

Splitting the square of homogeneous and elementary functions into their symmetric and anti-symmetric parts

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1 Introduction

The composition of polynomial representations of $GL_n(\mathbb{C})$ translate in terms of characters to the plethysm of symmetric functions. Although introduced in 1936 by Littlewood [Lit36], the plethysm of two symmetric functions f and g , denoted $f[g]$, still carries a lot of open problems.

One open problem is understanding the decomposition of $f[g]$ in the basis of Schur functions s_ν , since they correspond to the irreducible characters for polynomial representations of $GL_n(\mathbb{C})$. Basic properties of plethysm reduce the problem (slightly) to understanding the decomposition of f , and of $s_\mu[g]$, in the Schur basis.

This has applications in representation theory (and so chemistry, physics, crystallography, etc.), and in geometric complexity theory (see [Mul12] for an introduction).

For a specific symmetric function g , one wants to study g^n , which decomposes as a sum of plethysms $s_\mu[g]$ for $\mu \vdash n$, each with multiplicity f^μ counting standard tableaux of shape μ . Often, we also know the decomposition of g^n in the basis of Schur functions, so the real difficulty is to discriminate how many copies of s_λ contribute to a given plethysm $s_\mu[g]$. For small values of n , this is computable, but rather slow as the calculation of the coefficients appearing in these decompositions has been proven to be $\#P$ -hard [FI20].

In the case $n = 2$ that interests us, the plethysms $s_2[g]$ and $s_{11}[g]$ in g^2 are often referred to as the symmetric and anti-symmetric parts of the square.

For $g = s_\lambda$ a Schur function, Carré and Leclerc [CL93] gave a combinatorial solution to this problem. The decomposition of the plethysms of s_λ^n in the basis of Schur function remains however open for $n > 2$.

Our contribution is to give a combinatorial description in the case of two other linear bases of symmetric functions, namely the complete homogeneous symmetric functions h_λ and the elementary symmetric functions e_λ . As the construction for both is similar, we first describe our construction for h_λ and extended it to e_λ afterwards. We use a combinatorial description of h_λ in terms of λ -tuples of row tableaux (corresponding to tabloids), and use the RSK algorithm to describe the Q -tableaux indexing the copies of s_ν appearing in the decomposition of h_λ^2 in the basis of Schur functions. A sign statistic on these Q -tableaux determines to which plethysm the associated Schur function contributes. Our description of the RSK algorithm uses the product of tableaux $*$ of Lascoux and Schützenberger [LS81], formalized by Fulton in [Ful96], and rectification (Rect) of skew tableaux using jeu de taquin. Our proofs rely on this description, as well as basic properties of plethysm.

Our main results are the following. For conciseness, we combined here the analogous results for h_λ and e_λ .

Theorem 1 : Fix a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

- Let Q be a tableau of shape $\nu \vdash 2|\lambda|$ and content λ^2 . Let $Q^{(i)}$ be the subtableau of Q containing entries $2i - 1$ and $2i$, for $1 \leq i \leq \ell$ and $Q_i = \text{Rect}(Q^{(i)})$ of shape $(2\lambda_i - j_i, j_i)$. Define the sign of Q to be $\text{sign}(Q) = \prod_{i=1}^{\ell} (-1)^{j_i}$.

The copy of s_ν indexed by Q in h_λ^2 lies in $s_2[h_\lambda]$ if $\text{sign}(Q) = 1$, and in $s_{11}[h_\lambda]$ if $\text{sign}(Q) = -1$.

- Let \tilde{Q} be a conjugate tableau of shape $\nu \vdash 2|\lambda|$ and content λ^2 . Let $\tilde{Q}^{(i)}$ be the subtableau of \tilde{Q} containing entries $2i - 1$ and $2i$, for $1 \leq i \leq \ell$ and $\tilde{Q}_i = (\text{Rect}(\tilde{Q}^{(i)}))'$ of shape $(2^{\lambda_i - j_i}, 1^{2j_i})$. Define the (conjugate) sign of \tilde{Q} to be $\text{sign}_c(\tilde{Q}) = \prod_{i=1}^{\ell} (-1)^{j_i}$.

The copy of s_ν indexed by \tilde{Q} in e_λ^2 lies in $s_2[e_\lambda]$ if $\text{sign}_c(\tilde{Q}) = 1$, and in $s_{11}[e_\lambda]$ if $\text{sign}_c(\tilde{Q}) = -1$.

This gives us the following decomposition :

Theorem 2 : Consider the following numbers:

- $(K_{\lambda^2}^{\nu})^+ = \#\{Q \text{ tableau of shape } \nu \text{ and content } \lambda^2 \mid \text{sign}(Q) = 1\}$
- $(K_{\lambda^2}^{\nu})^- = \#\{Q \text{ tableau of shape } \nu \text{ and content } \lambda^2 \mid \text{sign}(Q) = -1\}$
- $(K_{\lambda^2}^{\nu'})^+ = \#\{\tilde{Q}' \text{ tableau of shape } \nu' \text{ and content } \lambda^2 \mid \text{sign}_c(\tilde{Q}') = 1\}$
- $(K_{\lambda^2}^{\nu'})^- = \#\{\tilde{Q}' \text{ tableau of shape } \nu' \text{ and content } \lambda^2 \mid \text{sign}_c(\tilde{Q}') = -1\}$

Then we have:

$$\begin{aligned} s_2[h_\lambda] &= \sum_{\nu} (K_{\lambda^2}^{\nu})^+ s_{\nu} \\ s_{11}[h_\lambda] &= \sum_{\nu} (K_{\lambda^2}^{\nu})^- s_{\nu} \\ s_2[e_\lambda] &= \sum_{\nu} (K_{\lambda^2}^{\nu'})^+ s_{\nu} \\ s_{11}[e_\lambda] &= \sum_{\nu} (K_{\lambda^2}^{\nu'})^- s_{\nu} \end{aligned}$$

In section 2, we recall definitions.

In section 3, we introduce all concepts necessary to state and prove our theorems for h_λ . We recall the product of tableaux of Lascoux, Schützenberger [LS81] and Fulton [Ful96], which uses jeu de taquin, and define the RSK algorithm using this product of tableaux. We show that it gives a bijection between λ -tuples of row tableaux and pairs of tableaux of the same shape ν , with the recording tableau Q having content λ . We finally define a sign statistic on tableaux of content λ^2 . We use basic properties of plethysm and symmetric functions, and properties of the RSK algorithm, to prove theorems 1 and 2 for h_λ .

In section 4, we translate our construction to e_λ .

Finally, in section 5, we show how our results could be generalized to higher plethysms. We also describe another construction using that of Carré-Leclerc, which gives an alternative proof of our results in terms of tuples of domino tableaux. This construction offers hope of generalization to higher plethysms in terms of ribbon tableaux.

2 Definitions

2.1 Tableaux

We recall here definitions and notations on tableaux. For a more detailed introduction, see [Ful96] or [Sag01].

