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Splitting of certain spaces CX

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There are a number of theorems to the effect that spaces of the form $\Omega^n\Sigma^n X$ split stably into wedges of simpler spaces when X is connected (by which we mean pathwise connected). The proofs generally proceed by exploitation of combinatorially manageable approximations for $\Omega^n\Sigma^n X$.

The earliest theorem of this kind is due to Milnor (18), who exploited the semi-simplicial version of the James construction on X to split $\Sigma\Omega\Sigma X$ into $\bigvee_{q\geq 1} \Sigma X^{[q]}$, where $X^{[q]}$ denotes the q-fold smash product of X with itself.

More recently, Kahn (10) proved that the suspension spectrum of $QX = \varinjlim \Omega^n\Sigma^n X$ splits as the wedge of the suspension spectra of the extended powers

$$D_q X = E\Sigma_i^+ \wedge_{\Sigma_q} X^{[q]},$$

where Σ_q is the symmetric group on q letters, $E\Sigma_q$ is a contractible space on which Σ_q acts freely, and Y^+ denotes the union of a space Y and a disjoint basepoint. He exploited Barratt's semisimplicial approximation to QX, and a proof of this splitting has also been given by Barratt and Eccles (1).

A bit later, Snaith (22) proved that the suspension spectrum of $\Omega^n\Sigma^n X$ for $1 \leqslant n$ splits as the wedge of the suspension spectra of the extended powers

$$D_{n,q} X = \mathscr{C}_{n,q}^+ \wedge_{\Sigma_q} X^{[q]},$$

where $\mathscr{C}_{n,q}$ is Boardman and Vogt's space of q-tuples of little n-cubes disjointly embedded in R^n. He exploited May's approximation $C_n X$ to $\Omega^n\Sigma^n X$. $C_n X$ comes with a filtration, and Snaith also proved that $\Sigma^t F_r C_n X$ splits as the wedge of the $\Sigma^t D_{n,q} X$ for $1 \leqslant q \leqslant r$ and a certain t depending on r and n. Snaith's stable splittings were later rederived, in the general context of operads, by Reedy (20).

These splittings have had a number of applications in homotopy theory. In particular, Mahowald (12) has recently made striking use of very special cases of Snaith's splittings.

The proofs of these splittings generally involve rather complicated combinatorial arguments. It is one purpose of the present paper to give transparently elementary proofs of all these results (modulo the approximation theorem for $\Omega^n\Sigma^n X$). The simplicity of our construction of the requisite splitting maps makes these decompositions considerably easier to work with. For example, the first author has obtained commutative diagrams exhibiting the relationship between the splitting maps and the actions of the little cubes operads on the spaces involved and the composition and smash products. Such explicit data together with the methods in (3) ought to lead to new calculations in homotopy theory.

Actually, the splittings of iterated loop spaces we have been discussing are only corollaries of the very general splitting theorems we shall obtain. The general theorems do not require X to be connected and do not depend on the approximation theorem. We are particularly interested in the resulting stable splittings of certain spaces $C(Y, X)$ built up from the configuration spaces of any space Y and the powers of any based space X. As explained in (5), such spaces play a key role in the calculation of Gelfands–Fuks cohomology and the cohomology of function spaces.

The splittings of Milnor and Barratt–Eccles were proved in a parametrized form, as splittings of $\Sigma(P \wedge \Omega\Sigma X)$ and the suspension spectrum of $P \wedge QX$ for an arbitrary based parameter space P and not just $P = S^0$. Such parametrized splittings will also be immediate corollaries of our general theorems.

We set up our basic definitional framework and collect various elementary ingredients needed for proofs in the first two sections. In Section 1 we introduce the notions of a 'coefficient system' $\mathscr{C}$ and of a 'Π-space' $\mathbf{X}$. The former is just a collection of spaces $\mathscr{C}_q$ related by degeneracy and permutation operations. The latter is just a sequence of spaces X_q having the same formal properties as the sequence of powers X^q of a based space X. We display a number of examples, but our lists are far from exhaustive. We shall pay particular attention to coefficient systems of configuration spaces, perhaps the main observation of the paper being that, quite aside from their intrinsic interest, these spaces provide a very convenient setting for the construction of the generalized James–Hopf invariant maps needed for the splitting theorems.

In Section 2 we associate a functor C on Π-spaces to any coefficient system $\mathscr{C}$ and discuss a number of examples, the most familiar one being the James construction M. While some of these functors C are related to iterated loop spaces via invariance properties and the approximation theorem, others seem far removed from any such connexion. Our main theorem will imply that $C\mathbf{X}$ splits stably as the wedge of the appropriate extended powers $D_q(\mathscr{C}, \mathbf{X})$ for any coefficient system $\mathscr{C}$ such that $\mathscr{C}_q$ is Σ_q-free for each q.

In Section 3 we give a geodesic argument from the definition of the James construction to the splitting of $\Sigma M\mathbf{X}$ and, for connected X, $\Sigma\Omega\Sigma X$. We refer to the result as the James–Milnor theorem because our proof uses nothing that was not already in James's paper (9) except knowledge of the homotopical behaviour of cofibration sequences. By a curious historical anachronism, this elementary material seems not to have been available at the time James was writing. Our proof here serves as a model for the more sophisticated splitting theorems to follow.

We generalize the James maps $MX \to MX^{[q]}$ to maps $C\mathbf{X} \to C'D_q(\mathscr{C}, \mathbf{X})$ for appropriately related coefficient systems $\mathscr{C}$ and $\mathscr{C}'$ in Section 4. Any $\mathscr{C}$ is suitably related to that coefficient system $\mathscr{N}$ such that, on spaces X, NX is the infinite symmetric power of X. We exploit this fact to give a simple homotopical proof that $C\mathbf{X}$ splits homologically as the wedge of the $D_q(\mathscr{C}, \mathbf{X})$ for any coefficient system $\mathscr{C}$ whatever. Taking $\mathscr{C}$ to be $\mathscr{N}$ itself, this gives a new proof of Steenrod's theorem (23) on the homological splitting of the symmetric powers of a space.

Returning to the homotopical splitting theorems, in section 5 we obtain canonical James maps, for reasonable $\mathscr{C}$, by means of suitable coefficient systems $\mathscr{C}'$ specified in terms of configuration spaces depending functorially on $\mathscr{C}$. We also study the passage from these James maps to James–Hopf maps $\Sigma^t C\mathbf{X} \to \Sigma^t D_q(\mathscr{C}, \mathbf{X})$. In favourable cases, we obtain good estimates on how small t can be, this information being of particular interest in the applications to n-fold loop spaces.

The product on MX was used to add up the James maps. We use an appropriate pairing defined in terms of configuration spaces to add up our canonical James maps in Section 6. Our general unstable and stable splitting theorems are proved in Sections 7 and 8. In all cases, we simply apply standard arguments to maps of cofibration sequences which drop out of the definition of the James maps and the procedures for their addition. Some technical results needed for the full strength of the stable splitting theorem are deferred until Section 9.

The basic results of this paper were originally obtained, in less general form, by the first and third authors (4), with a view towards homological applications which are discussed in (5) and will be presented in detail in (6).

CONTENTS

1. *Coefficient systems and* Π-*spaces*. Let $\mathscr{U}$ be the category of compactly generated weak Hausdorff spaces and let $\mathscr{T}$ be the category of non-degenerately based spaces in $\mathscr{U}$. Weak Hausdorff means that the diagonal is closed in the compactly generated product. All spaces are to be in $\mathscr{U}$ and all based spaces are to be in $\mathscr{T}$; our constructions will not take us out of these categories.

In (13), the second author constructed functors $C: \mathscr{T} \to \mathscr{T}$ associated to operads $\mathscr{C}$. The definition did not require all of the operad structure and could be given on more general objects than based spaces. Work of the first and third authors (4, 5, 6) and of the second author and Thomason (16) made clear that both generalizations are of considerable interest. Use of coefficient systems and Π-spaces will allow a reworking of the definition in proper generality. Both of these will be functors. Their domain categories are specified in the following definitions.

Definition 1·1. Define Λ to be the category of finite based sets $\mathbf{r} = \{0, 1, \ldots, r\}$ with basepoint 0 and their injective based functions. Say that an injection is ordered if $a < b$ implies $\phi(a) < \phi(b)$. Any morphism of Λ is the composite of a permutation and an ordered injection, and any ordered injection is a composite of the 'degeneracy operators' $\sigma_q: \mathbf{r} \to \mathbf{r}+1$ $(0 \leqslant q \leqslant r)$ specified by $\sigma_q(a) = a$ if $a \leqslant q$ and $\sigma_q(a) = a+1$ if $a > q$.

Definition 1·2. Define Π to be the category of finite based sets and based functions $\phi: \mathbf{r} \to \mathbf{s}$ such that $\phi^{-1}(b)$ has at most one element for $1 \leqslant b \leqslant s$; call ϕ a projection if $\phi^{-1}(b)$ has exactly one element. Clearly Λ is a subcategory of Π. A map $\phi: \mathbf{r} \to \mathbf{s}$ such that $\{a | \phi(a) > 0\}$ has q elements factors as the composite of a projection $\pi: \mathbf{r} \to \mathbf{q}$ and

and injection $\psi: \mathbf{q} \to \mathbf{s}$. If $\phi = \psi'\pi'$ is another such factorization, then there is a unique permutation $\tau: \mathbf{q} \to \mathbf{q}$ such that the following diagram commutes.

$$\begin{tikzcd}
 & \mathbf{q} \arrow[dd, "\tau"] \arrow[dr, "\psi"] & \\
\mathbf{r} \arrow[ur, "\pi"] \arrow[dr, "\pi'"'] & & \mathbf{s} \\
 & \mathbf{q} \arrow[ur, "\psi'"'] &
\end{tikzcd}$$

There is a unique such factorization for which ψ is ordered.

Definition 1·3. A coefficient system is a contravariant functor $\mathscr{C}: \Lambda \to \mathscr{U}$, written $\mathbf{r} \to \mathscr{C}_r$ on objects, such that $\mathscr{C}_0$ is a single point $*$. For an injection $\phi: \mathbf{r} \to \mathbf{s}$, write $\phi: \mathscr{C}_s \to \mathscr{C}_r$ on elements by $\phi(c) = c\phi$ for $c \in \mathscr{C}_s$. $\mathscr{C}$ is said to be Σ-free if the action of Σ_r on $\mathscr{C}_r$ is free for each $r \geq 1$. A map $g: \mathscr{C} \to \mathscr{C}'$ of coefficient systems is a natural transformation under Λ.

Examples 1·4. Writing $\mathscr{C}_r$ for $\mathscr{C}(r)$ of (13), 1·1, and using (13), 2·3, we see that operads naturally determine coefficient systems by neglect of structure. In particular, we have the following examples.

(i) $\mathscr{M}: \mathscr{M}_r = \Sigma_r$. For $\phi: \mathbf{r} \to \mathbf{s}$, $\phi: \Sigma_s \to \Sigma_r$ is specified as follows. For $\tau \in \Sigma_s$, there is a unique ordered injection $\phi': \mathbf{r} \to \mathbf{s}$ such that $\operatorname{Im} \phi' = \operatorname{Im}(\tau \circ \phi)$, and $\tau\phi \in \Sigma_r$ is the permutation such that $\phi' \circ (\tau\phi) = \tau \circ \phi$.

(ii) $\mathscr{N}: \mathscr{N}_r$ is a point; $\phi: \mathscr{N}_s \to \mathscr{N}_r$ is the only map possible.

(iii) $\mathscr{C}_n$, the nth little cubes operad (13), §4: $\mathscr{C}_{n,r}$ is the space of r-tuples of affine embeddings $I^n \to I^n$ with parallel axes and disjoint interiors. For $\phi: \mathbf{r} \to \mathbf{s}$, $\phi: \mathscr{C}_{n,s} \to \mathscr{C}_{n,r}$ is specified by

$$\langle c_1, \ldots, c_s \rangle \phi = \langle c_{\phi(1)}, \ldots, c_{\phi(r)} \rangle, \quad c_q: I^n \to I^n.$$

There are formally similar examples derived from spaces.

Example 1·5. A space $Y \in \mathscr{U}$ determines the contravariant projection functor $\mathscr{P}(Y): \Lambda \to \mathscr{U}$ with rth space Y^r, the map $\phi: Y^s \to Y^r$ determined by $\phi: \mathbf{r} \to \mathbf{s}$ being the projection

$$(y_1, \ldots, y_s)\phi = (y_{\phi(1)}, \ldots, y_{\phi(r)}).$$

It is not very useful to regard the functors $\mathscr{P}(Y)$ as coefficient systems because they are not Σ-free. By restriction, however, they yield the following basic coefficient systems $\mathscr{C}(Y)$. Let $\mathscr{I}$ denote the subcategory of $\mathscr{U}$ consisting of all spaces and all injective maps between them.

Example 1·6. For $Y \in \mathscr{I}$, define the configuration space $F(Y, r)$ to be the subspace of Y^r consisting of all $(y_1, \ldots, y_r)$ with $y_i \neq y_j$ for $i \neq j$. Observe that Σ_r acts freely on $F(Y, r)$ and define $B(Y, r) = F(Y, r)/\Sigma_r$. Let $\mathscr{C}(Y): \Lambda \to \mathscr{U}$ be the subfunctor of $\mathscr{P}(Y)$ with rth space $F(Y, r)$. Clearly $\mathscr{C}(?)$ specifies a functor from $\mathscr{I}$ to the category of coefficient systems.

We shall also need the following examples of maps of coefficient systems.

Examples 1·7. (i) By (13), pp. 24 and 34, $\pi_0\mathscr{C}_{1,r} = \Sigma_r$ and discretization specifies an augmentation $\epsilon: \mathscr{C}_1 \to \mathscr{M}$ of operads such that each $\epsilon_r: \mathscr{C}_{1,r} \to \mathscr{M}_r$ is a Σ_r-equivariant homotopy equivalence.

