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Spontaneous Breakdown of Symmetry and the Gauge Invariance in a Relativistic Field Theory*)

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The Nambu-Jona-Lasinio model in which the helicity current is coupled with a massless chiral gauge field, is discussed in the pair approximation. Our model is invariant under the constant as well as local gauge transformations. We have investigated the excitation spectrum of physical states in this model, under the requirement that the chiral gauge invariance is maintained at every stage. Symmetry breaking solutions are found in a noncovariant form. Contrary to previous assertions, massless fields are still present in our theory. Thus, the physical fields are a massive vector field, a massless vector field and a phase field. It is shown that the gauge transformations are carried by the phase field and the massless vector field, respectively.

§ 1. Introduction

In the last decade, theories involving spontaneous breakdown of symmetries have been one of the dominating topics of investigation in physics. Such theories have played a central role in our understanding of non-relativistic phenomena such as superfluidity, superconductivity and ferromagnetism.

The impressive success of the method of broken symmetries in non-relativistic problems led to the hope that analogous concepts might give an insight into some of the problems in the relativistic theory. In particular, a good deal of attention has been given to the conjectures[1),2)] that the observed symmetry violations in particle physics arise from an asymmetry of the vacuum state while the fundamental dynamical equations are exactly symmetrical. The study of field theoretical models which display spontaneous breakdown of symmetries was initiated by Nambu who proposed[1)] a relativistic model of the pion based on an analogy with the B.C.S. theory of superconductivity. It is well established[1),3),4)] by now that in a relativistic quantum field theory the occurrence of spontaneous symmetry breakdown implies the existence of massless bosons. However, the observed symmetry violations in particle physics do not seem to be connected in any direct way with the appearance of massless particles. This fact has motivated

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considerable interest in how to get around the assumptions of the Goldstone theorem avoiding the zero mass conclusion.

On the other hand it was suggested[6] that Goldstone particles may actually be a source of strength rather than weakness to theories of dynamical breakdown of symmetries. As a matter of fact it has been recognized[5],[6] that massless bosons not only are responsible for the degeneracy and polarization of the vacuum, but also play the crucial role of preserving the local conservation of currents derived from the invariance properties of the theory.

Further investigations in the domain of non-relativistic phenomena showed[7] that the introduction of Coulomb interaction drastically affects the Goldstone theorem in that the original massless boson conspires with the electromagnetic field to form an excitation mode of finite mass. This result was extended to relativistic theories[8] leading to the suggestion[9] that the coupling of gauge fields might reconcile spontaneous symmetry breaking with the absence of massless particles. It has been pointed out[10],[11] however, that if long range forces are present in the theory, the convergence properties of the integral defining the formal charge associated with a symmetry transformation can be drastically affected. Therefore, one of the main assumptions of the Goldstone theorem is destroyed and the theorem itself need not apply. However, if massless bosons completely disappear, it is pertinent to ask a question how the conservation law is guaranteed and the original invariance is maintained at every stage in the theory. In fact, as we have already mentioned, in absence of long range forces, the Goldstone bosons play the important role of carrying the symmetry transformations and preserving the invariance of the system when the theory is formulated in terms of asymptotic fields. Although the above question is a natural one, it has always been overlooked in spite of the fact that in many models involving long range interactions, the asymptotic fields are unchanged under the symmetry operation. This is the case, for example, in a recent treatment[12] of the Nambu model extended to the case in which the helicity current associated with the chiral gauge invariance of the theory, is coupled to an axial vector gauge field: the local and constant gauge transformations, which leave the lagrangian invariant, vanish entirely from the theory when expressed in terms of physical fields.

It is our intention here to investigate, by means of explicit dynamical calculations, the energy spectrum in the Nambu and Jona-Lasinio theory when a long range interaction is present among the Heisenberg fields. Dynamical effects are evaluated in the chain approximation under the requirements that the current conservation, the overall invariance of the theory and the algebraic relations, namely the so-called Goldstone commutator, be consistently maintained when the theory is formulated in terms of physical fields. We find that the internal consistency of the scheme demands the existence of massless fields which turn out to be the carriers of the symmetry transformations. We explicitly write the field equations for the asymptotic operators which are manifestly gauge invariant in spite of the fact that the axial vector field has acquired a finite physical mass. We also find that a manifestly covariant solution of the field equations is, strictly speaking, consistent only with a symmetric solution of the mass equation of the Nambu-Jona-Lasinio model, which is a solution with physical fermion mass equal to zero. A noncovariant solution involving a positive time-like unit vector[9),10)] is required in order to give an account of the constant chiral gauge transformation. This manifests itself at the level of asymptotic fields, as the gauge transformation of a phase field $B(x)$, which evidently plays now the role previously ascribed to the Goldstone boson in absence of long range interaction.

In § 2, we present the particle-anti-particle Bethe-Salpeter equation as well as the equation of the chiral gauge field in a form convenient for our purpose. We then solve, in § 3, these equations for both scattering and single particle states. In this respect, a convenient technique is proposed to express the relevant operator quantities of the theory in terms of physical fields. In order to show how our method works, we first discuss the Nambu-Jona-Lasinio model by putting the helicity charge to be zero. This reproduces the well-established results.[5)] We then proceed to show that the requirement of covariance leads to perturbative solutions in which the physical fermion mass vanishes, namely there is no spontaneous breakdown of symmetry.

Section 4 is devoted to show that the physical chiral gauge field satisfies a field equation which is gauge invariant in spite of the fact that it contains components with non-zero mass. Thus the physical fields are 1) the massive fermion, 2) the massive vector, 3) the massless transverse vector and 4) fields which carry constant and local gauges.

Appendix A is provided for a detailed discussion of the gauge invariant massive vector field. The field carrying the constant gauge can be quantized consistently. Various integrals and relevant commutators are given in Appendices B and C. It is interesting to see in Appendix D that the Goldstone commutator can be calculated directly from the Bethe-Salpeter kernel.

§ 2. Derivation of fundamental equations

We consider a system characterized by the Lagrangian

$$\mathcal{L}(x) = -\bar{\psi}(x)\gamma\partial\psi(x) + g[(\bar{\psi}(x)\psi(x))^2 + (i\bar{\psi}(x)\gamma_5\psi(x))^2]$$
$$-\tfrac{1}{4}F_{\mu\nu}(x)F_{\mu\nu}(x) + ej_{\mu5}(x)A_\mu(x) \tag{2·1}$$

with

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \tag{2·2}$$

$$j_{\mu5}(x) = i\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x). \tag{2·3}$$

The field equations can easily be derived:

$$(\gamma\partial+m)\psi(x)=2g[\psi(x)(\bar{\psi}(x)\psi(x))+i\gamma_5\psi(x)(i\bar{\psi}(x)\gamma_5\psi(x))]$$
$$+ie\gamma_\mu\gamma_5\psi(x)A_\mu(x)+m\psi(x)\equiv\eta(x), \qquad (2\cdot4)$$
$$\bar{\psi}(x)(-\gamma\overleftarrow{\partial}+m)=2g[(\bar{\psi}(x)\psi(x))\bar{\psi}(x)+i(i\bar{\psi}(x)\gamma_5\psi(x))\bar{\psi}(x)\gamma_5]$$
$$+ie\bar{\psi}(x)\gamma_\mu\gamma_5A_\mu(x)+m\bar{\psi}(x)\equiv\bar{\eta}(x), \qquad (2\cdot5)$$
$$[\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu]A_\nu(x)=-ej_{\mu5}(x). \qquad (2\cdot6)$$

These equations are obviously invariant under the local gauge transformation

$$\psi(x)\to e^{ie\Lambda(x)\gamma_5}\psi(x),$$
$$\bar{\psi}(x)\to\bar{\psi}(x)e^{ie\Lambda(x)\gamma_5}, \qquad (2\cdot7)$$
$$A_\mu(x)\to A_\mu(x)+\partial_\mu\Lambda(x)$$

and also are invariant under the constant gauge transformation

$$\psi(x)\to e^{i\alpha\gamma_5}\psi(x),$$
$$\bar{\psi}(x)\to\bar{\psi}(x)e^{i\alpha\gamma_5}. \qquad (2\cdot8)$$

The associated conservation law is

$$\partial_\mu j_{\mu5}(x)=0. \qquad (2\cdot9)$$

Using the equal-time commutator, we further obtain the Goldstone commutator

$$\langle0|[j_{\mu5}(x),\rho_5(x')]|0\rangle\delta(x_0-x_0')=2\delta_{\mu4}\langle0|\bar{\psi}(x)\psi(x)|0\rangle\delta^{(4)}(x-x'), \qquad (2\cdot10)$$

where

$$\rho_5(x)=i\bar{\psi}(x)\gamma_5\psi(x). \qquad (2\cdot11)$$

When the symmetry (2·8) is spontaneously broken, we can evaluate the right-hand side of (2·10) to obtain

$$\langle0|[j_{\mu5}(x),\rho_5(x')]|0\rangle\delta(x_0-x_0')=-\frac{m}{g}\delta_{\mu4}\delta^{(4)}(x-x') \qquad (2\cdot12)$$

where use has been made of the mass equation

$$\langle0|\bar{\psi}(x)\psi(x)|0\rangle=-\frac{1}{2}\frac{m}{g} \qquad (2\cdot13)$$

which is discussed in Appendix B.