Recall that a partition is a weakly decreasing vector of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$. We identify a partition λ to its diagram, the collection of boxes both top- and left-justified with λ_i boxes in its i^{th} row, counting top to bottom. We denote $\lambda^2 = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_\ell, \lambda_\ell)$ and λ' the conjugate of λ , where rows of λ become the columns of λ' . We denote also $|\lambda| = \sum_i \lambda_i$, $\lambda \vdash n$ if $\sum_i \lambda_i = n$ and $\ell(\lambda) = \ell$ the number of parts of λ .

A *semistandard tableau* t of shape λ is the filling of λ with positive integers, such that rows weakly increase from left to right, and columns strictly increase from top to bottom. We denote $\text{SSYT}(\lambda)$ the set of tableaux of shape λ . We use *tableau* to mean *semistandard tableau*, unless stated otherwise.

The *conjugate* of a tableau t of shape λ has shape λ' and is obtained by reflecting t along its main diagonal. A conjugate tableau then has strictly increasing rows and weakly increasing columns.

We allow tableaux to have skew shapes λ/μ , where the boxes of the partition μ are left blank. The content of a tableau t is the vector $(\alpha_1, \alpha_2, \dots, \alpha_k)$, where α_i counts the number of entries i in t .

Example 2.1 : The tableau

		1	1	2	2	3	4
	1	2	2	3	3		
2	3	3	3				

 has (skew) shape $(9, 7, 4)/(3, 2)$ and content $(3, 5, 6, 1)$.

$\text{SSYT}(n)$ correspond to row tableaux of length n , and $\text{SSYT}(1^n)$ to column tableaux of height n . We define λ -tuples of row tableaux to be elements of the cartesian product $T_{\leq}(\lambda) := \text{SSYT}(\lambda_1) \times \text{SSYT}(\lambda_2) \times \dots \times \text{SSYT}(\lambda_\ell)$, and we define λ -tuples of column tableaux to be elements of $T_{\wedge}(\lambda) := \text{SSYT}(1^{\lambda_1}) \times \text{SSYT}(1^{\lambda_2}) \times \dots \times \text{SSYT}(1^{\lambda_\ell})$. The content of a tuple of tableaux is the sum of the contents of each tableau forming the tuple.

Example 2.2 : The tuples s, t below are $(4, 3)^2$ -tuples of tableaux (row/column).

$$s = (s_1, s_2, s_3, s_4) = (\boxed{1\ 3\ 4\ 5}, \boxed{2\ 2\ 2\ 4}, \boxed{1\ 2\ 2}, \boxed{1\ 3\ 7}) \in T_{\leq}((4, 3)^2),$$

$$t = (t_1, t_2, t_3, t_4) = \left(\begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline 7 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 7 \\ \hline \end{array} \right) \in T_\lambda((4, 3)^2).$$

Their contents are $(4, 5, 2, 2, 1, 0, 1)$ and $(3, 2, 3, 2, 1, 0, 2, 1)$.

The content allows one to associate a monomial x^t with a tableau (or λ -tuple), where the power of x_i is given by α_i , the number of entries i in t .

To a tableau we can also associate a *reading word*, by reading off its entries from left to right and bottom to top. The reading word of a tuple of tableaux is the concatenation of the reading words of each tableau. For example, the tableau t in example 2.2 has reading word 54318742321731, and associated monomial $x^t = x_1^3 x_2^2 x_3^3 x_4^2 x_5 x_7^2 x_8$.

2.2 Symmetric functions

For an introduction to symmetric functions, see [Mac98] or [Sag01].

Recall that *symmetric functions* over \mathbb{Q} are (potentially infinite) polynomials $f(x)$ on formal variables $x = (x_1, x_2, x_3, \dots)$, such that permuting any two variables gives back the same polynomial. They form a well studied graded ring Λ for which many basis are known, including the following.

For each partition λ , the *Schur function* s_λ can be defined in terms of tableaux :

$$s_\lambda = \sum_{t \in \text{SSYT}(\lambda)} x^t.$$

The *homogeneous symmetric functions* h_λ can be defined in terms of λ -tuples of row tableaux :

$$h_\lambda = \sum_{t \in T_{\leq}(\lambda)} x^t = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_\ell} = s_{(\lambda_1)} s_{(\lambda_2)} \dots s_{(\lambda_\ell)}.$$

Similarly, the *elementary symmetric functions* e_λ can be defined in terms of λ -tuples of column tableaux:

$$e_\lambda = \sum_{t \in T_\lambda(\lambda)} x^t = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_\ell} = s_{(1^{\lambda_1})} s_{(1^{\lambda_2})} \dots s_{(1^{\lambda_\ell})}.$$

The *power sum symmetric functions* p_λ are defined as the product $p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_\ell}$, where

$$p_n = \sum_{i \geq 1} x_i^n.$$

The transition matrices between these bases are known, and important problems in representation theory correspond to decomposing a symmetric function in term of one of these bases. A widely studied problem is to find combinatorial descriptions of the coefficients appearing in the decomposition of the product of two symmetric functions, which is also a symmetric function. Generally we want the decomposition in the basis of the Schur functions, since they correspond to irreducible representations. The following propositions are well-known results about this problem, and can be found for example in [Mac98].

The Pieri rule describes the product of any Schur function by a homogeneous or elementary symmetric function :

Proposition 2.3 (Pieri rule) : For any partition λ and integer n , we have:

- $s_\lambda s_{(n)} = s_\lambda h_n = \sum_{\nu} s_\nu$, where the sum is over all partitions ν obtained by adding n boxes to λ , no two in the same column.
- $s_\lambda s_{(1^n)} = s_\lambda e_n = \sum_{\nu} s_\nu$, where the sum is over all partitions ν obtained by adding n boxes to λ , no two in the same row.

This allows us to describe the decomposition of h_n^2 and e_n^2 :

Proposition 2.4 :

$$(h_n)^2 = \sum_{j=0}^n s_{(2n-j,j)} \quad \text{and} \quad (e_n)^2 = \sum_{j=0}^n s_{(2n-j,1^{2j})}.$$

Iteratively applying the Pieri rule to get a description of h_λ^2 and e_λ^2 , we obtain:

Proposition 2.5 :

$$(h_\lambda)^2 = h_{\lambda^2} = \sum_{\nu \vdash 2|\lambda|} K_{\lambda^2}^\nu s_\nu \quad \text{and} \quad (e_\lambda)^2 = e_{\lambda^2} = \sum_{\nu \vdash 2|\lambda|} K_{\lambda^2}^{\nu'} s_\nu,$$

where the Kostka numbers $K_{\lambda^2}^\nu$ count the number of tableaux of shape ν and content λ^2 .

3 Associating a Q -tableau indexing a copy of s_ν in h_λ^2 to a plethysm

We will see that the tableaux of shape ν and content λ^2 counted by $K_{\lambda^2}^\nu$ in $h_\lambda^2 = \sum_\nu K_{\lambda^2 s_\nu}^\nu$ are Q -tableaux obtained through the RSK algorithm.

