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SPLITTING RING OF A MONIC SEPARABLE POLYNOMIAL

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In this short note we prove that if $S = R[x] = R[X]/\langle f(X)\rangle$ is separable over R, where $f(X)$ is a monic polynomial over R, then the embedding set up by Auslander and Goldman is the same as the splitting ring of f over R constructed by Barnard.

Throughout, the terms "ring", "algebra", and "ring homomorphism" are to be interpreted as in the category of commutative rings with identity. S is an algebra over the ring $R, f(X)$ is a monic polynomial of degree n over R, d_f is the discriminant of f, Z_i, W_i $(1 \leqq i \leqq n)$ are indeterminates over R, G is the symmetric group on n symbols, and $\epsilon(\sigma)$ is the signature of the permutation σ.

Auslander and Goldman [**1**, Theorem A.7, p. 399] show that if S is separable over R such that S is free of rank n as a module over R, then S can be embedded into a Galois extension Ω of R with group G. Their Ω is defined as follows: Let $\Gamma = \bigotimes^n S$ denote the n-fold tensor product of S over $R, E = \wedge^n S$ denote the n-th exterior power of S over $R, \pi: \bigotimes^n S \to \wedge^n S$ be the natural (R-module) homomorphism, I be the R-module conductor $(\ker \pi)$: $(\bigotimes^n S)$, (so I is an ideal of $\bigotimes^n S$ and is also an R-submodule of $\ker \pi$), and define $\Omega = (\bigotimes^n S)/I$. The group G acts on $\bigotimes^n S$ by permuting the n factors. Since $\pi\sigma(\xi) = \epsilon(\sigma)\pi(\xi)$ for $\xi \in \bigotimes^n S$ and $\sigma \in G, \ker \pi$ is stable under the action of G, hence so is I. Thus G acts on Ω. Since $\wedge^n S \approx \bigotimes^n S/\ker \pi$ is a free$R$-module (of rank 1), $R \cap \ker \pi = 0$, so that $R \cap I = 0$, and thus the restriction of the map $\Gamma \to \Omega = \Gamma/I$ to R is injective, i.e., Ω contains R. For $1 \leq i \leqq n$, let $p_i: S \to \bigotimes^n S$ be the R-algebra homomorphism defined by $p_i(s) = 1 \otimes \cdots \otimes 1 \otimes s \otimes 1 \otimes \cdots \otimes 1$ (the s occurring in the i-th place). Then it follows from the properties of the exterior algebra that for all $s \in S$,

$$(*) \qquad p_1(s) + \cdots + p_n(s) - \operatorname{trace}_{S/R}(\bar{s}) \in I$$

where $\bar{s}$ denotes the R-endomorphism of S defined by multiplication by s. Assume furthermore S is separable over R, then $t = \operatorname{trace}_{S/R}$ is nondegenerate ([**1**, Proposition A.4, p. 397]). It follows from $(*)$ and the non-degeneracy of t that the composite of the R-algebra homomorphisms $S \xrightarrow{p_1} \Gamma \longrightarrow \Omega$ gives an imbedding of S as an R-algebra into Ω. Then it can be shown that Ω is a Galois extension of R with group G ([**1**, line 14 of p. 400 to line 18 of p. 402]).

On the other hand, Barnard [2, §5, pp. 285–289] constructs a splitting ring R_f for a monic polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ of degree n over R. More specifically,

$$R_f = R[z_1, \cdots, z_n]$$

$$= R[Z_n, \cdots, Z_n]/\langle e_1 + a_{n-1}, e_2 - a_{n-2}, \cdots, e_n + (-1)^{n-1}a_0\rangle$$

where e_i $(1 \leqq i \leqq n)$ is the elementary symmetric polynomial of degree i in the indeterminates $Z_1, \cdots, Z_n$. The ring R_f is characterized by the following universal property: the polynomial f factors into the product of n linear factors over R_f, $f(X) = \Pi_{i=1}^n (X - z_i)$. And if A is an R-algebra over which f factors into the product of n linear factors, $f(X) = \Pi_{i=1}^n (X - a_i)$, then there is an R-algebra homomorphism $R_f \to A$ which maps z_i to a_i for $i = 1, \cdots, n$. As usual, such an R_f is unique up to isomorphism. The ring R_f contains R, is a free R-module of rank $n!$ and G acts on R_f by permuting the z_i's. Moreover, R_f contains $R[x] = R[X]/\langle f(X)\rangle$ as an R-subalgebra. It is also shown that R_f is a Galois extension of R with group G if and only if $\Pi_{i \neq j} (z_i - z_j)$ is a unit in R.

However, a moment's reflection will convince one that $\Pi_{i \neq j} (z_i - z_j)$ is d_f up to a sign. Recall d_f, the discriminant of f, is defined to be the discriminant of the basis $1, x, \cdots, x^{n-1}$ of $R[x]$ with respect to R, i.e., the determinant of the $n \times n$ matrix $(\mathrm{trace}_{R[x]/R}(x^{i-1}x^{j-1}))$ $1 \leqq i \leqq n$ $1 \leqq j \leqq n$.

For the remainder of the note, S will be $R[x] = R[X]/\langle f(X)\rangle$ and will be assumed to be separable over R or equivalently [5] d_f is a unit in R.