As was stated in the preceding section, our intention is to investigate the excitation spectrum of physical states in the above model under the condition that the conservation law (2·9) and the Goldstone commutator (2·12) are maintained so that the gauge symmetries (2·7) and (2·8) are recovered at the final stage. For this purpose, we introduce the fermion-anti-fermion Bethe-Salpeter amplitude

$$\chi_q^{\alpha\beta}(x,y)=\langle0|T(\psi_\alpha(x),\bar{\psi}_\beta(y))|q\rangle \qquad (2\cdot14)$$

and its conjugate

$$\bar{\chi}_q^{\alpha\beta}(x,y)=\langle q|T(\psi_\alpha(x),\bar{\psi}_\beta(y))|0\rangle\,, \tag{2·15}$$

where the state $|q\rangle$ is an eigenvector of the total energy-momentum vector P_μ, i.e.,

$$P_\mu|q\rangle=q_\mu|q\rangle\,. \tag{2·16}$$

The equations of motion (2·4) and (2·5) and the equal-time commutator then yield

$$\begin{aligned}(\gamma\partial_x+m)_{\alpha\alpha'}\chi_q^{\alpha'\beta'}(x,y)(-\gamma\overleftarrow{\partial}_y+m)_{\beta'\beta}\\ =-i\delta(x_0-y_0)(\gamma_4)_{\alpha\alpha'}\langle 0|\{\eta_{\alpha'}(x),\bar{\psi}_\beta(y)\}|q\rangle+\langle 0|T(\eta_\alpha(x),\bar{\eta}_\beta(y))|q\rangle\,.\end{aligned} \tag{2·17}$$

In pair approximation in which the last term can be neglected, this equation takes a particularly simple form: the use of the equal-time commutator reduces (2·17) to

$$\begin{aligned}(\gamma\partial_x+m)_{\alpha\alpha'}\chi_q^{\alpha'\beta'}(x,y)(-\gamma\overleftarrow{\partial}_y+m)_{\beta'\beta}\\ =-2ig\delta^{(4)}(x-y)(\gamma_5)_{\alpha\beta}\,\mathrm{Tr}[\gamma_5\chi_q(x,x)]-ie\delta^{(4)}(x-y)(i\gamma_\mu\gamma_5)_{\alpha\beta}\langle 0|A_\mu(x)|q\rangle\,.\end{aligned} \tag{2·18}$$

Integrating (2·18), we obtain

$$\begin{aligned}\chi_q(x,y)=\chi_q^{(0)}(x,y)-2ig\int d^4uS_c(x-u)\gamma_5S_c(u-y)\,\mathrm{Tr}[\gamma_5\chi_q(x,x)]\\ -ie\int d^4uS_c(x-u)i\gamma_\mu\gamma_5S_c(u-y)\langle 0|A_\mu(x)|q\rangle,\end{aligned} \tag{2·19}$$

where

$$(\gamma\partial_x+m)\chi_q^{(0)}(x,y)(-\gamma\overleftarrow{\partial}_y+m)=0\,, \tag{2·20}$$

$$(\gamma\partial_x+m)S_c(x-y)=\delta^{(4)}(x-y)\,. \tag{2·21}$$

The first term in the right-hand side of (2·19) represents two free fermions and vanishes if the state $|q\rangle$ is the bound state.

It proves convenient to introduce at this stage the new amplitudes $\chi_q(z)$ and $\bar{\chi}_q(z)$ by

$$\chi_q^{\alpha\beta}(x,y)=\frac{1}{(2\pi)^{3/2}}e^{iqX}\chi_q^{\alpha\beta}(z), \tag{2·22}$$

$$\bar{\chi}_q^{\alpha\beta}(x,y)=\frac{1}{(2\pi)^{3/2}}e^{-iqX}\bar{\chi}_q^{\alpha\beta}(z), \tag{2·23}$$

where

$$\begin{aligned}X&\equiv\tfrac{1}{2}(x+y),\\ z&\equiv x-y\,.\end{aligned} \tag{2·24}$$

We shall also introduce

$$\langle 0|A_\mu(x)|q\rangle = \frac{1}{(2\pi)^{3/2}} e^{iqx} a_\mu(q). \tag{2·25}$$

Equation (2·19) can then be rewritten as

$$\chi_q(z) = \chi_q^{(0)}(z) + 2g\,\mathrm{Tr}[\gamma_5\chi_q(0)]Q(z;q) + ea_\mu(q)Q_\mu(z;q) \tag{2·26}$$

with

$$Q(z;q) = -i\int d^4y S_c\left(\frac{z}{2}-y\right)\gamma_5 S_c\left(\frac{z}{2}+y\right)e^{iqy}, \tag{2·27}$$

$$Q_\mu(z;q) = -i\int d^4y S_c\left(\frac{z}{2}-y\right) i\gamma_\mu\gamma_5 S_c\left(\frac{z}{2}+y\right)e^{iqy}. \tag{2·28}$$

If we note the relations

$$\mathrm{Tr}[\gamma_5 Q(0;q)] = J^P(q^2), \tag{2·29}$$

$$\mathrm{Tr}[i\gamma_\mu\gamma_5 Q(0;q)] = J_\mu^{AP}(q) = -2mq_\mu I(q^2), \tag{2·30}$$

$$\mathrm{Tr}[\gamma_5 Q_\mu(0;q)] = J_\mu^{PA}(q) = 2mq_\mu I(q^2), \tag{2·31}$$

$$\mathrm{Tr}[i\gamma_\lambda\gamma_5 Q_\mu(0;q)] = J_{\lambda\mu}^A(q) = \left[\delta_{\mu\lambda} - \frac{1}{q^2}q_\mu q_\lambda\right]J^A(q^2) + 4m^2 I(q^2)\,\frac{q_\mu q_\lambda}{q^2}, \tag{2·32}$$

we can further reduce (2·26) to

$$C(q) = C^{(0)}(q) + 2gJ^P(q^2)C(q) + 2iemI(q^2)q_\mu a_\mu(q), \tag{2·33}$$

$$D_\lambda(q) = D_\lambda^{(0)}(q) + eJ^A(q^2)\left[\delta_{\lambda\nu} - \frac{q_\lambda q_\nu}{q^2}\right]a_\nu(q)$$
$$+ 4imI(q^2)q_\lambda\left[gC(q) - iem\frac{1}{q^2}q_\nu a_\nu(q)\right], \tag{2·34}$$

where

$$C(q) \equiv \mathrm{Tr}[i\gamma_5\chi_q(0)], \tag{2·35}$$

$$D_\lambda(q) \equiv \mathrm{Tr}[i\gamma_\lambda\gamma_5\chi_q(0)] \tag{2·36}$$

and similarly

$$C^{(0)}(q) \equiv \mathrm{Tr}[i\gamma_5\chi_q^{(0)}(0)], \tag{2·37}$$

$$D_\lambda^{(0)}(q) \equiv \mathrm{Tr}[i\gamma_\lambda\gamma_5\chi_q^{(0)}(0)]. \tag{2·38}$$

The quantities such as $J^P(q^2)$, $J^A(q^2)$ and $I(q^2)$ are given integrals coming from fermion-anti-fermion loops. Their definitions, explicit form and properties can be found in Appendix B. In particular, we note that when the gauge symmetry is spontaneously broken, namely, $m\neq 0$, it holds

$$J^P(q^2) = \frac{1}{2g} - q^2 I(q^2) \tag{2·39}$$

by virtue of the mass equation (2·13). Furthermore, $J^A(q^2)$ is a monotonically increasing function of q^2 in the region $q^2>-4m^2$, and $I(q^2)$ is nonsingular at $q^2=0$.

The above two equations (2·33) and (2·34) are not sufficient to determine the unknowns $C(q)$, $D_\lambda(q)$ and $a_\lambda(q)$. We note, however, the relations

$$\langle 0|j_{\mu 5}(x)|q\rangle = -\frac{1}{(2\pi)^{3/2}}e^{iqx}D_\mu(q), \qquad (2·40)$$

$$\langle 0|\rho_5(x)|q\rangle = -\frac{1}{(2\pi)^{3/2}}e^{iqx}C(q), \qquad (2·41)$$

which follow from the definitions (2·35), (2·36), (2·3) and (2·11). Equations (2·40), (2·25) and (2·6) then yield

$$[q^2\delta_{\mu\nu}-q_\mu q_\nu]a_\nu(q) = -eD_\mu(q). \qquad (2·42)$$

The set of equations (2·33) and (2·34) supplemented by (2·42) can be solved for the unknowns $C(q)$, $D_\mu(q)$ and $a_\mu(q)$.

