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SPLITTING THE TANGENT BUNDLE(1)

BY

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ABSTRACT. We determine those unoriented cobordism classes which can be realized by a manifold whose tangent bundle splits into a sum of real (complex) line bundles.

The main result of this paper answers a question raised by Robert Stong: A closed manifold M of positive dimension n is cobordant to a manifold N whose tangent bundle is a sum of line bundles iff the Euler characteristic of M is even. This has a nice algebraic consequence since N is the splitting space of its tangent bundle, namely, the Stiefel-Whitney classes of N are the elementary symmetric polynomials of n classes in $H^1(N; \mathbf{Z}_2)$. Note that the Euler characteristic mod 2 is just the top Stiefel-Whitney number.

In the stable range we have: Every manifold is cobordant to a manifold with the property that the Whitney sum of its tangent bundle and a trivial line bundle is a sum of line bundles.

We also get similar results for weakly complex manifolds, and ask that the splitting be in terms of complex line bundles.

1. Constructions. Here we construct manifolds which arise as projective bundles and which will be used in the following section to prove the main theorems.

All objects and morphisms are smooth. Associated to a real vector bundle $\xi \longrightarrow M$ is the fibre bundle $RP(\xi) \xrightarrow{\pi} M$ of lines in the fibres of ξ. The tangent bundle $\tau RP(\xi)$ of $RP(\xi)$ is well known. Let λ denote the canonical line bundle over $RP(\xi)$, that is

$$\lambda = \{(v, x) \in \xi \times RP(\xi) \,|\, v \in x\}$$

then

$$(1) \qquad \tau RP(\xi) = \pi^*(\tau M) \oplus \phi$$

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where ϕ is the tangent bundle along the fibres of π. ϕ may be interpreted as follows:

$$\phi \oplus 1 = \pi^*(\xi) \otimes_R \lambda.$$

Here **n** denotes the trivial n-dimensional vector bundle.

If, in particular, $\tau M = \eta \oplus 1$ then

$$\tau RP(\xi) = \pi^*(\eta) \oplus (\pi^*(\xi) \otimes_R \lambda). \tag{2}$$

Lemma 1.1. *Suppose τM and ξ split into a sum of line bundles. If n of the line bundles, in the splitting of τM, are trivial, $n > 0$, then $\tau RP(\xi)$ splits into a sum of line bundles, $n-1$ of which are trivial.*

Proof. Note, that in (2), $\pi^*\xi \otimes_R \lambda$ is again a sum of line bundles. □

We construct some manifolds: Let $RP(n_1, \cdots, n_t) = RP(\lambda_{n_1} \oplus \cdots \oplus \lambda_{n_t})$ where $\lambda_{n_i} \to RP^{n_1} \times \cdots \times RP^{n_i} \times \cdots \times RP^{n_t}$ is the pullback of the canonical line bundle over the real projective space $RP^{n_i} = RP(\mathbf{n}_i + 1)$. $RP(n_1, \cdots, n_t)$ is a closed manifold of dimension $n_1 + \cdots + n_t + t - 1$.

Define the *Stong generators* X^n, $n \neq 2^s - 1$, as follows:

(a) If $n = 4s - 2$, $s \geq 0$,

$$X^n = RP(0, 0, 0, \underbrace{1, \cdots, 1}_{2s}).$$

(b) If $n = 4s$, $s \geq 1$,

$$X^n = RP(0, \underbrace{1, \cdots, 1}_{2s}).$$

(c) Let λ be the canonical line bundle over $RP(0, 1)$;

$$X^5 = RP(\lambda \oplus 3).$$

(d) If $n = 2^p(2q+1) - 1$, $p > 0$, $q > 0$, $n \neq 5$,

$$X^n = RP(0, \underbrace{1, \cdots, 1}_{2^p q - 1}, 2^p).$$

Stong [4] proved that the unoriented cobordism class $[RP(n_1, \cdots, n_t)]$, $t \geq 2$, is indecomposable iff $\sum_{i=1}^n \binom{n+t-2}{n_i} \equiv 1 \bmod 2$, where $n = n_1 + \cdots + n_t$. Thus by [6] the collection $\{[X^n]\}$ generates the unoriented cobordism ring $\mathfrak{N}_*$. (We show that $[X^5]$ is indecomposable in Proposition 1.3.)