3.1 Product of tableaux, jeu de taquin and RSK

We define the RSK algorithm in terms of product of tableaux, which will be central to our construction. The product of tableaux was introduced by Lascoux and Schützenberger in the setting of the plactic monoid [LS81]. Its elements correspond to tableaux: a tableau is identified with the Knuth-equivalence class of the reading word of the tableau. The product of tableaux was more formally defined by Fulton in [Ful96] using *jeu de taquin*, so that it corresponds to the product on words (concatenation) in the plactic monoid.

Jeu de taquin gives a way to rectify a skew tableau of shape λ/μ to a straight tableau. A jeu de taquin slide starts at an inner corner of λ/μ and exchanges the empty cell with a non-empty adjacent cell, respecting constraints on the entries of the rows and columns, until the empty cell lies on the outer border. The rectified tableau is independent of the order of the slides.

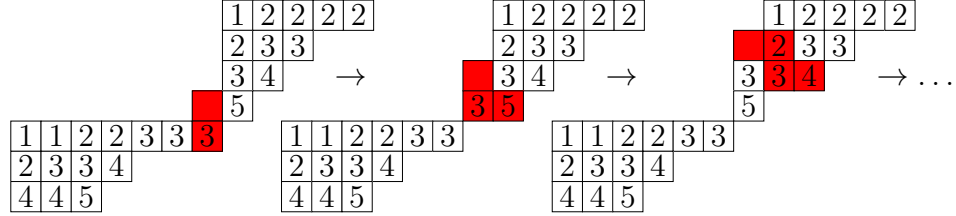
Let's define the product of two tableaux t_1, t_2 , of respective shape μ, ν :

1. Construct the skew tableau $t_1 * t_2$ by placing t_1 below and left of t_2 , into the skew shape $(\mu_1 + \nu_1, \mu_1 + \nu_2, \dots, \mu_1 + \nu_\ell, \mu_1, \mu_2, \dots, \mu_k)/(\mu_1^\ell)$.
2. The product T of t_1, t_2 is the rectification of $t_1 * t_2$ using jeu de taquin.

Example 3.1 : For $t_1 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ \hline 2 & 3 & 3 & 4 & & & \\ \hline 4 & 4 & 5 & & & & \\ \hline \end{array}$ and $t_2 = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 2 & 2 \\ \hline 2 & 3 & 3 & & \\ \hline 3 & 4 & & & \\ \hline 5 & & & & \\ \hline \end{array}$,

$t_1 * t_2 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 2 & 2 & & \\ \hline 2 & 3 & 3 & & & & \\ \hline 3 & 4 & & & & & \\ \hline 5 & & & & & & \\ \hline 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ \hline 2 & 3 & 3 & 4 & & & \\ \hline 4 & 4 & 5 & & & & \\ \hline \end{array}$, and $T = \text{Rect}(t_1 * t_2) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ \hline 3 & 4 & 4 & 4 & & & & & \\ \hline 4 & 5 & & & & & & & \\ \hline 5 & & & & & & & & \\ \hline \end{array}$.

Three jeu de taquin slides are illustrated below.



Proposition 3.2 : Let t_1, t_2 be tableaux of respective shape μ, ν . The word obtained by the concatenation of $\text{word}(t_1)$ and $\text{word}(t_2)$ lies in the plactic class of the reading word of $T = \text{Rect}(t_1 * t_2)$.

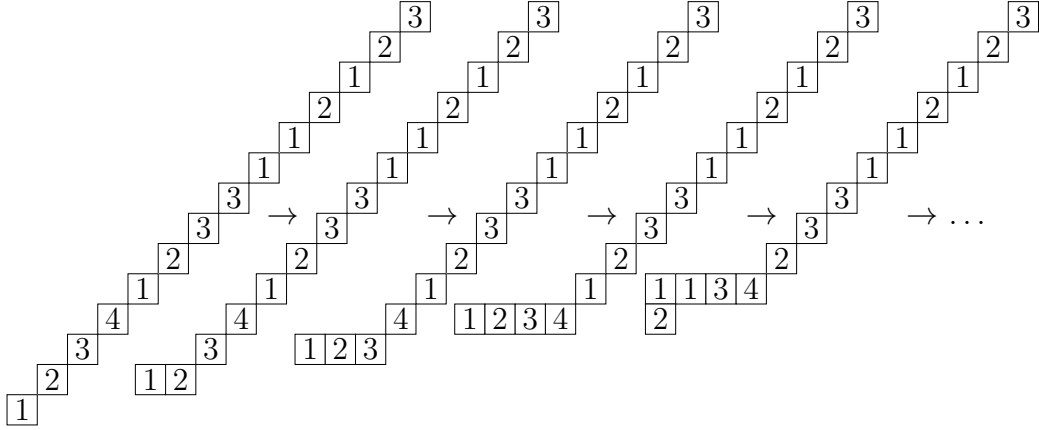
Idea of proof. Applying jeu de taquin on a skew tableau takes its reading word onto a Knuth-equivalent reading word. Since the reading word of $t_1 * t_2$ is equal to $\text{word}(t_1)\text{word}(t_2)$, we have the desired result. \blacksquare

The RSK algorithm [Knu70] is closely linked to the plactic monoid, since all words in the same (plactic) class will have the same insertion tableau under RSK. Recall that RSK establishes a bijection between biwords $W = \begin{pmatrix} u_1 & u_2 & \dots & u_k \\ v_1 & v_2 & \dots & v_k \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ and pairs of tableaux (P, Q) of the same shape μ , for $\mu \vdash k$, with $\text{content}(P) = \text{content}(v)$, and $\text{content}(Q) = \text{content}(u)$. Recall that the bi-letters $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ of a biword are ordered such that the u_i 's weakly increase, and for $u_i = u_{i+1}$, $v_i \leq v_{i+1}$.

Note that words w are inserted by RSK into pairs (P, Q) by considering $u = 123 \dots \text{length}(w)$ and $v = w$. Then the tableau Q is standard.

Our alternate definition of RSK using product of tableaux goes as follows. Let P_i the tableau obtained after the insertion of v_i , with $P_0 = \emptyset$. Then $P_{i+1} = \text{Rect}(P_i * \boxed{v_{i+1}})$, and $P = P_k$. The tableau Q records the order in which cells are added, with entry u_i in position $\text{shape}(P_{i+1})/\text{shape}(P_i)$.

Remark 3.3 : This process can be seen as rectifying $v_1 * v_2 * \dots * v_k$ one cell after the other from left to right to yield the tableaux P_i , as illustrated below. Recording the added cells with the u_i 's yields the tableaux Q_i .



3.2 Associating a λ^2 -tuple of row tableaux to a Q -tableau

Proposition 3.4 : λ -tuples of row tableaux with content α are in bijection with pairs of tableaux of the same shape (P_t, Q_t) , where the content of P_t is α and the content of Q_t is λ .

