Spontaneous Hole-Clump Pair Creation

H.L. Berk, B.N. Breizman, J. Candy,* M. Pekker, and N.V. Petviashvili

Institute for Fusion Studies, The University of Texas at Austin, Austin, TX 78712

(February 9, 1999)

Abstract

A nonlinear theory is presented for the spontaneous formation of a hole-clump pair in the phase space of a system whose equilibrium is just above the linear threshold for instability. The first case studied takes the damping to be a purely linear response with the nonlinear instability drive due to a single wave-particle resonance with particles that have an inverted distribution function. Analytic results shows that the hole and clump can each support a Bernstein-Greene-Kruskal nonlinear wave, with the trapping frequency of particles comparable to the linear growth rate without dissipation. The power that is dissipated to the background plasma is balanced by the energy extracted from the inverted equilibrium distribution by the moving phase space structures. This motion produces frequency sweeping of the fields. The second case studied has no extrinsic dissipation. Instead, a second resonance is taken, which affects a different species of particles whose distribution function decreases with energy. For an electrostatic interaction, we consider cases for which the mass ratio of the destabilizing to stabilizing species is: (i) much less than unity; (ii) equal to unity; (iii) much greater than unity. Case (i) gives results that are similar to the linear dissipation model, while cases (ii)

*present address: General Atomics, San Diego, CA 92186-5608 e-mail: candy@gav.gat.com

and (iii) saturate without any frequency sweeping. However, in case (ii), the saturated level is proportional to the total linear growth rate, while the saturation level in (iii) is nearly the same as the saturation level in a system where the stabilizing species is not present. In the third case we show that frequency sweeping can reappear in the problem of a collisionless destabilizing heavy species, with collisions affecting the stabilizing light species. When the collision frequency is relatively large, so that the light species in effect have a linear response, the problem reverts to the first case. More subtle and speculative explanations are given to explain why, at lower collisionality, holes and clumps also emerge.

52.35–g, 52.35.Mw, 52.40.Mj

I. INTRODUCTION

Recently it was observed that an unstable mode, driven by resonant particles, can grow explosively to a level that remains finite even at the instability threshold, where the mode linear growth rate, γ, becomes vanishingly small.[1] The quantity γ is the difference between the fast particle drive, $\gamma_{\rm L}$, and the damping due to background dissipation, $\gamma_{\rm d}$. In this problem the wave taps the free energy of an inverted distribution function to overcome the dissipation. Specific examples of this effect include the bump-on-tail instability[2] and the excitation of Alfvén waves in plasmas of interest in fusion research.[3]

An important intrinsic feature of the explosive growth is that the mode frequency shifts in time from its value at the instability threshold. By the time the amplitude grows to a level where the trapped particle nonlinear bounce frequency, $\omega_{\rm b}$, reaches the value $\gamma_{\rm L}$, the frequency shifts also become comparable to $\gamma_{\rm L}$. At this moment, the explosive growth described in Ref. 1 stops, and the mode saturates. We have developed a code that confirms the expected saturation level, but also reveals a surprising effect: the saturated mode lasts much longer than the damping time associated with the background dissipation. In addition, the mode frequency keeps shifting after the mode saturation is reached.[4,5] These numerical results and the underlying physics were reported in a short communication.[4] The initial explosive phase leads to the formation of a hole[6–8]-clump[9,10] pair in phase space. In the bump-on-tail instability, the hole produces an upward shift of the frequency and the clump a downward shift of the frequency.

The purpose of this paper is to elaborate on results already reported in Ref. 4, as well as to extend the scope of the problem to two kinetic species. The paper has the following structure: in Sec. II, we summarize a previously developed formalism[11,12] for treating the kinetic instability problem. Further, in Sec. III, we reproduce the principal results for the single-species problem with externally-imposed dissipation[4] and present them with appropriate figures (in Ref. 4, incorrect illustrations were inadvertently published).