Let $X^n = RP(\xi)$ where $\xi \to M$ is the bundle defining X^n. If n is even, $n \neq 2$ then ξ is a sum of line bundles and M is of the form $S^1 \times \cdots \times S^1 = (S^1)^k$, $k \geq 2$. If n is odd, $n \neq 5$, then ξ is a sum of line bundles and M is of the form $(S^1)^k RP^l$, $k \geq 3$. Applying Lemma 1.1 gives most of:

Proposition 1.2. *The Stong generators X^n satisfy*:

(a) *τX^n is isomorphic to a sum of line bundles if $n \neq 2$.*

(b) *One of the line bundles in* (a) *is trivial if $n \neq 5$.*

(c) *The only odd-dimensional product* M^{2k+1} *in the Stong generators with Stiefel-Whitney number* $w_2 w_{2k-1} \neq 0$ *is of the form* $X^2 \cdots X^2 X^5$.

Proof. It remains to prove (a) for $n = 5$ and part (c). We prove (c). (Part (a) for $n = 5$ will follow from Proposition 1.3(a).) A computation will show that $X^2 \cdots X^2 X^5$ ($k - 2$ factors of X^2) has nonvanishing Stiefel-Whitney number $w_2 w_{2k-1}$. We want to prove that no other product in the Stong generators has this property. It follows from (1) and the analysis prior to this proposition that, if n is even, $n \neq 2$, then $w_1 w_{n-1} = w_n[X^n] = 0$. If n is odd, $n \neq 5$, then all Stiefel-Whitney numbers associated with w_{n-2}, w_{n-1}, and w_n vanish. Thus M^{2k+1} cannot be divisible by an odd Stong generator X^n if $n > 5$, so X^5 divides M^{2k+1}. But $w_1 w_4 = w_5[X^5] = 0$ so no Stong generator X^n, $n \neq 2, 5$, can divide M^{2k+1} and X^5 divides M^{2k+1} only once. □

Some more manifolds need to be constructed. Denote a bundle ξ over M by (M, ξ). We use induction now:

$$(M^1, \lambda_1) = (RP^1, \lambda), \qquad (M^k, \lambda_k) = (RP(\lambda_{k-1} \oplus 1), \lambda)$$

where λ is the canonical line bundle. Define $Y^{k+3} = RP(\lambda_k \oplus 3)$.

Cohomology is Z_2-cohomology. Using the Leray-Hirsch theorem, [1, p. 61] and induction, one shows:

(3) $$H^*(M^k) = \mathbf{Z}_2[a_1, \cdots, a_k]$$

mod the relations $a_1^2 = 0$ and $a_{i-1} a_i = a_i^2$, $2 \leq i \leq k$, where $a_i \in H^1(M^k)$ for all i.

(4) $$H^*(Y^{k+3}) = H^*(M^k)[b]$$

mod the relation $a_k b^3 = b^4$ where $b \in H^1(Y^{k+3})$.

(5) $$w(Y^{k+3}) = (1 + a_1) \cdots (1 + a_{k-1})(1 + a_k + b)(1 + b)^3.$$

If $\sigma_i(t)$ denotes the ith elementary symmetric polynomial in variables $a_1, \cdots, a_t$, then it follows from (5) that

$$w_2(Y) = \sigma_2(k) + a_k b,$$

$$w_{k+1}(Y) = a_k b^2 \sigma_{k-2}(k-1) + a_1 \cdots a_k b.$$

Since the a_i's live in the cohomology of a k-manifold, any homogeneous polynomial in the a_i's of degree $> k$ vanishes so

$$w_2 w_{k+1}(Y) = a_k^2 b^3 \sigma_{k-2}(k-1) = \binom{k-1}{k-2} a_1 \cdots a_k b^3.$$

We have proved part (b) of:

Proposition 1.3. *If* k *is even,* $k \geq 2$, *then* Y^{k+3} *has the following properties*:
(a) τY *is isomorphic to a sum of line bundles.*
(b) $w_2 w_{k+1}[Y] \neq 0$.