(ii) By (13), 4·8, the map $g_r: \mathscr{C}_{n,r} \to F(R^n, r)$ which sends all embedded cubes to their centre points is a Σ_r-equivariant homotopy equivalence. Clearly these maps specify a morphism $g: \mathscr{C}_n \to \mathscr{C}(R^n)$ of coefficient systems. Note that $\mathscr{C}(R^n)$ is not an operad.

We now turn to the complementary notion of a Π-space.

Definition 1·8. A Π-space is a covariant functor $\mathbf{X}: \Pi \to \mathscr{T}$, written $\mathbf{r} \to X_r$ on objects, such that $X_0 = (*)$ and for each subset s of $\{1, \dots, r+1\}$, the inclusion $\bigcap_{q+1 \in s} \sigma_q X_r \to X_{r+1}$ is a Σ_s-cofibration, where Σ_s is the group of permutations $r+1 \to r+1$ which fix all letters not in s. Write $\phi: X_r \to X_s$ on elements by $\phi(x) = \phi x$ for $x \in X_r$. A map $f: \mathbf{X} \to \mathbf{X}'$ of Π-spaces is a natural transformation under Π. The notion of a Λ-space is defined in the same way, and Π-spaces determine Λ-spaces by neglect of projections.

The projections are irrelevant in the next section, in which we shall work with Λ-spaces, but are essential to all of our later work.

Examples 1·9. (i) A space $X \in \mathscr{T}$ determines the Π-space $\Pi \to \mathscr{T}$ specified by $\mathbf{r} \to X^r$ on objects, the map $\phi: X^r \to X^s$ determined by $\phi: \mathbf{r} \to \mathbf{s}$ being given by

$$\phi(x_1, \dots, x_r) = (x'_1, \dots, x'_s),$$

where $x'_{\phi(a)} = x_a$ if $\phi(a) > 0$ and $x_b = *$ if $b > 0$ and $b \notin \operatorname{Im} \phi$. The non-degeneracy of the basepoint of X implies the cofibration condition required of a Π-space by (13), A. 4.

(ii) Any functor on based spaces with good behaviour with respect to cofibrations extends to Π-spaces by application of the given functor to rth spaces and to the maps determined by morphisms in Π. Examples include wedges, finite smash products, and the based function space $F(K, ?)$ for a compact space K.

(iii) Given a based space P and a Π-space $\mathbf{X}$, there is a Π-space $P \wedge \mathbf{X}$ with rth space $P \wedge X_r$ for all r, the map $P \wedge X_r \to P \wedge X_s$ determined by $\phi: \mathbf{r} \to \mathbf{s}$ being $1 \wedge \phi$.

Remarks 1·10. In (16), May and Thomason set up an axiomatic foundation for infinite loop space theory. Their work made clear that the basic objects of study in that subject are not just spaces but rather Π-spaces $\mathbf{X}$ such that the maps $X_n \to X_1^n$ determined by the n projections $\mathbf{n} \to \mathbf{1}$ are equivalences. This condition fails for the examples $P \wedge \mathbf{X}$ and is not needed in the present paper. The cofibration condition in Definition 1·8 is written in the form appropriate for Λ-spaces; for Π-spaces, it is equivalent to the more conceptual form given in (16), 1·2 (3).

2. *The spaces* $C\mathbf{X}$. We can now define the spaces we wish to study.

Definition 2·1. Let $\mathscr{C}$ be a coefficient system and $\mathbf{X}$ be a Λ-space. Define $C\mathbf{X}$ to be the 'coend'

$$\int_\Lambda \mathscr{C}_r \times X_r = \coprod_{r \geq 0} \mathscr{C}_r \times X_r/(\sim),$$

where the equivalence relation is specified by

$$(c\phi, x) \sim (c, \phi x) \quad \text{for} \quad c \in \mathscr{C}_s, \quad \phi: \mathbf{r} \to \mathbf{s}, \quad \text{and} \quad x \in X_r.$$

Let $F_s CX$ be the image of

$$\coprod_{r=0}^{s} \mathscr{C}_r \times X_r$$

in $C\mathbf{X}$ and give $F_s C\mathbf{X}$ the quotient topology. Then give $C\mathbf{X}$ the topology of the union of the $F_s CX$. Thus a collection of maps $f_r: \mathscr{C}_r \times X_r \to Z$ extends to a map $f: C\mathbf{X} \to Z$ if and only if the following diagram commutes for each injection $\phi: \mathbf{r} \to \mathbf{s}$:

$$\begin{tikzcd}
\mathscr{C}_s \times X_r \arrow[r, "1 \times \phi"] \arrow[d, "\phi \times 1"'] & \mathscr{C}_s \times X_s \arrow[d, "f_s"] \\
\mathscr{C}_r \times X_r \arrow[r, "f_r"] & Z
\end{tikzcd}$$

Write $[c, x]$ for the image in $C\mathbf{X}$ of a point (c, x) in $\mathscr{C}_r \times X_r$.

The cofibration condition of Definition 1·8, which is no real restriction in view of the whiskering construction of (16), app. B, is precisely what is needed to ensure the validity of the following generalization of (13), 2·6; see Boardman and Vogt (2), p. 234.

LEMMA 2·2. *With basepoint* $* = [*, *]$, $C\mathbf{X}$ *is a well-defined space in* $\mathscr{T}$, *and the construction is functorial in* $\mathscr{C}$ *and* $\mathbf{X}$. *Each inclusion* $F_{r-1} C\mathbf{X} \to F\, C\mathbf{X}$ *is a cofibration and the following are pushout diagrams, where*

$$\sigma X_{r-1} = \bigcup_{q=0}^{r-1} \sigma_q X_{r-1}$$

and $f(c, \sigma_q x) = [c\sigma_q, x]$ *for* $c \in \mathscr{C}_r$ *and* $x \in X_{r-1}$:

$$\begin{tikzcd}
\mathscr{C}_r \times_{\Sigma_r} \sigma X_{r-1} \arrow[r, "f"] \arrow[d, "\cap"'] & F_{r-1} C\mathbf{X} \arrow[d] \\
\mathscr{C}_r \times_{\Sigma_r} X_r \arrow[r] & F_r C\mathbf{X}
\end{tikzcd}$$

We fix the following notations for use throughout the paper.

Notations 2·3. (i) Let $X_{[r]}$ denote the quotient $X_r/\sigma X_{r-1}$ and let $D_r(\mathscr{C}, \mathbf{X})$ denote the quotient

$$\mathscr{C}_r^+ \wedge_{\Sigma_r} X_{[r]} = F_r C\mathbf{X}/F_{r-1} C\mathbf{X}.$$

Abbreviate $D_r(\mathscr{C}, \mathbf{X})$ to $D_r \mathbf{X}$ when $\mathscr{C}$ is clear from context.

(ii) When $\mathbf{X}$ arises as in Examples 1·9(i) from a space X, write CX for $C\mathbf{X}$ and $D_r(\mathscr{C}, X)$ for $D_r(\mathscr{C}, \mathbf{X})$. Of course, $X_{[r]}$ is here just the r-fold smash product $X^{[r]}$.

(iii) If $\mathscr{C}_1$ has a given basepoint, 1 say, let $\eta: X \to CX$ be the map specified by $\eta(x) = [1, x]$. It is natural in X and in $\mathscr{C}$. Let η also denote the natural map $X \to \Omega^n\Sigma^n X$ (adjoint to the identity map of $\Sigma^n X$).

Examples 2·4. The following special cases result from the coefficient systems specified in Examples 1·4 and 1·6.

(i) MX is the James reduced product, or free topological monoid, generated by X. Here permutations are unnecessary since $\mathscr{M}_r \times_{\Sigma_r} X_r = X_r$, and $M\mathbf{X}$ is constructed from $\amalg X_r$ by identifying ϕx and x for all ordered injections $\phi: \mathbf{r} \to \mathbf{s}$ and all $x \in X_r$.

(ii) NX is the infinite symmetric product, or free commutative topological monoid, generated by X.

(iii) $C_n X$ is an approximation to $\Omega^n\Sigma^n X$. A natural map $\alpha_n: C_n X \to \Omega^n\Sigma^n X$ such that $\alpha_n \circ \eta = \eta: X \to \Omega^n\Sigma^n X$ is specified in (13), 5·2. When X is connected (that is, as was intended and needed in (13), pathwise connected), α_n is a weak equivalence by (13), 6·1. Moreover, α_n is an H-map, indeed a $\mathscr{C}_n$-map (13), 5·2.

(iv) For a space Y and Π-space $\mathbf{X}$, let $C(Y, \mathbf{X})$ denote the space obtained by application of Definition 2·1 to the configuration space coefficient system $\mathscr{C}(Y)$. Similarly, abbreviate $D_q(\mathscr{C}(Y), \mathbf{X})$ to $D_q(Y, \mathbf{X})$. For spaces X, these functors $C(Y, X)$ are studied homologically in (5) and (6).

The reader who wishes to concentrate on these examples of CX for spaces X need only read subscripts as superscripts (X_r as X^r, etc.) in what follows; no mathematical simplification will ensue. Use of general Π-spaces allows the following observation.

Example 2·5. Via the correspondence $[c, p \wedge x] \leftrightarrow p \wedge [c, x]$, the spaces $C(P \wedge \mathbf{X})$ and $P \wedge C\mathbf{X}$ are homeomorphic, naturally in $\mathscr{C}$, P, and $\mathbf{X}$. Similarly $D_q(\mathscr{C}, P \wedge \mathbf{X})$ is homeomorphic to $P \wedge D_q(\mathscr{C}, \mathbf{X})$. Thus any natural equivalence between suitable suspensions of $C\mathbf{X}$ and $\bigvee_{q \geq 1} D_q(\mathscr{C}, \mathbf{X})$ for Π-spaces $\mathbf{X}$ specializes to yield natural equivalences between suitable suspensions of $P \wedge CX$ and $\bigvee_{q \geq 1} P \wedge D_q(\mathscr{C}, X)$ for based spaces X and P.

We record the following two homotopy invariance properties of $C\mathbf{X}$ for Λ-spaces $\mathbf{X}$.

LEMMA 2·6. *Let $f: \mathbf{X} \to \mathbf{X}'$ be a map of Λ-spaces.*

(i) *If $\mathscr{C}$ is Σ-free and each $f_j: X_j \to X'_j$ is a weak equivalence, then $Cf: C\mathbf{X} \to C\mathbf{X}'$ is a weak equivalence.*

(ii) *If each $f_j: X_j \to X'_j$ is a Σ_j-equivariant equivalence, then $Cf: C\mathbf{X} \to C\mathbf{X}'$ is an equivalence.*

LEMMA 2·7. *Let $g: \mathscr{C} \to \mathscr{C}'$ be a map of coefficient systems.*

(i) *If $\mathscr{C}$ and $\mathscr{C}'$ are Σ-free and each $g_j: \mathscr{C}_j \to \mathscr{C}'_j$ is a weak equivalence, then $g: C\mathbf{X} \to C'\mathbf{X}$ is a weak equivalence.*

(ii) *If each $g_j: \mathscr{C}_j \to \mathscr{C}'_j$ is a Σ_j-equivariant equivalence, then $g: C\mathbf{X} \to C'\mathbf{X}$ is an equivalence.*

Proofs. These are based on inductive use of the pushout diagrams of Lemma 2·2 and, for parts (i), the long exact homotopy sequences of covering projections. The latter show that the maps induced on the left sides of the diagrams are weak equivalences after passage to orbits with respect to Σ_r because they are so before passage to orbits. The invariance of pushouts of cofibrations under equivalence is well known. Their invariance under weak equivalence is also true, but apparently not in the literature. Proofs will appear in (15), I. 3·4 and III. 8·2. Observe that the conclusions of the lemmas are also true when restricted to each finite filtration and that, in parts (ii), Cf and g admit filtration preserving homotopy inverses.

By Examples 1·7 and the approximation theorem, the previous lemma has the following consequences. These will be used in the passage from combinatoric analysis of spaces $C\mathbf{X}$ to the various splitting theorems.

PROPOSITION 2·8. *In the following natural diagram, ϵ is always an equivalence and α_1 is a weak equivalence if X is connected:*

$$MX \xleftarrow{\epsilon} C_1 X \xrightarrow{\alpha_1} \Omega\Sigma X.$$

PROPOSITION 2·9. *In the following natural diagram, g is always an equivalence and α_n is a weak equivalence if X is connected:*

$$C(R^n, X) \xleftarrow{g} C_n X \xrightarrow{\alpha_n} \Omega^n\Sigma^n X.$$

3. *The James–Milnor theorem.* Practically every working homotopy theorist has his own favourite elementary proof of Milnor's splitting of $\Sigma\Omega\Sigma X$. While ours does not appear in print, it ought not to be new since it uses nothing that was not already available in the 1950s. It has some significant advantages (explained in Remarks 3·9).

We agree to write Δ, ι, and π generically for diagonal maps and for canonical inclusions and quotient maps.

We begin with some combinatorics, essentially following James (9), §2, but keeping track of permutations as in Barratt and Eccles (1), §4, in preparation for the work in the next section. These observations will serve to show that the James maps and their generalizations are well-defined. Recall Definitions 1·1 and 1·2 and let

$$(i,j) = (i+j)!/i!j!.$$

Combinatorics 3·1. Fix $q \geqslant 1$ and fix an injection $\phi: \mathbf{r} \to \mathbf{s}$, where $r \geqslant q$.

(i) For an injection $\psi: \mathbf{q} \to \mathbf{r}$, define the 'inverse' projection $\psi^{-1}: \mathbf{r} \to \mathbf{q}$ by

$$\psi^{-1}(b) = a \quad \text{if} \quad b = \psi(a) \quad \text{and} \quad \psi^{-1}(b) = 0 \quad \text{if} \quad b \notin \operatorname{Im} \psi.$$

(ii) Let R be the set of ordered injections $\mathbf{q} \to \mathbf{r}$ and note that R has $m = (r-q, q)$ elements. Give R the reverse lexicographic ordering, so that $\psi < \psi'$ if $\psi(a) < \psi'(a)$ for the largest a such that $\psi(a) \neq \psi'(a)$, and write $R = \{\psi_1, \ldots, \psi_m\}$. Similarly, let $S = \{\omega_1, \ldots, \omega_n\}$, $n = (s-q, q)$, be the ordered set of ordered injections $\mathbf{q} \to \mathbf{s}$.

