Commun. Math. Phys. 182, 303 – 318 (1996)

Communications in Mathematical Physics

Splitting of the Family Index

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Received: 29 January 1996 / Accepted: 23 April 1996

Abstract: We establish a general splitting formula for index bundles of families of Dirac type operators. Among the applications, our result provides a positive answer to a question of Bismut and Cheeger [BC2].

Introduction

Due to the increasing influence of the topological quantum field theory (cf. [A]), it becomes very important to study the behavior of natural analytic invariants under the splitting of manifolds. The first splitting formula, for the most fundamental invariants—the index of Dirac operators, was proved by Atiyah–Patodi–Singer [APS], in relation with their index theorem for manifolds with boundary. There is also an alternative approach surrounding the "Bojarski conjecture". For the latter see the book of Booss–Wojciechowski [BW1] for a thorough discussion.

In this paper, we treat the index bundles associated to families of Dirac type operators in this framework. Although our results should play a role in the topological field theory, our original motivation is in fact to answer a question of Bismut and Cheeger [BC2, Remark 2.3].

There are family versions of the Atiyah–Patodi–Singer index theorem due to Bismut–Cheeger [BC1, BC2] and subsequently Melrose–Piazza [MP1, MP2] among others. These formulas do imply the splitting formula for the Chern character of index bundles but fail to work on the level of K-theory. It is the purpose of this paper to establish the desired splitting formula in full generality on the K-theoretic level.

For a family of Dirac operators acting on a closed manifold Z and parametrized by B, the index bundle lives in either the K or K^1 group of the parameter space B, depending on the parity of dim Z (See [AS1 and AS2]). Now let Y be a (family of) closed hypersurface in Z separating it into two pieces Z_1, Z_2. Since Z_1 and Z_2 are now (families of) manifolds with boundary, we impose a boundary condition as in [MP1, 2] by choosing a spectral section ($cl(1)$-spectral section if $\dim Y$ is even). This is a generalization of the (now classical) Atiyah–Patodi–Singer boundary condition to the family situation. Thus let P_1, P_2 be two spectral sections (again,

$cl(1)$-spectral sections if $\dim Y$ is even). These give rise to well-defined index bundles $\mathrm{ind}(D_{Z_1}, P_1)$, $\mathrm{ind}(D_{Z_2}, 1 - P_2)$. Moreover, according to [MP1, 2], the difference $[P_1 - P_2]$ also defines an element of $K^{\dim Z}(B)$.

Theorem 1.1. *The following identity holds in* $K^{\dim Z}(B)$,

$$\mathrm{ind} D_Z = \mathrm{ind}(D_{Z_1}, P_1) + \mathrm{ind}(D_{Z_2}, 1 - P_2) + [P_1 - P_2]. \tag{1.8}$$

In their study of the family index for manifolds with boundary [BC2], Bismut–Cheeger obtained a certain additivity property for the Chern character of the index bundle. They then raise the question of whether the additivity already holds in K-theory. Theorem 1.1 provides an affirmative answer to their question.

The general idea here is to reduce the problem to the (finite) cylinder, although our methods of proof are quite different in the two cases where $\dim Z$ is even, resp. odd. In the first case we make essential use of Caldéron projections, similar to the approach of Bojarski as developed in [BW1]. Roughly speaking, the Caldéron projection, say, defined by (Z_1, D_{Z_1}), is the projection onto the boundary values of harmonic spinors on Z_1. Thus contribution to the index bundle from Z_1 is effectively encoded in this projection, which lives in Y. This explains why the index bundle of the whole manifold is actually the same as that of a collar of the separating hypersurface, with the Caldéron projections as the boundary conditions. However, Caldéron projections are definitely not spectral sections, and for this reason, we introduce the notion of a generalized spectral section and deal with the splitting problem in this general framework.

Another point we would like to make is that, unlike the case of a single operator, the computation on the cylinder is far from immediate. In fact we use the cylinder to construct a natural deformation (or, homotopy) of a generalized spectral section and employ a perturbation argument of Atiyah–Singer [AS1] to construct a "majorizing" generalized spectral section.

In the second case, the reduction to the cylinder is accomplished by adapting a beautiful idea of Bunke [Bu] in his approach to the splitting formula for η invariants. In this case we do not have to go to generalized spectral sections and the "majorizing spectral section" is constructed in [MP1, 2]. It thus turns out that the splitting problem for family indices is much simpler than it was expected.

There is also a very interesting approach to splitting formulas of index bundles from the symplectic point of view due to Nicolaescu [N]. Although his splitting formula does not reach the generality as we stated in this paper, Nicolaescu mainly concentrates on the real case (vs. the complex case treated here) which is more complicated.

The rest of this paper is organized as follows. In Sect. 1, we introduce the notations and state the main splitting theorem of this paper. In Sect. 2, we give a proof of the splitting formula for the even dimensional fiber case and, in the process, prove several results which seem to have some interest of their own. Theorem 2.13 generalizes the relative index theorem of [MP1]. And Theorem 2.14 relates the index bundle of a family of Dirac operators on manifolds with boundary to the index bundle of some Toeplitz type operators on the boundary. Finally in Sect. 3, we give the proof of the result for the odd dimensional fiber case.

1. A Splitting Formula for the Families Index

Let $\pi_Z : Z \to M \to B$ be a smooth fibration with compact base and connected closed fibers. We assume that TZ is spin and carries a fixed spin structure.