§ 3. Covariant solutions

We wish to find various solutions of Eqs. (2·33), (2·34) and (2·42). It is instructive, however, to see within our formalism how in the Nambu-Jona-Lasinio model the Goldstone particle appears and the symmetry (2·8) is fully recovered at the level of the physical fields.

(a) *When $e=0$:*

Our model reduces to that of Nambu-Jona-Lasinio, if we put $e=0$. We obtain from (2·33) and (2·34)

$$D_\lambda(q) = D_\lambda^{(0)}(q) + 4igmq_\lambda I(q^2)C(q) \qquad (3·1)$$

and

$$C(q) = C^{(0)}(q) + 2gJ^P(q^2)C(q), \qquad (3·2)$$

which can be cast, when $m\neq 0$, into the form

$$2gq^2I(q^2)C(q) = C^{(0)}(q) \qquad (3·3)$$

by the aid of (2·39). From the helicity current conservation (2·9) and the relation (2·40), it follows

$$q_\lambda D_\lambda(q) = 0. \qquad (3·4)$$

Multiplying (3·1) by q_λ and using (3·4) and (3·3), we obtain

$$q_\lambda D_\lambda^{(0)}(q) + 4igmq^2I(q^2)C(q) = q_\lambda D_\lambda^{(0)}(q) + 2imC^{(0)}(q) = 0. \qquad (3·5)$$

If we denote the contribution of two free fermion state to $C(q)$ and $D_\lambda(q)$ by $C(q,F)$ and $D_\lambda(q,F)$, respectively, Eq. (3·3) gives

$$C(q, F) = \frac{1}{2gq^2 I(q^2)} C^{(0)}(q). \tag{3·6}$$

The substitution of (3·6) into (3·1) yields

$$D_\lambda(q, F) = D_\lambda^{(0)}(q) + 2imq_\lambda \frac{1}{q^2} C^{(0)}(q)$$
$$= \left(\delta_{\lambda\nu} - \frac{1}{q^2} q_\lambda q_\nu\right) D_\nu^{(0)}(q) \tag{3·7}$$

by virtue of (3·5).

We now introduce the massive free fermion field $\phi(x)$ satisfying

$$(\gamma\partial + m)\phi(x) = 0 . \tag{3·8}$$

We then transform (3·7) and (3·6) into coordinate space and express their contribution to $\rho_5(x)$ and $j_{\mu 5}(x)$:

$$\rho_5^F(x) = -\frac{1}{2g\Box I(-\Box)} (i\bar{\phi}(x)\gamma_5\phi(x)), \tag{3·9}$$

$$j_{\mu 5}^F(x) = \left(\delta_{\mu\nu} - \frac{1}{\Box}\partial_\mu\partial_\nu\right)(i\bar{\phi}(x)\gamma_\nu\gamma_5\phi(x)), \tag{3·10}$$

which agree with the relations previously obtained in Ref. 5).

The bound state contribution $C(q, B)$ can easily be obtained by dropping the inhomogeneous terms $C^{(0)}(q)$ from (3·3):

$$q^2 I(q^2) C(q, B) = 0 . \tag{3·11}$$

Due to the fact that $I(q^2)$ is not singular at $q^2 = 0$, there exists a bound state with the vanishing mass. The bound state contribution to $D_\lambda(q)$ is extracted from (3·1) by omitting $D_\lambda^{(0)}(q)$:

$$D_\lambda(q, B) = 4igmq_\lambda I(q^2) C(q, B). \tag{3·12}$$

The solutions $C(q, B)$ and $D_\lambda(q, B)$ of homogeneous equations can be normalized in two ways. The first method is a straightforward application of the technique put forward by Lurié et al.[18] We shall here apply the formula

$$i\partial_\mu\langle 0|T(j_{\mu 5}(x), \rho_5(x'))|0\rangle = \langle 0|[j_{45}(x), \rho_5(x')]|0\rangle\delta(x_0 - x_0')$$
$$= -2\langle 0|T(\psi(x), \bar{\psi}(x'))|0\rangle\delta^{(4)}(x - x')$$
$$= -\frac{m}{g}\delta^{(4)}(x - x'). \tag{3·13}$$

The derivation of this relation is given in Appendix D. The first method gives the same result as that obtained by (3·13). To make use of (3·13), let us introduce a massless boson field $B(x)$ such that

$$[B(x), B(x')] = iD(x - x'), \tag{3·14}$$

$$\Box B(x)=0. \tag{3·15}$$

Then, the boson contribution to $\rho_5(x)$ is

$$\rho_5{}^B(x)=aB(x). \tag{3·16}$$

Correspondingly, the boson contribution to $j_{\mu 5}(x)$ is, due to (3·12)

$$j_{\mu 5}^B(x)=4gmaI(0)\partial_\mu B(x), \tag{3·17}$$

where a is a real constant to be determined by (3·13).

Since we have, as is shown in Appendix C,

$$\langle 0|[j_{\mu 5}^F(x),\rho_5{}^F(x')]|0\rangle=0, \tag{3·18}$$

we obtain

$$\begin{aligned}\langle 0|[j_{45}(x),\rho_5(x')]|0\rangle\delta(x_0-x_0') &= \langle 0|[j_{45}^B(x),\rho_5{}^B(x')]|0\rangle\delta(x_0-x_0')\\ &=4gma^2I(0)\langle 0|[\partial_4 B(x),B(x')]|0\rangle\delta(x_0-x_0')\\ &=-4gma^2I(0)\delta^{(4)}(x-x')\end{aligned} \tag{3·19}$$

which must agree with (3·13). Hence,

$$a=\frac{1}{2g}\frac{1}{\sqrt{I(0)}}. \tag{3·20}$$

We thus arrive at

$$\rho_5{}^B(x)=\frac{1}{2g}\frac{1}{\sqrt{I(0)}}B(x), \tag{3·21}$$

$$j_{\mu 5}^B(x)=2m\sqrt{I(0)}\,\partial_\mu B(x). \tag{3·22}$$

The generator of the transformation (2·8) can be constructed formally:

$$\begin{aligned}G&\equiv\int d\sigma_\mu(x)j_{\mu 5}(x)\\ &=2m\sqrt{I(0)}\int d\sigma_\mu(x)\partial_\mu B(x).\end{aligned} \tag{3·23}$$

The fermion part of $j_{\mu 5}(x)$ does not contribute to G due to the presence of the projection operator in (3·10). The operator

$$U_\alpha\equiv\exp i\alpha G \tag{3·24}$$

then transforms $\phi(x)$ and $B(x)$ as

$$U_\alpha{}^{-1}\phi(x)U_\alpha=\phi(x), \tag{3·25}$$

$$U_\alpha{}^{-1}B(x)U_\alpha=B(x)+2m\sqrt{I(0)}\,\alpha. \tag{3·26}$$

We see that the original constant gauge (2·8) is now carried by the massless field $B(x)$. It is a simple matter to demonstrate that the current expressed in terms of the physical fields is indeed conserved.

(b) *When $e\neq 0$:*

In this case, equations to be solved are

$$D_\lambda(q)=D_\lambda^{(0)}(q)+eJ^A(q^2)\left(\delta_{\lambda\nu}-\frac{1}{q^2}q_\lambda q_\nu\right)a_\nu(q)$$
$$+4imI(q^2)q_\lambda\left[gC(q)-iem\frac{1}{q^2}q_\nu a_\nu(q)\right], \tag{3·27}$$

$$2gq^2I(q^2)C(q)=C^{(0)}(q)+2imeI(q^2)q_\nu a_\nu(q), \tag{3·28}$$

$$[q^2\delta_{\lambda\nu}-q_\lambda q_\nu]a_\nu(q)=-eD_\lambda(q). \tag{3·29}$$

The second equation follows from (2·33) and (2·34), which is valid only when $m\neq 0$. If we multiply (3·27) by q^2 and substitute (3·28) and (3·29) into the resultant equation, we obtain

$$[q^2+e^2J^A(q^2)]D_\lambda(q)=q^2D_\lambda^{(0)}(q)+2imq_\lambda C^{(0)}(q)$$
$$=(q^2\delta_{\lambda\nu}-q_\lambda q_\nu)D_\nu^{(0)}(q) \tag{3·30}$$

by virtue of (3·5). Hence, the fermion contribution to $D_\lambda(q)$ can easily be obtained:

$$D_\lambda(q,F)=\frac{1}{q^2+e^2J^A(q^2)}[q^2\delta_{\lambda\nu}-q_\lambda q_\nu]D_\nu^{(0)}(q). \tag{3·31}$$

The solution of the homogeneous equation, denoted by $D_\lambda(q,A)$, satisfies

$$[q^2+e^2J^A(q^2)]D_\lambda(q,A)=0 \tag{3·32}$$

which obviously admits a non-trivial solution since the equation

$$M^2=e^2J^A(-M^2) \tag{3·33}$$

has one and only one root due to the fact that $J^A(z)$ is a monotonic function in the range $z<-4m^2$. Hence we may write

$$(q^2+M^2)D_\lambda(q,A)=0. \tag{3·34}$$

Considering the relation (3·4), we would be tempted to identify $D_\lambda(q,A)$ with a physical Proca field with the mass M. This would then mean that the original gauge transformation has completely vanished from our theory.