(ii) If $\omega_j \in S$ and Im ω_j is not contained in Im ϕ, then $(\omega_j^{-1} \circ \phi)(b) \neq 0$ for at most $q-1$ values b and therefore $\omega_j^{-1} \circ \phi$ factors as the composite of a projection $\mathbf{r} \to \mathbf{p}$ and an injection $\mathbf{p} \to \mathbf{q}$ for some $p < q$.

(iv) If $\omega_j \in S$ and $\operatorname{Im} \omega_j$ is contained in Im ϕ, then there is exactly one $\psi_i \in R$ such that $\operatorname{Im}(\phi \circ \psi_i) = \operatorname{Im} \omega_j$. Rewrite $\omega_j = \chi_i$ and $j = \bar{\phi}(i)$. Then $\psi_i \to \chi_i$ specifies an injection $R \to S$ and $\bar{\phi}: \mathbf{m} \to \mathbf{n}$ specifies the corresponding injection in Λ.

(v) If ϕ is ordered, then $\chi_i = \phi \circ \psi_i$. In general, there is a unique permutation $\tau_i\colon \mathbf{q} \to \mathbf{q}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{q} \arrow[r, "\psi_i"] \arrow[d, "\tau_i"'] & \mathbf{r} \arrow[d, "\phi"] \\ \mathbf{q} \arrow[r, "\chi_i"] & \mathbf{s} \end{tikzcd}$$

(vi) Moreover, the following diagram also commutes:

$$\begin{tikzcd} \mathbf{q} \arrow[d, "\tau_i"'] & \mathbf{r} \arrow[l, "\psi_i^{-1}"'] \arrow[d, "\phi"] \\ \mathbf{q} & \mathbf{s} \arrow[l, "\chi_i^{-1}"'] \end{tikzcd}$$

We have the following immediate consequence of these observations.

LEMMA 3·2. *Let* $\mathbf{X}$ *be a* Π*-space and let* $x \in X_r$ *and* $\omega_j \in S$. *If* $\operatorname{Im}\omega_j$ *is not contained in* $\operatorname{Im}\phi$, *then* $(\omega_j^{-1}\circ\phi)(x) \in \sigma X_{q-1}$. *On the other hand, if* $j = \bar{\phi}(i)$ *so that* $\omega_j = \chi_i$, *then*

$$(\omega_j^{-1}\circ\phi)(x) = (\tau_i \circ \psi_j^{-1})(x).$$

Thus if $\pi\colon X_q \to X_{[q]}$ is the quotient map, then, in $(X_{[q]})^n$,

$$\mathop{\times}_{j=1}^{n} \pi(\omega_j \circ \phi)(x) = \bar{\phi}\left(\mathop{\times}_{i=1}^{m} \pi(\tau_i \circ \psi_i^{-1})(x)\right).$$

Moreover, each $\tau_i = 1$ if ϕ is ordered.

Retaining the notations above, we have the following definition of the James maps (compare (9), 2·5).

Definition 3·3. Let $\mathbf{X}$ be a Π-space. For $r < q$, let $j_{qr}\colon X_r \to (X_{[q]})^0$ be the trivial map. For $r \geqslant q$, define $j_{qr}\colon X_r \to (X_{[q]})^m$ by

$$j_{qr}(x) = (\pi(\psi_1^{-1}x), \ldots, \pi(\psi_m^{-1}x)).$$

If $\phi\colon \mathbf{r} \to \mathbf{s}$ is an ordered injection, then the following diagram commutes by the lemma above:

$$\begin{tikzcd} X_r \arrow[r, "\phi"] \arrow[d, "j_{qr}"'] & X_s \arrow[d, "j_{qs}"] \\ (X_{[q]})^m \arrow[r, "\bar{\phi}"] & (X_{[q]})^n \end{tikzcd}$$

Therefore, by Examples 2·4 (i), the j_{qr} together specify a well-defined map

$$j_q: M\mathbf{X} \to MX_{[q]}.$$

Having defined the James maps, our next task is to add them up. MX is a monoid, and we let μ denote its (iterated) product. Note that μ is only defined (and only needed) for spaces X and not for general Π-spaces.

Definition 3·4. Define $k_r: M\mathbf{X} \to M\left(\bigvee_{q=1}^{r} X_{[q]}\right)$ to be the composite

$$M\mathbf{X} \xrightarrow{\Delta} (M\mathbf{X})^r \xrightarrow{\prod_{q=1}^{r} j_q} \prod_{q=1}^{r} MX_{[q]} \xrightarrow{\prod_{q=1}^{r} M\iota} \prod_{q=1}^{r} M\left(\bigvee_{q=1}^{r} X_{[q]}\right) \xrightarrow{\mu} M\left(\bigvee_{q=1}^{r} X_{[q]}\right).$$

Continue to write k_r for its restriction to any $F_s M\mathbf{X} \subset M\mathbf{X}$.

The following observation is the crux of the James–Milnor theorem.

PROPOSITION 3·5. *The following diagram commutes for $r \geqslant 1$.*

$$\begin{tikzcd}
F_{r-1}M\mathbf{X} \arrow[r, "\iota"] \arrow[d, "k_{r-1}"'] & F_r M\mathbf{X} \arrow[r, "\pi"] \arrow[d, "k_r"] & X_{[r]} \arrow[d, "\eta"] \\
M\left(\bigvee_{q=1}^{r-1} X_{[q]}\right) \arrow[r, "\iota"] & M\left(\bigvee_{q=1}^{r} X_{[q]}\right) \arrow[r, "\pi"] & MX_{[r]}
\end{tikzcd}$$

In particular, $k_1 = \eta: F_1\mathbf{X} = X_{[1]} \to MX_{[1]}$.

Proof. The left square commutes because $j_r(x) = *$ for $x \in F_{r-1}M\mathbf{X}$ and $*$ is the identity element for the product. The right square commutes because $M\pi \circ k_r$ is induced by $j_{rr} = \pi: X_r \to X_{[r]}$.

The left squares allow the following definition.

Definition 3·6. Define $k_\infty: M\mathbf{X} \to M(\bigvee_{q\geqslant 1} X_{[q]})$ by passage to limits over r from the maps k_r.

Now assume given H-maps $\beta: MX \to \Omega\Sigma X$ which are natural up to homotopy and satisfy $\beta\eta \simeq \eta: X \to \Omega\Sigma X$. For a given map $f: X' \to MX$, let $\tilde{f}: \Sigma X' \to \Sigma X$ denote the adjoint of the composite $\beta \circ f: X' \to \Omega\Sigma X$. Of course, $\tilde{\eta} \simeq 1$.

THEOREM 3·7. *For all Π-spaces $\mathbf{X}$ and for all $r \geqslant 1$ (including $r = \infty$),*

$$\tilde{k}_r: \Sigma F_r M\mathbf{X} \to \bigvee_{q=1}^{r} \Sigma X_{[q]}$$

is a homotopy equivalence. Moreover, $\tilde{k}_r$ is the sum over q of the maps

$$h_q = \tilde{j}_q: \Sigma M\mathbf{X} \to \Sigma X_{[q]}.$$

Proof. Since $\tilde{k}_1 \simeq 1$, we may assume inductively that $\tilde{k}_{r-1}$ is an equivalence in the homotopy commutative diagram

$$\begin{tikzcd}
X_{[r]} \arrow[r, "\partial"] & \Sigma F_{r-1}M\mathbf{X} \arrow[r, "\Sigma\iota"] \arrow[d, "\tilde{k}_{r-1}"'] & \Sigma F_r M\mathbf{X} \arrow[r, "\Sigma\pi"] \arrow[d, "\tilde{k}_r"] & \Sigma X_{[r]} \arrow[d, equal] \\
& \bigvee_{q=1}^{r-1} \Sigma X_{[q]} \arrow[r, "\Sigma\iota"] & \bigvee_{q=1}^{r} \Sigma X_{[q]} \arrow[r, "\Sigma\pi"] & \Sigma X_{[r]}
\end{tikzcd}$$

A chase gives a left homotopy inverse $\Sigma F_r M\mathbf{X} \to \Sigma F_{r-1} M\mathbf{X}$ to $\Sigma\iota$. Thus $\partial \simeq 0$ in the top cofibration sequence, and it follows that $\tilde{k}_r$ is an equivalence (e.g. by (15), I. 1·12). The conclusion for $r = \infty$ follows by passage to colimits (e.g. by (15), I, 3·5). Since β is an H-map, the last statement is immediate from Definitions 3·4 and 3·6.

Finally, assume further that β is a weak equivalence when X is connected. By Example 2·5, the theorem then has the following immediate consequence.

COROLLARY 3·8. *For all based spaces P and connected based spaces X, $\tilde{k}_\infty$ is an equivalence and $\Sigma(1 \wedge \beta)$ is a weak equivalence in the diagram*

$$\Sigma(P \wedge \Omega\Sigma X) \xleftarrow{\Sigma(1\wedge\beta)} \Sigma(P \wedge MX) \xrightarrow{\tilde{k}_\infty} \bigvee_{q\geqslant 1} \Sigma(P \wedge X^{[q]}).$$

Remarks 3·9. The maps h_q or, when β is an equivalence, the composites

$$h_q(\Sigma\beta)^{-1}\colon \Sigma\Omega\Sigma X \to \Sigma X^{[q]}$$

are called James–Hopf maps. While there exist other simple proofs that $\Sigma\Omega\Sigma X$ splits, they generally deal only with the obvious quotient maps $F_q MX \to X^{[q]}$ and not with possible extensions of these maps to all of MX. For the deeper applications, it is vital to have the splitting given in terms of explicitly described globally defined James–Hopf maps. For example, study of the fibres of these maps is essential to setting up the EHP-sequence and analysing the double suspension.

Suitable maps β were already given by James (9). A construction of β appropriate for purposes of generalization is obtained by choosing a homotopy inverse ϵ^{-1} to ϵ of Proposition 2·8 and setting $\beta = \alpha_1 \circ \epsilon^{-1}$. The standard construction is obtained by choosing a homotopy inverse $(M\pi)^{-1}$ to $M\pi$ in the following diagram and setting $\beta = r \circ \bar{\eta} \circ (M\pi)^{-1}$.

$$MX \xleftarrow{M\pi} MX' \xrightarrow{\bar{\eta}} \Lambda\Sigma X \xrightarrow{r} \Omega\Sigma X.$$

Here Λ is the associative Moore loop space functor, r is the natural retraction,

$$X' = X \vee I \quad \text{and} \quad \eta\colon X' \to \Lambda\Sigma X$$

is obtained by using the whisker to extend the natural unbased inclusion $X \to \Lambda\Sigma X$ to a based inclusion ($\eta(t)$ being the trivial loop of length $1-t$), $\bar{\eta}$ is the map of monoids obtained from η by the freeness of MX', and $\pi: X' \to X$ is the equivalence obtained by collapsing the whisker.

4. *Generalized James maps; the homological splitting theorem.* We begin our generalization of the program carried out for $\mathscr{M}$ in the previous section by generalizing the James maps. We then digress to use a special case to prove a very general homological splitting theorem. Recall Notations 2·3 and return to the notations established in Combinatorics 3·1. Assume given two coefficient systems $\mathscr{C}$ and $\mathscr{C}'$,

Definition 4·1. Fix $q \geq 1$. A James system $\{\xi_{qr} \mid r \geq q\}: \mathscr{C} \to \mathscr{C}'$ is a sequence of maps $\xi_{qr}: \mathscr{C}_r \to \mathscr{C}'_m$, $m = (r-q, q)$, such that the following diagram commutes for each injection $\phi: \mathbf{r} \to \mathbf{s}$, where $n = (s-q, q)$ and $\bar{\phi}$ is as given in Combinatorics 3·1 (iv).

$$\begin{tikzcd}
\mathscr{C}_r \arrow[r, "\xi_{qr}"] & \mathscr{C}'_m \\
\mathscr{C}_s \arrow[u, "\phi"] \arrow[r, "\xi_{qs}"] & \mathscr{C}'_n \arrow[u, "\bar{\phi}"']
\end{tikzcd}$$

Definition 4·2. Let $\{\xi_{qr}\}: \mathscr{C} \to \mathscr{C}'$ be a James system and let $\mathbf{X}$ be a Π-space. For $r < q$, let $j_{qr}: \mathscr{C}_r \times X_r \to \{*\} = \mathscr{C}'_0 \times D_q(\mathscr{C}, \mathbf{X})^0$ be the trivial map. For $r \geq q$, define $j_{qr}: \mathscr{C}_r \times X_r \to \mathscr{C}'_m \times D_q(\mathscr{C}, \mathbf{X})^m$ by

$$j_{qr}(c, x) = (\xi_{qr}(c), [c\psi_1; \psi_1^{-1}x], \ldots, [c\psi_m; \psi_m^{-1}x]),$$

where $\{\psi_i\}$ is the ordered set of ordered injections $\mathbf{q} \to \mathbf{r}$. For an injection $\phi: \mathbf{r} \to \mathbf{s}$, the following diagram commutes.

$$\begin{tikzcd}
\mathscr{C}_r \times X_r \arrow[d, "j_{qr}"'] & \mathscr{C}_s \times X_r \arrow[l, "\phi \times 1"'] \arrow[r, "1 \times \phi"] & \mathscr{C}_s \times X_s \arrow[d, "j_{qs}"] \\
\mathscr{C}'_m \times D_q(\mathscr{C}, \mathbf{X})^m \arrow[r] & C'D_q(\mathscr{C}, \mathbf{X}) & \mathscr{C}'_n \times D_q(\mathscr{C}, \mathbf{X})^n \arrow[l]
\end{tikzcd}$$

Indeed, for $c \in \mathscr{C}_s$ and $x \in X_r$,

$$j_{qr}(\phi \times 1)(c, x) = \left(\xi_{qr}(c\phi), \mathop{\times}_{i=1}^{m} [c\phi\psi_i; \psi_i^{-1}x]\right)$$

while Lemma 3·2 implies that

$$j_{qs}(1 \times \phi)(c, x) = \left(\xi_{qs}(c), \bar{\phi}\left(\mathop{\times}_{i=1}^{m} [c\chi_i; (\tau_i \circ \psi_i^{-1})(x)]\right)\right).$$

Here the functoriality of $\mathscr{C}$ applied to the diagram of Combinatorics 3·1 (v) and the definition of $D_q(\mathscr{C}, \mathbf{X})$ imply

$$[c\phi\psi_i, \psi_i^{-1}x] = [c\chi_i\tau_i, \psi_i^{-1}x] = [c\chi_i, (\tau_i \circ \psi_i^{-1})(x)],$$

hence the claimed commutativity follows directly from the definitions of $C'D_q(\mathscr{C}, \mathbf{X})$ and of a James system. Therefore the j_{qr} together specify a well-defined map

$$j_q: C\mathbf{X} \to C'D_q(\mathscr{C}, \mathbf{X}).$$

Exercise 4·3. Construct functions $\xi_{qr}\colon \Sigma_r \to \Sigma_m$ which give a James system $\mathscr{M} \to \mathscr{M}$. Verify that the resulting generalized James map agrees with the James map of Definition 3·3.