Let $\pi_Y : Y \to M' \to B$ be a subfibration of π_Z such that for each $b \in B$, $Y_b = \pi_Y^{-1}(b)$ is a closed hypersurface of $Z_b = \pi_Z^{-1}(b)$, cutting Z_b into two disjoint pieces

$$Z_b = Z_{1,b} \cup Z_{2,b}, \quad \text{with} \quad Z_{1,b} \cap Z_{2,b} = Y_b. \tag{1.1}$$

Thus we have also two induced fibrations of manifolds with boundary $\pi_{Z_i} : Z_i \to M_i \to B$, $i = 1, 2$, with the intersection fibration π_Y.

We identify a neighborhood of M' in M with $\pi' : [-2,2] \times Y \to U = [-2,2] \times M' \to B$, with $\{0\} \times Y = Y$. Also we specify the sides of M_i, $i = 1,2$ by requiring that $M_1 \cap [-2,0] \times M' = [-2,0] \times M'$.

We fix on TY the orientation induced from TZ_1. Then TY carries a canonically induced spin structure.

As usual we require product structures on the collar. Thus, let g^{TY} be a metric on TY. Choose the metric g^{TZ} on TZ so that

$$g^{TZ}|_U = dt^2 + g^{TY}, \tag{1.2}$$

where t is the parameter of $[-2,2]$. Let $S(TZ)$ be the spinor bundle of (TZ, g^{TZ}). If $\dim Z$ is even, then one also has the canonical splitting $S(TZ) = S_+(TZ) \oplus S_-(TZ)$.

Also, let $\pi_E : E \to M$ be a complex vector bundle over M and g^E be a metric on E such that $g^E|_U$ does not depend on t. Let ∇^E be a unitary connection on (E, g^E) such that $\nabla^E|_U$ does not depend on t.

These data give rise to the family of (twisted) Dirac operators, D_{Z_b} $(b \in B)$ acting on $\Gamma(S(TZ_b) \otimes E_b)$. If $\dim Z$ is even, we use the same notation D_{Z_b} to denote its restriction on $\Gamma(S_+(TZ_b) \otimes E_b)$. Let $D_{Z_{i,b}}$, $i = 1, 2$ be the restriction of D_{Z_b} on $Z_{i,b}$.

Let $S(TY)$ be the spinor bundle of (TY, g^{TY}). If $\dim Y$ is even, then the Clifford action $c(\frac{\partial}{\partial t})$ on the boundary of M_1 determines the canonical $\mathbf{Z}_2$ splitting $S(TY) = S_+(TY) \oplus S_-(TY)$.

For any $b \in B$, let D_{Y_b} be the (twisted) Dirac operator acting on $\Gamma(S(TY_b) \otimes E|_{Y_b})$ canonically constructed from g^{TY}, g^E and ∇^E. Again in case $\dim Y$ is even, we use the same notation D_{Y_b} for its restriction on $\Gamma(S_+(TY_b) \otimes E|_{Y_b})$.

We therefore obtain four smooth families of Dirac operators D_Z, D_{Z_1}, D_{Z_2} and D_Y over B, among which two have well-defined index bundles ([AS1, AS2]) (a suspension construction is involved for the odd dimensional family):

$$\operatorname{ind} D_Z \in K^{\dim Z}(B) \tag{1.3}$$

and

$$\operatorname{ind} D_Y \in K^{\dim Y}(B). \tag{1.4}$$

Now since π_Y is a boundary family, by the cobordism invariance of family indices (cf. [Sh, MP1 and MP2]), one has

$$\operatorname{ind} D_Y = 0. \tag{1.5}$$

It follows from [MP1, MP2] that there exist ($cl(1)$- if $\dim Y$ is even) spectral sections of D_Y (see also Sect. 2 for more details).

Let $P_i, i = 1, 2$ be two ($cl(1)$- if $\dim Y$ is even) spectral sections of D_Y. Then by [MP1, MP2], one obtains two families of elliptic (self-adjoint if $\dim Z$ is odd) boundary problems: $(D_{Z_1}, P_1) = \{(D_{Z_{1,b}}, P_{1,b})\}_{b \in B}$ and $(D_{Z_2}, 1 - P_2) = \{(D_{Z_{2,b}}, 1 - P_{2,b})\}_{b \in B}$. And according to [MP1, MP2, AS1 and AS2], one gets well-defined index bundles

$$\operatorname{ind}(D_{Z_1}, P_1) \in K^{\dim Z}(B),$$
$$\operatorname{ind}(D_{Z_2}, 1 - P_2) \in K^{\dim Z}(B). \tag{1.6}$$

On the other hand, there is also a K-group element naturally associated to P_1, P_2 denoted by ([MP1, MP2])

$$[P_1 - P_2] \in K^{\dim Z}(B). \tag{1.7}$$

The main result of this paper can now be stated as follows.

Theorem 1.1. *The following identity holds in* $K^{\dim Z}(B)$,

$$\operatorname{ind} D_Z = \operatorname{ind}(D_{Z_1}, P_1) + \operatorname{ind}(D_{Z_2}, 1 - P_2) + [P_1 - P_2]. \tag{1.8}$$

Remark 1.2. Equation (1.8) on the cohomological level is immediate from the index formulas in [MP1 and MP2].