Proof. (Note that $X^5 = Y^5$.) We show (a) by induction. The tangent bundle of $M^1 = S^1$ is simply 1. Suppose we have shown that $\tau M^{k-1} = 1 \oplus \theta$ where θ is a sum of line bundles. $M^k \xrightarrow{\pi} M^{k-1}$ is a fibre bundle and $\tau M^k = \pi^* \tau M^{k-1} \oplus \phi$ by (1) where ϕ is the tangent bundle along the fibres of π. But ϕ is a line bundle; thus $\tau M^k = 1 \oplus \pi^*\theta \oplus \phi$ where $\pi^*\theta \oplus \phi$ is a sum of line bundles. It now follows from Lemma 1.1 that τY^{k+3} is a sum of line bundles. □

2. The main theorems. Given an n-manifold M, denote the evaluation map $H^n(M) \longrightarrow \mathbf{Z}_2$ by $x[M]$ for any $x \in H^n(M)$. The total Wu class $v(M) = 1 + v_1 + \cdots + v_{[n/2]}$ and the total Stiefel-Whitney class $w(M) = 1 + w_1 + \cdots + w_n$ of M are related by the Steenrod squaring operation $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \cdots + \mathrm{Sq}^n$: $\mathrm{Sq}\, v = w$. Moreover, given $x \in H^k(M)$ then $\mathrm{Sq}^{n-k}(x)[M] = v^{n-k}x[M]$. See [2].

Proposition 2.1. *Let M^{kn} be a closed manifold of dimension kn. Suppose that $w(M) = \Pi_{i=1}^n (1 + z_i)$ where each z_i is in $H^k(M)$ and has the property that $\mathrm{Sq}^j z_i = 0$ for $0 < j < k$; then $w_{kn}[M] = 0$.*

Proof. Note that $v_k = w_k = z_1 + \cdots + z_n$ and $w_{kn} = z_1 \cdots z_n$.

$$\begin{aligned} w_{kn}[M] &= z_1 \cdots z_n[M] \\ &= (z_2 + \cdots + z_n + v_k)(z_2 \cdots z_n)[M] \\ &= (z_2 + \cdots + z_n)(z_2 \cdots z_n)[M] + \mathrm{Sq}^k(z_2 \cdots z_n)[M] = 0. \quad \square \end{aligned}$$

If, in particular, τM splits into a sum of (real, complex, or quaternionic) line bundles, then the Euler characteristic of M is even.

Theorem 2.2. *Every manifold is unoriented cobordant to a manifold M with the property that $\tau M \oplus 1$ splits into a sum of line bundles.*

Proof. It suffices to prove that products of the Stong generators have this property. By 1.2(a) we only have to prove it for manifolds of the form $(X^2)^k$. We induct on k: $\tau X^2 \oplus 1 = \tau RP^2 \oplus 1$ is a sum of line bundles. Suppose the theorem is true for $(X^2)^j$, $j < k$. If $\omega = (n_1, \cdots, n_t)$ is a partition of $2n$, let $X^\omega = X^{n_1} \ldots X^{n_t}$. RP^{2k} can be uniquely expressed in terms of the Stong generators

$$[RP^{2k}] = [X^2]^k + \sum_W [X^\omega]$$

for some finite set W. Note that $w_{2k}[X^\omega] = 0$ for all $\omega \in W$; thus, if $(X^2)^j$ divides X^ω, then $j < k$. By induction and 1.2(a) for every $\omega \in W$, X^ω is cobordant to a manifold M with the property that the sum of τM and a trivial line bundle splits into a sum of line bundles. But $\tau RP^{2k} \oplus 1$ is isomorphic to a sum of line bundles, so the theorem holds for $(X^2)^k$. □

Theorem 2.3. *A class $\alpha \in \mathfrak{N}_n$, $n > 0$, contains a manifold $M \in \alpha$ whose tangent bundle splits into a sum of line bundles iff $w_n(\alpha) = 0$.*

Proof. Necessity follows from 2.1. It suffices to prove the theorem for all products of the Stong generators. From 1.2(b) and 2.2 it follows that, except for $(X^2)^k$ and $(X^2)^k(X^5)^l$, all products of the Stong generators satisfy the theorem.