We shall return to our main line of development in the next section, after exploiting the following obvious example to obtain a homological splitting theorem applicable to arbitrary coefficient systems $\mathscr{C}$. Write $D_q\mathbf{X}$ for $D_q(\mathscr{C}, \mathbf{X})$ in the rest of this section. The following statements are precise analogues of 3·3–3·6.

Example 4·4. For any $q \geq 1$ and any $\mathscr{C}$, the unique maps ξ_{qr} from the spaces $\mathscr{C}_r$ to the points $\mathscr{N}_m$ specify a James system $\mathscr{C} \to \mathscr{N}$. There result James maps $j_q\colon C\mathbf{X} \to ND_q\mathbf{X}$.

Let μ denote the (iterated) product of the monoid NX, for spaces X.

Definition 4·5. Define $k_r\colon C\mathbf{X} \to N\left(\bigvee_{q=1}^{r} D_q\mathbf{X}\right)$ to be the sum of the r composites

$$C\mathbf{X} \xrightarrow{j_q} ND_q\mathbf{X} \xrightarrow{N\iota} N\left(\bigvee_{q=1}^{r} D_q\mathbf{X}\right).$$

That is, $k_r = \mu\left(\prod_{q=1}^{r} N\iota \circ j_q\right)\Delta$. Continue to write k_r for its restriction to any $F_sC\mathbf{X} \subset C\mathbf{X}$.

PROPOSITION 4·6. *The following diagram commutes for $r \geq 1$.*

$$\begin{tikzcd}
F_{r-1}C\mathbf{X} \arrow[r, "\iota"] \arrow[d, "k_{r-1}"] & F_rC\mathbf{X} \arrow[r, "\pi"] \arrow[d, "k_r"] & D_r\mathbf{X} \arrow[d, "\eta"] \\
N\left(\bigvee_{q=1}^{r-1} D_q\mathbf{X}\right) \arrow[r, "N\iota"] & N\left(\bigvee_{q=1}^{r} D_q\mathbf{X}\right) \arrow[r, "N\pi"] & ND_r\mathbf{X}
\end{tikzcd}$$

Definition 4·7. Define $k_\infty\colon C\mathbf{X} \to N\left(\bigvee_{q\geq 1} D_q\mathbf{X}\right)$ by passage to limits over r from the maps k_r.

We can apply N to the diagram above and then apply the natural (monad) product $\mu\colon NNX \to NX$ induced by the iterated products $(NX)^q \to NX$ to obtain the following consequence of the proposition.

COROLLARY 4·8. *The following diagram commutes for $r \geq 1$.*

$$\begin{tikzcd}
NF_{r-1}C\mathbf{X} \arrow[r, "N\iota"] \arrow[d, "\tilde{k}_{r-1}"] & NF_rC\mathbf{X} \arrow[r, "N\pi"] \arrow[d, "\tilde{k}_r"] & ND_r\mathbf{X} \arrow[d, equal] \\
N\left(\bigvee_{q=1}^{r-1} D_q\mathbf{X}\right) \arrow[r, "N\iota"] & N\left(\bigvee_{q=1}^{r} D_q\mathbf{X}\right) \arrow[r, "N\pi"] & ND_r\mathbf{X}
\end{tikzcd}$$

where $\tilde{k}_r = \mu \circ Nk_r$ is the sum over q of $\tilde{j}_q = \mu \circ N(N\iota \circ j_q)$.

To obtain the full strength of the homological consequences of the corollary, we recall the homotopical formulation of ordinary homology theory. Most of the following result is proved in Dold and Thom (8), and new proofs will appear in (15). Observing that N is a continuous and thus homotopy preserving functor on spaces and verifying the dimension axiom for S^1, one sees that only part (i) is really substantive. The rest could well be taken to be true by definition in a homotopical development of homology theory (see (15).)

For an Abelian group G, let MG be a degree one Moore space, so that MG is connected and
$$\tilde{H}_*(MG; Z) = H_1(MG; Z) = G.$$

PROPOSITION 4·9. *The infinite symmetric product functor N on spaces satisfies the following properties, where X is connected.*

(i) For a cofibration $X \to Y$, $N\pi: NY \to N(Y/X)$ is a quasifibration with fibre NX.

(ii) $\pi_q NX$ is naturally isomorphic to $\tilde{H}_q(X; Z)$.

(iii) The inverse of the connecting isomorphism of the homotopy exact sequence of $NX \to NTX \to N\Sigma X$, where $TX = I \wedge X$, is the suspension isomorphism
$$\Sigma_*: \tilde{H}_q(X; Z) \to \tilde{H}_{q+1}(\Sigma X; Z).$$

(iv) $\pi_{q+1} N(MG \wedge X)$ is naturally isomorphic to $\tilde{H}_q(X; G)$.

(v) If $\gamma: G' \to G$ is realized on H_1 by $\tilde{\gamma}: MG' \to MG$, then $\pi_{q+1} N(\tilde{\gamma} \wedge 1)$ is the induced homomorphism $\gamma_*: \tilde{H}_q(X; G') \to \tilde{H}_q(X; G)$.

(vi) If γ is a monomorphism, $G'' = G/G'$, MG'' is taken as the cofibre of $\tilde{\gamma}$, and $\tilde{\beta}: MG'' \to \Sigma MG'$ is the resulting quotient map, then
$$\pi_{q+1} N(\tilde{\beta} \wedge 1): \pi_{q+1} N(MG'' \wedge X) \to \pi_{q+1} N(\Sigma MG' \wedge X) \simeq \pi_{q+1} N(MG' \wedge \Sigma X)$$
is the composite $\Sigma_* \circ \beta: \tilde{H}_q(X; G'') \to \tilde{H}_q(\Sigma X; G')$, where β is the Bockstein operation associated to $0 \to G' \to G \to G'' \to 0$.

THEOREM 4·10. *For all coefficient systems $\mathscr{C}$, Π-spaces $\mathbf{X}$, Abelian groups G, and for all $r \geq 1$ (including $r = \infty$), $\tilde{H}_*(F_r C\mathbf{X}; G)$ is isomorphic to $\sum_{q=1}^{r} \tilde{H}_*(D_q \mathbf{X}; G)$. These isomorphisms are natural in $\mathscr{C}$, $\mathbf{X}$, and G and commute with Bockstein operations.*

Proof. When each X_q is connected, the diagrams of Corollary 4·8 display comparisons of quasifibrations. Thus, by induction on r and passage to colimits,
$$\tilde{k}_{r*}: \pi_* NF_r C\mathbf{X} = \tilde{H}_*(F_r C\mathbf{X}; Z) \to \sum_{q=1}^{r} \tilde{H}_*(D_q \mathbf{X}; Z) = \pi_* N\left(\bigvee_{q=1}^{r} D_q \mathbf{X}\right)$$
is an isomorphism in this case. The generalizations to the non-connected case and to arbitrary G are applications of the use of parametrized splittings as explained in Example 2·5. For Π-spaces $\mathbf{X}$, we have Π-spaces $\Sigma\mathbf{X} = S^1 \wedge \mathbf{X}$ and $T\mathbf{X} = I \wedge \mathbf{X}$. The dotted composite in the following diagram is an isomorphism.

$$\begin{tikzcd}
\tilde{H}_*(F_r C\mathbf{X}; Z) \arrow[rr, dashed] \arrow[d, "\Sigma_*"] & & \sum_{q=1}^{r} \tilde{H}_*(D_q C\mathbf{X}; Z) \\
\tilde{H}_*(\Sigma F_r C\mathbf{X}; Z) = \tilde{H}_*(F_r C\Sigma\mathbf{X}; Z) \arrow[r, "\tilde{k}_{r*}"] & \sum_{q=1}^{r} \tilde{H}_*(D_q C\Sigma\mathbf{X}: Z) = & \sum_{q=1}^{r} \tilde{H}_*(\Sigma D_q \mathbf{X}; Z) \arrow[u, "\Sigma_*^{-1}"']
\end{tikzcd}$$

By a diagram chase from the homotopical description of Σ_* and naturality applied to the maps $\mathbf{X} \to T\mathbf{X} \to \Sigma\mathbf{X}$ of Π-spaces, we see that the dotted arrow coincides with the earlier isomorphism $\tilde{k}_{r*}$ when each X_q is connected. Upon replacing $\mathbf{X}$ by $MG \wedge \mathbf{X}$, we see that the case of general coefficient groups G follows directly from the case $G = Z$.

When $\mathscr{C}$ is Σ-free, this result will be superceded by our later stable splitting of $F_r C\mathbf{X}$. Its force lies in its applicability to non-Σ-free coefficient systems such as the projection functors $\mathscr{P}(Y)$. For Y a point, $\mathscr{P}(Y) = \mathscr{N}$ and our homological splitting of the r-fold symmetric powers $F_r NX$ recovers a result originally due to Steenrod (23) (see also Nakaoka (19) and Dold (7)). This example shows that we could not hope for a stable splitting without some restriction on $\mathscr{C}$, since it is easy to see that $H_*(F_r NX; Z_p)$ does not split as a module over the Steenrod algebra. For example, when X is the mod p Moore space $S^n \cup_p TS^n$, $NX = K(Z_p, n)$.

5. *The canonical James maps*; *James–Hopf maps*. There is an obvious generic procedure for constructing a coefficient system $\mathscr{C}'$ and maps ξ_{qr} as in Definition 4·1 from any given coefficient system $\mathscr{C}$.

Example 5·1. Let $\mathscr{B}_q = \mathscr{C}_q/\Sigma_q$ and define $\xi_{qr}: \mathscr{C}_r \to (\mathscr{B}_q)^m$ by

$$\xi_{qr}(c) = (\pi(c\psi_1), \ldots, \pi(c\psi_m)), \pi: \mathscr{C}_q \to \mathscr{B}_q.$$

For an injection $\phi: \mathbf{r} \to \mathbf{s}$, the equation $\phi\psi_i = \chi_i \tau_i$ and the fact that $\pi(c\chi_i\tau_i) = \pi(c\chi_i)$ for $c \in \mathscr{C}_s$ imply the commutativity of the following diagram by comparison of Example 1·5 and Combinatorics 3·1.

$$\begin{tikzcd}
\mathscr{C}_r \arrow[r, "\xi_{qr}"] & (\mathscr{B}_q)^m \\
\mathscr{C}_s \arrow[u, "\phi"] \arrow[r, "\xi_{qs}"] & (\mathscr{B}_q)^n \arrow[u, "\bar{\phi}"']
\end{tikzcd}$$

This example has the defect that the coefficient system given by powers of $\mathscr{B}_q$ is not Σ-free. We remedy this by fiat.

Definition 5·2. The coefficient system $\mathscr{C}$ is said to be separated if the maps

$$\xi_{qr}: \mathscr{C}_r \to (\mathscr{B}_q)^m$$

take values in the configuration spaces $F(\mathscr{B}_q, m)$ for each $q \geq 1$. Taking $\mathscr{C}' = \mathscr{C}(\mathscr{B}_q)$ in Definition 4·2, there result well-defined James maps

$$j_q: C\mathbf{X} \to C(\mathscr{B}_q, D_q(\mathscr{C}, \mathbf{X})).$$

Examples 5·3. (i) The little cubes coefficient systems of Examples 1·4 (iii) are all separated. For an ordered r-tuple $c \in \mathscr{C}_{n,r}$ of little n-cubes, the $\pi(c\psi_i)$ are the m distinct unordered sub q-tuples of c.

(ii) Similarly, the configuration space coefficient systems $\mathscr{C}(Y)$ of Example 1·6 are all separated.

(iii) If $\mathscr{C}'$ is any coefficient system and $\mathscr{C}''$ is a separated coefficient system, then $\mathscr{C} = \mathscr{C}' \times \mathscr{C}''$ is a separated coefficient system.

Remarks 5·4. If $\mathscr{C}'$ is Σ-free and each $\mathscr{C}''_q$ is contractible in the last example, for example if $\mathscr{C}''$ is $\mathscr{C}_\infty$ or $\mathscr{C}(R^\infty)$, then the projections $C\mathbf{X} \to C'\mathbf{X}$ and

$$D_q(\mathscr{C}, \mathbf{X}) \to D_q(\mathscr{C}', \mathbf{X})$$

are weak equivalences by Lemma 2·7(i). Thus any natural equivalence between suitable suspensions of $C\mathbf{X}$ and $\bigvee_{q \geq 1} D_q(\mathscr{C}, \mathbf{X})$ for separated coefficient systems $\mathscr{C}$ implies an analogous natural weak equivalence for Σ-free coefficient systems $\mathscr{C}'$.