Proof. Consider the biword W_t built from a λ -tuple of row tableaux $t = (t_1, t_2, \dots, t_k)$ using the bi-letters $\begin{pmatrix} i \\ j \end{pmatrix}$, for j an entry in t_i . The top word of W_t is $1^{\lambda_1} 2^{\lambda_2} \dots \ell^{\lambda_k}$, and the bottom word is the reading word of t .

The map $t \mapsto W_t$ is a bijection between λ -tuples of row tableaux of content α and biwords such that the top word has content λ , and the bottom word has content α and is the reading word of the λ -tuples. By RSK, these biwords are in bijection with pairs (P_t, Q_t) of tableaux with the desired properties. ■

Example 3.5 : If $t = (\boxed{1\ 2\ 3\ 4}, \boxed{1\ 2\ 3\ 3}, \boxed{1\ 1\ 2}, \boxed{1\ 2\ 3})$, then $W_t =$

$$\left(\begin{array}{cccc} 1111 & 2222 & 333 & 444 \\ 1234 & 1233 & 112 & 123 \end{array} \right) \text{ and } (P, Q) = \left(\begin{array}{cccccc} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & & & \\ \boxed{3} & \boxed{3} & & & & & \\ \boxed{4} & & & & & & \end{array}, \begin{array}{cccccc} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{4} \\ \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{4} & & \\ \boxed{3} & \boxed{3} & & & & & \\ \boxed{4} & & & & & & \end{array} \right).$$

Remark 3.6 : In the case of a biword associated to a λ -tuple of row tableaux, the rectifying process of the RSK algorithm can be greatly sped up as illustrated below, by first rectifying together the letters in the same line tableau t_i .

Since all letters of the reading word of t_i correspond to entries i in Q , the skew tableau $t_1 * t_2 * \dots * t_\ell$ can be rectified recursively row by row, recording the added cells in the subtableaux of Q with entries 1 up to i .

$$\begin{array}{ccccccc}
& & & & \boxed{1} \boxed{2} \boxed{3} & & \\
& & & & & & \boxed{1} \boxed{2} \boxed{3} \\
& & & & & & & & \boxed{1} \boxed{1} \boxed{2} & & & & & & & & & & \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{2} \boxed{3} \\
& & & & & & & & & & & & & & & & & & & \boxed{2} \boxed{2} \boxed{2} \boxed{3} \\
& & & & & & & & & & & & & & & & & & & \boxed{3} \boxed{3} \\
& & & & & & & & & & & & & & & & & & & \boxed{4} \\
\boxed{1} \boxed{2} \boxed{3} \boxed{4} & & \boxed{1} \boxed{2} \boxed{3} \boxed{3} & \rightarrow & \boxed{1} \boxed{1} \boxed{2} \boxed{3} \boxed{3} & & \boxed{1} \boxed{1} \boxed{2} & \rightarrow & \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{2} & & \boxed{2} \boxed{2} \boxed{3} \boxed{3} & \rightarrow & \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{2} \boxed{2} \boxed{4} \\
& & & & \boxed{2} \boxed{3} \boxed{4} & & & & \boxed{2} \boxed{2} \boxed{3} \boxed{3} & & \boxed{3} \boxed{4} & & & & & & & & & \boxed{2} \boxed{2} \boxed{2} \boxed{3} \boxed{4} \\
& \boxed{3} \boxed{3} \\
& \boxed{4} \\
& \boxed{4} \\
Q_4 = \boxed{1} \boxed{1} \boxed{1} \boxed{1} & & Q_8 = \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{2} \boxed{2} & & Q_{11} = \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{2} \boxed{2} & & Q = \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{2} \boxed{2} \boxed{4} \\
& & & & \boxed{2} \boxed{2} \boxed{2} & & \boxed{2} \boxed{2} \boxed{2} \boxed{3} & & \boxed{2} \boxed{2} \boxed{2} \boxed{3} \boxed{4} \\
& & & & & & \boxed{3} \boxed{3} & & \boxed{3} \boxed{3} \\
& \boxed{4}
\end{array}$$

Remark 3.7 : In the case of a λ^2 -tuple of row tableaux, we have a third way to rectify $t_1 * t_2 * \dots * t_{2\ell-1} * t_{2\ell}$ into P_t : each pair of tableaux of respective length λ_i can be first rectified together into tableaux T_i . They correspond to insertion tableaux for the sub-biwords $\left(\begin{array}{cc} 2i-1 \dots 2i-1 & 2i \dots 2i \\ \text{word}(t_{2i-1}) & \text{word}(t_{2i}) \end{array} \right)$.

We can recover the associated recording tableaux Q_i directly from Q :

Corollary 3.8 : Let t be a λ^2 -tuple of row tableaux, W_t the associated biword and $(P, Q) = \text{RSK}(W_t)$. Let

$$(T_i, Q_i) = \text{RSK} \left(\begin{array}{cc} 2i-1 \dots 2i-1 & 2i \dots 2i \\ \text{word}(t_{2i-1}) & \text{word}(t_{2i}) \end{array} \right),$$

and $Q^{(i)}$ the subtableau of Q with entries $2i-1$ and $2i$, for $1 \leq i \leq \ell = \ell(\lambda)$. Then $Q_i = \text{Rect}(Q^{(i)})$.

This follows from a result found in chapter 5 of [Ful96].

We can now define a sign statistic on the Q -tableaux which will allow us to discriminate in which plethysm the associated copy of s_ν lies.

Let Q be a tableau of shape $\nu \vdash 2|\lambda|$ and content λ^2 , for a fixed partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$. Let $Q^{(i)}$ be the subtableau of Q with entries $2i-1$ and $2i$. Then $Q_i = \text{Rect}(Q^{(i)})$ is of shape $(2\lambda_i - j_i, j_i)$, for $1 \leq i \leq \ell$. Define the sign of Q to be $\text{sign}(Q) = \prod_{i=1}^{\ell} (-1)^{j_i}$.

Theorem 1 (Part 1) : The copy of s_ν indexed by Q in h_λ^2 lies in $s_2[h_\lambda]$ if $\text{sign}(Q) = 1$, and in $s_{11}[h_\lambda]$ if $\text{sign}(Q) = -1$.

This is proved in the next section, where we show that this sign statistic is well defined, and can be used to discriminate contribution to a plethysm. We also have :

Theorem 2 (Part 1) : Consider the following numbers:

- $(K_{\lambda^2}^\nu)^+ = \#\{Q \text{ tableau of shape } \nu \text{ and content } \lambda^2 \mid \text{sign}(Q) = 1\}$
- $(K_{\lambda^2}^\nu)^- = \#\{Q \text{ tableau of shape } \nu \text{ and content } \lambda^2 \mid \text{sign}(Q) = -1\}$

Then we have:

$$s_2[h_\lambda] = \sum_{\nu} (K_{\lambda^2}^\nu)^+ s_\nu$$

$$s_{11}[h_\lambda] = \sum_{\nu} (K_{\lambda^2}^\nu)^- s_\nu$$

Example 3.9 : Let's see how theorem 1 in action. The four tableaux of shape $\nu = (3, 2, 1)$ and content $(2, 1)^2 = (2, 2, 1, 1)$ are illustrated below. We thank Franco Saliola for providing the figure. The computed signs tell us that $s_2[h_{21}]$ and $s_{11}[h_{21}]$ both contain two copies of s_{321} .