Remark 1.3. In the case where $\dim Z$ is even and that $\ker D_{Y_b}$ are of constant dimension, Theorem 1.1 provides a positive answer to a question of Bismut and Cheeger [BC2, Remark 2.3].

Remark 1.4. Theorem 1.1 holds in fact for general Dirac type operators as our proof in the next two sections works in such a generality. For the simplicity of our presentation, we content ourselves with the pure (twisted) Dirac operators case.

2. Splitting: The Even Dimensional Case

In this section we present a proof of Theorem 1.1 for the case where $\dim Z$ is even. It turns out that, for our purpose, we need a generalized version of the concept of the spectral section of Melrose and Piazza [MP1]. This is because we want to emphasize the role of Caldéron projections.

We make the same assumptions and use the same notation as in Sect. 1. We also make the extra assumption that $\dim Z$ is even in this section.

This section is organized as follows. In a), we introduce generalized spectral sections and present their basic properties. In b), we prove a family version of the Bojarski theorem. In c), we prove a family index theorem for cylinders which has been used in b). In d), we prove a relative family index theorem and use it to complete the proof of Theorem 1.1.

a) Generalized spectral sections. We first recall the definition of the spectral section for the family D_Y in [MP1]. By definition a spectral section P of D_Y is a continuous family $P = \{P_b\}_{b \in B}$ of self-adjoint zeroth order pseudodifferential projections of $L^2(S(TY_b) \otimes E|_{Y_b})$ such that for any $b \in B$, P_b is a finite dimensional perturbation of the Atiyah–Patodi–Singer [APS] projection $P_{\geqq, b}$ of D_{Y_b}. In particular they all have the same principal symbol. In view of (1.5), the existence of P is clear via [MP1].

We now give the definition of what we call generalized spectral sections.

Definition 2.1. *A generalized spectral section Q of D_Y is a continuous family of self-adjoint zeroth order pseudodifferential projections $Q = \{Q_b\}_{b\in B}$ with $Q_b : L^2(S(TY_b)\otimes E|_{Y_b}) \to L^2(S(TY_b)\otimes E|_{Y_b})$, such that the principle symbol of Q is the same as that of a spectral section P.*

Remark 2.2. As in [MP1], the generalized spectral section can be defined for any family of self-adjoint elliptic first order pseudodifferential operators with vanishing index bundle.

Remark 2.3. There are in fact many equivalent definitions of generalized spectral sections. For example one can use the language of the infinite dimensional Grassmannian discussed in [BW1]. Also one can use the infinite dimensional Lagrangian subspaces as used by Nicolaescu in [N].

One of the most important examples of generalized spectral sections is the following.

Example 2.4. For any $b \in B$, let $C_{i,b}$, $i = 1, 2$ be the Calderon projection on $L^2(S(TY_b)\otimes E_{Y_b})$ associated to $D_{Z_{i,b}}$ (see [BW1] for more details). Then C_1, $1 - C_2$ are both generalized spectral sections of D_Y (Compare also with [BW1 and N]).

Remark 2.5. In the above example one should be careful about the different orientations on the boundary induced from two bounding pieces. Also, the continuity of Calderon projections follows easily from the construction, see [Se] (cf. also [BW1, N]).

We now state some basic properties of generalized spectral sections.

Let Q_1, Q_2 be two generalized spectral sections of D_Y. For any $b \in B$, set

$$T_b(Q_1,Q_2) = Q_{2,b}Q_{1,b} : Q_1L^2(S(TY_b)\otimes E|_{Y_b}) \to Q_2L^2(S(TY_b)\otimes E|_{Y_b}). \tag{2.1}$$

Then $T(Q_1,Q_2) = \{T_b(Q_1,Q_2)\}_{b\in B}$ forms a continuous family of Fredholm operators over B. Thus according to Atiyah–Singer [AS1], it determines an element

$$[Q_1 - Q_2] = \operatorname{ind} T(Q_1,Q_2) \in K(B). \tag{2.2}$$

In particular, if Q_1 and Q_2 are two spectral sections in the sense of [MP1], then one verifies easily that $[Q_1 - Q_2]$ is the same difference element as defined by Melrose–Piazza [MP1] (see also (1.7)).

Definition 2.6. *Two generalized spectral sections Q_1, Q_2 of D_Y are said to be homotopic to each other if there is a continuous curve of generalized spectral sections P_u, $0 \leqq u \leqq 1$, of D_Y such that $Q_1 = P_0$, $Q_2 = P_1$.*

The following two properties follow easily from some elementary arguments concerning Fredholm families.

Proposition 2.7. *If Q_i, $i = 1, 2, 3$, are three generalized spectral sections of D_Y such that Q_1 and Q_2 are homotopic to each other, then one has*

$$[Q_1 - Q_3] = [Q_2 - Q_3] \quad \text{in} \quad K(B). \tag{2.3}$$

Proposition 2.8. *If Q_i, $i = 1,2,3$ are three generalized spectral sections of D_Y, then the following identity holds in $K(B)$,*

$$[Q_1 - Q_2] + [Q_2 - Q_3] = [Q_1 - Q_3]. \tag{2.4}$$

Homotopic generalized spectral sections can be constructed in the following way, which will be used in our proof of the splitting theorem. Given a generalized spectral section Q of D_Y, let $V = [0, 1] \times Y$ be the cylinder and D_V the Dirac operator on V. Consider the subspace (for simplicity we are suppressing the bundles here)

$$H_1 = \{u|_{\{1\}\times Y} : u \in H^{\frac{1}{2}}(V),\ D_V u = 0,\ Q_0(u|_{\{0\}\times Y}) = 0\}.$$

Let $\mathrm{I} - Q_1$ denote the orthogonal projection from $L^2(Y)$ onto H_1.