In Sec. IV we replace the external linear dissipation with a kinetic dissipation arising from a stabilizing population of particles with a distribution that decreases with energy. This population will be treated self-consistently together with the inverted population that destabilizes the wave. We choose parameters so that the linear drive slightly exceeds the wave damping. Specifically, we examine the evolution of charged particles interacting with a one-dimensional electrostatic wave. We assume that the absolute values of the charges are equal for the two species, and examine the effect of varying mass ratio.

When the instability drive is associated with the light component, and the stabilizing component is much heavier, one initially finds essentially the same results as in the one-species case with external, linear damping. The reason is that the stabilizing species responds linearly for some time after the drive of the destabilizing species becomes nonlinear. However, later in time even the stabilizing species may exhibit a nonlinear response which affects the rate at which frequency changes.

In the opposite limit, where the instability drive is associated with the more massive species, the mode frequency does not change, but the wave rapidly evolves into a nonlinear explosive regime and saturates at a level that is independent of the presence of the stabilizing component. When the two species have equal or nearly-equal masses, there is again no frequency shift, but the mode amplitude remains at a low level that is proportional to the increment by which the system exceeds the linear instability threshold.

In Sec. V we show that with a heavy collisionless drive, frequency sweeping can be restored when collisions are taken into account for the light stabilizing species.

In Sec. VI we summarize our work.

II. FUNDAMENTAL EQUATIONS

We first outline the derivation of the fundamental equations used in the particle simulation code of Ref. 4. These equations are given below [see Eqs. (13) and (17)]. We will use the same notation as in Ref. 12 and follow the standard perturbation procedure for weakly interacting (slowly evolving) modes, assuming that the polarization and spatial structure of the mode are determined by the linear properties of the background plasma.

The electric field of the mode can then be represented in the form

$$E(\boldsymbol{r},t) = 2\,\mathrm{Re}\,\left[C(t)e^{-i\omega_0 t}\boldsymbol{e}(\boldsymbol{r},\omega_0)\right]. \tag{1}$$

In Eq. (1), ω_0 is the unperturbed mode frequency, $C(t)$ is a slowly-varying complex amplitude, and $\boldsymbol{e}(\boldsymbol{r},\omega_0)$ is the mode eigenvector normalized to unity by volume averaging,

$$\frac{1}{V}\int d\boldsymbol{r}\,\boldsymbol{e}\cdot\boldsymbol{e}^* = 1. \tag{2}$$

The frequency ω_0 is a solution of the linear dispersion relation, excluding the energetic particle contribution and the dissipative part of the plasma response. By using Maxwell's equations with a given plasma dielectric tensor, and taking the dot product of the energetic particle current $\boldsymbol{J}$ with $\boldsymbol{e}(\boldsymbol{r},\omega_0)$, we find that the amplitude $C(t)$ satisfies the equation

$$i\mathrm{G}_\omega\left(\frac{d}{dt}+\gamma_\mathrm{d}\right)C(t) = e^{i\omega_0 t}\int d\boldsymbol{r}\,\boldsymbol{e}^*(\boldsymbol{r},\omega_0)\cdot\boldsymbol{J}(\boldsymbol{r},t), \tag{3}$$

where γ_d is the mode damping rate in the absence of energetic particles and G_ω is a purely imaginary quantity related to the mode energy, w, by

$$w = \mathrm{Im}\,G_w|C|^2. \tag{4}$$

The right-hand side of Eq. (3) can be rewritten in terms of the energetic-particle distribution function f, giving

$$i\mathrm{G}_\omega\left(\frac{d}{dt}+\gamma_\mathrm{d}\right)C(t) = \mathrm{q}e^{i\omega_0 t}\int d\Gamma\,\boldsymbol{e}^*(\boldsymbol{r},\omega_0)\cdot\boldsymbol{v}(\Gamma)f(\Gamma,t), \tag{5}$$

where $\Gamma = (\boldsymbol{r},\boldsymbol{p})$ is the phase space position, q is the charge, and $\boldsymbol{v}$ is the velocity.