Q	$Q^{(1)}$	$\text{Rect}(Q^{(1)})$	$Q^{(2)}$	$\text{Rect}(Q^{(2)})$	$\text{sign}(Q)$	contributes to
$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 4 & & \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 2 & & \\ \hline & & \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 2 & & \\ \hline & & \\ \hline \end{array}$	$\begin{array}{ c } \hline 3 \\ \hline 4 \\ \hline \end{array}$	$\begin{array}{ c } \hline 3 \\ \hline 4 \\ \hline \end{array}$	$(-1)^{1+1} = 1$	$s_2[h_{21}]$
$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 2 & & \\ \hline & & \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 2 & & \\ \hline & & \\ \hline \end{array}$	$\begin{array}{ c } \hline 4 \\ \hline 3 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & 4 \\ \hline \end{array}$	$(-1)^{1+0} = -1$	$s_{11}[h_{21}]$
$\begin{array}{ c c c } \hline 1 & 1 & 3 \\ \hline 2 & 2 & \\ \hline 4 & & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$	$\begin{array}{ c } \hline 3 \\ \hline 4 \\ \hline \end{array}$	$\begin{array}{ c } \hline 3 \\ \hline 4 \\ \hline \end{array}$	$(-1)^{2+1} = -1$	$s_{11}[h_{21}]$
$\begin{array}{ c c c } \hline 1 & 1 & 4 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$	$\begin{array}{ c } \hline 4 \\ \hline 3 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & 4 \\ \hline \end{array}$	$(-1)^{2+0} = 1$	$s_2[h_{21}]$

3.3 Proof of theorem 1

In order to prove Theorem 1, we need to introduce basic properties of plethysm.

The plethysm of two symmetric functions can be defined using the following properties and power sum symmetric functions [Mac98].

Let f, g, h be symmetric functions :

- $p_k[p_m] = p_{km}$
- $p_k[f + g] = p_k[f] + p_k[g]$
- $p_k[f \cdot g] = p_k[f] \cdot p_k[g]$
- $(f + g)[h] = f[h] + g[h]$
- $(f \cdot g)[h] = f[h] \cdot g[h]$.

The proof that the sign $(-1)^{j_i}$ of each Q_i is well defined rests on the following result introduced by Littlewood in [Lit50].

Proposition 3.10 : The plethysms $s_2[h_n]$, $s_{11}[h_n]$ in h_n^2 decompose into the Schur basis as

$$s_2[h_n] = h_2[h_n] = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} s_{(2n-2j, 2j)},$$

$$s_{11}[h_n] = e_2[h_n] = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} s_{(2n-(2j+1), (2j+1))}.$$

Idea of proof. The proof follows directly from the transition matrix between the different basis, and basis properties of plethysm. It can be found in chapter 1, section 8, of Macdonald [Mac98]. For completeness, we include it here. The following formula is proved in MacDonalld [Mac98] :

$$p_2[h_n] = \sum_{j=0}^n (-1)^j s_{(2n-j, j)}$$

For $s_2[h_n]$, as $s_2 = h_2 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2$, we have:

$$\begin{aligned}
h_2[h_n] &= \left(\frac{1}{2}p_1^2 + \frac{1}{2}p_2 \right) [h_n] \\
&= \frac{1}{2}p_1^2[h_n] + \frac{1}{2}p_2[h_n] \\
&= \frac{1}{2}h_n^2 + \frac{1}{2}p_2[h_n] \\
&= \frac{1}{2} \sum_{i=0}^n s_{(2n-i,i)} + \frac{1}{2} \sum_{i=0}^n (-1)^i s_{(2n-i,i)} \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} s_{(2n-2j,2j)}
\end{aligned}$$

The other expression is obtained in the same way. ■

Applying this result in our context, we obtain:

Corollary 3.11 : Let $t = (t_1, t_2)$ be a $(n)^2$ -tuple of row tableaux, W_t the associated biword and (P, Q) the associated pair of tableaux of the same shape $(2n-j, j)$, with $0 \leq j \leq n$. The associated monomial $x^P = x^t$ appears in $s_{(2n-j,j)}$ in h_n^2 , which contributes to $s_2[h_n]$ if $\text{sign}(Q) = (-1)^j$ is positive, and to $s_{11}[h_n]$ if $\text{sign}(Q)$ is negative.

The other ingredient needed to prove our theorem is the following result:

Proposition 3.12 : Let g_1, g_2, \dots, g_n be any symmetric functions. Then:

$$(g_1 g_2 \dots g_n)^2 = s_2[g_1 g_2 \dots g_n] + s_{11}[g_1 g_2 \dots g_n],$$

where

$$\begin{aligned}
s_2[g_1 g_2 \dots g_n] &= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I| \text{ even}}} \prod_{i \in I} s_{11}[g_i] \cdot \prod_{j \in I^c} s_2[g_j], \\
s_{11}[g_1 g_2 \dots g_n] &= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I| \text{ odd}}} \prod_{i \in I} s_{11}[g_i] \cdot \prod_{j \in I^c} s_2[g_j].
\end{aligned}$$

Proof. As $g^2 = s_2[g] + s_{11}[g]$, we have

$$\begin{aligned} (g_1 g_2 \dots g_n)^2 &= g_1^2 g_2^2 \dots g_n^2 \\ &= \prod_{i=1}^n (s_2[g_i] + s_{11}[g_i]) \end{aligned}$$

We will prove the result by induction on n .

For general symmetric functions g_1, g_2 , it is possible to prove that $s_2[g_1 g_2] = s_2[g_1]s_2[g_2] + s_{11}[g_1]s_{11}[g_2]$, while $s_{11}[g_1 g_2] = s_{11}[g_1]s_2[g_2] + s_2[g_1]s_{11}[g_2]$, using only basic properties of plethysm (see for example [Mac98]).

Suppose now we have the result for n symmetric functions, and we show it holds for $n + 1$ symmetric functions g_1, g_2, \dots, g_{n+1} .