Henceforward, we generally restrict attention to separated coefficient systems $\mathscr{C}$ and to the canonical James maps of Definition 5·2. By the previous remarks, this results in no real loss of generality. We again agree to abbreviate $D_q(\mathscr{C}, \mathbf{X})$ to $D_q\mathbf{X}$.

For the applications to the splittings of iterated loop spaces, one important new feature of our James maps is that they are defined over all of $C\mathbf{X}$ and not just over some finite filtrations (compare Remarks 3·9). The rest of this section is concerned with another new feature, namely a procedure for the study of how many suspensions are needed in order to obtain James–Hopf maps from the James maps. We have the following generic construction.

Definition 5·5. Suppose given an injective map $e_q: \mathscr{B}_q \to R^t$. Define the James–Hopf map

$$h_q: \Sigma^t C\mathbf{X} \to \Sigma^t D_q\mathbf{X}$$

determined by e_q to be the adjoint of the composite

$$\bar{\jmath}_q: C\mathbf{X} \xrightarrow{j_q} C(\mathscr{B}_q, D_q\mathbf{X}) \xrightarrow{C(e_q,1)} C(R^t, D_q\mathbf{X}) \xrightarrow{g^{-1}} C_t D_q\mathbf{X} \xrightarrow{\alpha_t} \Omega^t\Sigma^t D_q\mathbf{X},$$

where g^{-1} is a chosen homotopy inverse to the equivalence g of Proposition 2·9.

Example 5·6. If $\mathscr{C} = \mathscr{C}(Y)$ for an n-dimensional paracompact manifold Y, then $\mathscr{B}_q = B(Y, q)$ is a qn-dimensional manifold and so embeds in R^{2qn}.

Of course, the embedding dimension of $B(Y, q)$ may be much less than $2qn$. The precise estimate is of considerable interest for the understanding of the splitting of $\Omega^n\Sigma^n X$. The following lemma gives at least the improvement to $(2q-1)n$ in the cases relevant to that application.

LEMMA 5·7. *Specify a free Σ_q-action on $F(R^n - \{0\}, q-1)$ by letting Σ_{q-1} act in the natural way and letting the transposition τ_{1q} act by*

$$(y_1, \ldots, y_{q-1})\tau_{1q} = (-y_1, y_2 - y_1, \ldots, y_{q-1} - y_1).$$

Give R^n the trivial Σ_q-action and $R^n \times F(R^n - \{0\}, q-1)$ the diagonal Σ_q-action. Then $F(R^n, q)$ is Σ_q-equivariantly diffeomorphic to $R^n \times F(R^n - \{0\}, q-1)$.

Proof. The requisite diffeomorphism $f: F(R^n, q) \to R^n \times F(R^n - \{0\}, q-1)$ is specified by

$$f(x_1, \ldots, x_q) = \left(\sum_{i=1}^{q} x_i, x_1 - x_q, \ldots, x_{q-1} - x_q\right).$$

Its inverse is given explicitly by

$$f^{-1}(z, y_1, \ldots, y_{q-1}) = (y_1 + w, \ldots, y_{q-1} + w, w), \quad \text{where} \quad w = \frac{1}{q}\left(z - \sum_{i=1}^{q-1} y_i\right).$$

The equivariance of f is easily checked and implies that we have specified a well-defined free action by all of Σ_q on $F(R^n - \{0\}, q-1)$.

When $q = 2$, it is easy to construct a Σ_2-equivariant diffeomorphism from $R^n - \{0\}$ to $R^+ \times S^{n-1}$, where τ_{12} acts trivially on R^+, antipodally on S^{n-1}, and diagonally on their product. This implies the first part of the following result.

PROPOSITION 5·8. *$B(R^n, 2)$ is diffeomorphic to $R^{n+1} \times RP^{n-1}$. Its embedding dimension is $2n + k$, where k is the geometric dimension of the stable normal bundle of RP^{n-1}. Thus $k = 0$ if $n = 2, 4,$ or 8 and $k \leqslant n-2$ is the immersion codimension of RP^{n-1} otherwise.*

Proof. For $k \geqslant 0$, $RP^{n-1} = \{0\} \times RP^{n-1}$ certainly embeds in R^{2n+k}. The normal bundle ν of such an embedding has dimension $n+1+k$ and is thus stable. Clearly ν is the Whitney sum of a k-dimensional bundle and a trivial bundle if and only if the given embedding is the restriction of an embedding $e: R^{n+1} \times RP^{n-1} \to R^{2n+k}$. Indeed, such an embedding e induces a splitting of ν, and a splitting of ν yields an embedding of

$$R^{n+1} \times RP^{n-1}$$

in the total space of ν and thus in R^{2n+k}.

COROLLARY 5·9. *There is a James–Hopf map*

$$h_2: \Sigma^4 C(R^2, \mathbf{X}) \to \Sigma^4 D_2(R^2, \mathbf{X}).$$

Remarks 5·10. With $\bar{\jmath}_q$ as in Definition 5·5, computation of

$$\bar{\jmath}_{q*}: H_* C(R^n, \mathbf{X}) \to H_* \Omega^t \Sigma^t D_q(R^n, \mathbf{X})$$

can be used to determine a lower bound on the possible value of t, the calculations being facilitated by mapping further to $QD_q(R^\infty, \mathbf{X})$. The idea is that $\bar{\jmath}_{q*}$ may involve homology operations the definition of which requires at least t loops. Taking homology with Z_2 coefficients, Kirley (11) demonstrated that $t \geqslant 2^{m+1}$ is required when $q = 2^m$ and $n \geqslant 2$. Thus Corollary 5·9 is best possible. However, our positive estimate on t for $h_2: \Sigma^t C(R^n, \mathbf{X}) \to \Sigma^t D_2(R^n, \mathbf{X})$ increases with n whereas Kirley's negative estimate is constant at 4. More extensive homological calculations should yield sharper negative estimates and are essential to a really complete understanding of our splittings.

Kirley's calculation just cited is purely 2-primary. Away from 2, we can show that Proposition 5·8 is far from best possible. We first explain the principle behind such improved local estimates. Note that, in Example 5·6, any two embeddings of Y in R^{2qn+1} are isotopic and therefore induce homotopic James maps. We shall prove more general stable uniqueness statements in Section 9. In particular, suppose we happen to be given a space Z and injections

$$R^s \xleftarrow{d} Z \xrightarrow{i} \mathscr{B}_q \xrightarrow{e} R^t.$$

Proposition 9·9 then gives that the following diagram is homotopy commutative for any Λ-space $\mathbf{X}$.

$$\begin{tikzcd} C(Z,\mathbf{X}) \arrow[r, "C(d,1)"] \arrow[d, "C(i,1)"'] & C(R^s,\mathbf{X}) \arrow[dr, "i_1"] & \\ & & C(R^{s+t},\mathbf{X}) \\ C(\mathscr{B}_q,\mathbf{X}) \arrow[r, "C(e,1)"] & C(R^t,\mathbf{X}) \arrow[ur, "i_2"'] & \end{tikzcd}$$

Here i_1 and i_2 come from the embeddings of R^s and R^t in R^{s+t} as the first s and last t coordinates. It may happen that $C(i,1)$ is a local equivalence at some set of primes and that $s < t$. In such cases, we get improved localized desuspensions of the stable James–Hopf maps.

PROPOSITION 5·11. *Let p be a prime. Upon localization away from the primes $q \leqslant p$, there is a James–Hopf map*

$$h_p: \Sigma^{np} C(R^n,\mathbf{X}) \to \Sigma^{np} D_p(R^n,\mathbf{X})$$

when n is even.

Proof. We apply the considerations above to $C(B(R^n,p), D_p(R^n,\mathbf{X}))$. Let

$$Z = S^{n-1} \times R^{n(p-1)+1}.$$

Certainly Z embeds in R^{np}. By the calculations of the first and third authors (5), (6), there is an embedding $i: Z \to B(R^n,p)$ such that $C(i,1): C(Z,X) \to C(B(R^n,p),X)$ induces an isomorphism on q'-local homology for all spaces X and all primes $q' > p$. At least after double suspension, $C(i,1)$ thus becomes an equivalence upon localization away from $q \leqslant p$.

6. *Adding up the James maps.* The following external product μ relating the spaces $C(Y,X)$ will substitute for the monoid product of the James construction in the derivation of our general splitting theorems. Again μ will only be defined and needed for spaces X and not for general Π-spaces.

Definition 6·1. Given unbased spaces $Y_1, \ldots, Y_r$, a based space X, and numbers $m_1, \ldots, m_r$, let $m = m_1 + \ldots + m_r$ and define

$$\mu: F(Y_1,m_1) \times X^{m_1} \times \ldots \times F(Y_r,m_r) \times X^{m_r} \to F\left(\coprod_{q=1}^{r} Y_q, m\right) \times X^m$$

by
$$\mu(y_1,x_1,\ldots,y_r,x_r) = (y_1,\ldots,y_r,x_1,\ldots,x_r).$$

Here y_q is an ordered m_q-tuple of distinct points of Y_q, hence $(y_1,\ldots,y_r)$ is an ordered m-tuple of distinct points of $\coprod Y_q$. Clearly the maps μ together determine a map

$$\mu: \prod_{q=1}^{r} C(Y_q,X) \to C\left(\coprod_{q=1}^{r} Y_q, X\right)$$

such that

$$\mu(z_1,\ldots,z_r) = \mu(z_1,\ldots,z_{q-1},z_{q+1},\ldots,z_r) \quad \text{if} \quad z_q = * \in F_0 C(Y_q,X).$$

With $r = 2$ and $Y_1 = Y_2 = Y$, any injection $i\colon Y \amalg Y \to Y$ induces an internal product

$$C(Y,X) \times C(Y,X) \to C(Y,X)$$

by composition with μ. If i restricts on each component Y to a map homotopic through injections to the identity map of Y, then $C(Y,X)$ is an H-space. In particular,

$$C(R^n \times Y, X)$$

is an H-space for $n \geqslant 1$ and any Y.

Remarks 6·2. Much more is true. Think of R^n as the interior of I^n. Then a point of $\mathscr{C}_{n,j}$ is an embedding of the disjoint union of j copies of R^n in R^n and so determines an embedding of the disjoint union of j copies of $R^n \times Y$ in $R^n \times Y$. By composition with μ, these embeddings yield a well-defined continuous map

$$\theta_{nj}\colon \mathscr{C}_{n,j} \times C(R^n \times Y, X)^j \to C(R^n \times Y, X).$$

In view of the evident associativity law satisfied by the maps μ, a trivial calculation shows that the θ_{nj} together specify an action (in the sense of (13), 1·2 and 1·3) of the little cubes operad $\mathscr{C}_n$ on $C(R^n \times Y, X)$. Moreover, the equivalence $g\colon C_n X \to C(R^n, X)$ of Proposition 2·9 is a map of $\mathscr{C}_n$-spaces. The second author conjectured and J. Caruso [unpublished] recently proved that $C(R^n \times Y, X)$ is weakly equivalent as a $\mathscr{C}_n$-space to $\Omega^n C(Y, \Sigma^n X)$ when X is connected.

Given a separated coefficient system $\mathscr{C}$, we proceed to use μ to add up the James maps of Definition 5·2. The data W, d, e_q, and i in the following definition will be supplied, naturally in $\mathscr{C}$, by Lemma 8·1. However, we want the extra generality so as to be able to exploit the particular James–Hopf maps discussed in the previous section. The role of the homotopy d will shortly become apparent.

Definition 6·3. Assume given a contractible space W with contracting homotopy $d\colon 0 \simeq 1$, where 0 is the constant map at a basepoint $0 \in W$, and assume given injective maps

$$e_q\colon \mathscr{B}_q \to W \quad \text{for} \quad 1 \leqslant q \leqslant r \quad \text{and} \quad i\colon \coprod_{q=1}^{r} W \to W.$$

For a Π-space $\mathbf{X}$, define $k_r\colon C\mathbf{X} \to C\left(W, \bigvee_{q=1}^{r} D_q\mathbf{X}\right)$ by commutativity of the diagram

$$\begin{tikzcd}
C\mathbf{X} \arrow[r, "\Delta"] \arrow[d, "k_r"'] & (C\mathbf{X})^r \arrow[r, "\prod_{q=1}^{r} j_q"] & \prod_{q=1}^{r} C(\mathscr{B}_q, D_q\mathbf{X}) \arrow[d, "\prod_{q=1}^{r} C(e_q, \iota)"] \\
C\left(W, \bigvee_{q=1}^{r} D_q\mathbf{X}\right) & C\left(\coprod_{q=1}^{r} W, \bigvee_{q=1}^{r} D_q\mathbf{X}\right) \arrow[l, "C(i,1)"'] & \prod_{q=1}^{r} C\left(W, \bigvee_{q=1}^{r} D_q\mathbf{X}\right) \arrow[l, "\mu"']
\end{tikzcd}$$

That is, in the context of Definition 6·1, k_r is the sum of the composites

$$C(e_q, \iota) j_q\colon C\mathbf{X} \to C\left(W, \bigvee_{q=1}^{r} D_q\mathbf{X}\right).$$

Continue to write k_r for its restriction to any $F_s C\mathbf{X} \subset C\mathbf{X}$, and note that $k_{r'}$ can be defined similarly for all $r' < r$.

The following analogue of Proposition 3·5 is the heart of our work. Let

$$\eta: X \to C(W, X)$$

be determined as in Notations 2·3 from the basepoint $0 \in W = \mathscr{C}_1(W)$.