We now need to find $f(\Gamma,t)$ from the kinetic equation

$$\frac{\partial f}{\partial t}+\frac{\partial H}{\partial \boldsymbol{p}}\cdot\frac{\partial f}{\partial \boldsymbol{r}}-\frac{\partial H}{\partial \boldsymbol{r}}\cdot\frac{\partial f}{\partial \boldsymbol{p}} = St\,f+Q, \tag{6}$$

in which the Hamiltonian H splits into $H = H_0+H_1$, such that H_0 determines the equilibrium orbits and H_1 represents the perturbation from the mode. The right-hand side of Eq. (6)

takes into account the particle source Q and the collision operator St. For phase space coordinates, we choose action-angle variables of the unperturbed motion, (I_i, ξ_i), with $i = 1, 2, 3$. This choice can always be made if the unperturbed orbits are integrable. The Hamiltonian H can then be written as $H = H_0(I_1, I_2, I_3) + H_1$ with

$$H_1 = 2\,\mathrm{Re}\ C(t)e^{-i\omega_0 t}\sum_{\ell} V_\ell(I_1, I_2, I_3)e^{i\ell_1\xi_1+i\ell_2\xi_2+i\ell_3\xi_3}. \tag{7}$$

Here, ℓ represents a triad of integers (ℓ_1, ℓ_2, ℓ_3), and $V_\ell(I_1, I_2, I_3)$ are matrix elements that can be calculated in a standard way (see Ref. 13).

Each term in the perturbed Hamiltonian represents a resonance that can be treated separately, so long as the mode amplitude remains sufficiently low so that resonances do not overlap. For nearly resonant particles, we can also express H_1 in terms of the perturbed electric field, given by Eq. (1). The result is

$$H_1 = 2q\,\mathrm{Re}\ \left(i\,C(t)\frac{\boldsymbol{v}\cdot\boldsymbol{e}}{\omega_0}\,e^{-i\omega_0 t}\right). \tag{8}$$

The location of a resonance is determined by the condition

$$\Omega \equiv \ell_1\,\tfrac{\partial H_0}{\partial I_1} + \ell_2\,\tfrac{\partial H_0}{\partial I_2} + \ell_3\,\tfrac{\partial H_0}{\partial I_3} = \omega_0. \tag{9}$$

It is then natural to make a canonical transformation to a new set of action-angle variables, such that one of the new angles is $\xi = \ell_1\xi_1 + \ell_2\xi_2 + \ell_3\xi_3$, with I the corresponding action. For particles nearly resonant with this action-angle pair, the Hamiltonian reduces to the one-dimensional form

$$H = H_0(I) + 2\,\mathrm{Re}\ C(t)V(I)e^{i(\xi-\omega_0 t)}, \tag{10}$$

where the dependence on the other two new actions is suppressed, since they appear as parameters in the new Hamiltonian. In absence of collisions, the motion of resonant particles satisfies the pendulum equation

$$\frac{d^2\xi}{dt^2} + \omega_\mathrm{b}^2\sin(\xi - \omega_0 t - \xi_0) = 0, \tag{11}$$

where ξ_0 is a phase determined by $C(t)$, and

$$\omega_{\rm b}^2 \equiv \left| 2CV(I)\,\frac{\partial\Omega}{\partial I}\right|_{I=I_r} \tag{12}$$

can be interpreted as the square of the bounce frequency of a trapped particle. In Eq. (12), I_r is the value of the action at resonance: $\Omega(I_r) = \omega_0$.

Near resonance, the kinetic equation is

$$\frac{\partial f}{\partial t} + \Omega\,\frac{\partial f}{\partial \xi} - 2\,{\rm Re}\,\left[iC(t)V\left(\frac{\partial\Omega}{\partial I}\right) e^{i(\xi-\omega_0 t)}\right]_{I=I_r}\frac{\partial f}{\partial\Omega} = St\, f + Q. \tag{13}$$

where we have chosen $\Omega(I)$ as a new independent variable to replace the action, I.