Recall that the product of symmetric functions g_1, g_2, \dots, g_n is symmetric.

$$\begin{aligned} s_2[g_1 g_2 \dots g_n g_{n+1}] &= s_2[g_1 g_2 \dots g_n]s_2[g_{n+1}] + s_{11}[g_1 g_2 \dots g_n]s_{11}[g_{n+1}] \\ &= \left(\sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I| \text{ even}}} \prod_{i \in I} s_{11}[g_i] \cdot \prod_{j \in I^c} s_2[g_j] \right) s_2[g_{n+1}] \\ &\quad + \left(\sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I| \text{ odd}}} \prod_{i \in I} s_{11}[g_i] \cdot \prod_{j \in I^c} s_2[g_j] \right) s_{11}[g_{n+1}] \\ &= \sum_{\substack{I \subseteq \{1, \dots, n+1\} \\ |I| \text{ even}}} \prod_{i \in I} s_{11}[g_i] \cdot \prod_{j \in I^c} s_2[g_j]. \end{aligned}$$

The proof for $s_{11}[g_1 g_2 \dots g_n g_{n+1}]$ is similar. ■

Proof of Theorem 1. As seen in proposition 3.10, the shape $(2\lambda_i - j_i, j_i)$ of the Q_i and the sign $(-1)^{j_i}$ determine to which plethysm the associated Schur function $s_{(2\lambda_i - j_i, j_i)}$ contributes in $h_{\lambda_i}^2$. Q then encodes that the associated Schur function s_ν appears in the product $s_{(2\lambda_1 - j_1, j_1)} s_{(2\lambda_2 - j_2, j_2)} \dots s_{(2\lambda_\ell - j_\ell, j_\ell)}$ in $h_\lambda^2 = h_{\lambda_1}^2 h_{\lambda_2}^2 \dots h_{\lambda_\ell}^2 = \prod_{i=1}^\ell (s_2[h_{\lambda_i}] + s_{11}[h_{\lambda_i}])$.

Let the set $I = \{ i \mid j_i \text{ odd} \}$ record the anti-symmetric parts that have been picked. The sign of Q relies only on the parity of the cardinality of I : $\text{sign}(Q) = \prod_i \text{sign}(Q_i) = \prod_i (-1)^{j_i} = (-1)^{|I|}$.

Therefore, the sign of such a tableau Q is well defined and indicates the participation of the associated Schur function to a plethysm. ■

We can use the same ideas to show a more general result. Let μ be another partition, and consider the product $s_\mu h_\lambda^2$. Iteratively applying the Pieri rule, as in proposition 2.5, we have

$$s_\mu h_\lambda^2 = \sum_{\nu \vdash 2|\lambda| + |\mu|} K_{\lambda^2}^{\nu/\mu} s_\nu,$$

where $K_{\lambda^2}^{\nu/\mu}$ counts the number of tableaux of shape ν/μ and content λ^2 . Using the very same ideas and notations developed in this section, we obtain the following corollary:

Corollary 3.13 : Let Q be a tableau of shape ν/μ and content λ^2 , for $\nu \vdash 2|\lambda| + |\mu|$, and fixed partitions μ and $\lambda = (\lambda_1, \dots, \lambda_\ell)$. Let $Q^{(i)}$ be the subtableau of Q with entries $2i - 1$ and $2i$. Then, $Q_i = \text{Rect}(Q^{(i)})$ is of shape $(2\lambda_i - j_i, j_i)$, for $1 \leq i \leq \ell$. Define the sign of Q to be $\text{sign}(Q) = \prod_{i=1}^{\ell} (-1)^{j_i}$.

The copy of s_ν indexed by Q in $s_\mu h_\lambda^2$ lies in $s_\mu(s_2[h_\lambda])$ if $\text{sign}(Q) = 1$, and in $s_\mu(s_{11}[h_\lambda])$ if $\text{sign}(Q) = -1$.

4 Associating a \widetilde{Q} -tableau indexing a copy of s_ν in e_λ^2 to a plethysm

All of the above can be translated into the realm of elementary symmetric functions. We have seen that the definition of elementary symmetric functions is very similar to that of homogeneous symmetric functions. Their Pieri rules are conjugate, and so applying them iteratively give conjugate tableaux.

Therefore $e_\lambda^2 = \sum_{\nu} K_{\lambda^2}^{\nu'} s_\nu$, where $K_{\lambda^2}^{\nu'}$ is the number of tableaux of conjugate shape ν' and content λ^2 .

We use a variation of the RSK algorithm to define the bijection of λ -tuples of column tableaux and pairs of "tableaux" of the same shape. This variant is called the RSK' algorithm in [Kra05], but to avoid confusion with the conjugate, we call it the $\widetilde{\text{RSK}}$ algorithm.

It works in the same way as RSK, but is defined on *Burge words* : biwords \widetilde{W} such that no bi-letters appear twice, and with its bi-letters $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ ordered such that the u_i weakly increase (as before), but for $u_i = u_{i+1}$, $v_i > v_{i+1}$.

The pair of "tableaux" of the same shape $(\tilde{P}, \tilde{Q}) = \widetilde{\text{RSK}}(\tilde{W}_t)$ obtained is such that the conjugate \tilde{Q}' of \tilde{Q} is a semistandard tableau. We call \tilde{Q} a conjugate tableau. The tableau \tilde{Q}' then has shape conjugate to that of \tilde{P} . The $\widetilde{\text{RSK}}$ algorithm establishes a bijection between Burge words and pairs tableaux/conjugate tableaux of the same shape. Note that the bijection between λ -tuples of column tableaux and Burge words follows the same construction we have seen before, with the top word being λ , and the bottom word being the reading words of the column tableaux.

Example 4.1 : Let $t = \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 5 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \\ \hline 6 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} \right)$ be a $(5, 3)^2$ -tuple of column tableaux.

Then $\tilde{W}_t = \left(\begin{array}{cccc} 11111 & 22222 & 333 & 444 \\ 75321 & 86431 & 532 & 421 \end{array} \right)$ and $(\tilde{P}, \tilde{Q}) = \left(\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 3 & 5 & \\ \hline 4 & 6 & & \\ \hline 5 & 8 & & \\ \hline 7 & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & \\ \hline 1 & 2 & & \\ \hline 1 & 2 & & \\ \hline 4 & & & \\ \hline \end{array} \right).$

We have that

Corollary 4.2 : Let t a λ^2 -tuple of column tableaux, \tilde{W}_t the associated Burge word and $(\tilde{P}, \tilde{Q}) = \widetilde{\text{RSK}}(\tilde{W}_t)$. Let $(\tilde{T}_i, \tilde{Q}_i) = \widetilde{\text{RSK}} \left(\begin{array}{cc} 2i-1 \dots 2i-1 & 2i \dots 2i \\ \text{word}(t_{2i-1}) & \text{word}(t_{2i}) \end{array} \right)$, and $\tilde{Q}^{(i)}$ the subtableau of \tilde{Q} containing entries $2i-1$ and $2i$, for $1 \leq i \leq \ell(\lambda)$.

Then $\tilde{Q}_i' = \text{Rect}((\tilde{Q}^{(i)})')$.