PROPOSITION 6·4. *Consider the following diagram, where $r \geqslant 1$.*

$$\begin{tikzcd} F_{r-1}C\mathbf{X} \arrow[r, "\iota"] \arrow[d, "k_{r-1}"'] & F_rC\mathbf{X} \arrow[r, "\pi"] \arrow[d, "k_r"] & D_r\mathbf{X} \arrow[d, "g_r"] \\ C\left(W, \bigvee_{q=1}^{r-1} D_q\mathbf{X}\right) \arrow[r, "C(1,\iota)"] & C\left(W, \bigvee_{q=1}^{r} D_q\mathbf{X}\right) \arrow[r, "C(1,\pi)"] & C(W, D_r\mathbf{X}) \end{tikzcd}$$

The left square commutes and there exists a map g_r homotopic to η which makes the right square commute. In particular,

$$k_1 = g_1 \simeq \eta: F_1 C\mathbf{X} = D_1\mathbf{X} \to C(W, D_1\mathbf{X}).$$

Proof. Since $j_r(x) = *$ for $x \in F_{r-1}C\mathbf{X}$, the commutativity of the left square is immediate from the definitions. The composite $C(1,\pi) \circ k_r$ is induced by j_r and, specifically, by $j_{rr}: \mathscr{C}_r \times X_r \to \mathscr{C}_r \times D_r\mathbf{X}$. By Definitions 4·2 and 5·2,

$$j_{rr}(c, x) = (\pi(c), [c, x]) \quad \text{for} \quad c \in \mathscr{C}_r \quad \text{and} \quad x \in X_r.$$

Let $g_r = G_{r,1}$, where $G_r: D_r\mathbf{X} \times I \to C(W, D_r, \mathbf{X})$ is the homotopy of η specified by

$$G_r([c, x], t) = [d(e\pi(c), t), x].$$

Clearly g_r makes the right square commute, as required.

The left squares allow the following definition.

Definition 6·5. Suppose given injective maps $e_q: \mathscr{B}_q \to W$ for all $q \geqslant 1$ and $i: \coprod_{q \geqslant 1} W \to W$. Define k_r for all $r \geqslant 1$ by the use of restrictions if i and define

$$k_\infty: C\mathbf{X} \to C(W, \bigvee_{q \geqslant 1} D_q\mathbf{X})$$

by passage to limits over r from the maps k_r.

7. *The unstable splitting theorem.* Before turning to the stable splitting of CX, we illustrate the idea with a splitting of $\Sigma^t F_r CX$ for suitable finite r and t. This is of independent interest since the inclusion $F_r CX \to CX$ is a homology isomorphism in degrees less than $q(r+1)-1$ if X is $(q-1)$-connected.

For each $t \geqslant 1$, embed the disjoint union of countably many copies of R^t in R^t by mapping the qth copy homeomorphically to $(q-1, q) \times R^{t-1}$. This gives $i: \coprod_{q \geqslant 1} R^t \to R^t$. For our unstable theorem, we fix $r \geqslant 1$ and assume given an injection $e_q: \mathscr{B}_q \to R^t$ for $1 \leqslant q \leqslant r$. Clearly Definition 6·3 then applies with $W = R^t$.

Choose filtration preserving homotopy inverses g^{-1} to the equivalences g of Proposition 2·9. There result composite maps

$$C(R^t, X) \xrightarrow{g^{-1}} C_t X \xrightarrow{\alpha_t} \Omega^t \Sigma^t X,$$

and these are clearly natural up to homotopy as X varies. Since g and α_t are H-maps, so is $\alpha_t g^{-1}$. By Examples 2·4 (iii), $\alpha_t g^{-1}\eta \simeq \eta: X \to \Omega^t\Sigma^t X$.

For a given map $f: X' \to C(R^t, X)$, we agree to write $\tilde{f}: \Sigma^t X' \to \Sigma^t X$ for the adjoint of the composite $\alpha_t g^{-1} f$; in particular, $\tilde{\eta} \simeq 1$. With these assumptions and notations, our unstable splitting theorem reads as follows.

THEOREM 7·1. *For all* Π-*spaces* $\mathbf{X}$,

$$\tilde{k}_r: \Sigma^t F_r C\mathbf{X} \to \bigvee_{q=1}^{r} \Sigma^t D_q \mathbf{X}$$

is a homotopy equivalence. Moreover, $\tilde{k}_r$ *is the sum over* q *of restrictions of James–Hopf maps*

$$h_q: \Sigma^t C\mathbf{X} \to \Sigma^t D_q \mathbf{X}.$$

Proof. The diagram of Proposition 6·4 gives rise to a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma^{t-1} D_r \mathbf{X} \arrow[r, "\partial"] & \Sigma^t F_{r-1} C\mathbf{X} \arrow[r, "\Sigma^t \iota"] \arrow[d, "\tilde{k}_{r-1}"'] & \Sigma^t F_r C\mathbf{X} \arrow[r, "\Sigma^t \pi"] \arrow[d, "\tilde{k}_r"'] & \Sigma^t D_r \mathbf{X} \arrow[d, equal] \\
 & \bigvee_{q=1}^{r-1} \Sigma^t D_q \mathbf{X} \arrow[r, "\Sigma^t \iota"] & \bigvee_{q=1}^{r} \Sigma^t D_q \mathbf{X} \arrow[r, "\Sigma^t \pi"] & \Sigma^t D_r \mathbf{X}
\end{tikzcd}$$

and similarly with r replaced by r' for $r' < r$. That $\tilde{k}_r$ is an equivalence follows by precisely the same inductive argument as was used to prove Theorem 3·8. The maps h_q are specified in Definition 5·5. Since $\alpha_t g^{-1}$ is an H-map and k_r is specified as a sum in Definition 6·3, the last statement is clear.

Examples 7·2. The theorem applies to $\mathscr{C} = \mathscr{C}(Y)$ for any space Y such that $\mathscr{B}(Y, r)$ maps injectively to R^t. In particular, it applies to any n-manifold Y with $t \leqslant 2rn$ taken as the embedding dimension of $B(Y, r)$.

Remarks 7·3. $F_r\ C(R^n, X)$ is equivalent to $F_r C_n X$. Here Snaith (22) obtained an analogous splitting of $\Sigma^t F_r C_n X$, but with somewhat different methods and a wholly different estimate on t. Indeed, our reading of his work gives the value

$$t = (2 + i(i + \epsilon - 1))n + i + 1 \quad \text{if} \quad r = 2i + \epsilon \quad \text{with} \quad \epsilon = 0 \text{ or } 1.$$

When $r \geqslant 9$, this value of t is greater than $2rn$. With a few exceptions when $n = 2$, this t is less than $2rn$ when $r \leqslant 8$ and $n \geqslant 2$. However, the main point of comparison is that his splitting is not given by James–Hopf maps. That is, there is no obvious way (even after any further finite number of suspensions) to extend his splitting maps

$$\Sigma^t F_r C_n X \to \Sigma^t D_q(\mathscr{C}_n, X)$$

over all of $\Sigma^t C_n X$.

One approach to stabilization would be to exploit the following observations.

Remarks 7·4. Continue to write e_q for its composite with the standard inclusion $j: R^t \to R^v$ for $v \geqslant t$. Suppose that $r < s$ and there are injections $e_q: \mathscr{B}_q \to R^v$ for $r < q \leqslant s$. There results a commutative diagram

$$\begin{tikzcd}
F_r C\mathbf{X} \arrow[r, "k_r"] \arrow[d, "\iota"'] & C\left(R^t, \bigvee_{q=1}^{r} D_q\mathbf{X}\right) \arrow[d, "C(j,\iota)"] \\
F_s C\mathbf{X} \arrow[r, "k_s"] & C\left(R^v, \bigvee_{q=1}^{s} D_q\mathbf{X}\right)
\end{tikzcd}$$

For any space X, (13), 4·8 and 5·2, imply that the following diagram commutes, where the maps σ are the natural suspension inclusions.

$$\begin{tikzcd}
C(R^t, X) \arrow[d, "C(j,1)"'] & C_t X \arrow[l, "g"'] \arrow[r, "\alpha_t"] \arrow[d, "\sigma"] & \Omega^t\Sigma^t X \arrow[d, "\sigma"] \\
C(R^v, X) & C_v X \arrow[l, "g"'] \arrow[r, "\alpha_v"] & \Omega^v\Sigma^v X
\end{tikzcd}$$

Upon choosing homotopy inverses g^{-1}, we deduce that the following stability diagram is homotopy commutative.

$$\begin{tikzcd}
\Sigma^v F_r C\mathbf{X} \arrow[r, "\Sigma^{v-t}\tilde{k}_r"] \arrow[d, "\iota"'] & \bigvee_{q=1}^{r} \Sigma^v D_q\mathbf{X} \arrow[d, "\iota"] \\
\Sigma^v F_s C\mathbf{X} \arrow[r, "\tilde{k}_s"] & \bigvee_{q=1}^{s} \Sigma^v D_q\mathbf{X}
\end{tikzcd}$$

It follows easily that if $\mathscr{B}_q$ injects to R^{t_q} for $q \geqslant 1$ and an increasing sequence $\{t_q\}$, then the resulting maps

$$\tilde{k}_r: \Sigma^{t_r}(F_r C\mathbf{X}) \to \bigvee_{q=1}^{r} \Sigma^{t_r}(D_q \mathbf{X})$$

induce a weak equivalence from the suspension spectrum of $C\mathbf{X}$ to the suspension spectrum of $\bigvee_{q \geqslant 1} D_q \mathbf{X}$. However, this stable equivalence, like those of Snaith (22), would only be well-defined and natural modulo $\lim^1$ terms, the ambiguity arising from first passing from the k_r to the $\tilde{k}_r$ and then to spectra rather than first passing from the k_r to k_∞ and then to a spectrum level $\tilde{k}_\infty$. Moreover, there are interesting examples, such as $\mathscr{C}(R^\infty)$ and $\mathscr{C}_\infty$, for which the requisite injections fail to exist.

8. *The stable splitting theorem.* Before stating our stable theorem, we must fix conventions on spectra. As usual when dealing with iterated loop spaces, spectra are best defined as sequences of spaces E_i with E_i homeomorphic to ΩE_{i+1}. Maps are sequences $E_i \to E'_i$ compatible with the given homeomorphisms. This gives a category of spectra $\mathscr{S}$ which admits a homotopy category $h\mathscr{S}$. The stable category $H\mathscr{S}$ is obtained from $h\mathscr{S}$ by formally inverting its weak equivalences. Similarly, $H\mathscr{T}$ is obtained from the homotopy category $h\mathscr{T}$ by inverting its weak equivalences. Because a functor on $\mathscr{T}$ or $\mathscr{S}$ which inverts weak equivalences is necessarily homotopy-preserving, $H\mathscr{T}$ and $H\mathscr{S}$ can equally well be constructed by formally inverting the weak equivalences of $\mathscr{T}$ and $\mathscr{S}$, without mention of homotopy categories.

For $X \in \mathscr{T}$, let $Q_\infty X = \{Q\Sigma^i X\} \in \mathscr{S}$. Then Q_∞ gives functors $\mathscr{T} \to \mathscr{S}$, $h\mathscr{T} \to h\mathscr{S}$, and $H\mathscr{T} \to H\mathscr{S}$, the last being the appropriate stabilization functor from spaces to the stable category. On all three levels, Q_∞ is the free functor adjoint to the zeroth space functor. Via loops on evaluation maps, we have a natural map of spectra $\xi: Q_\infty E_0 \to E_0$. For a map $f: X \to E_0$ of spaces, $\tilde{f} = \xi \circ Q_\infty f: Q_\infty X \to E$ is the unique map of spectra such that $\tilde{f}_0 \circ \eta = f$. If $E = Q_\infty X'$ and f factors through the inclusion $\Omega^t \Sigma^t X' \to QX'$, say via $f_t: X \to \Omega^t \Sigma^t X'$ with adjoint $\tilde{f}_t: \Sigma^t X \to \Sigma^t X'$, then $\tilde{f} = \Omega^t Q_\infty \tilde{f}_t$. In particular, $\tilde{\eta}: Q_\infty X \to Q_\infty X$ is the identity map.

(See (14), II and (15), for details on the material above; the reader who prefers to translate to the equivalent stable categories of Boardman or Adams should have no trouble doing so.)

Our general construction of stable James–Hopf maps depends on the following result, which will be proved in the next section.

LEMMA 8·1. *There is a functor W from coefficient systems to based spaces (with basepoint 0) together with the following data.*

(i) *A natural contracting homotopy $d: 0 \simeq 1$ on W.*

(ii) *A natural injection $e_q: \mathscr{B}_q = \mathscr{C}_q/\Sigma_q \to W\mathscr{C}$ for each $q \geqslant 1$.*

(iii) *A natural injection $i: \coprod_{q \geqslant 1} W\mathscr{C} \to W\mathscr{C}$.*

(iv) *A natural injection $\iota: R^\infty \to W\mathscr{C}$ compatible with the maps i. Moreover, $F(R^\infty, r)$ and $F(W\mathscr{C}, r)$ are contractible spaces for all r, and $C(\iota, 1): C(R^\infty, \mathbf{X}) \to C(W\mathscr{C}, \mathbf{X})$ is a weak equivalence for all Π-spaces $\mathbf{X}$ and an H-map when $\mathbf{X}$ comes from a space X.*

Here the last clause will be immediate from Lemma 2·7 (i) and Definition 6·1. By Proposition 2·9 we have the following natural maps in $H\mathscr{T}$, where $C(\iota, 1), g$, and α_∞ are all H-maps.

$$C(W\mathscr{C}, X) \xrightarrow{C(\iota,1)^{-1}} C(R^\infty, X) \xrightarrow{g^{-1}} C_\infty X \xrightarrow{\alpha_\infty} QX.$$

For a given map $f: X' \to C(W\mathscr{C}, X)$ in $H\mathscr{T}$, we agree to write $\tilde{f}: Q_\infty X' \to Q_\infty X$ for the map in $H\mathscr{S}$ induced by freeness from the composite of f and the displayed map

$$C(W\mathscr{C}, X) \to QX.$$

By Examples 2·4 (iii) and the discussion above, for $\eta: X \to C(W\mathscr{C}, X)$, $\tilde{\eta} = 1$ in $H\mathscr{S}$.