Proposition 2.9. *Q_1 defines a generalized spectral section which is homotopic to Q_0.*

Proof. We first show that Q_1 defines a generalized spectral section. Let

$$H(D_V) = \{u|_{\partial V} : \quad u \in H^{\frac{1}{2}}(V),\ D_V u = 0\}$$

be the Cauchy data space of D_V and C the Caldéron projection. Also, denote $H(Q_0) = \mathrm{Im} Q_0 \times L^2(Y)$. If we denote P the projection of $L^2(Y) \times L^2(Y)$ onto the intersection $H(D_V) \cap H(Q_0)$, then the second component is precisely $\mathrm{I} - Q_1$. Therefore it suffices to show that P is a pseudodifferential operator of order 0, with the correct principal symbol.

Consider the map

$$R: \ L^2(Y) \times L^2(Y) \longrightarrow L^2(Y) \times L^2(Y) \times L^2(Y),$$

taking (u_1, u_2) to $((\mathrm{I} - C)(u_1, u_2), Q_0 u_1)$. This is a pseudodifferential operator of order 0 whose kernel is precisely the space $H(D_V) \cap H(Q_0)$. By a result of Seeley [Se], P is also a pseudodifferential operator of order 0. Moreover one can compute its principal symbol from that of R. Since the principal symbol of R is given by that of a spectral section, one verifies the same for P.

Now, for $0 \leqq t \leqq 1$ define

$$H_t = \{u|_{\{t\}\times Y} : \quad u \in H^{\frac{1}{2}}(V),\ D_V u = 0,\ Q_0(u|_{\{0\}\times Y}) = 0\},$$

and Q_t accordingly. By the above discussion, for each t, Q_t is a generalized spectral section. To see that this defines a continuous family connecting Q_0 and Q_1, note that H_t is the intersection of the Cauchy data space of $V_t = [0,\ t] \times Y$ with the space $\mathrm{Im} Q_0 \times L^2(Y)$. Since the Cauchy data space varies continuously with the parameter, so is H_t. □

b) Family index: A Bojarski type theorem. Recall from Example 2.4 that C_1, C_2 are the families of Caldéron projections of D_{Z_1}, D_{Z_2} respectively. Thus C_1 and $1 - C_2$ forms two generalized spectral sections of D_Y.

The main result of this subsection is the following generalization to fibrations of the Bojarski theorem stated in [BW1, Theorem 24.1].

Theorem 2.10. *The following identity holds in $K(B)$,*

$$\mathrm{ind}D_Z = [C_1 - (1 - C_2)].$$

Proof. In what follows, we will suppress the subscript b for $b \in B$ most of the time. All the constructions below are understood fiberwise unless stated otherwise. Also, while being fiberwise, all the procedures will be continuous with respect to $b \in B$, as will be clear from the context. We will not keep repeating this every time.

Let $Z_1(-1)$, $Z_2(1)$ be defined by

$$\begin{aligned} Z_1(-1) &= Z_1 - (-1,0] \times Y, \\ Z_2(1) &= Z_2 - [0,1) \times Y. \end{aligned} \tag{2.6}$$

Let $C_1(-1)$, $C_2(1)$ be the Calderon projections of $D_{Z_1(-1)}$, $D_{Z_2(1)}$ respectively. From the product structures on the collar,

$$D_Z = c\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t} + D_Y\right) \quad \text{on} \quad U. \tag{2.7}$$

We denote $[-1,1] \times Y = V$. Let $H^1(V, C_1(-1), C_2(1))$ be the Hilbert space defined by

$$\begin{aligned} H^1(V, C_1(-1), C_2(1)) &= \{u \in H^1(V, (S_+(TZ) \otimes E)|_V) : \\ (1 - C_1(-1))u|_{\{-1\}\times Y} &= 0, (1 - C_2(1))u|_{\{1\}\times Y} = 0\}. \end{aligned} \tag{2.8}$$

Then by the unique continuation property of Dirac type operators and by the definition of Calderon projections (see [BW1] for both), for any $u \in H^1(V, C_1(-1), C_2(1))$, there is a unique element $\tilde{u} \in H^1(Z, S_+(TZ) \otimes E)$ such that

$$\begin{aligned} &\tilde{u}|_V = u, \\ &D_{Z_1(-1)}\tilde{u}|_{Z_1(-1)} = 0, D_{Z_2(1)}\tilde{u}|_{Z_2(1)} = 0. \end{aligned} \tag{2.9}$$

In this way $H^1(V, C_1(-1), C_2(1))$ embeds as a Hilbert subspace of $H^1(Z, S_+(TZ) \otimes E)$. We still note this subspace by $H^1(V, C_1(-1), C_2(1))$. Of critical importance here is to note that this embedding is again continuous with respect to $b \in B$.

Let $H^\perp$ be the orthogonal complement of $H^1(V, C_1(-1), C_2(1))$ in $H^1(Z, S_+(TZ) \otimes E)$. Denote R the orthogonal projection from $H^1(Z, S_+(TZ) \otimes E)$ to $H^1(V, C_1(-1), C_2(1))$.