Proof. In order to get this result, we need to adapt the proof of Fulton's result ([Ful96], chapter 5) to $\widetilde{\text{RSK}}$. We can invert all bi-letters of a Burge word and rearrange them in lexicographical order (so that it is a biword). This gives a bottom word such that its reverse is Knuth-equivalent to the reading words of both \tilde{Q}_i' and $(\tilde{Q}^{(i)})'$. It is the same strategy that Fulton uses in his proof of the RSK case. The final rectification is applied on the conjugate of the conjugate tableau $\tilde{Q}^{(i)}$, because jeu de taquin is not well defined otherwise. ■

This allows us to also define a sign statistic on the \tilde{Q} -tableaux which will allow us to discriminate in which plethysm the associated copy of s_ν lies :

Theorem 1 (Part 2) : Let \tilde{Q} be a conjugate tableau of shape $\nu \vdash 2|\lambda|$ and content λ^2 , for a fixed partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$. Let $\tilde{Q}^{(i)}$ be the subtableau of \tilde{Q} with entries $2i-1$ and $2i$, for $1 \leq i \leq \ell$. Then $\tilde{Q}_i = (\text{Rect}(\tilde{Q}^{(i)}))'$ is of shape $(2^{\lambda_i-j_i}, 1^{2j_i})$. Define the (conjugate) sign of \tilde{Q} to be $\text{sign}_c(\tilde{Q}) = \prod_{i=1}^{\ell} (-1)^{j_i}$.

The copy of s_ν indexed by \tilde{Q} in e_λ^2 lies in $s_2[e_\lambda]$ if $\text{sign}_c(\tilde{Q}) = 1$, and in $s_{11}[e_\lambda]$ if $\text{sign}_c(\tilde{Q}) = -1$.

This give us the following theorem, second part of theorem 2.

Theorem 2 (Part 2) : Consider the following numbers:

- $(K_{\lambda^2}^{\nu'})^+ = \#\{\tilde{Q}' \text{ tableau of shape } \nu' \text{ and content } \lambda^2 \mid \text{sign}_c(\tilde{Q}) = 1\}$
- $(K_{\lambda^2}^{\nu'})^- = \#\{\tilde{Q}' \text{ tableau of shape } \nu' \text{ and content } \lambda^2 \mid \text{sign}_c(\tilde{Q}) = -1\}$

Then we have:

$$s_2[e_\lambda] = \sum_{\nu} (K_{\lambda^2}^{\nu'})^+ s_\nu$$

$$s_{11}[e_\lambda] = \sum_{\nu} (K_{\lambda^2}^{\nu'})^- s_\nu$$

The proof that the sign of each Q_i is well-defined rests on the following result analogous to that seen in section 3.3, also introduced initially by Littlewood.

Proposition 4.3 : The plethysms $s_2[e_n]$, $s_{11}[e_n]$ in e_n^2 decompose into the Schur basis as

$$s_2[e_n] = h_2[e_n] = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} s_{(2^{n-2j}, 1^{4j})},$$

$$s_{11}[e_n] = e_2[e_n] = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} s_{(2^{n-(2j+1)}, 1^{2(2j+1)})}.$$

Proof. The proof is very similar as the one in section 3.3 and relies on the following formulas also proven in MacDonal [Mac98] :

$$p_2[e_n] = \sum_{j=0}^n (-1)^j s_{2^{n-j}, 1^{2j}}$$

$$e_2 = \frac{1}{2}p_1^2 - \frac{1}{2}p_2$$

■

Proof of part 2 of Theorem 2. As seen in 4.3, the shape $(2^{\lambda_i - j_i}, 1^{2j_i})$ of the \widetilde{Q}_i and the (conjugate) sign $(-1)^{j_i}$ determine in which plethysm the associated Schur function $s_{(2^{\lambda_i - j_i}, 1^{2j_i})}$ contributes to in $e_{\lambda_i}^2$.

For a given conjugate tableau \widetilde{Q} of shape ν and content λ^2 , this holds for each \widetilde{Q}_i , and \widetilde{Q} then encodes that the associated Schur function s_ν appears in the product $s_{(2^{\lambda_1 - j_1}, 1^{2j_1})} s_{(2^{\lambda_2 - j_2}, 1^{2j_2})} \cdots s_{(2^{\lambda_\ell - j_\ell}, 1^{2j_\ell})}$ in $e_\lambda^2 = e_{\lambda_1}^2 e_{\lambda_2}^2 \cdots e_{\lambda_\ell}^2 = \prod_{i=1}^{\ell} (s_2[e_{\lambda_i}] + s_{11}[e_{\lambda_i}])$.

Let the set $I = \{ i \mid j_i \text{ odd} \}$ record the anti-symmetric parts that have been picked. The (conjugate) sign of \widetilde{Q} relies only on the parity of the cardinality of I as before : $\text{sign}_c(\widetilde{Q}) = \prod_i \text{sign}_c(\widetilde{Q}_i) = \prod_i (-1)^{j_i} = (-1)^{|I|}$. ■

As in section 3, we can generalize these ideas to the product $s_\mu e_{\lambda^2}$:

Corollary 4.4 : Let \widetilde{Q} be a conjugate tableau of shape ν/μ and content λ^2 , for $\nu \vdash 2|\lambda| + |\mu|$, and fixed partitions μ and $\lambda = (\lambda_1, \dots, \lambda_\ell)$. Let $\widetilde{Q}^{(i)}$ the subtableau of \widetilde{Q} with entries $2i - 1$ and $2i$, for $1 \leq i \leq \ell$ and $\widetilde{Q}_i = (\text{Rect}(\widetilde{Q}^{(i)}))'$ of shape $(2^{\lambda_i - j_i}, 1^{2j_i})$. Define the (conjugate) sign of \widetilde{Q} to be $\text{sign}_c(\widetilde{Q}) = \prod_{i=1}^{\ell} (-1)^{j_i}$.

The copy of s_ν indexed by \widetilde{Q} in $s_\mu e_\lambda^2$ lies in $s_\mu(s_2[e_\lambda])$ if $\text{sign}_c(\widetilde{Q}) = 1$, and in $s_\mu(s_{11}[e_\lambda])$ if $\text{sign}_c(\widetilde{Q}) = -1$.

5 What's next?

We are of course interested in investigating higher plethysms.

5.1 Generalizing this approach

To be able to generalize our approach, we need to use some known results, which are implicit in [Mac98].

Proposition 5.1 : For any symmetric function g ,

$$g^n = \sum_{\mu \vdash n} f^\mu s_\mu[g],$$

where f^μ is the number of standard tableaux of shape μ .

Proof. We have that $p_1[g] = g$ for any symmetric function g . Moreover, $p_1^n = f^\mu s_\mu$, which follows directly from the RSK algorithm: the indices picked in each copy of p_1 in $p_1^n = (x_1 + x_2 + x_3 + \dots)^n$ form a word of length n , which are in bijection with pairs of tableaux of the same shape $\mu \vdash n$, with the recording tableau being standard.