The crucial point in our model lies however in the relation (3·29) which relates the physical field to the matrix element of the helicity current. If we substitute (3·29) into (3·34), we have

$$(q^2+M^2)[q^2\delta_{\lambda\nu}-q_\lambda q_\nu]a_\nu(q,A)=0, \tag{3·35}$$

which reads in configuration space

$$(\Box-M^2)[\Box\delta_{\lambda\nu}-\partial_\lambda\partial_\nu]A_\nu^{(0)}(x)=0. \tag{3·36}$$

This implies that the physical field $A_\lambda^{(0)}(x)$ satisfies the gauge invariant massive field equation. Such a field equation was investigated previously by Palmer and one of us (Y.T.) in a different context.[14] Appendix A is provided to discuss this field. We here emphasize that the massive field $A_\mu^{(0)}(x)$ does not satisfy the Proca equation. It is worth pointing out that the gauge transformation of $A_\mu^{(0)}(x)$ does not require the simultaneous transformation of the fermion field, whereas the gauge transformation of the interpolating field $A_\mu(x)$ must be accompanied by that of the fermion field $\psi(x)$, as was seen in (2·7). Thus, we see that the original local gauge carried by $A_\mu(x)$ and $\psi(x)$ is now carried solely by the asymptotic vector field $A_\mu^{(0)}(x)$.

The helicity current in terms of physical fields now takes the form

$$j_{\mu 5}(x) = j_{\mu 5}^F(x) + j_{\mu 5}^A(x), \tag{3·37}$$

with

$$j_{\mu 5}^F(x) = \frac{1}{\Box - e^2 J^A(-\Box)}[\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu](i\bar{\psi}(x)\gamma_\nu\gamma_5\psi(x)), \tag{3·38}$$

$$j_{\mu 5}^A(x) = -\frac{1}{e}[\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu]A_\nu^{(0)}(x) \tag{3·39}$$

and also

$$\rho_5(x) = -\frac{1}{2g\Box I(-\Box)}(i\bar{\psi}(x)\gamma_5\psi(x)) + \frac{em}{g}\frac{1}{\Box}\partial_\mu A_\mu(x). \tag{3·40}$$

It should be noted that the last term in (3·40) is written in terms of the interpolating field. However, we can arrange the gauge in such a way that

$$A_\mu(x) = A_\mu^{(0)}(x) - \frac{e}{\Box}j_{\mu 5}^F(x), \tag{3·41}$$

as will be shown shortly. In this case, we can separate $\rho_5(x)$ into two parts given by

$$\rho_5^F(x) = -\frac{1}{2g\Box I(-\Box)}(i\bar{\psi}(x)\gamma_5\psi(x)) \tag{3·42}$$

and

$$\rho_5^A(x) = \frac{em}{g}\frac{1}{\Box}\partial_\mu A_\mu^{(0)}(x) \tag{3·43}$$

on account of the relation

$$\partial_\mu j_{\mu 5}^F(x) = 0\,. \tag{3·44}$$

To justify (3·41), we observe

$$\begin{aligned}[\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu]A_\nu(x) &= -ej_{\mu 5}(x)\\ &= -ej_{\mu 5}^F(x) - ej_{\mu 5}^A(x),\end{aligned} \tag{3·45}$$

which enables us to write

$$[\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu](A_\nu(x)-A_\nu^{(0)}(x))=-ej_{\mu5}^F(x) \tag{3·46}$$

due to (3·39). Hence,

$$A_\mu(x)=A_\mu^{(0)}(x)-\frac{e}{\Box}j_{\mu5}^F(x)+\partial_\mu\phi(x), \tag{3·47}$$

with an arbitrary function $\phi(x)$. However, the gauge of $A_\mu(x)$ and $A_\mu^{(0)}(x)$ is arbitrary and therefore the last term in (3·47) can be absorbed into $A_\mu(x)$ or $A_\mu^{(0)}(x)$. Thus, we arrive at (3·41).

Since the asymptotic field $A_\mu^{(0)}(x)$ satisfies (3·36), it is reasonable to assume, *on the ground of covariance*, the commutation relation

$$[A_\mu^{(0)}(x), A_\nu^{(0)}(x')]=i\alpha\delta_{\mu\nu}\Delta(x-x')+i\partial_\mu\partial_\nu f(x-x') \tag{3·48}$$

with an arbitrary constant α and an arbitrary function $f(x-x')$. Consequently, we have

$$[(\Box\delta_{\lambda\mu}-\partial_\lambda\partial_\mu)A_\mu^{(0)}(x), \partial_\nu' A_\nu^{(0)}(x')]=0\,, \tag{3·49}$$

which yields

$$\langle 0|[j_{\mu5}(x), \rho_5(x')]|0\rangle=\langle 0|[j_{\mu5}^F(x), \rho_5^F(x')]|0\rangle+\langle 0|[j_{\mu5}^A(x), \rho_5^A(x')]|0\rangle$$
$$=0 \tag{3·50}$$

due to (C·39), (3·39) and (3·43). The relation (3·50) is compatible with (2·12) only when $m=0$. Namely the physical fermion field must have the vanishing mass. We must recall however that the various relations used above are based on the assumption $m\neq 0$. If we wish to consider the case $m=0$, we have to start all over again from (2·33), (2·34) and (2·42) without using the relation such as (2·39). We then obtain

$$D_\lambda(q, F)=\frac{1}{1+e^2\hat{J}^A(q^2)}D_\lambda^{(0)}(q), \tag{3·51}$$

$$D_\lambda(q, A)=0\,, \tag{3·52}$$

$$C(q, F)=\frac{1}{1-2gJ^P(q^2)}C^{(0)}(q), \tag{3·53}$$

$$C(q, A)=0\,, \tag{3·54}$$

and in particular, $a_\mu(q, A)$ satisfies

$$[q^2\delta_{\lambda\nu}-q_\lambda q_\nu]a_\nu(q, A)=0\,, \tag{3·55}$$

where $\hat{J}^A(q^2)$ and $J^P(q^2)$ in (3·51) and (3·53) can be obtained by putting $m=0$ in (B·29) and (B·3), respectively. The above solutions (3·51)$\sim$(3·55) coincide with the perturbative solutions with the pair approximation, as we have expected.

§ 4. Noncovariant solutions

The argument presented in the preceding section indicates that the manifestly covariant solutions exist only when the symmetry is *not* spontaneously broken. In view of the fact that the Goldstone commutator is a manifestation of the completeness of solutions, this implies that the asymptotic fields obtained are not complete, and an extra mode must appear in a noncovariant form, as long as we assume $m\neq 0$. Moreover, the noncovariant mode must be of long range, for, otherwise, the generator of the constant gauge transformation would vanish identically due to the factor $\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu$ in (3·39).*)

The various relations leading to (3·47) are valid under $m\neq 0$. However, the commutation relation (3·48) for the field $A_\mu^{(0)}(x)$ should now be abandoned. To obtain the more general commutation relation, we seek a solution of the form

$$A_\mu^{(0)}(x)=a_\mu(x)+U_\mu(x)+\partial_\mu\chi(x)+n_\mu\frac{1}{\nabla^2}\pi(x), \tag{4·1}$$

where $a_\mu(x)$ and $U_\mu(x)$ are transverse massless and the Proca fields, respectively. They satisfy

$$n_\mu a_\mu(x)=\partial_\mu a_\mu(x)=\Box a_\mu(x)=0\,, \tag{4·2}$$

$$\partial_\mu U_\mu(x)=(\Box-M^2)\,U_\mu(x)=0\,. \tag{4·3}$$

The field $\chi(x)$ is arbitrary provided that

$$\chi(x)\to 0, \qquad |\boldsymbol{x}|\to\infty\,. \tag{4·4}$$

The substitution of (4·1) into (3·36) gives**)

$$n_\lambda(\Box-M^2)\pi(x)-\partial_\lambda{}^s(\Box-M^2)\frac{1}{\nabla^2}(n\partial)\pi(x)=0\,. \tag{4·5}$$

Introducing the phase field $B(x)$ by

$$(n\partial)\pi(x)=M^2\nabla^2B(x), \tag{4·6}$$

we obtain

$$n_\lambda(\Box-M^2)\pi(x)-M^2(\Box-M^2)\partial_\lambda{}^sB(x)=0\,, \tag{4·7}$$

which can be split into two equations

$$(\Box-M^2)\pi(x)=0\,, \tag{4·8}$$

$$(\Box-M^2)\nabla^2B(x)=0\,. \tag{4·9}$$

*) The existence of a noncovariant long range term can explicitly be demonstrated by showing that the set of equations (3·27), (3·28) and (3·29) admit a solution under the constraint $\boldsymbol{q}\cdot\boldsymbol{a}(q)=0$.[15] See also 3).