On the other hand, $L^2(Z, S_-(TZ) \otimes E)$ has the natural orthogonal splitting

$$\begin{aligned} L^2(Z, S_-(TZ) \otimes E) &= L^2(V, (S_-(TZ) \otimes E)|_V) \\ &\oplus L^2(Z_1(-1), (S_-(TZ) \otimes E)|_{Z_1(-1)}) \oplus L^2(Z_2(1), (S_-(TZ) \otimes E)|_{Z_2(1)}). \end{aligned} \tag{2.10}$$

Let P be the orthogonal projection from $L^2(Z, S_-(TZ) \otimes E)$ to $L^2(V, (S_-(TZ) \otimes E)|_V)$.

Now consider the Fredholm operator

$$D_Z : H^1(Z, S_+(TZ) \otimes E) \to L^2(Z, S_-(TZ) \otimes E).$$

First of all one has clearly that

$$(1-P)D_Z R = 0. \tag{2.11}$$

Thus D_Z has the following decomposition:

$$D_Z = (1-P)D_Z(1-R) + PD_Z R + PD_Z(1-R). \tag{2.12}$$

The following lemma is critical to our proof.

Lemma 2.11. *The operator* $(1-P)D_Z(1-R)$ *is invertible.*

Proof. (i) Let $\alpha \in H^1(Z, S_+(TZ)\otimes E)$ be such that

$$(1-P)D_Z(1-R)\alpha = 0. \tag{2.13}$$

By (2.11), one gets

$$(1-P)D_Z\alpha = 0. \tag{2.14}$$

By definition this means that $D_Z\alpha = 0$ on $Z_1(-1)\cup Z_2(1)$, which implies that $\alpha \in \mathrm{Im}R$. Thus one has $\ker((1-P)D_Z(1-R)) = 0$.

(ii) Let $\beta \in L^2(Z, S_-(TZ)\otimes E)$ be such that

$$\langle (1-P)D_Z(1-R)\alpha, \beta\rangle = 0 \tag{2.15}$$

for any $\alpha \in H^1(Z, S_+(TZ)\otimes E)$. Once again by (2.11) and also by the self-adjointness of P, one gets

$$\langle D_Z\alpha, (1-P)\beta\rangle = 0. \tag{2.16}$$

This implies $(1-P)\beta \in \ker D_Z^*$. Now D_Z^* is also a Dirac operator while $(1-P)\beta|_V = 0$. By the unique continuation property of Dirac operators one sees $(1-P)\beta = 0$. Now as D_Z is a Fredholm operator while an orthogonal projection maps a closed subspace to a closed subspace, one sees again from (2.11) that $\mathrm{Im}((1-P)D_Z(1-R))$ is closed. Thus we get $\mathrm{coker}((1-P)D_Z(1-R)) = 0$.

Combining (i), (ii), we finish the proof of the lemma. □

Now for any $u \in [0,1]$, set

$$D_Z(u) = (1-P)D_Z(1-R) + PD_Z R + uPD_Z(1-R). \tag{2.17}$$

By the invertibility of $(1-P)D_Z(1-R)$, one verifies directly that for each $u \in [0,1]$, $D_Z(u)$ is a Fredholm operator.

In summary, what we have is a continuous curve of continuous families of Fredholm operators $D_Z(u) = \{D_{Z_b}(u)\}_{b\in B}$, $u \in [0,1]$. Thus by the homotopy invariance of the index bundle one gets

$$\begin{aligned}\mathrm{ind}D_Z &= \mathrm{ind}D_Z(1) = \mathrm{ind}D_Z(0) = \mathrm{ind}(1-P)D_Z(1-R) + \mathrm{ind}PD_Z R\\ &= \mathrm{ind}PD_Z R = \mathrm{ind}(D_V, C_1(-1), 1-C_2(1)),\end{aligned} \tag{2.18}$$

where $(D_V, C_1(-1), 1-C_2(1))$ is the corresponding family of elliptic boundary problems on the cylinders $V = \{V_b\}_{b\in B}$.

The evaluation of the left cylindrical family index will be carried out in the next subsection.

c). The family index for cylinders. In this subsection we prove the following result.

Theorem 2.12. *Let P, Q be two generalized spectral sections of D_Y. Let (D_V, P, Q) be the family of elliptic boundary problems defined by*

$$\mathrm{Dom}(D_V, P, Q) = \{u \in H^1(V, (S_+(TZ) \otimes E)|_V) : \\ (1-P)u|_{\{-1\}\times Y} = 0,\ Qu|_{\{1\}\times Y} = 0\}. \tag{2.19}$$

Then the following identity holds in $K(B)$,

$$\mathrm{ind}(D_V, P, Q) = [P - Q]. \tag{2.20}$$

Proof. (i) If the kernels of the family of the Fredholm operators $T_b(P, Q)$, $b \in B$, defined in (2.1) have constant dimension, then (2.20) can be verified directly with the help of Proposition 2.9 and Proposition 2.7.

(ii) In the general case, by applying the procedure in [AS1] to the family of Fredholm operators

$$T(1-Q, 1-P) : (1-Q)L^2(Y, S(TY) \otimes E|_Y) \to (1-P)L^2(Y, S(TY) \otimes E|_Y), \tag{2.21}$$

one constructs easily a generalized spectral section R of D_Y such that

$$(1-Q)L^2(Y, S(TY) \otimes E|_Y) \subset (1-R)L^2(Y, S(TY) \otimes E|_Y), \tag{2.22}$$

and that the family of Fredholm operators $T(1-R, 1-P)$ has vanishing cokernels, which implies that the family $T(P, R)$ has vanishing cokernels.