Using plethysm rules, we have:

$$\begin{aligned} g^n &= (p_1[g])^n \\ &= p_1^n[g] \\ &= \sum_{\mu \vdash n} f^\mu s_\mu[g] \end{aligned}$$

■

We can also use the Pieri rule recursively (or $\text{RSK}/\widetilde{\text{RSK}}$) to get the decomposition of h_λ^n and e_λ^n into the basis of Schur functions:

Proposition 5.2 : $(h_\lambda)^n = h_{\lambda^n} = \sum_{\nu \vdash n|\lambda} K_{\lambda^n}^\nu s_\nu$, and $(e_\lambda)^n = e_{\lambda^n} = \sum_{\nu \vdash n|\lambda} K_{\lambda^n}^{\nu'} s_\nu$,

where the Kostka numbers $K_{\lambda^n}^\nu$ count the number of tableaux of shape ν and content λ^n .

All of the above constructions can be generalized to $n > 2$.

For a tableau Q of shape ν and content λ^n , we can consider the sub-tableaux $Q^{(i)}$ of Q with entries $ni-(n-1), ni-(n-2), \dots, ni$, for $1 \leq i \leq \ell(\lambda)$, and their rectifications Q_i . If $\nu_i = \text{shape}(Q_i)$, and $\nu = \text{shape}(Q)$, then the Schur function s_ν occurs in the product $\prod_i s_{\nu_i}$, each s_{ν_i} indexed by Q_i in $h_{\lambda_i}^2$. Same thing goes to describe the tableaux counted by $K_{\lambda^n}^{\nu'}$ using \widetilde{RSK} .

We would like to identify a statistic on the tableaux Q_i and Q to determine to which plethysms of $h_{\lambda_i}^n$ and h_λ^n the associated Schur functions s_{ν_i} and s_ν contribute, and similarly for the plethysms of $e_{\lambda_i}^n$ and e_λ^n . However, it proves to be extremely difficult.

For $n = 3$, we have that $g^3 = s_3[g] + 2s_{21}[g] + s_{111}[g]$. The changing multiplicities already make things a bit complicated. For h_n^3 , and e_n^3 , we have closed formulas for the number of copies of s_ν that lie in $s_3[h_n], s_{111}[h_n]$ (resp. $s_3[e_n]$ and $s_{111}[e_n]$), studied by Chen [Che82] and Thrall [Thr42]. The left-over Schur functions would then be dispatched into the two copies of $s_{21}[h_n]$ (resp. $s_{21}[e_n]$). This can help pinpoint the right statistic to consider. We would like to refine this by understanding more precisely which copies of s_ν lie in each plethysm, eventually also distinguishing between the two copies of $s_{21}[h_n]$ or $s_{21}[e_n]$, but we are far from this result.

In the next section, we describe another strategy which might prove successful and which allows to recover our results for $n = 2$.

5.2 Generalizing using ribbon tableaux

The title of this article refers explicitly to that of Carré and Leclerc [CL93], where they describe the product of two Schur functions in terms of domino tableaux. They showed that the number of Yamanouchi domino tableaux of a certain shape I and content λ give the multiplicity $c_{\mu\nu}^\lambda$ of s_λ in $s_\mu s_\nu$. When $\mu = \nu$, then $I = (2\mu)^2 = (2\mu_1, 2\mu_1, 2\mu_2, 2\mu_2, \dots, 2\mu_\ell, 2\mu_\ell)$. A Yamanouchi domino tableau has a Yamanouchi (or reverse lattice) reading word: the content of each suffix is a partition. Domino tableaux are read off by scanning rows left to right and bottom to top, reading dominoes on their first scanning.

For a square s_μ^2 , Carré and Leclerc showed that the parity of the cospin of the Yamanouchi domino tableaux determine whether the associated Schur function s_λ contributes to $s_2[s_\mu]$ or $s_{11}[s_\mu]$, with the cospin equal the number of horizontal dominoes divided by two. Note that the cospin is more generally defined as the maximal number of vertical dominoes that can pave I minus the number of vertical dominoes in the domino tableau, divided by two.

For $h_n^2 = s_n^2$, $I = (2n, 2n)$. The only pavings and fillings of I that result in Yamanouchi domino tableaux consist of a sequence of $2n - 2j$ vertical dominos filled with ones, followed by j pairs of stacked horizontal dominoes, filled with a one for the top one, and a two for the bottom one. The weight of such a Yamanouchi tableau is then $\mu = (2n - j, j)$, and its cospin is j . By the result of Carré and Leclerc, there is a unique copy of s_μ in the decomposition of $h_n^2 = s_n^2$, and it contributes to the symmetric part of the square if j is even, and to its anti-symmetric part if j is odd.

For $e_n^2 = s_{1^n}^2$, $I = (2^{2n})$. The only pavings and fillings of I that result in Yamanouchi domino tableaux, built top to bottom, consists of a sequence of $n - j$ couples of vertical dominoes filled with ones, then two's, etc. till $n - j$, followed by $2j$ horizontal dominoes, filled with $j + 1$, etc. till $n + j$. The weight of such a Yamanouchi tableau is then $\mu = (2^{n-j}, 1^{2j})$, and its cospin is j . Therefore, by the result of Carré and Leclerc, there is a unique copy of s_μ in the decomposition of $e_n^2 = s_{1^n}^2$, and it contributes to the symmetric part of the square if j is even, and to its anti-symmetric part if j is odd.

The Yamanouchi domino tableaux are illustrated below.

1 1 1 1 : 1 1 1 1	1 1 1 1 : 1 1 $\frac{1}{2}$...	1 1 $\frac{1}{2}$: $\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$: $\frac{1}{2}$ $\frac{1}{2}$
$\mu = (2n)$	$\mu = (2n - 1, 1)$...	$\mu = (n + 1, n - 1)$	$\mu = (n, n)$
1 1	1 1	...	1 1	1
2 2	2 2	...	2	2
:	3	3
n-1 n-1	n-1 n-1
n n	n n	...	2n-4	2n-3
	n+1	...	2n-3	2n-2
		...	2n-2	2n-1
		...	2n-1	2n
$\mu = (2^n)$	$\mu = (2^{n-1}, 1^2)$...	$\mu = (2^1, 1^{2(n-1)})$	$\mu = (1^{2n})$
<i>cospin</i> = 0	<i>cospin</i> = 1	...	<i>cospin</i> = $n - 1$	<i>cospin</i> = n

This gives an alternate proof of proposition 3.10.

This could be generalized to higher plethysms. We know, thanks to Stanton and White [SW85], that r -tuples of tableaux are in bijection with r -ribbon tableaux (domino tableaux are 2-ribbon tableaux). In particular, $(m)^n$ -tuples of row (resp. column) tableaux are in bijection with r -ribbon tableaux of shape $(m \cdot n)^n$ (resp. $(n)^{m \cdot n}$). Could we then define a Yamanouchi-like property and a cospin-like statistic that could help solve the plethystic decomposition of h_m^n and e_m^n ? Could this be extended even to h_λ^n , e_λ^n or s_λ^n ?

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