Since $w_{2k}[(X^2)^k] \neq 0$ it remains to show that $(X^2)^k(X^5)^l$ is cobordant to a manifold whose tangent bundle splits into a sum of line bundles. τX^5 splits into a sum of line bundles so we only need to prove the theorem for $(X^2)^k X^5$.

We induct on k: Suppose it is true for $(X^2)^j X^5$, $j < k$. Y^{2k+5} of Proposition 1.3 is uniquely expressed in terms of the Stong generators

$$[Y^{2k+5}] = [(X^2)^k X^5] + \sum_W [X^\omega]$$

for some finite set W of partitions. From 1.2(c) and 1.3(b) it follows that $w_2 w_{2k+3}[X^\omega] = 0$ for all $\omega \in W$. Thus, if $(X^2)^j X^k$ divides X^ω, then $j < k$. By induction and 1.2(a), for every $\omega \in W$, X^ω is cobordant to a manifold whose tangent bundle splits into a sum of line bundles. From 1.3(a) it follows that this is also true for $(X^2)^k X^5$. □

Corollary 2.4. *A class $\alpha \in \mathfrak{N}_n$, $n > 0$, contains a manifold M whose Stiefel-Whitney classes are the elementary symmetric polynomials of classes $t_1, \cdots, t_n \in H^1(M)$ iff $w_n(\alpha) = 0$.* □

It follows from Theorem 2.3 that every odd-dimensional manifold is cobordant to a manifold whose tangent bundle splits into a sum of line bundles. The following is the best one can say for all even-dimensional manifolds.

Corollary 2.5. *Every even-dimensional manifold M^{2k} is cobordant to a manifold whose tangent bundle is isomorphic to a sum of 2-dimensional vector bundles.*

Proof. If $w_{2k}[M] = 0$, then we are done by 2.3. Otherwise M is cobordant to a sum of $(X^2)^k$ and a manifold N^{2k} with $w_{2k}[N] = 0$. But $\tau(X^2)^k$ is isomorphic to a sum of 2-dimensional bundles. □

Remark. Note that none of the manifolds we constructed is orientable, so the question as to whether there exists an orientable manifold M^n, $n > 0$, which does not bound and whose tangent bundle splits into a sum of line bundles is still open. Nevertheless, one may get some necessary conditions. Since the first Pontryagin class of a line bundle is zero, all Pontryagin classes of M are torsion. $H^n(M; \mathbf{Z})$ is free, so all Pontryagin numbers vanish. Thus, $[M] \in \mathrm{Tor}\, \Omega^{SO}$. Moreover, it follows from [7] that $\mathrm{Tor}\, \Omega^{SO}$ is contained in the ideal of $\mathfrak{N}$ generated by all odd-dimensional classes.

3. The complex case. Unfortunately, it is very hard to determine those classes in the complex cobordism ring Ω_*^U which can be realized by a manifold whose tangent bundle splits into a sum of complex line bundles. But such determination is accessible in $\mathfrak{N}_*$.

One may define $CP(n_1, \cdots, n_t)$ as in §1 and go on to define the complex Stong generators CX^n of real dimension $2n$. It follows from [3] and [1, p. 64] that these manifolds generate the image of $\Omega_*^U \to \mathfrak{N}_*$.

The tangent bundle of CX^n does not split into a sum of line bundles. We remedy this as follows: According to [4] there exists a map $\sigma: S^1 \times S^1 \to S^2$ which is bordant to the identity $1: S^2 \to S^2$ (e.g., the identification map which collapses to a point the complement of an open disc in the torus). Therefore, if M fibres over $S^2 \times N$, then $[M] = [(\sigma \times 1)^*M]$ where $(\sigma \times 1)^*M$ is the pullback of M along $\sigma \times 1$: $S^1 \times S^1 \times N \to S^2 \times N$.