Two forms of the collision operator in Eq. (13) are of interest. The first is a diffusive one that follows from an orbit averaging of the Fokker-Planck collision operator for nearly resonant particles[11,12]

$$St\, f + Q = \nu_{\rm eff}^3\,\frac{\partial^2}{\partial\Omega^2}(f - F), \tag{14}$$

where $F(I)$ is the equilibrium distribution. The explicit expression for $\nu_{\rm eff}^3$ in Eq. (14) is given in Ref. 12. The second form we use in this work is the Krook collision operator, which is a model particularly convenient for δf simulations. This is

$$St\, f + Q = -\nu_r(f - F), \tag{15}$$

where ν_r is the reconstitution rate of the particle distribution. It has been previously noted in Ref. 14 that the two operators produce similar physical results if $\nu_r \sim \nu_{\rm eff}^3/(\omega_{\rm b}^2 + \nu_r^2)$.

The function $C(t)$ that enters Eq. (13) satisfies Eq. (5). In terms of the coordinate pair (Ω, ξ), the phase space volume element becomes

$$d\Gamma = d\Gamma_\perp\, d\xi\, d\Omega\,\frac{dI}{d\Omega}, \tag{16}$$

where $d\Gamma_\perp$ represents the volume element corresponding to the remaining actions and angles. The evolution equation can then be written as

$$\left(\frac{d}{dt} + \gamma_{\rm d}\right) C(t) = \frac{\omega_0}{{\rm G}_\omega}\sum_\ell \int d\Gamma_\perp\,\frac{d\xi\, d\Omega}{|d\Omega/dI|}\, V_\ell^* e^{-i(\xi-\omega_0 t)}\, f. \tag{17}$$

Equation (13) for f, together with Eq. (17) for $C(t)$ form a closed set of fundamental equations for the system. From this point onwards, the ℓ-summation over resonances, as well as the integration over $d\Gamma_\perp$, will usually be suppressed for brevity. Note that Eq. (13) is a kinetic equation in just one spatial dimension. The sum over particles in the wave equation, Eq. (17), involves summation over $\Gamma_\perp$ and ℓ. Of course, for a true one-dimensional problem, this summation is not present.

III. EVIDENCE FOR HOLE-CLUMP FORMATION

This section summarizes work that was originally published in Ref. 4, where Eqs. (13) and (17) were solved numerically for the case of the electrostatic bump-on-tail instability. The key features of this problem are common to many other kinetic instabilities, if one describes the evolution in terms of the bounce frequency of resonant particles. Thus it is natural to define the amplitude

$$A(t) = 2\,C(t)\left(V\,\frac{\partial\Omega}{\partial I}\right)_{I=I_r}, \tag{18}$$

with $|A(t)| = \omega_{\rm b}^2$ [see Eq. (12)]. The input parameters for a simulation are the growth rate induced by energetic particles, $\gamma_{\rm L}$, the externally imposed damping rate, $\gamma_{\rm d}$, and the effective collision frequency $\nu_{\rm eff}$ (with the diffusive collision operator). The growth rate, $\gamma_{\rm L}$, is related to the slope of the equilibrium distribution function according to

$$\gamma_{\rm L} = \frac{2\pi^2\omega_0}{{\rm Im}\,G_w}\,|V|^2\,\frac{\partial F}{\partial\Omega}\bigg|_{\Omega=\omega_0}. \tag{19}$$

For the bump-on-tail problem, this expression reduces to

$$\gamma_{\rm L} = \frac{2\pi^2{\rm q}^2\omega_{\rm pe}}{k^2}\,\frac{\partial F}{\partial v}\bigg|_{v=\omega_{\rm pe}/k}, \tag{20}$$

where k is the wavenumber, v is the particle velocity, and $\omega_{\rm pe}$ is the electron plasma frequency.

The quantity $\nu_{\rm eff}^3$ is roughly given by $\nu_{\rm eff}^3 = \nu_{90^\circ}\omega_{\rm pe}^2$, where ν_{90° is the particle 90° pitch angle scattering rate.