Now set $V_1 = [-1, 0] \times Y$, $V_2 = [0, 1] \times Y$. Let R' be the generalized spectral section of D_Y determined by

$$(1-R')H^{1/2}(\{1\} \times Y, S(TY) \otimes E|_Y) \\ = \{u|_{\{1\}\times Y} : u \in H^1(V_2, (S_+(TZ) \otimes E)|_{V_2}), D_Z|_{V_2}u = 0, Ru|_{\{0\}\times Y} = 0\}. \tag{2.23}$$

It is clear that R' is well-defined by (2.23). Furthermore, the two generalized spectral sections R and R' are homotopic to each other.

Now from (2.23), (2.22) and the unique continuation property of Dirac operators, one sees immediately that there is a canonical embedding of

$$H^1(V_1, P, Q) = \{u \in H^1(V_1, (S_+(TZ) \otimes E)|_{V_1}) : \\ (1-P)u|_{\{-1\}\times Y} = 0, Qu|_{\{0\}\times Y} = 0\} \tag{2.24}$$

into

$$H^1(V, P, R') = \{u \in H^1(V, (S_+(TZ) \otimes E)|_V) : \\ (1-P)u|_{\{-1\}\times Y} = 0, R'u|_{\{1\}\times Y} = 0\}. \tag{2.25}$$

We can then apply the deformation trick (2.17) here to this new pairing. By using (2.17), one in fact deduces easily that

$$\mathrm{ind}(D_V, P, R') = \mathrm{ind}(D_{V_1}, P, Q) + \mathrm{ind}\, T(1-R, 1-Q). \tag{2.26}$$

Also since R' is homotopic to R, by the homotopy invariance of the index bundle and by (i), one finds

$$\mathrm{ind}(D_V, P, R') = \mathrm{ind}(D_V, P, R) = [P - R]. \tag{2.27}$$

From (2.26), (2.27), one then gets

$$\begin{aligned}\mathrm{ind}(D_{V_1}, P, Q) &= [P - R] - \mathrm{ind}T(Q, R) \\ &= [P - Q]. \end{aligned} \tag{2.28}$$

On the other hand, one clearly has

$$\mathrm{ind}(D_{V_1}, P, Q) = \mathrm{ind}(D_V, P, Q). \tag{2.29}$$

The proof of Theorem 2.12 is completed by combining (2.28) and (2.29). □
We now complete the proof of Theorem 2.10.
From Theorem 2.12, we have

$$\mathrm{ind}(D_V, C_1(-1), 1 - C_2(1)) = [C_1(-1) - (1 - C_2(1))]. \tag{2.30}$$

Now $C_1(-1)$ and $C_2(1)$ are homotopic to C_1, C_2 respectively, so we get

$$\mathrm{ind}(D_V, C_1(-1), 1 - C_2(1)) = [C_1 - (1 - C_2)]. \tag{2.31}$$

By (2.31), (2.18), the proof of Theorem 2.10 is completed. □

d) A relative index theorem and the proof of Theorem 1.1. In order to deduce Theorem 1.1 from Theorem 2.10, we now state a relative index theorem.

Let P be a generalized spectral section of D_Y. By now it is clear that (D_{Z_1}, P) defines a continuous family of elliptic boundary problems over B and thus determines an index bundle

$$\mathrm{ind}(D_{Z_1}, P) \in K(B). \tag{2.32}$$

Let now Q be another generalized spectral section of D_Y. Then one has the following relative index theorem which extends the corresponding relative index theorem of Melrose and Piazza [MP1] for spectral sections.

Theorem 2.13. *The following identity holds in $K(B)$,*

$$\mathrm{ind}(D_{Z_1}, P) - \mathrm{ind}(D_{Z_1}, Q) = [Q - P]. \tag{2.33}$$

The proof of (2.33) follows in fact from the following general index formula.

Theorem 2.14. *The following identity holds in $K(B)$,*

$$\mathrm{ind}(D_{Z_1}, P) = [C_1 - P]. \tag{2.34}$$

Proof. By combining the methods in b), c), one deduces easily that

$$\mathrm{ind}(D_{Z_1}, P) = \mathrm{ind}(V_1, C_1, P) = [C_1 - P]. \tag{2.35}$$

Theorem 2.14 follows. □

Proof of Theorem 2.13. One has

$$\begin{aligned}\operatorname{ind}(D_{Z_1},P)-\operatorname{ind}(D_{Z_1},Q)&=[C_1-P]-[C_1-Q]\\&=[Q-P],\end{aligned}\tag{2.36}$$

which is exactly (2.33). □

We can now prove Theorem 1.1. In fact we can prove a generalized version so that P_i, $i=1,2$ in (1.8) can be assumed to be generalized spectral sections.