Now let $\mathscr{C}$ be any Σ-free coefficient system. Following Remarks 5·4, let

$$\tilde{\mathscr{C}} = \mathscr{C} \times \mathscr{C}(R^\infty),$$

so that $\tilde{C}$ is a separated coefficient system and the projection π_1: $\tilde{C}\mathbf{X} \to C\mathbf{X}$ is a weak equivalence for all Π-spaces $\mathbf{X}$. By abuse of notation, let k_r denote the following composite in $H\mathscr{T}$,

$$C\mathbf{X} \xrightarrow{\pi_1^{-1}} \tilde{\mathscr{C}}\mathbf{X} \xrightarrow{k_r} C\left(W\tilde{\mathscr{C}}, \bigvee_{q=1}^{r} D_q(\tilde{\mathscr{C}}, \mathbf{X})\right) \xrightarrow{C(1, \vee D_q(\pi_1, 1))} C\left(W\tilde{\mathscr{C}}, \bigvee_{q=1}^{r} D_q \mathbf{X}\right),$$

where $D_q\mathbf{X} = D_q(\mathscr{C}, \mathbf{X})$ and the middle map k_r is obtained by application of Definitions 6·3 and 6·5 to $\tilde{\mathscr{C}}$ and $W\tilde{\mathscr{C}}$. The adjoint construction above then gives

$$\tilde{k}_r: Q_\infty C\mathbf{X} \to Q_\infty\left(\bigvee_{q=1}^{r} D_q \mathbf{X}\right).$$

With these notations, our stable splitting theorem reads as follows.

THEOREM 8·2. *For all Σ-free coefficient systems $\mathscr{C}$, Π-spaces $\mathbf{X}$, and $r \geq 1$ (including $r = \infty$),*

$$\tilde{k}_r: Q_\infty F_r C\mathbf{X} \to \bigvee_{q=1}^{r} Q_\infty D_q \mathbf{X}$$

is an isomorphism in the stable category. Moreover, $\tilde{k}_r$ is the sum over q of restrictions of stable James–Hopf maps

$$h_q^s: Q_\infty C\mathbf{X} \to Q_\infty D_q \mathbf{X}.$$

The h_q^s, and thus also k_r, are natural with respect to maps of coefficient systems and maps of Π-spaces.

Proof. For r finite, the diagram of Proposition 6·4 (applied to $\tilde{\mathscr{C}}$) and a trivial diagram chase give the following commutative diagram in $H\mathscr{S}$.

$$\begin{tikzcd}
Q_\infty F_{r-1} C\mathbf{X} \arrow[r, "Q_\infty \iota"] \arrow[d, "\tilde{k}_{r-1}"'] & Q_\infty F_r C\mathbf{X} \arrow[r, "Q_\infty \pi"] \arrow[d, "\tilde{k}_r"'] & Q_\infty D_r \mathbf{X} \arrow[d, equal] \\
\bigvee_{q=1}^{r-1} Q_\infty D_q \mathbf{X} \arrow[r, "\iota"] & \bigvee_{q=1}^{r} Q_\infty D_q \mathbf{X} \arrow[r, "\pi"] & Q_\infty D_r \mathbf{X}
\end{tikzcd}$$

Here we have used that Q_∞ commutes with wedges. Since Q_∞ also preserves cofiberings, $\tilde{k}_r$ is an isomorphism by induction and the five lemma. Since Q_∞ commutes with colimits of sequences of cofibrations and since k_∞ for $\widetilde{\mathscr{C}}$ is the colimit of the k_r, the conclusion for $r = \infty$ follows easily from the conclusion for $r < \infty$. The maps h_q^s are obtained by application of the adjoint (tilde) construction above to the composites

$$C\mathbf{X} \xrightarrow{\pi_1^{-1}} \widetilde{C}\mathbf{X} \xrightarrow{j_q} C(\widetilde{\mathscr{B}}_q, D_q(\widetilde{\mathscr{C}}, \mathbf{X})) \xrightarrow{C(e_q,\, \iota \circ D_q(\pi_1, 1))} C\left(W\widetilde{\mathscr{C}}, \bigvee_{q=1}^{r} D_q\mathbf{X}\right),$$

and the fact that $\tilde{k}_r$ is the sum of the h_q^s is immediate from Definition 6·3. Naturality with respect to maps of Π-spaces and injections of Σ-free coefficient systems is evident. The full naturality in $\mathscr{C}$ is not at all obvious and will be proved in the next section.

Remarks 8·3 (i) For separated coefficient systems $\mathscr{C}$ the h_q^s can be defined without use of $\widetilde{\mathscr{C}}$, and for particular coefficient systems like $\mathscr{C}(Y)$ for a manifold Y we have described other James maps $C\mathbf{X} \to QD_q\mathbf{X}$ in $H\mathscr{T}$. All such maps factor as composites

$$C\mathbf{X} \xrightarrow{f} C(R^\infty, D_q\mathbf{X}) \xrightarrow{g^{-1}} C_\infty D_q\mathbf{X} \xrightarrow{\alpha_\infty} QD_q\mathbf{X}$$

for suitable maps f in $H\mathscr{T}$. We shall prove in the next section that, for a given $\mathscr{C}$, all maps f that arise from any variant of our basic constructions are actually equal in $H\mathscr{T}$. Note too that, since $D_q(\mathscr{C}, \mathbf{X})$ is a functor of $\mathscr{C}$, $\alpha_\infty g^{-1}$ is certainly natural and thus the naturality statement left unproved in the theorem reduces to consideration of the naturality of f.

(ii) When f in (i) factors through $C(R^t, D_q\mathbf{X})$, the middle diagram of Remarks 7·4 and our discussion of the relationship between maps of spaces and maps of spectra shows that h_q^s is determined by its zeroth map and that the latter is determined by the unstable James–Hopf map h_q adjoint to

$$C\mathbf{X} \xrightarrow{f} C(R^t, D_q\mathbf{X}) \xrightarrow{g^{-1}} C_t D_q\mathbf{X} \xrightarrow{\alpha_t} \Omega^t\Sigma^t D_q\mathbf{X}$$

via

$$(h_q^s)_0 = \Omega^t Q h_q \colon QC\mathbf{X} \cong \Omega^t Q\Sigma^t C\mathbf{X} \to \Omega^t Q\Sigma^t D_q\mathbf{X} \cong QD_q\mathbf{X}.$$

In particular, $(h_q^s)_0\eta \colon C\mathbf{X} \to QD_q\mathbf{X}$ factors through $\Omega^t\Sigma^t D_q\mathbf{X}$.

While the theorem depends on formal properties of α_∞, it does not depend on the approximation theorem. However, we may specialize to $\mathscr{C}_n$ or $\mathscr{C}(R^n)$ and quote that result to obtain the following sharpening of Snaith's stable decompositions (22) to parametrized splittings which are well-defined and natural in the stable category. As in the introduction, let $D_{n,q}X = D_q(\mathscr{C}_n, X)$.

COROLLARY 8·4. *For all based spaces P and connected based spaces X and all $n \geqslant 1$ (including $n = \infty$, when $\Omega^\infty\Sigma^\infty X$ is to be interpreted as QX), the following natural maps are isomorphisms in the stable category.*

$$Q_\infty(P \wedge \Omega^n\Sigma^n X) \xleftarrow{Q_\infty(1 \wedge \alpha_n)} Q_\infty(P \wedge C_n X) \xrightarrow{\tilde{k}_\infty} \bigvee_{q \geqslant 1} Q_\infty(P \wedge D_{n,q}X).$$

For $m < n$, the following compatibility diagram commutes.

$$\begin{tikzcd}
Q_\infty(P \wedge \Omega^m\Sigma^m X) \arrow[d, "Q_\infty(1\wedge\sigma)"'] & Q_\infty(P \wedge C_m X) \arrow[l, "Q_\infty(1\wedge\alpha_m)"'] \arrow[r, "\tilde{k}_\infty"] \arrow[d, "Q_\infty(1\wedge\sigma)"] & \bigvee_{q\geqslant 1} Q_\infty(P \wedge D_{m,q} X) \arrow[d, "\vee Q_\infty(1\wedge\sigma)"] \\
Q_\infty(P \wedge \Omega^n\Sigma^n X) & Q_\infty(P \wedge C_n X) \arrow[l, "Q_\infty(1\wedge\alpha_n)"'] \arrow[r, "\tilde{k}_\infty"] & \bigvee_{q\geqslant 1} Q_\infty(P \wedge D_{n,q} X)
\end{tikzcd}$$

Moreover, these assertions remain true with $C_n X, D_{n,q} X$, and α_n replaced by

$$C(R^n, X), \quad D_q(R^n, X), \quad \text{and} \quad \alpha_n g^{-1}.$$

Here the diagram follows from naturality and the diagrams of Remarks 7·4. By the claims of Remarks 8·3 and by Lemma 5·7, the splittings of $C_n X$ and $C(R^n, X)$ agree under g and the latter splitting is determined by James–Hopf maps

$$h_q: \Sigma^t\Omega^n\Sigma^n X \to \Sigma^t D_q(R^n, X)$$

with $t \leqslant (2q-1)n$, a better estimate on t being given by Proposition 5·8 when $q = 2$. We reiterate that earlier methods failed to give any such destabilized splitting maps. The results of Kirley (11) quoted in Remarks 5·10 show that, for $n \geqslant 2$, the entire splitting of $Q_\infty \Omega^n\Sigma^n X$ cannot be improved to a splitting of $\Sigma^t\Omega^n\Sigma^n X$ for any finite t.

Finally, we point out that, by Remarks 6·2, specialization to $\mathscr{C}(R^n \times Y)$ for arbitrary spaces Y gives splittings of other interesting n-fold loop spaces.

9. *The existence and uniqueness of stable James–Hopf maps.* We must prove Lemma 8·1 and the claims in Remarks 8·3 (i). We begin with a general result about the homotopy types of configuration spaces.

LEMMA 9·1. *Let W be a space which admits an injection $j: W \times I \to W$ and a homotopy $h: j_0 \simeq 1$ through injections. Then the inclusion $i: F(W, r) \to W^r$ is a homotopy equivalence for all r. In particular, $F(W, r)$ is contractible if W is contractible.*

Proof. Define $f: W^r \to F(W, r)$ by

$$f(w_1, \ldots, w_r) = \left(j(w_1, 1), j(w_2, \tfrac{1}{2}), \ldots, j\left(w_r, \frac{1}{r}\right)\right)$$

and define $K: W^r \times [-1, 1] \to W^r$ by letting

$$K(w_1, \ldots, w_r, t) = (K_1(w_1, t), K_2(w_2, t), \ldots, K_r(w_r, t)),$$

where

$$K_q(w, t) = \begin{cases} j(w, t/q) & \text{if} \quad 0 \leqslant t \leqslant 1, \\ h(w, -t) & \text{if} \quad -1 \leqslant t \leqslant 0. \end{cases}$$

Then K_{-1} is the identity map and $K_1 = i \circ f$. Moreover, $K(z, t) \in \operatorname{Im} i$ if $r \in \operatorname{Im} i$, hence K restricts to a homotopy $1 \simeq f \circ i$ on $F(W, r)$.

Examples 9·2. The hypothesis of the lemma is satisfied by the following spaces W.

(i) $W = V^\infty$, where V is a non-zero real topological vector space; the requisite maps j and h are specified by

$$j((v_1, v_2, \ldots, v_k, \ldots), t) = (tv, v_1, v_2, \ldots, v_k, \ldots) \quad (0 \neq v \in V),$$

and

$$h((v_1, v_2, \ldots, v_k, \ldots), t) = (tv_1, tv_2 + (1-t)\,v_1, \ldots, tv_k + (1-t)\,v_{k-1}, \ldots).$$

(ii) $W = W' \times Y$, where W' satisfies the hypothesis and Y is arbitrary.

Now the following simple construction proves Lemma 8·1.

Construction 9·3. Let $Z\mathscr{C}$ be the wedge of the unreduced cones on the spaces $\mathscr{B}_q$, with cone points 0 as common basepoint. Let $W\mathscr{C} = R^\infty \times Z\mathscr{C}$ with basepoint $(0, 0)$. Multiplication of points of R^∞ and of each cone coordinate by t at time t gives $d\colon 0 \simeq 1$. Let $e_q\colon \mathscr{B}_q \to W\mathscr{C}$ embed $\mathscr{B}_q$ as the product of $\{0\}$ and the base of the qth cone. Let

$$\iota\colon R^\infty \to W\mathscr{C} \quad \text{embed} \quad R^\infty \quad \text{as} \quad R^\infty \times \{0\}$$

and let

$$i\colon \coprod_{q \geqslant 1} R^\infty \to R^\infty \quad \text{determine} \quad i\colon \coprod_{q \geqslant 1} W\mathscr{C} \to W\mathscr{C} \quad \text{via} \quad i(r, z) = (i(r), z).$$

We turn to the naturality and uniqueness assertions of the previous section (and warn the reader that the preprint version gave stronger statements than are actually correct). As explained in Remarks 8·3(i), we need only study the uniqueness and naturality of the relevant maps $C\mathbf{X} \to C(R^\infty, D_q\mathbf{X})$. We shall do this by exploiting the following special property of $\mathscr{C}(R^\infty)$.

LEMMA 9·4. *Let* $\mathscr{C} = \mathscr{C}(R^\infty) \times \mathscr{C}(R^\infty)$. *Then the projections* π_1 *and* π_2 *from* $C\mathbf{X}$ *to* $C(R^\infty, \mathbf{X})$ *are homotopic for all* Λ*-spaces* $\mathbf{X}$.