It will be seen from the numerical results that a saturated level $\omega_{\rm b} \sim \gamma_{\rm L}$ is obtained, with $\delta\omega \equiv \omega - \omega_0$ much larger than $\gamma_{\rm L}$. Eqs. (13) and (17) permit a solution where $A(t)$ is real, and this special case was examined in our first numerical study.[4] Figure 1 shows a numerical solution for $A(t)/\gamma_{\rm L}^2$ as a function of $\gamma_{\rm L}\, t$. Initially, the amplitude increases exponentially in time, and then goes into the explosive phase described analytically in Ref. 1 [see Eq. (21) below for more details]. The explosive solution leads to saturation at a level $A(t)/\gamma_{\rm L}^2 \sim 1$. When this level is reached, the instability drive does not deplete as might be expected. Instead the envelope of the mode amplitude remains roughly constant in the time interval $1 \ll \gamma_{\rm L} t \ll (\gamma_{\rm L}/\nu_{\rm eff})^3$, and $A(t)$ oscillates with increasing frequency. Figure 2 shows the upshifted frequency spectrum, $\delta\omega$, as a function of time; an equal-strength downshifted spectrum, $-\delta\omega$, also forms. (These two figures are also presented in the errata[5] for Ref. 4). The most intensive component is the one with the largest frequency shift, but appreciable satellite bands are also generated.

In Figs. 3a and 3b we show respectively the spatially-averaged distribution function and wave spectrum as a function of time. The depression/enhancement of the average distribution function coincides with the upshifted/downshifted frequency, which suggests that phase space structures in the form of holes (the upshifted component) and clumps (the downshifted component) have been spontaneously created. This inference was verified in the phase space contour plot of Ref. 4, which directly demonstrated that early in time the value of the distribution function in the trapping regions of the hole and clump does not change as they move in phase space. Only later in time, when $\nu_{\rm eff}^3\, t/\gamma_{\rm L}^2 \sim 1$ does the value of the distribution at the hole and clump change, causing the phase space structures to dissipate.

A. Creation of holes and clumps

The numerical results can be understood as follows: in Ref. 1 it was found that the evolution of a sufficiently small real amplitude $A(t)$ is described by the nonlinear threshold equation

$$\frac{dA}{dt} - \gamma A = -\frac{\gamma_{\rm L}}{2}\int_0^{t/2} dz\, z^2 A(t-z) \int_0^{t-2z} dx\, K(x,z)\, A(t-z-x) A^*(t-2z-x), \qquad (21)$$

with

$$K(x,z) \equiv \begin{cases} \exp\left[-\nu_{\rm eff}^3 z^2 (2z/3 + x)\right], & \text{for Eq. (14)} \\ \exp\left[-\nu_r (2z + x)\right], & \text{for Eq. (15).} \end{cases} \qquad (22)$$

In the limit of weak collisions ($K \to 1$) we find that for any positive γ, the amplitude A exhibits an explosive self-similar oscillatory behavior with a divergence in a finite time: $A \propto (t_0 - t)^{-5/2}$. This solution produces upshifted and downshifted sideband frequencies that diverge as $(t_0 - t)^{-1}$. Although the solution of the reduced cubic equation, Eq. (21), remains valid only so long as $|A| \ll \gamma_{\rm L}^2$, the divergent oscillations indicate that the mode splits into several spectral components. In the full-scale simulations, described in the previous section, these components evolve into two Bernstein-Greene-Kruskal (BGK) modes,[15] which maintain a hole and a clump in the particle distribution function. Apparently, the explosive behavior sets up the precursor needed to achieve these BGK modes.

Note that the full-scale simulations with a code described in Ref. 16 accurately reproduce the transition from linear instability to the oscillatory growth of the mode amplitude predicted by Eq. (21). As seen from Fig. 4, the two solutions agree until the mode amplitude reaches the applicability limit of the threshold equation.