Proof of Theorem 1.1. It is clear that Theorem 2.14 also holds for D_{Z_2}. Thus one has by Theorem 2.10,

$$\begin{aligned}\operatorname{ind}D_Z&=[C_1-(1-C_2)]=[C_1-P_1]+[P_2-(1-C_2)]+[P_1-P_2]\\&=\operatorname{ind}(D_{Z_1},P_1)+\operatorname{ind}(D_{Z_2},1-P_2)+[P_1-P_2].\end{aligned}\tag{2.37}$$

The proof of Theorem 1.1 is completed. □

Remark 2.15. Theorem 2.14 is closely related to Theorem 6.2 in Nicolaescu [N]. The difference is that in [N], Nicolaescu works on the real case. It is also assumed in [N] that all the operators D_{Z_b}, $b\in B$, have the same symbol.[1]

3. The Case of Odd Dimensional Fibers

In this section we give a proof of Theorem 1.1 for the case where $\dim Z$ is odd. We make the same assumptions and use the same notation as in Sect. 1. We assume in all this section that $\dim Z$ is odd.

The method we present in this section is actually much simpler than that of Sect. 2. It is based on a beautiful idea of Bunke [Bu] (cf. also [DF]).

Let us first recall the definition of the $cl(1)$-spectral sections which appeared in the statement of Theorem 1.1.

Recall that since $\dim Z$ is odd, the Clifford action $c(\frac{\partial}{\partial t})$ determines the $\mathbf{Z}_2$-splitting

$$S(TY)=S_+(TY)\oplus S_-(TY).\tag{3.1}$$

Denote by σ the action of this Clifford action on $S(TY)$.

By definition [MP2], a $cl(1)$-spectral section P of D_Y is simply a spectral section of D_Y satisfying that

$$\sigma P+P\sigma=\sigma.\tag{3.2}$$

The principal advantage is that if P is a $cl(1)$-spectral section of D_Y, then for any $b\in B$, the elliptic boundary problem

$$\begin{aligned}(D_{Z_{1,b}},P_b):H^1(Z_{1,b},P_b)&=\{u\in H^1(Z_{1,b},(S(TZ)\otimes E)|_{Z_{1,b}}\ :\ P_bu|_{Y_b}=0\}\\&\to L^2(Z_{1,b},(S(TZ)\otimes E)|_{Z_{1,b}})\end{aligned}\tag{3.3}$$

is self-adjoint (cf. [BW2, MP2]).

[1] We are informed by a referee that in a subsequent work Nicolaescu considers the case of family with varying principal symbols

Similarly, the elliptic boundary problem

$$(D_{Z_{2,b}}, 1 - P_b) : H^1(Z_{2,b}, 1 - P_b) = \{u \in H^1(Z_{2,b}, (S(TZ) \otimes E)|_{Z_{2,b}} : \\ (1 - P_b)u|_{Y_b} = 0\} \to L^2(Z_{2,b}, (S(TZ) \otimes E)|_{Z_{2,b}}) \quad (3.4)$$

is also self-adjoint for any $b \in B$.

Thus we obtain two continuous families of self-adjoint Fredholm operators (D_{Z_1}, P) and $(D_{Z_2}, 1 - P)$. According to [AS2], they determine the index bundles

$$\mathrm{ind}(D_{Z_1}, P) \in K^1(B), \\ \mathrm{ind}(D_{Z_2}, 1 - P) \in K^1(B). \quad (3.5)$$

Also recall that the family D_Z determines an index bundle

$$\mathrm{ind} D_Z \in K^1(B). \quad (3.6)$$

Let now Q be another $cl(1)$-spectral section of D_Y. Then there is a well-defined element $[P - Q] \in K^1(B)$, which is constructed through a suspension argument and then appeals Definition (2.2) between resulting Fredholm operators, see [MP2] for more details.

For convenience, we recall the statement of Theorem 1.1 as follows.

Theorem 3.1. *The following identity holds in* $K^1(B)$,

$$\mathrm{ind}(D_Z) = \mathrm{ind}(D_{Z_1}, P) + \mathrm{ind}(D_{Z_2}, 1 - Q) + [P - Q]. \quad (3.7)$$

Proof. As has been said, we will adapt an idea of Bunke [Bu] to prove (3.7).

Set

$$Z_1(1) = Z_1 \cup [0, 1] \times Y, \quad Z_2(-1) = Z_2 \cup [-1, 0] \times Y, \\ V = [-1, 1] \times Y. \quad (3.8)$$

Let $\xi \in C^\infty([-1, 1])$ be such that

$$\xi|_{[-1,-1/2]} = 1, \; 0 \leqq \xi|_{[-1/2,1/2]} \leqq 1, \; \xi|_{[1/2,1]} = 0 \quad (3.9)$$

and that

$$\gamma = (1 - \xi^2)^{1/2} \quad (3.10)$$

is also smooth. Clearly ξ extends to $Z_1(1)$ by equaling 1 on $Z_1(-1) = Z_1 - (-1, 0] \times Y$. It also extends to $Z_2(-1)$ by equaling 0 on $Z_2(1) = Z_2 - [0, 1) \times Y$. Thus γ also extends to $Z_1(1)$ and $Z_2(-1)$.