We will show that there is a $\varphi : \Omega \to R_f$ which is both an R-algebra and a G-module homomorphism. To establish this, let us first observe that there is an R-algebra isomorphism

$$\otimes^n S \approx R[W_1, \cdots, W_n]/\langle f(W_1), \cdots, f(W_n)\rangle$$

where for $g(x) \in S = R[x]$, $p_i(g(x))$ goes to the coset of $g(W_i)$ $(1 \leqq i \leqq n)$. Here p_i, as before, denotes the ith injection: $S \to \otimes^n S$. On the other hand, there is another description of I. Put $x_i = x^{i-1}$, $t = \mathrm{trace}_{S/R}$, and let the $n \times n$ matrix (λ_{ij}) be the adjoint matrix of$(t(x_ix_j))$; let

$$y_j = (\lambda_{j1}x_1 + \lambda_{j2}x_2 + \cdots + \lambda_{jn}x_n)d_f^{-1} \quad (1 \leqq j \leqq n).$$

Then $t(x_iy_j) = \delta_{ij}$ $(1 \leqq i, j \leqq n)$ [5]. By $\alpha(\xi)$ will be meant the (contravariant) skew-symmetrization of ξ, i.e., $\alpha(\xi) = \Sigma_{\sigma \in G}\epsilon(\sigma)\sigma(\xi)$ if $\xi \in \otimes^n S$. Then I is precisely the principal ideal generated by

$\alpha(x_1\otimes\cdots\otimes x_n)\alpha(y_1\otimes\cdots\otimes y_n)-1\otimes\cdots\otimes 1$ [**1**, p. 401]. Let $s_1,\cdots,s_n\in S$; then $\alpha(s_1\otimes\cdots\otimes s_n)=\det(p_i(s_j))$. This may be verified by expanding as an alternating sum of n ! terms; these terms are precisely those in the sum $\Sigma_{\sigma\in G}\epsilon(\sigma)\sigma(s_1\otimes\cdots\otimes s_n)$ [**1**, p. 401]. Accordingly $\alpha(x_1\otimes\cdots\otimes x_n)=\det(p_i(x_j))$ and $\alpha(y_1\otimes\cdots\otimes y_n)=\det(p_i(y_j))=d_f^{-1}\det(p_i(x_j))$ by taking $\det(\lambda_{ij})=d_f^{n-1}$ into account. Hence I is the principal ideal generated by $(\det(p_i(x_j)))^2-d_f$. If follows that the image of I in $R[W_1,\cdots,W_n]$, under the aforementioned isomorphism $\otimes^n S\approx R[W_1,\cdots,W_n]/\langle f(W_1),\cdots,f(W_n)\rangle$, is the principal ideal generated by $[\det(W_i^{j-1})]^2-d_f$. Note, however, it is well-known that $\det(W_i^{j-1})$, a so-called Vandermonde determinant of the sequence $(W_1,\cdots,W_n)$, has the value $\Pi_{i>j}(W_i-W_j)$. Consequently, this map induces an isomorphism

$$\Omega\approx R[W_1,\cdots,W_n]\Big/\Big\langle f(W_1),\cdots,f(W_n),d_f-\Big(\prod_{i>j}(W_i-W_j)\Big)^2\Big\rangle$$

and therefore, since $f(z_1)=0,\cdots,f(z_n)=0$, $d_f=(\Pi_{i>j}(z_i-z_j))^2$, there is an R-algebra homomorphism $\varphi:\Omega\to R_f$ which takes the coset of W_i to z_i $(1\leqq i\leqq n)$. Obviously such an φ preserves the G-action. Therefore $\Omega\approx R_f$ by [**3**, Theorem 3.4, p.12]. This establishes our assertion.

REMARKS. (1) As a matter of fact, we have also proved the following proposition: If S is separable over R, then the surjective R-algebra homomorphism from $R[w_1,\cdots,w_n]=R[W_1,\cdots,W_n]/\langle f(W_1),\cdots f(W_n),\ d_f-(\Pi_{i>j}(W_i-W_j))^2\rangle$ to $R_f=R[z_1,\cdots,z_n]$ is an isomorphism. This is not necessarily true if S is not separable over R. For example, take R to be the field of real numbers and $f(X)=X^2+2X+1$, then $R[W_1,W_2]/\langle f(W_1),f(W_2),(W_2-W_1)^2\rangle$ has dimension 3 over R while R_f has dimension 2 over R.

(2) Recently, Andy Magid has pointed out that the splitting ring constructed by Barnard is the same as the "free splitting ring" constructed by Nagahara in [**4**, pp. 150–152].

REFERENCES

1. M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc., **97** (1960), 367–409.

2. A. D. Barnard, *Commutative rings with operators* (*Galois theory and ramification*), Proc. London Math. Soc., (3) **28** (1974), 274–290.

3. S. U. Chase, D. K. Harrison and A. Rosenberg, *Galois theory and Galois cohomology of commutative rings*, Mem. Amer. Math. Soc., No. 52 (1965) (third printing 1969), 1–19.

4. T. Nagahara, *On separable polynomials over a commutative ring II*, Math. J. of Okayama Univ., **15** (1971/72), 149–162.
5. S. S. Wang, *Separable algebras and free cubic extensions over commutative rings*, Ph.D. thesis, Cornell University, 1975.

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