leqq C(t^{\rho(1-\beta)} + t^{-\rho\gamma+(1-\beta)/\alpha})$$

and (9) again is established by taking $0 < \rho < 1/\alpha$.

For $0 < \alpha + \beta < 1$, (12c) and a similar argument lead to

$$\int_0^1 |h(x,t)|\, dF(x) \leqq C(t^{\rho} m'(t,\beta) + t^{-\rho\gamma+(1-\beta)/\alpha}).$$

As $t \to \infty$, $m'(t,\beta) \sim C t^{(1-\beta)/\alpha - 1}$ so (9) follows by taking $0 < \rho < 1$.

Returning to the analysis of (8), we have shown that the last two terms in (8) are dominated by $C t^{2(1-\beta)/\alpha - \theta}$ as $t \to \infty$ and $0 < \theta < \min(1, 1/\alpha)$ depends on F and α. There are two cases to consider $0 \leqq \beta < 1$ and $1 < \beta$. (For $\beta = 1$, $L(t,\beta) \equiv 1$ and there is nothing to show.)

For $0 \leqq \beta < 1$, we can find a positive function, $B(\cdot)$, such that $B(t) \geqq -2m(t,\beta)\, m'(t,\beta) + \int_0^1 f(x,t)\, h(x,t)\, dF(x)$, $B'(t) > 0$ and $B(t) \sim C t^{2(1-\beta)/\alpha - \theta}$ as $t \to \infty$. By a simple comparison test $\sigma(t) \leqq w(t)$ if $w(t)$ satisfies

$$w'(t) = -w(t) + \int_0^1 x^{2\beta-1} w(x^{\alpha} t)\, d\hat{F}(x) + B(t) \tag{14}$$

with $w(0) > 0$ but small enough so that

$$w'(0) = w(0) \int_0^1 (x^{2\beta-1} - 1)\, d\hat{F}(x) + B'(0) > 0$$

$w'(t) > 0$ for if $\inf\{t: w'(t) = 0\} = s < \infty$, then

$$w''(s) = \int_0^1 x^{2\beta+\alpha-1} w'(x^{\alpha} s)\, d\hat{F}(x) + B'(s) > 0$$

which is a contradiction. Replacing $w'(t)$ by zero and making the substitutions $t = e^{\alpha u}$, $y = -\log x$, $g(u) = e^{(2\beta-1)u} w(e^{\alpha u})$ and $h(u) = e^{(2\beta-1)u} B(e^{\alpha u})$ in (14), we obtain

$$g(u) \leqq \int_0^{\infty} g(u-y)\, d\hat{G}(y) + h(u).$$

By the argument developed in SI and used to analyze (3.3) of this paper, we have

$$g(u) \leqq J(u) + H(u)$$

where

$$\lim_{u \to \infty} J(u) = \hat{\mu}^{-1} \int_0^{\infty} g(T-s)(1 - \hat{G}(s))\, ds < \infty \tag{15}$$

and

$$H(u) \sim \hat{\mu}^{-1} \int_T^u h(s)\, ds \sim C e^{(1-\alpha\theta)u} \quad \text{as} \quad u \to \infty.$$

This shows that $w(e^{\alpha u}) \leqq C e^{(2-2\beta-\alpha\theta)u}$ as $u \to \infty$ which in turn shows $\sigma(t) \leqq C t^{2(1-\beta)/\alpha-\theta}$ as $t \to \infty$. This completes the proof for the case $0 \leqq \beta < 1$.

For the case $1 < \beta$, we let $B(t)$ and $w(t)$ be as above except that $B'(t) < 0$ and $B''(t) > 0$.

$$w'(0) = w(0) \int_0^1 (x^{2\beta-1} - 1)\, d\hat{F}(x) + B(0)$$

and

$$w''(0) = w'(0) \int_0^1 (x^{2\beta+\alpha-1} - 1)\, d\hat{F}(x) + B'(0)$$

so $w'(0) < 0$ and $w''(0) > 0$ if $w(0)$ is taken sufficiently large. Using the simple argument by contradiction given above, we conclude that $w'(t) < 0$ and $w''(t) > 0$ for $t \geqq 0$. By the same argument that led to (3.6), we have

$$-w'(t) = |w'(t)| \leqq 2t^{-1} w(t/2) \leqq C t^{-1}. \tag{16}$$

We next make the substitutions $t = e^{\alpha u}$, $y = -\log x$, $g(u) = e^{(2\beta-1)u} w(e^{\alpha u})$, $h(u) = e^{(2\beta-1)u} B(e^{\alpha u})$ and $k(u) = -e^{(2\beta-1)u} w'(e^{\alpha u})$ in (14) and obtain

$$g(u) = \int_0^\infty g(u-y)\, d\hat{G}(y) + h(u) + k(u)$$

then

$$g(u) = J(u) + H(u) + K(u)$$

where $J(u)$ and $H(u)$ satisfy (15) and

$$K(u) \sim \hat{\mu}^{-1} \int_T^u e^{(2\beta-1)s} |w'(e^{\alpha s})|\, ds \quad \text{as} \quad u \to \infty.$$

We are done if we can show

$$e^{(2\beta-1)s} |w'(e^{\alpha s})| \leqq C e^{(1-\alpha\theta)s} \quad \text{as} \quad s \to \infty. \tag{17}$$

By (16), $|w'(e^{\alpha s})| \leqq C e^{-\alpha s}$ so (17) holds if $1 < \beta$ and $(2\beta - 1) - \alpha \leqq 1 - \alpha\theta$. Thus, (17) holds for $1 < \beta \leqq 1 + \frac{\alpha}{2}(1-\theta)$.

If $1 < \beta_1 < \beta_2$ and $w(\cdot, \beta_i)$ and $B(\cdot, \beta_i)$, $i = 1$ and 2, denote the corresponding functions satisfying (14), and if $w(0, \beta_2) < w(0, \beta_1)$ and $B(t, \beta_2) < B(t, \beta_1)$, then a comparison test shows $w(t, \beta_2) \leqq w(t, \beta_1)$ for $t \geqq 0$. So if $\beta > 1 + \frac{\alpha}{2}(1-\theta)$, then

$$w(t, \beta) \leqq w\left(t, 1 + \frac{\alpha}{2}(1-\theta)\right) \sim C t^{-1}$$

and

$$|w'(t,\beta)| \leqq C t^{-2}$$

by (16). So (17) holds if $\beta > 1 + \frac{\alpha}{2}(1-\theta)$ and $(2\beta-1) - 2\alpha \leqq 1 - \alpha\theta$. Thus, (17) holds for $1 < \beta \leqq 1 + \frac{\alpha}{2}(2-\theta)$. Clearly (17) holds for $\beta > 1$ by induction.

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