Another interesting observation from both the solution of Eq. (21) and the full-scale simulations is that, if the initial perturbation is sufficiently strong, the explosive onset of the hole and clump is possible even in linearly stable systems. This is illustrated in Fig. 5, which shows the evolution of a seed perturbation in the barely-stable regime ($\gamma = -0.1\gamma_{\rm L}$). We see that the mode amplitude first decreases due to linear damping but then the nonlinear effects take over and produce oscillatory growth that eventually leads to hole-clump formation accompanied by frequency sweeping. By a straightforward dimensional analysis, one can show that the required amplitude of the seed perturbation has to satisfy the condition

$$\omega_{\rm b}^2 \gtrsim (\nu_{\rm eff} + \gamma)^{5/2} / \gamma_{\rm L}^{1/2}.$$

The mode will then take off and grow explosively to a level $\omega_b \sim \gamma_L$.

B. Adiabatic regime

Our simulations show that the frequency separation between the two BGK modes increases in time. This separation is typically greater than ω_b and γ_L, which allows us to treat the two modes independently. In addition, particles trapped in the potential well of each mode can be described in the adiabatic approximation provided that the mode evolution is sufficiently slow (which is the case for the simulations discussed herein).

This observation allows us to develop an analytic picture for the evolution of the phase space structures. The analytic method is a generalization of a procedure described in Ref. 17, where the particle-to-wave energy transfer was calculated for energetic particles that were affected by drag (mathematically, the effect of drag is equivalent to frequency sweeping).

In order to describe a single BGK mode, we make the replacement

$$\mathrm{Re}\ \left[A(t)e^{i(\xi-\omega_0 t)}\right] \longrightarrow -\omega_b^2 \cos\left(\xi - \omega_0 t - \int_0^t dt'\, \delta\omega(t')\right). \tag{23}$$

This is not an equality; rather, is shows that we can neglect the upshifted component when treating the downshifted component, and vice versa. We use this ansatz to compute the distribution function, f, in the trapping regions. In the passing regions, we take simply $f = F(\Omega)$. Now we introduce a Hamiltonian for the particle motion in the potential well of a single BGK mode; this is

$$\mathcal{H} = \frac{p^2}{2} - \omega_b^2(\cos q - \alpha q), \tag{24}$$

with $\alpha \equiv \dot{\delta\omega}/\omega_b^2$. The coordinate and momentum have the definitions

$$q \equiv \xi - \omega_0 t - \int_0^t dt'\, \delta\omega(t'), \tag{25}$$

$$p \equiv \Omega - \omega_0 - \delta\omega(t). \tag{26}$$

It is then appropriate to construct the adiabatic invariant for particle motion in the slowly-changing field of the mode,

$$J(\mathcal{H},t) \equiv \frac{1}{2\pi}\oint dq\, p(\mathcal{H},q,t), \tag{27}$$

where care must be taken in Eq. (27) and below to choose the proper root for the function $p(\mathcal{H},q,t)$ in accordance with Eq. (26). It follows from the equations of motion that $\dot{q} = p(\mathcal{H},q,t)$ and $\dot{J} = \tau(\dot{\mathcal{H}} - \langle\dot{\mathcal{H}}\rangle)$, where

$$\dot{\mathcal{H}} = q\,\ddot{\delta\omega} - \frac{d(\omega_{\rm b}^2)}{dt}\cos q. \tag{28}$$

The angle brackets indicate a bounce average at fixed (J,t) according to

$$\langle f\rangle \equiv \frac{1}{2\pi\tau}\oint\frac{dq}{p}\,f, \text{with } \tau \equiv \frac{\partial J}{\partial\mathcal{H}} = \frac{1}{2\pi}\oint\frac{dq}{p}. \tag{29}$$

We can now write the kinetic equation as

$$\frac{\partial f}{\partial t} + \tau(\dot{\mathcal{H}} - \langle\dot{\mathcal{H}}\rangle)\frac{\partial f}{\partial J} + p\frac{\partial f}{\partial q} = St\, f + Q, \tag{30}$$

such that $f = f(J,q,t,\sigma)$, where $\sigma \equiv p/|p|$ is binary variable which denotes the sign of the momentum. Further $f(\sigma = 1) = f(\sigma = -1)$ at $p = 0$.