Set

$$H = L^2(Z, S(TZ) \otimes E) \oplus L^2(V, (S(TZ) \otimes E)|_V), \\ H_0 = L^2(Z_1(1), (S(TZ) \otimes E)|_{Z_1(1)}) \oplus L^2(Z_2(-1), (S(TZ) \otimes E)|_{Z_2(-1)}). \quad (3.11)$$

Following Bunke [Bu], we define the operators a, b, c, $d : H \to H_0$ fiberwise by

(i) a is the multiplication by ξ on Z followed by the transfer to $Z_1(1)$,
(ii) b is the multiplication by ξ on V followed by the transfer to $Z_2(-1)$,

(iii) c is the multiplication by γ on Z followed by the transfer to $Z_2(-1)$,
(iv) d is the multiplication by γ on V followed by the transfer to $Z_1(1)$. (3.12)

Set as in [Bu],

$$W = a + b + c - d : H \to H_0. \tag{3.13}$$

The fact of critical importance is the following

Lemma 3.2. ([Bu]). *The operator* $W : H \to H_0$ *is unitary.*

Another important observation is that if we consider the elliptic boundary problems $(D_{Z_1(1)}, P)$, $(D_{Z_2(-1)}, 1 - Q)$ and $(D_V, (1-Q)|_{\{-1\}\times Y}, P_{\{1\}\times Y})$, then one finds that ([Bu])

$$W^* \mathrm{dom}\{(D_{Z_1(1)}, P) \oplus (D_{Z_2(-1)}, 1 - Q)\}$$
$$= \mathrm{dom}\{D_Z \oplus (D_V, (1-Q)|_{\{-1\}\times Y}, P|_{\{1\}\times Y})\}. \tag{3.14}$$

Thus the two operators

$$D_+ := D_Z \oplus (D_V, (1-Q)|_{\{-1\}\times Y}, P|_{\{1\}\times Y}) \tag{3.15}$$

and

$$D_- := W^*((D_{Z_1(1)}, P) \oplus (D_{Z_2(-1)}, 1 - Q))W \tag{3.16}$$

has the same domain and furthermore one verifies easily that the operator

$$G := D_+ - D_- \tag{3.17}$$

is a bounded operator (cf. [Bu]).

Clearly all the above procedures are continuous with respect to $b \in B$. Thus, by the trivial linear homotopy, one gets

$$\mathrm{ind} D_+ = \mathrm{ind} D_- \quad \text{in} \quad K^1(B). \tag{3.18}$$

Combining (3.15)–(3.18), we get

$$\mathrm{ind} D_Z = \mathrm{ind} W^*((D_{Z_1(1)}, P) \oplus (D_{Z_2(-1)}, 1 - Q))W$$
$$-\mathrm{ind}(D_V, (1-Q)|_{\{-1\}\times Y}, P_{\{1\}\times Y})$$
$$= \mathrm{ind}(D_{Z_1}, P) + \mathrm{ind}(D_{Z_2}, 1 - Q) - \mathrm{ind}(D_V, (1-Q)|_{\{-1\}\times Y}, P|_{\{1\}\times Y}). \tag{3.20}$$

Theorem 3.1 will then follow from (3.20) and the following result.

Theorem 3.3. *The following identity holds in* $K^1(B)$,

$$\mathrm{ind}(D_V, (1-Q)|_{\{-1\}\times Y}, P_{\{1\}\times Y}) = [Q - P]. \tag{3.21}$$

Proof. Set $V_1 = [-1,0] \times Y$, $V_2 = [0,1] \times Y$. Then by cutting V at $\{0\} \times Y$ and adapting Bunke's trick to the cylinder V in view of this cutting, one easily deduces that for any third $cl(1)$-spectral section R of D_Y, one has the following identity between index bundles of families of elliptic boundary problem for cylinders,

$$\begin{aligned} \operatorname{ind}(D_V, (1-Q)|_{\{-1\}\times Y}, P_{\{1\}\times Y}) &= \operatorname{ind}(D_{V_1}, (1-Q)|_{\{-1\}\times Y}, R_{\{0\}\times Y}) \\ &\quad + \operatorname{ind}(D_{V_2}, (1-R)|_{\{0\}\times Y}, P|_{\{1\}\times Y}). \end{aligned} \tag{3.22}$$

Now as in [MP2], one can assume R satisfying $RP = P$ and $RQ = Q$. With this R, one verifies directly by identifying the virtual index bundles that

$$\operatorname{ind}(D_{V_1}, (1-Q)|_{\{-1\}\times Y}, R|_{\{0\}\times Y}) = [Q - R] \tag{3.23}$$

and

$$\operatorname{ind}(D_{V_2}, (1-R)|_{\{0\}\times Y}, P_{\{1\}\times Y}) = [R - P]. \tag{3.24}$$

By (3.22)–(3.24), one then gets

$$\operatorname{ind}(D_V, (1-Q)|_{\{-1\}\times Y}, P|_{\{1\}\times Y}) = [Q - R] + [R - P] = [Q - P]. \tag{3.25}$$

This proves Theorem 3.3. The proof of Theorem 3.1 is thus also completed. □

Remark 3.4. If we take $B = S^1$, then we get a splitting formula for spectral flows.

Remark 3.5. As in Sect. 2, one can also define generalized $cl(1)$-spectral sections and prove Theorem 1.1 for them. We leave this easy modification to the interested reader.

Remark 3.6. The proof of Theorem 1.1 presented above also works for the case of even dimensional fibers. It thus turns out that the splitting problem for index bundles is surprisingly simple. We still include the proof in Sect. 2 here in this paper, just because we think it does have some interest of its own. Also it would still be interesting if one could modify the method in Sect. 2 to cover the case of odd dimensional fibers.

Acknowledgements The work of the first author was partially supported by the Alfred P. Sloan Foundation. The work of the second author was partially supported by the Chinese National Science Foundation. He would also like to thank the Courant Institute for its hospitality. Both authors thank the referees for some very helpful comments.

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Communicated by S.-T. Yau