**) n_μ is a positive time-like vector and

$$\partial_\lambda{}^s\equiv\partial_\lambda+n_\lambda(n\partial),$$

$$\nabla^2\equiv\partial_\lambda{}^s\partial_\lambda{}^s\,.$$

We observe here an important fact that all the equations involving the phase field $B(x)$ are invariant under

$$B(x) \to B(x) + \text{constant}. \tag{4·10}$$

The relations

$$[\Box\delta_{\lambda\nu} - \partial_\lambda\partial_\nu] A_\nu^{(0)}(x) = M^2 U_\lambda(x) + n_\lambda \pi(x) - M^2 \partial_\lambda^s B(x), \tag{4·11}$$

$$\partial_\mu A_\mu^{(0)}(x) = \frac{M^2}{\nabla^2} \nabla^2 B(x) + \Box\chi(x), \tag{4·12}$$

which can easily be derived, enable Eqs. (3·39) and (3·43) to be rewritten as

$$j_{\mu5}^A(x) = -\frac{1}{e}[M^2 U_\mu(x) + n_\mu \pi(x) - M^2 \partial_\mu^s B(x)], \tag{4·13}$$

$$\rho_5^A(x) = \frac{em}{g}\left[\frac{1}{\Box}\frac{M^2}{\nabla^2}\nabla^2 B(x) + \chi(x)\right]$$

$$= \frac{em}{g}\left[\frac{1}{\nabla^2}\nabla^2 B(x) + \chi(x)\right] \tag{4·14}$$

by the aid of (4·9). The conservation of $j_{\mu5}^A(x)$ follows at once from (4·6) and (4·3).

The Goldstone commutator of $j_{\mu5}^A(x)$ and $\rho_5^A(x)$ is

$$[j_{\mu5}^A(x), \rho_5^A(x')]\delta(x_0 - x_0') = -\frac{m}{g}\frac{1}{\nabla'^2}\nabla'^2 [n_\mu \pi(x), B(x')]\delta(x_0 - x_0')$$

$$= i n_\mu \frac{m}{g} \delta^{(4)}(x - x') \tag{4·15}$$

on account of (A·27). Combining (4·15) with (C·12), we obtain

$$\langle 0|[j_{\mu5}(x), \rho_5(x')]|0\rangle \delta(x_0 - x_0') = i n_\mu \frac{m}{g} \delta^{(4)}(x - x'), \tag{4·16}$$

which agrees with (2·12).

As was pointed out before, all the equations involving $B(x)$ are invariant under (4·10). This fact makes it plausible that the constant gauge transformation is now carried by the field $B(x)$. That it is really so can be justified formally as follows: We first rewrite using (4·13) the generator of the constant gauge transformation in terms of physical fields

$$G = \int d\sigma_\mu(x) j_{\mu5}(x) = \int d\sigma_\mu(x) j_{\mu5}^A(x)$$

$$= -\frac{1}{e}\int d\sigma_\mu(x)[M^2 U_\mu(x) + n_\mu \pi(x) - M^2 \partial_\mu^s B(x)]. \tag{4·17}$$

The canonical transformation

$$U_\alpha \equiv \exp i\alpha G \tag{4·18}$$

induces

$$U_\alpha^{-1}\phi(x)U_\alpha = \phi(x), \tag{4·19}$$

$$U_\alpha^{-1}B(x)U_\alpha = B(x) + \frac{\alpha}{e}. \tag{4·20}$$

We see therefore that the constant gauge transformation (2·8) is taken over by the phase field $B(x)$.

Finally, it is worth remarking that Eqs. (3·36) and (4·9) have their counterparts in the theory of superconductivity.[16),17)] This shows that the well-known analogy between the theory of superconductivity and the Nambu-Jona-Lasinio theory holds even in the presence of long range forces.

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Appendix A

A gauge invariant massive vector field

There are at least two ways in which the Maxwell field can be generalized to a massive field. The Maxwell equations can be written as

$$\partial_\mu F_{\mu\nu}(x) = 0, \tag{A·1}$$

$$\partial_\lambda F_{\mu\nu}(x) + \partial_\mu F_{\nu\lambda}(x) + \partial_\nu F_{\lambda\mu}(x) = 0. \tag{A·2}$$

Equation (A·2) implies that there exists a vector field $A_\mu(x)$ such that

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{A·3}$$

Consequently, (A·1) can be written as

$$[\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu]A_\nu(x) = 0. \tag{A·4}$$

If we adopt a prescription

$$\Box \to \Box - M^2 \tag{A·5}$$

to obtain a massive field equation, we arrive at

$$[(\Box - M^2)\delta_{\mu\nu} - \partial_\mu\partial_\nu]U_\nu(x) = 0, \tag{A·6}$$

which is known as the Proca equation. On the other hand, we can generalize differently: Multiply (A·2) by ∂_λ. Then on account of (A·1), we obtain

$$\Box F_{\mu\nu}(x) = 0. \tag{A·7}$$

We now adopt the prescription (A·5) to obtain

$$(\Box - M^2)\, W_{\mu\nu}(x) = 0\,, \tag{A·8}$$

with

$$W_{\mu\nu}(x) = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x), \tag{A·9}$$

where $V_\mu(x)$ is a vector field. Equation (A·9) has an advantage of being gauge invariant. Indeed, as was shown in Ref. 14), the field $V_\mu(x)$ agrees with the Proca field $U_\mu(x)$ *when the gauge is fixed.*

It is interesting to note that the field $W_{\mu\nu}(x)$ satisfying (A·8) and (A·9) is equivalent to the skew-symmetric divergenceless field satisfying the Klein-Gordon equation, i.e.,

$$(\Box - M^2)\psi_{\mu\nu}(x) = 0\,, \tag{A·10}$$

$$\psi_{\mu\nu}(x) + \psi_{\nu\mu}(x) = 0\,, \tag{A·11}$$

$$\partial_\mu \psi_{\mu\nu}(x) = 0\,. \tag{A·12}$$

The fields $\psi_{\mu\nu}(x)$ and $W_{\mu\nu}(x)$ are mutually dual. Equations (A·10)$\sim$(A·12) are derivable from a Lagrangian, as was shown in 14). We observe here that the vector field $V_\mu(x)$ is a product of the Euler-Lagrange equations (A·11) and (A·12). It is very difficult to construct a Lagrangian which leads to equations (A·8) and (A·9) directly. If we set

$$\mathcal{L}(x) = -\frac{1}{4M^2}\partial_\lambda(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x))\partial_\lambda(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x))$$

$$-\frac{1}{4}(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x))(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)) \tag{A·13}$$

and take a variation with respect to $V_\nu(x)$, we obtain, instead of (A·8) and (A·9),

$$(\Box - M^2)\partial_\mu W_{\mu\nu}(x) = 0\,, \tag{A·14}$$

$$W_{\mu\nu}(x) = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x) \tag{A·15}$$

or

$$(\Box - M^2)(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\, V_\nu(x) = 0\,. \tag{A·16}$$

The field $V_\mu(x)$ satisfying (A·16) has attractive features that (i) it is completely gauge invariant, and (ii) it is identical to the asymptotic field appearing in the Nambu-Jona-Lasinio field interacting with a massless pseudovector field. The quantization of the Proca field is quite straightforward, whereas that of the Maxwell field causes some difficulty.