Say $n_1 = 1$, then $CP(\lambda_{n_1} \oplus \cdots \oplus \lambda_{n_t})$ fibres over $S^2 \times N$ where N is a product of complex projective spaces. Thus

$$(\sigma \times 1)^* CP(\lambda_{n_1} \oplus \cdots \oplus \lambda_{n_t}) = CP(\sigma^*\lambda_{n_1} \oplus \cdots \oplus \lambda_{n_t})$$

where $\sigma^*\lambda_{n_1}$ is again a complex line bundle and $(\sigma \times 1)^*CP(n_1, \cdots, n_t)$ fibres over $S^1 \times S^1 \times N$.

Denote the almost complex manifold $(\sigma \times 1)^*CX^n$ by $C_\sigma X^n$.

Proposition 3.1. *The complex Stong generators $C_\sigma X^n$ have the following properties*:

(a) *$\tau C_\sigma X^n$ is isomorphic to a sum of complex line bundles.*

(b) *One of the line bundles from* (a) *is trivial if $n \neq 5$.*

(c) *The only product $M^{2(2k+1)}$ in the complex Stong generators with $w_4 w_{4k-2}[M] \neq 0$ is of the form $(C_\sigma X^2)^t C_\sigma X^5$.*

Proof. For $n \neq 2, 5$, $C_\sigma X^n$ fibres over $(S^1)^2 \times (S^2)^t \times CP^k$. If π denotes the projection onto CP^k and λ the canonical line bundle over CP^k then

$$\tau((S^1)^2 \times (S^2)^t \times CP^k) = t + 1 \oplus \pi^* \tau CP^k = t \oplus (k+1)\pi^*\lambda.$$

With this in hand, the proofs of (a) and (b) are identical to the real case.

Part (c) follows from Proposition 1.2(c) since, according to [3],

$$w_4 w_{2k-4}[C_\sigma X^{n_1} \cdots C_\sigma X^{n_t}] = w_4 w_{2k-4}[CX^{n_1} \cdots CX^{n_t}]$$
$$= w_2 w_{k-2}[X^{n_1} \cdots X^{n_t}]$$

where $k = n_1 + \cdots + n_t$. □

We may complexify our construction of Y (Proposition 1.3) and get CY which by [4] fibres over S^2. τCY may not split, but σ^*CY does. Let us again identify $C_\sigma Y$ with σ^*CY.

Proposition 3.2. *If k is even, $k \geq 2$, then the almost complex manifold $C_\sigma Y^{k+3}$ of dimension $2k+6$ has the following properties*:

(a) $\tau C_\sigma Y$ *is isomorphic to a sum of line bundles.*
(b) $w_4 w_{2k+2}[C_\sigma Y] \neq 0$. □

Theorem 3.3. *Every weakly complex manifold is unoriented cobordant to an almost complex manifold with the property that the Whitney sum of its tangent bundle and a trivial complex line bundle splits into a sum of complex line bundles.*

Proof. Note that the Stong generators $C_\sigma X^n$ generate the image of $\Omega_*^U \to \mathfrak{N}_* = (\mathfrak{N}_*)^2$. The proof is the same as in the real case. □

Remark. By using CX^n instead of $C_\sigma X^n$ we can actually get a stronger result: Every class in the image of $\Omega_*^U \to \mathfrak{N}_*$ can be realized by a complex manifold M with the property that $\tau M \oplus 1$ is a sum of complex line bundles.

Theorem 3.4. *A class α in the image of $\Omega_{2n}^U \to \mathfrak{N}_{2n}$, $n > 0$, can be represented by an almost complex manifold whose tangent bundle splits into a sum of complex line bundles iff $w_{2n}(\alpha) = 0$.* □

Corollary 3.5. *A class α in the image of $\Omega_{2n}^U \to \mathfrak{N}_{2n}$, $n > 0$, can be represented by an almost complex manifold M whose Chern classes are the elementary symmetric polynomials of classes $t_1, \cdots, t_n$ in $H^2(M; \mathbf{Z})$ iff $w_{2n}(\alpha) = 0$.* □

Corollary 3.6. *Every weakly complex manifold M^{4n} of dimension $4n$ is unoriented cobordant to a manifold whose tangent bundle is isomorphic to a sum of 2-dimensional complex vector bundles.* □

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