Now, we transform from f to a new function g which measures the deviation from the unperturbed distribution function F at the center of the well:

$$g(J,q,t,\sigma) \equiv f(J,q,t,\sigma) - F(\omega_0 + \delta\omega). \tag{31}$$

The equation for g is then

$$Dg + p\frac{\partial g}{\partial q} = -\frac{\partial F}{\partial t}, \tag{32}$$

where

$$\mathcal{D} \equiv \frac{\partial}{\partial t} + \tau(\dot{\mathcal{H}} - \langle\dot{\mathcal{H}}\rangle)\frac{\partial}{\partial J} - St. \tag{33}$$

To solve Eq. (32) we note that for slow time variation of the fields, the dependence of g on q must be weak. This motivates the introduction of a small parameter ϵ, such that

$$\epsilon \equiv \max\left(\ddot{\delta\omega}/\omega_{\mathrm{b}}^3,\, \dot{\omega}_{\mathrm{b}}/\omega_{\mathrm{b}}^2,\, \omega_{\mathrm{b}}/\delta\omega\right) \ll 1. \tag{34}$$

Then, we expand g in powers of ϵ according to $g = g_0 + g_1 + \cdots$, where g_1 is $\mathcal{O}(\epsilon)$, and we take as $\mathcal{O}(\epsilon)$ both the source term $\partial F/\partial t$ and the differential operator $\mathcal{D}$. To lowest order, Eq. (32) is just

$$p\frac{\partial g_0}{\partial q} = 0 \quad \text{so that} \quad g_0 = g_0(J,t). \tag{35}$$

Note here that g_0 has no σ dependence, as required by the continuity condition at the particle turning points. Next, to $\mathcal{O}(\epsilon)$, we have

$$\mathcal{D}g_0 + p\frac{\partial g_1}{\partial q} = -\frac{\partial F}{\partial t}. \tag{36}$$

Two important results follow from this equation. First, subtracting the $p < 0$ branch from the $p > 0$ branch, we find

$$|p|\,\frac{\partial}{\partial q}[g_1(J,q,t,+1) + g_1(J,q,t,-1)] = 0. \tag{37}$$

Next, when bounce-averaged, Eq. (36) becomes

$$\frac{\partial g_0}{\partial t} - \langle St\, g_0\rangle = -\frac{\partial F}{\partial t}. \tag{38}$$

We will return to these results shortly. Now, we analyze the amplitude equation, Eq. (17). For a system with one degree-of-freedom, this can be written in terms of the coordinates (q,p) as

$$\left(\frac{d}{dt} + \gamma_{\mathrm{d}}\right) A(t) = \frac{-2i\omega_0|V|^2}{\mathrm{Im}\, G_w} \int dq\, dp\, e^{-iq - i\int_0^t dt'\delta\omega(t')}\, f(q,p,t). \tag{39}$$

We can replace f in Eq. (39) with g, since the equilibrium piece, F, does not contribute to the integral. In addition, the integration over p and q can be transformed to an integration over the action, J, and the conjugate angle, ψ, according to

$$dp\, dq = dJ d\psi. \tag{40}$$

Then, taking the real and imaginary parts of the resulting equation, we are left with the compact result

$$\left\{ \begin{array}{c} -\delta\omega \\ \\ \gamma_{\rm d} \end{array} \right\} = \frac{2\gamma_{\rm L}}{\pi\omega_{\rm b}^2 F'(\omega_0)} \int_0^{J_{\rm max}} dJ \left\{ \begin{array}{c} \langle g\ \cos q\rangle \\ \\ \langle g\ \sin q\rangle \end{array} \right\}, \tag{41}$$