In order to obtain the most general solution of (A·16), we separate the $1/\nabla^2$ singularity and put

$$V_\mu(x) = a_\mu(x) + n_\mu \frac{1}{\nabla^2}\pi(x) + \partial_\mu \chi(x) + U_\mu(x)\,. \tag{A·17}$$

The field $\pi(x)$ and the phase field defined by

$$M^2\nabla^2 B(x) = (n\partial)\pi(x) \tag{A·18}$$

satisfy

$$(\Box - M^2)\pi(x) = 0\,, \tag{A·19}$$

$$(\Box - M^2)\nabla^2 B(x) = 0\,. \tag{A·20}$$

It is interesting to note that (A·20) can be expressed in an integral form

$$\Box B(x) = -\frac{M^2}{4\pi}\int d^3x' \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\nabla'^2 B(x') \tag{A·21}$$

which is exactly the same as that obtained by Leplae et al. in discussion of the superconductivity.[17] It should also be noted that Eqs. (A·18), (A·20) and (A·21) are all invariant under

$$B(x) \to B(x) + \text{constant}\,. \tag{A·22}$$

Equations (A·18), (A·19) and (A·20) are equivalent to

$$M^2\nabla^2 B(x) = (n\partial)\pi(x), \tag{A·23}$$

$$M^2\nabla^2 (n\partial) B(x) = (\nabla^2 - M^2)\pi(x). \tag{A·24}$$

To quantize the fields $B(x)$ and $\pi(x)$, we demand that Eqs. (A·23) and (A·24) are consistent with

$$i\dot{B}(x) = [B(x), H_B], \tag{A·25}$$

$$i\dot{\pi}(x) = [\pi(x), H_B]. \tag{A·26}$$

It is not difficult to see that

$$[B(x), \pi(x')]_{x_0=x_0'} = i\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{A·27}$$

$$H_B = \frac{M^2}{2}\int d^3x \nabla B(x)\cdot\nabla B(x) + \frac{1}{8\pi M^2}\int d^3x d^3x' \times\left\{\nabla\pi(x)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\nabla'\pi(x') + M^2\pi(x)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\pi(x')\right\} \tag{A·28}$$

fulfill the required condition. The quantization of $a_\mu(x)$ and $U_\mu(x)$ is well known.

Finally, we show that (A·8) is equivalent to the relativistic London equation[16]

$$\boldsymbol{E} = \lambda^2\left[\frac{\partial \boldsymbol{j}}{\partial t} + \nabla\rho\right], \tag{A·29}$$

$$\boldsymbol{H} = -\lambda^2\nabla\times\boldsymbol{j}\,, \tag{A·30}$$

$$\nabla\cdot\boldsymbol{H} = 0\,, \tag{A·31}$$

$$\nabla\cdot\boldsymbol{E}=\rho\,, \tag{A·32}$$

$$\nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{H}}{\partial t}=0\,, \tag{A·33}$$

$$\nabla\times\boldsymbol{H}-\frac{\partial\boldsymbol{E}}{\partial t}=\boldsymbol{j}\,. \tag{A·34}$$

From (A·34), (A·30) we obtain

$$\nabla\times(\nabla\times\boldsymbol{H})-\frac{\partial}{\partial t}\nabla\times\boldsymbol{E}=-\frac{1}{\lambda^2}\boldsymbol{H}\,. \tag{A·35}$$

The use of (A·33) into (A·35) gives

$$\left(\Box-\frac{1}{\lambda^2}\right)\boldsymbol{H}=0\,. \tag{A·36}$$

Similarly, from (A·34), (A·32) and (A·29) we obtain

$$\left(\Box-\frac{1}{\lambda^2}\right)\boldsymbol{E}=0\,. \tag{A·37}$$

The two equations (A·36) and (A·37) can be combined to give

$$\left(\Box-\frac{1}{\lambda^2}\right)F_{\mu\nu}=0\,. \tag{A·38}$$

Multiplying (A·38) by ∂_μ, we arrive at

$$\left(\Box-\frac{1}{\lambda^2}\right)(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)A_\nu(x)=0\,. \tag{A·39}$$

Appendix B

A table of integrals for the Nambu-Jona-Lasinio model

The relevant integrals in the Nambu-Jona-Lasinio model come from the analytic expression of various fermion-anti-fermion loops. The (ps) (ps) fermion loop corresponds to

$$J^P(q^2)=-\frac{i}{(2\pi)^4}\int d^4p\;\mathrm{Tr}\left[\gamma_5 S_c\left(p+\frac{1}{2}q\right)\gamma_5 S_c\left(p-\frac{1}{2}q\right)\right] \tag{B·1}$$

where

$$S_c(p)=-\frac{i\gamma\cdot p-m}{p^2+m^2-i\varepsilon}\,. \tag{B·2}$$

The dependence on q^2 of $J^P(q^2)$ can be established on the ground of Lorentz invariance and can be explicitly given by rewriting (B·1) in the form of a dispersive integral

$$J^P(q^2)=\int_{4m^2}^{\infty} dk^2 \frac{k^2\rho(k^2)}{q^2+k^2-i\varepsilon}, \tag{B·3}$$

where

$$\rho(k^2)=\frac{1}{8\pi^2}\left[1-4\frac{m^2}{k^2}\right]^{1/2}\theta(\Lambda^2-k^2). \tag{B·4}$$

A cutoff has been introduced in (B·4) since the integral (B·1) is quadratically divergent.

It is well known that[1),5)] the dynamics of the Nambu-Jona-Lasinio model, in addition to the "normal" perturbative solution with vanishing physical fermion mass, also allow the existence of a massive physical fermion, provided the self-consistency condition

$$m=-2ig\,\mathrm{Tr}\,S_c(0)=-\frac{8igm}{(2\pi)^4}\int^{\Lambda^2}\frac{d^4p}{p^2+m^2} \tag{B·5}$$

is satisfied with $m\neq0$.

It proves useful for practical purposes to use the expression of $J^P(q^2)$ to cast (B·5) into the form

$$1=2gJ^P(0), \tag{B·6}$$

i.e.,

$$1=2g\int_{4m^2}^{\infty}\rho(k^2)\,dk^2. \tag{B·7}$$

Taking now into account (B·6), or equivalently (B·7), we can write $J^P(q^2)$ as

$$J^P(q^2)=\frac{1}{2g}-q^2I(q^2), \tag{B·8}$$

where $I(q^2)$ is easily seen to be, from (B·3) and (B·4);

$$I(q^2)=\int_{4m^2}^{\infty}dk^2\frac{\rho(k^2)}{q^2+k^2-i\varepsilon}. \tag{B·9}$$

It is worth noticing that $I(q^2)$ is not singular at $q^2=0$. We shall also introduce the analytic expressions of (ps)(pv), (pv)(ps) and (pv)(pv) fermion loops and their corresponding dispersive forms, namely,

$$\begin{aligned}J_\mu^{PA}(q)&=-\frac{i}{(2\pi)^4}\int d^4p\,\mathrm{Tr}\left[\gamma_5S_c\left(p+\frac{1}{2}q\right)i\gamma_\mu\gamma_5S_c\left(p-\frac{1}{2}q\right)\right]\\&=2mq_\mu I(q^2),\end{aligned} \tag{B·10}$$

$$\begin{aligned}J_\mu^{AP}(q)&=-\frac{i}{(2\pi)^4}\int d^4p\,\mathrm{Tr}\left[i\gamma_\mu\gamma_5S_c\left(p+\frac{1}{2}q\right)\gamma_5S_c\left(p-\frac{1}{2}q\right)\right]\\&=-2mq_\mu I(q^2),\end{aligned} \tag{B·11}$$

$$J^A_{\mu\nu}(q) = -\frac{i}{(2\pi)^4}\int d^4p\,\mathrm{Tr}\Big[i\gamma_\mu\gamma_5 S_c\Big(p+\frac{1}{2}q\Big)i\gamma_\nu\gamma_5 S_c\Big(p-\frac{1}{2}q\Big)\Big]$$

$$= J^A(q^2)\left[\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right]+4m^2 I(q^2)\frac{q_\mu q_\nu}{q^2}, \qquad \text{(B·12)}$$

where

$$J^A(q^2) = -\int_{4m^2}^{\infty} dk^2\frac{\zeta(k^2)}{q^2+k^2-i\varepsilon}+\int_{4m^2}^{\infty} dk^2\xi(k^2), \qquad \text{(B·13)}$$

with

$$\zeta(k^2) = \frac{1}{12\pi^2}\left[1-\frac{4m^2}{k^2}\right]^{1/2}(k^2-4m^2)\theta(\Lambda^2-k^2), \qquad \text{(B·14)}$$

$$\xi(k^2) = \frac{1}{12\pi^2}\left[1-\frac{4m^2}{k^2}\right]^{1/2}\left(1-\frac{2m^2}{k^2}\right)\theta(\Lambda^2-k^2). \qquad \text{(B·15)}$$

Observe that

$$\frac{dJ^A(-z)}{dz} = -\int_{4m^2}^{\infty} dk^2\frac{\zeta(k^2)}{(z-k^2)^2}<0 \qquad \text{for } z<4m^2. \qquad \text{(B·16)}$$