Proof. Consider the following diagram of coefficient systems.

$$\begin{tikzcd}
\mathscr{C}(R^\infty) \arrow[d, "\mathscr{C}(i_1)"'] & \mathscr{C}(R^\infty) \times \mathscr{C}(R^\infty) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[dl, "p_1"'] \arrow[d, "q"] \arrow[dr, "p_2"] & \mathscr{C}(R^\infty) \arrow[d, "\mathscr{C}(i_2)"] \\
\mathscr{C}(R^\infty) & \mathscr{C}(R^\infty) & \mathscr{C}(R^\infty)
\end{tikzcd}$$

Here i_1 and i_2 are the linear isometries $R^\infty \to R^\infty$ specified on the standard basis

$$\{e_j \mid j \geqslant 1\} \quad \text{by} \quad e_j \to e_{2j-1} \quad \text{and} \quad e_j \to e_{2j},$$

p_1 and p_2 are defined to be $\mathscr{C}(i_1) \circ \pi_1$ and $\mathscr{C}(i_2) \circ \pi_2$, and q is induced by the linear isometry $R^\infty \oplus R^\infty \to R^\infty$ specified by $e'_j \to e_{2j-1}$ and $e''_j \to e_{2j}$ on the bases for the two copies of R^∞. Since the space of linear isometries $R^\infty \to R^\infty$ is contractible (14), I. 1. 3, there is a path of isometries from the identity map of R^∞ to i_1. Therefore $\pi_1 \simeq p_1$ through maps of coefficient systems. Similarly $\pi_2 \simeq p_2$. Further, $p_1 \simeq q$ through maps of coefficient systems since the linear maps $R^\infty \oplus R^\infty \to R^\infty$ specified by $e_j' \to e_{2j-1}$ and $e_j'' \to te_{2j}$ $(0 \leqslant t \leqslant 1)$, induce maps $h_t\colon \mathscr{C} \to \mathscr{C}(R^\infty)$ of coefficient systems with $h_0 = p_1$ and $h_1 = q$. Similarly, $p_2 \simeq q$. For each fixed Λ-space $\mathbf{X}$, the functor $\mathscr{C} \to C\mathbf{X}$ from coefficient systems to spaces is easily seen to be continuous, and it follows that

$$\pi_1 \simeq p_1 \simeq q \simeq p_2 \simeq \pi_2\colon C\mathbf{X} \to C(R^\infty, \mathbf{X}).$$

Proposition 9·5. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be Σ-free coefficient systems and let $f_i\colon \mathscr{C} \to \mathscr{C}_i$ be maps of coefficient systems. Let $\tilde{\mathscr{C}}_i = \mathscr{C}_i \times \mathscr{C}(R^\infty)$. For any Λ-space $\mathbf{X}$, the following diagram commutes in $H\mathscr{T}$.*

$$\begin{tikzcd}
 & C_1\mathbf{X} \arrow[rr, "\pi_1^{-1}"] & & \tilde{C}_1\mathbf{X} \arrow[dr, "\pi_2"] & \\
C\mathbf{X} \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & & & C(R^\infty, \mathbf{X}) \\
 & C_2\mathbf{X} \arrow[rr, "\pi_1^{-1}"] & & \tilde{C}_2\mathbf{X} \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

In particular, f_1 is homotopic to f_2 if $\mathscr{C}_1 = \mathscr{C}(R^\infty) = \mathscr{C}_2$.

Proof. The projections $\pi_1\colon \tilde{C}_i\mathbf{X} \to C_i\mathbf{X}$ are weak equivalences by Lemma 2·7 (i). Since $f_i = \pi_i f$, where $f = (f_1, f_2)\colon \mathscr{C} \to \mathscr{C}_1 \times \mathscr{C}_2$, we may assume without loss of generality that $\mathscr{C} = \mathscr{C}_1 \times \mathscr{C}_2$ and $f_i = \pi_i$. Observe that the following diagram of coefficient systems commutes, where π and π' are obtained by transposing the middle two coordinates and then projecting on the first two or last two coordinates.

$$\begin{tikzcd}
\mathscr{C}_1 \times \mathscr{C}_2 \arrow[d, "\pi_1"] & \mathscr{C}_1 \times \mathscr{C}(R^\infty) \times \mathscr{C}_2 \times \mathscr{C}(R^\infty) \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, "\pi_1"] & \mathscr{C}(R^\infty) \times \mathscr{C}(R^\infty) \arrow[d, "\pi_1"] \\
\mathscr{C}_1 & \mathscr{C}_1 \times \mathscr{C}(R^\infty) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathscr{C}(R^\infty)
\end{tikzcd}$$

Similarly with $\mathscr{C}_1$ replaced by $\mathscr{C}_2$ in the bottom row. Abbreviate $\mathscr{D} = \tilde{\mathscr{C}}_1 \times \tilde{\mathscr{C}}_2$ and $\mathscr{C}' = \mathscr{C}(R^\infty) \times \mathscr{C}(R^\infty)$. By the diagram above, the following is a commutative diagram

$$\begin{tikzcd}
C\mathbf{X} \arrow[d, "\pi_1"] & D\mathbf{X} \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, "\pi_1"] & C'\mathbf{X} \arrow[d, "\pi_1"] \\
C_1\mathbf{X} & \tilde{C}_1\mathbf{X} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C(R^\infty, \mathbf{X})
\end{tikzcd}$$

By the lemma, $\pi_1 \simeq \pi_2 : C'\mathbf{X} \to C(R^\infty, \mathbf{X})$. The result follows, since the symmetric diagram shows that the maps $\pi_2 \pi_1^{-1} \pi_i : C\mathbf{X} \to C(R^\infty, \mathbf{X})$ in $H\mathscr{T}$ are both equal to $\pi_1 \pi' \pi^{-1}$.

This result combines with the following observation to yield a uniqueness assertion for James maps $C\mathbf{X} \to C(R^\infty, D_q\mathbf{X})$. Note first that the composite of a James system $\mathscr{C} \to \mathscr{C}'$ with maps $\bar{\mathscr{C}} \to \mathscr{C}$ and $\mathscr{C}' \to \bar{\mathscr{C}}'$ of coefficient systems yields a James system $\bar{\mathscr{C}} \to \bar{\mathscr{C}}'$.

LEMMA 9·6. *If $\{\xi'_{qr}\} : \mathscr{C} \to \mathscr{C}_1$ and $\{\xi''_{qr}\} : \mathscr{C} \to \mathscr{C}_2$ are James systems, then*

$$\{(\xi'_{qr}, \xi''_{qr})\} : \mathscr{C} \to \mathscr{C}' = \mathscr{C}_1 \times \mathscr{C}_2$$

is a James system and the following diagram is commutative for any Π-space $\mathbf{X}$.

$$\begin{tikzcd} & & C_1 D_q\mathbf{X} \\ C\mathbf{X} \arrow[urr, "j_q"] \arrow[r, "j_q"] \arrow[drr, "j_q"'] & C' D_q\mathbf{X} \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ & & C_2 D_q\mathbf{X} \end{tikzcd}$$

The previous results focus attention on the passage from $C\mathbf{X}$ to $C(R^\infty, \mathbf{X})$ via the projections of $\tilde{\mathscr{C}} = \mathscr{C} \times \mathscr{C}(R^\infty)$. We show next that this is the only way to get from $C\mathbf{X}$ to $C(R^\infty, \mathbf{X})$ by use of maps induced from maps of Σ-free coefficient systems. To see this, consider the collection of diagrams of Σ-free coefficient systems of the form

$$\mathscr{C} \xleftarrow{e_1} \mathscr{C}_1 \xrightarrow{f_1} \mathscr{C}_2 \leftarrow \ldots \to \mathscr{C}_{2k-2} \xleftarrow{e_k} \mathscr{C}_{2k-1} \xrightarrow{f_k} \mathscr{C}(R^\infty) \quad (k \geqslant 1), \tag{*}$$

where $\mathscr{C}$ is fixed, the $\mathscr{C}_i$ are variable, and the e_i are weak equivalences (in the sense that each $\mathscr{C}_{i,r} \to \mathscr{C}_{i-1,r}$ is a weak equivalence). For any Λ-space $\mathbf{X}$, there results a diagram

$$C\mathbf{X} \xleftarrow{e_1} C_1\mathbf{X} \xrightarrow{f_1} C_2\mathbf{X} \leftarrow \ldots \to C_{2k-2}\mathbf{X} \xleftarrow{e_k} C_{2k-1}\mathbf{X} \xrightarrow{f_k} C(R^\infty, \mathbf{X}) \tag{**}$$

in which the e_i are weak equivalences by Lemma 2·7 (i). Thus (**) may be viewed as a map $C\mathbf{X} \to C(R^\infty, \mathbf{X})$ in $H\mathscr{T}$. We have the following uniqueness assertion.

PROPOSITION 9·7. *There is one and only one map $C\mathbf{X} \to C(R^\infty, \mathbf{X})$ in $H\mathscr{T}$ of the form (**), namely*

$$C\mathbf{X} \xleftarrow{\pi_1} \tilde{C}\mathbf{X} \xrightarrow{\pi_2} C(R^\infty, \mathbf{X}).$$

Proof. We require homotopy pullbacks. Given a diagram

$$\mathscr{C}_1 \xrightarrow{f} \mathscr{C}_2 \xleftarrow{e} \mathscr{C}_3$$

of Σ-free coefficient systems with e a weak equivalence, we construct a homotopy commutative diagram of coefficient systems

$$\begin{tikzcd}
\mathscr{P} \arrow[r, "f'"] \arrow[d, "e'"'] & \mathscr{C}_3 \arrow[d, "e"] \\
\mathscr{C}_1 \arrow[r, "f"] & \mathscr{C}_2
\end{tikzcd}$$

with $\mathscr{P}$ Σ-free and e' a weak equivalence by setting

$$\mathscr{P}_r = \{(c_1, \gamma_2, c_3) | \gamma_2(0) = f(c_1) \quad \text{and} \quad \gamma_2(1) = e(c_3)\} \subset \mathscr{C}_{1,r} \times \mathscr{C}^I_{2,r} \times \mathscr{C}_{3,r}$$

and letting e'_r and f'_r be the evident projections. By the functoriality of homotopy pullbacks (on the space level), $\mathscr{P}$ inherits a structure of contravariant functor on Λ from the $\mathscr{C}_i$. By induction on k, it follows from the diagram

$$\begin{tikzcd}
 & & \mathscr{P} \arrow[dl, "e_1'"'] \arrow[dr, "f_1'"] & & \\
\mathscr{C} & \mathscr{C}_1 \arrow[l, "e_1"'] \arrow[r, "f_1"] & \mathscr{C}_2 & \mathscr{C}_3 \arrow[l, "e_2"'] \arrow[r, "f_2"] & \mathscr{C}_4 \; \cdots
\end{tikzcd}$$

that $(**)$ will equal $\pi_2\pi_1^{-1}$ in $H\mathscr{T}$ provided that this is so when $k = 1$. Thus assume given a diagram of Σ-free coefficient systems

$$\begin{tikzcd}
\mathscr{C} & \mathscr{C}' \arrow[l, "e"'] \arrow[r, "f"] & \mathscr{C}(R^\infty)
\end{tikzcd}$$

with e a weak equivalence and consider the following diagram, where $\mathscr{P}$ is the homotopy pullback of π_1 and e.

$$\begin{tikzcd}
 & \mathscr{C}' \arrow[dr, "e"] \arrow[drr, "f"] & & \\
\mathscr{P} \arrow[ur, "\pi_1'"] \arrow[dr, "e'"'] & & \mathscr{C} & \mathscr{C}(R^\infty) \\
 & \tilde{\mathscr{C}} \arrow[ur, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

The maps π_1, π_1', e, and e' are all weak equivalences. In $H\mathscr{T}$,

$$f\pi_1' = \pi_2 e' : P\mathbf{X} \to C(R^\infty, \mathbf{X})$$

by the last statement of Proposition 9·5. A chase of the diagram gives

$$fe^{-1} = \pi_2\pi_1^{-1}: C\mathbf{X} \to C(R^\infty, \mathbf{X}),$$

hence the conclusion.

In particular, the maps $C(\iota, 1)^{-1}: C(W\mathscr{C}, \mathbf{X}) \to C(R^\infty, \mathbf{X})$ used in the previous section are equal to $\pi_2\pi_1^{-1}$ in $H\mathscr{T}$. The naturality in $\mathscr{C}$ of the maps h_q^s of Theorem 8·2 now follows from Lemma 9·6, Proposition 9·5, and an easy diagram chase, and similarly for the uniqueness assertions of Remarks 8·3 (i).

Remarks 9·8. Our entire sequence of results beginning with Lemma 9·4 applies equally well upon restriction to finite filtrations. Indeed, all maps in sight are filtration-preserving except the James maps themselves, and j_q takes $F_r C\mathbf{X}$ to $F_{(r-q,q)} C'D_q\mathbf{X}$.

While the full strength of our results requires use of R^∞, we do have the following unstable analogue of Proposition 9·5.

PROPOSITION 9·9. *Let $f_1: \mathscr{C} \to \mathscr{C}(R^s)$ and $f_2: \mathscr{C} \to \mathscr{C}(R^t)$ be maps of coefficient systems. Then the following diagram is homotopy commutative, where i_1 and i_2 are induced by the inclusions of R^s and R^t in R^{s+t} as the first s and last t coordinates.*

$$\begin{tikzcd}
 & C(R^s, \mathbf{X}) \arrow[dr, "i_1"] & \\
C\mathbf{X} \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & C(R^{s+t}, \mathbf{X}) \\
 & C(R^t, \mathbf{X}) \arrow[ur, "i_2"'] &
\end{tikzcd}$$

Proof. As in the proof of Proposition 9·5, we may assume that

$$\mathscr{C} = \mathscr{C}(R^s) \times \mathscr{C}(R^t) \quad \text{and} \quad f_i = \pi_i.$$

Let $q: \mathscr{C} \to \mathscr{C}(R^{s+t})$ be induced by the identity map of R^{s+t}. As in the proof of Lemma 9·4, $i_1\pi_1 \simeq q \simeq i_2\pi_2$.

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