Hence, $J^A(-z)$ is a decreasing function of z. However we have

$$\lim_{z\to-\infty} J^A(-z) = \int_{4m^2}^{\infty}\xi(k^2)\,dk^2 \equiv \xi \qquad \text{(B·17)}$$

and

$$\lim_{z\to 4m^2} J^A(-z) = \frac{m^2}{6\pi^2}\int_{4m^2}^{\Lambda^2}\left[1-\frac{4m^2}{k^2}\right]^{1/2}\frac{dk^2}{k^2}>0. \qquad \text{(B·18)}$$

Thus

$$0<J^A(-z)\leq\xi. \qquad \text{(B·19)}$$

From (B·16) follows that

$$Z_3 \equiv \left[1-e^2\frac{dJ^A(-z)}{dz}\right]^{-1}_{z=M^2} \qquad \text{(B·20)}$$

satisfies

$$0<Z_3<1. \qquad \text{(B·21)}$$

If we use the identity

$$\frac{1}{q^2+k^2-i\varepsilon} = P\frac{1}{q^2+k^2}+i\pi\delta(q^2+k^2), \qquad \text{(B·22)}$$

we obtain from (B·13)

$$\mathrm{Im}\,J^A(q^2) = -\pi\zeta(-q^2). \qquad \text{(B·23)}$$

As a last remark, we notice the following obvious indetities:

$$J_\mu^{PA}(q)+J_\mu^{AP}(q)=0\,, \tag{B·24}$$

$$J^A(0)-4m^2I(0)=0\,, \tag{B·25}$$

$$q_\mu J_{\mu\nu}^A(q)=4m^2q_\nu I(q^2), \tag{B·26}$$

$$J_{\mu\nu}^A(q)q_\nu=4m^2q_\mu I(q^2). \tag{B·27}$$

The identity (B·25), in particular, allows an alternative expression of $J^A(q^2)$ which follows immediately from the definition (B·13):

$$J^A(q^2)=4m^2I(q^2)+q^2\hat{J}^A(q^2), \tag{B·28}$$

where

$$\hat{J}^A(q^2)=\frac{1}{12\pi^2}\int_{4m^2}^{A^2}dk^2\frac{[1-4m^2/k^2]^{1/2}}{q^2+k^2}\left[1+\frac{2m^2}{k^2}\right]. \tag{B·29}$$

Appendix C

Commutation relations of fermion current

The fermion contribution to the current is given by

$$j_{\mu5}^F(x)=\frac{1}{\Box-e^2J^A(-\Box)}(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\,(i\bar{\phi}(x)\gamma_\nu\gamma_5\phi(x)). \tag{C·1}$$

We first note that the factor $1/(z-e^2J^A(-z))$ is complex due to the cut along the real axis in z-plane ($z\geq 4m^2$). Thus, $1/(z-e^2J^A(-z))$ in front of the pair creation term is hermitian conjugate of that in front of the pair annihilation term. We shall prove in this appendix the following relations:

$$\langle 0|[j_{i5}^F(x),j_{j5}^F(y)]|0\rangle=0\,, \tag{C·2}$$

$$\langle 0|[j_{45}^F(x),j_{45}^F(y)]|0\rangle=0\,, \tag{C·3}$$

$$\langle 0|[\partial_4 j_{45}^F(x),j_{i5}^F(y)]|0\rangle=0\,, \tag{C·4}$$

$$\langle 0|[j_{45}^F(x),j_{i5}^F(y)]|0\rangle=R_1\partial_i\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{C·5}$$

$$\langle 0|\left[\frac{1}{\Box}\partial_4 j_{45}^F(x),j_{45}^F(y)\right]|0\rangle=-R_2\nabla^2\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{C·6}$$

$$\langle 0|[\partial_4 j_{45}^F(x),j_{45}^F(y)]|0\rangle=-R_1\nabla^2\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{C·7}$$

$$\langle 0|\left[\frac{1}{\Box}j_{i5}^F(x),\frac{1}{\Box}j_{j5}^F(y)\right]|0\rangle=0\,, \tag{C·8}$$

$$\langle 0|\left[\frac{1}{\Box}j_{i5}^F(x),\frac{1}{\Box}\partial_4 j_{j5}^F(y)\right]|0\rangle=R_2\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{y})-R_3\partial_i\partial_j\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{C·9}$$

$$\langle 0|\left[\frac{1}{\Box}j_{\mu5}^F(x),\frac{1}{\Box}j_{\nu5}^F(y)\right]|0\rangle=-iR_3(n_\mu\partial_\nu^s+n_\nu\partial_\mu^s)\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{C·10}$$

$$\langle 0|\left[\frac{1}{\Box}j^F_{\mu 5}(x),\ \frac{1}{\Box}\partial_4 j^F_{\nu 5}(y)\right]|0\rangle=\{R_2(\delta_{\mu\nu}+n_\mu n_\nu)-R_3\partial_\mu{}^s\partial_\nu{}^s\}\delta(\boldsymbol{x}-\boldsymbol{y}),\tag{C·11}$$

which are valid at $x_0=y_0$. It will also be proved that at any x and y

$$\langle 0|[i\bar{\phi}(x)\gamma_5\phi(x),j^F_{\mu 5}(y)]|0\rangle=0\,,\tag{C·12}$$

$$\langle 0|\left[j^F_{45}(x),\ \frac{1}{\Box}\left(\delta_{ij}-\frac{1}{\nabla^2}\partial_i\partial_j\right)j^F_{j5}(y)\right]|0\rangle=0\,.\tag{C·13}$$

The constants appearing in the right-hand side of the above equations are given in terms of the renormalization constant

$$Z_3\equiv\left[1-e^2\frac{dJ^A(-z)}{dz}\right]^{-1}_{z=M^2}\tag{C·14}$$

and

$$\xi\equiv\lim_{|z|\to\infty}J^A(-z),\tag{C·15}$$

as

$$R_1=\xi-Z_3\frac{M^2}{e^2}\,,\tag{C·16}$$

$$R_2=\frac{1}{e^2}(1-Z_3),\tag{C·17}$$

$$R_3=-\frac{1}{e^2}\left\{\frac{Z_3}{M^2}-\frac{1}{e^2J^A(0)}\right\}.\tag{C·18}$$

In order to prove the above relations, we introduce the spectral representation and take various operations involved.

Let us define

$$\begin{aligned}G^c_{\mu\nu}(x-y)&\equiv\langle 0|T(i\bar{\phi}(x)\gamma_\mu\gamma_5\phi(x),i\bar{\phi}(y)\gamma_\nu\gamma_5\phi(y)|0\rangle\\&=-\mathrm{Tr}\left[i\gamma_\mu\gamma_5\frac{1}{i}S_c(x-y)i\gamma_\nu\gamma_5\frac{1}{i}S_c(y-x)\right].\end{aligned}\tag{C·19}$$

As was shown in Appendix B, we have the spectral representation

$$\begin{aligned}G^c_{\mu\nu}(x-y)=&\,i\left(\delta_{\mu\nu}-\frac{1}{\Box}\partial_\mu\partial_\nu\right)\int_{4m^2}^{\infty}dk^2\zeta(k^2)\Delta_c(x-y;k)\\&+i4m^2\frac{1}{\Box}\partial_\mu\partial_\nu\int_{4m^2}^{\infty}dk^2\rho(k^2)\Delta_c(x-y;k)\\&-i\left(\delta_{\mu\nu}-\frac{1}{\Box}\partial_\mu\partial_\nu\right)\int_{4m^2}^{\infty}dk^2\xi(k^2)(\Box-k^2)\Delta_c(x-y;k),\end{aligned}\tag{C·20}$$

where

$$\zeta(k^2) \equiv \frac{1}{12\pi^2}\left(1-\frac{4m^2}{k^2}\right)^{1/2}(k^2-4m^2)\theta(\Lambda^2-k^2), \tag{C·21}$$

$$\rho(k^2) \equiv \frac{1}{8\pi^2}\left(1-\frac{4m^2}{k^2}\right)^{1/2}\theta(\Lambda^2-k^2), \tag{C·22}$$

$$\xi(k^2) \equiv \frac{1}{12\pi^2}\left(1-\frac{4m^2}{k^2}\right)^{1/2}\left(1-\frac{2m^2}{k^2}\right)\theta(\Lambda^2-k^2). \tag{C·23}$$

Hence the spectral representation of the commutator

$$G_{\mu\nu}(x-y) = \langle 0|[i\bar{\phi}(x)\gamma_\mu\gamma_5\phi(x), i\bar{\phi}(y)\gamma_\nu\gamma_5\phi(y)]|0\rangle \tag{C·24}$$

is given by

$$G_{\mu\nu}(x-y) = i\left(\delta_{\mu\nu}-\frac{1}{\Box}\partial_\mu\partial_\nu\right)\int_{4m^2}^{\infty}dk^2\zeta(k^2)\Delta(x-y;k)$$
$$+i4m^2\frac{1}{\Box}\partial_\mu\partial_\nu\int_{4m^2}^{\infty}dk^2\rho(k^2)\Delta(x-y;k). \tag{C·25}$$

Using (C·1) and (C·25), we have

$$\langle 0|[j^F_{\mu5}(x), j^F_{\nu5}(y)]|0\rangle = \frac{1}{|\Box-e^2J^A(-\Box)|^2}(\Box\delta_{\mu\sigma}-\partial_\mu\partial_\sigma)(\Box\delta_{\nu\rho}-\partial_\nu\partial_\rho)G_{\sigma\rho}(x-y)$$
$$= i\frac{\Box}{|\Box-e^2J^A(-\Box)|^2}(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\int_{-\infty}^{\infty}dk^2\zeta(k^2)\Delta(x-y;k). \tag{C·26}$$

If we use the well-known relations

$$\Delta(x-y;k) = 0, \tag{C·27}$$

$$\frac{\partial}{\partial x_0}\Delta(x-y;k) = -\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{C·28}$$

at $x_0=y_0$, and the continuity equation

$$\partial_\mu j^F_{\mu5}(x) = 0, \tag{C·29}$$

Eqs. (C·2), (C·3), (C·4) and (C·8) follow at once.

The proof of (C·5) is as follows: From (C·26), we have

$$\langle 0|[j^F_{45}(x), j^F_{i5}(y)]|0\rangle = -i\frac{\Box}{|\Box-e^2J^A(-\Box)|^2}\int_{4m^2}^{\infty}dk^2\zeta(k^2)\partial_4\partial_i\Delta(x-y;k)$$
$$= -i\int_{4m^2}^{\infty}dk^2\frac{k^2}{|k^2-e^2J^A(-k^2)|^2}\zeta(k^2)\partial_4\partial_i\Delta(x-y;k). \tag{C·30}$$

Hence, at equal time $x_0=y_0$, we have

$$\langle 0|[j^F_{45}(x), j^F_{i5}(y)]|0\rangle_{x_0=y_0} = R_1\partial_i\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{C·31}$$

where

$$R_1=\int_{4m^2}^{\infty} dk^2 \frac{k^2\zeta(k^2)}{|k^2-e^2J^A(-k^2)|^2}\,. \tag{C·32}$$

It now remains to show that the integration (C·32) gives (C·16). For this purpose, we note (B·23), i.e.,

$$\zeta(k^2)=-\frac{1}{\pi}\,\mathrm{Im}\,J^A(-k^2). \tag{C·33}$$

Substituting (C·33) into (C·32), we obtain

$$\begin{aligned} R_1&=-\frac{1}{\pi}\int_{4m^2}^{\infty} dk^2 \frac{k^2\,\mathrm{Im}\,J^A(-k^2)}{|k^2-e^2J^A(-k^2)|^2} \\ &=-\frac{1}{\pi}\frac{1}{e^2}\int_{4m^2}^{\infty} dk^2k^2\,\mathrm{Im}\left[\frac{1}{k^2-e^2J^A(-k^2)}\right] \\ &=-\frac{1}{e^2}\frac{1}{2\pi i}\int_{c_1} dz \frac{z}{z-e^2J^A(-z)}\,. \end{aligned} \tag{C·34}$$

The contour c_1 is indicated in Fig. 1. If we add the contour c_2 indicated in Fig. 2, we can evaluate the integration (C·34). The result depends on the pole in the circle c_2 and the value at $|z|\to\infty$. Thus, we arrive at

$$R_1=\xi-\frac{M^2}{e^2}Z_3\,, \tag{C·35}$$

with

$$\xi=\lim_{|z|\to\infty} J^A(-z), \tag{C·36}$$

$$Z_3=\left[1-e^2\frac{dJ^A(-z)}{dz}\right]_{z=M^2}^{-1}. \tag{C·37}$$

This establishes the relation (C·5). The proof of (C·6), (C·7), (C·9), (C·10) and (C·11) is similar.

To prove (C·12), we make use of the relation

$$i\bar{\psi}(x)\gamma_5\psi(x)=\frac{1}{2m}\partial_\mu(i\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)). \tag{C·38}$$

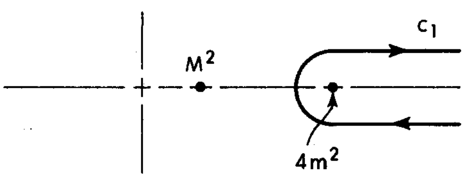

Fig. 1. The contour c_1.

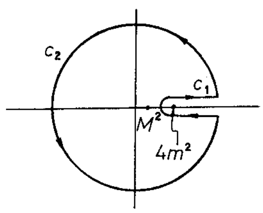

Fig. 2. The contour c_2.

Hence

$$\langle 0|[i\bar{\phi}(x)\gamma_5\phi(x), j^F_{\mu 5}(y)]|0\rangle = \frac{1}{2m}\frac{1}{\Box - e^2 J^A(-\Box)}(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\partial_\rho G_{\rho\nu}(x-y)$$
$$= 0 \qquad (C\cdot 39)$$

as a consequence of (C·25).

Lastly, Eq. (C·13) follows directly from (C·26).

Appendix D

Derivation of the Goldstone commutator from the Bethe-Salpeter kernel

In this appendix we show that the vacuum expectation value of the Goldstone commutator is directly related to the Bethe-Salpeter kernel. The Bethe-Salpeter kernel is defined by

$$\langle 0|T(\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\psi_\sigma(x_3)\bar{\psi}_\rho(x_4))|0\rangle \equiv K_{\alpha\beta\sigma\rho}(x_1, x_2, x_3, x_4). \qquad (D\cdot 1)$$

Using the equal-time commutator

$$\{\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\}\delta(t_1-t_2) = (\gamma_4)_{\alpha\beta}\delta^{(4)}(x_1-x_2), \qquad (D\cdot 2)$$

we can derive

$$\begin{aligned}\partial_{1\mu}K_{\alpha\beta\sigma\rho}(x_1, x_2, x_3, x_4) &= \langle 0|T(\partial_{1\mu}\psi_\alpha(x_1), \bar{\psi}_\beta(x_2), \psi_\sigma(x_3), \bar{\psi}_\rho(x_4))|0\rangle \\ &\quad - i\delta^{(4)}(x_1-x_2)(\gamma_4)_{\alpha\beta}\langle 0|T(\psi_\sigma(x_3), \bar{\psi}_\rho(x_4))|0\rangle\delta_{\mu 4} \\ &\quad - i\delta^{(4)}(x_1-x_4)(\gamma_4)_{\alpha\rho}\langle 0|T(\bar{\psi}_\beta(x_2), \psi_\sigma(x_3))|0\rangle\delta_{\mu 4} \end{aligned} \qquad (D\cdot 3)$$

and

$$\begin{aligned}\partial_{2\mu}K_{\alpha\beta\sigma\rho}(x_1, x_2, x_3, x_4) &= \langle 0|T(\psi_\alpha(x_1), \partial_{2\mu}\bar{\psi}_\beta(x_2), \psi_\sigma(x_3), \bar{\psi}_\rho(x_4))|0\rangle \\ &\quad + i\delta^{(4)}(x_2-x_1)(\gamma_4)_{\alpha\beta}\langle 0|T(\psi_\sigma(x_3), \bar{\psi}_\rho(x_4))|0\rangle\delta_{\mu 4} \\ &\quad + i\delta^{(4)}(x_2-x_3)(\gamma_4)_{\sigma\rho}\langle 0|T(\bar{\psi}_\rho(x_4), \psi_\alpha(x_1))|0\rangle\delta_{\mu 4}. \end{aligned} \qquad (D\cdot 4)$$

If we put $x_1=x_2=x$ and $x_3=x_4=x'$, and multiply by $(i\gamma_\mu\gamma_5)_{\alpha\beta}$ and $(i\gamma_5)_{\rho\sigma}$, we obtain

$$\begin{aligned}\partial_\mu[(i\gamma_\mu\gamma_5)_{\beta\alpha}K_{\alpha\beta\sigma\rho}(x, x, x', x')(i\gamma_5)_{\rho\sigma}] &= \partial_\mu\langle 0|T(j_{\mu 5}(x), \rho_5(x'))|0\rangle \\ &= -i\delta(t-t')\langle 0|[j_{45}(x), \rho_5(x')]|0\rangle \\ &= 2i\delta^{(4)}(x-x')\langle 0|T(\psi_\rho(x), \bar{\psi}_\rho(x))|0\rangle, \end{aligned} \qquad (D\cdot 5)$$

where use has been made of

$$j_{\mu 5}(x) = i\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x), \qquad (D\cdot 6)$$

$$\rho_5(x) = i\bar{\psi}(x)\gamma_5\psi(x) \qquad (D\cdot 7)$$

and

$$\partial_\mu j_{\mu 5}(x) = 0. \qquad (D\cdot 8)$$

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