where $F' \equiv \partial F/\partial\Omega$. Eq. (41) has been expressed in terms of $\gamma_{\rm L}$ as defined in Eq. (19). Furthermore, we have omitted a term proportional to $\dot{\omega}_{\rm b}$, since by assumption $\dot{\omega}_{\rm b}/\omega_{\rm b} \ll \gamma_{\rm d}$. To simplify Eq. (41) further, we need to evaluate the indicated averages. For the upper line, the averaging is straightforward:

$$\langle g\ \cos q\rangle = g_0 \langle \cos q\rangle + \mathcal{O}(\epsilon). \tag{42}$$

For the lower line, the derivation is more subtle. We can proceed by first showing that $\langle g\,(\sin q + \alpha)\rangle = \mathcal{O}(\epsilon^2)$. To see this, we use Eq. (24) and integrate by parts in q to find

$$\begin{aligned} \langle g\,(\sin q + \alpha)\rangle &= -\frac{1}{2\pi\tau\omega_{\rm b}^2} \oint dq\, g \frac{\partial}{\partial q} p[H(J), q] \\ &= \frac{1}{\pi\tau}\frac{1}{\omega_{\rm b}^2} \int_{q_-}^{q_+} dq\, |p| \frac{\partial}{\partial q}[g(J,q,t,+1) + g(J,q,t,-1)] \\ &= \mathcal{O}(\epsilon^2), \end{aligned} \tag{43}$$

where $q_\pm$ are the turning points (i.e. solutions of $p(q,J,t) = 0$). That the integral is $\mathcal{O}(\epsilon^2)$ follows directly from Eq. (37). Thus, we have the asymptotic result

$$\langle g\ \sin q\rangle = -\alpha \langle g\rangle + \mathcal{O}(\epsilon^2) = -\alpha g_0 + \mathcal{O}(\epsilon\alpha) + \mathcal{O}(\epsilon^2), \tag{44}$$

making it possible to write Eq. (41) in the simplified form

$$\left\{ \begin{array}{c} \delta\omega \\ \\ \gamma_{\rm d} \end{array} \right\} = -\frac{2\gamma_{\rm L}}{\pi\omega_{\rm b}^2 F'(\omega_0)} \int_0^{J_{\rm max}} dJ \left\{ \begin{array}{c} \langle \cos q\rangle g_0 \\ \\ \alpha g_0 \end{array} \right\}. \tag{45}$$

We note that the lower line in Eq. (45) can also be derived from physically intuitive arguments like that used in Ref. 4, where the power released by an adiabatic structure was calculated. The intuitive derivation seemed to require $\dot{\omega}_{\rm b} \ll \delta\dot{\omega}$, whereas the present derivation does not have this restriction.

We now consider the collisionless limit, where the solution for g_0 is

$$g_0(t) = F(\omega_0) - F(\omega_0 + \delta\omega). \tag{46}$$

This expression is an immediate consequence of the bounce-averaged result in Eq. (38). g_0 can now be removed from the integrand in Eq. (45), and the action integrals evaluated to yield

$$\int_0^{J_{\max}} dJ\ \langle \cos qg_0\rangle = \frac{8g_0\omega_{\rm b}}{\pi}\left(1-\alpha^2\right)^{7/4} c_1(\alpha), \tag{47}$$

$$\int_0^{J_{\max}} dJ\, \alpha g_0 = \frac{8g_0\omega_{\rm b}}{\pi}\left(1-\alpha^2\right)^{5/4} \alpha\, c_2(\alpha). \tag{48}$$

Above, we have introduced the $\mathcal{O}(1)$ functions c_1 and c_2. These have the asymptotic forms

$$(c_1, c_2) \simeq \begin{cases} (1/3, 1) & \text{for } \alpha \ll 1; \\ \\ (-3/5, 3/5) & \text{for } 1-\alpha^2 \ll 1. \end{cases} \tag{49}$$

By combining both components of Eq. (45), we find the solutions for $\omega_{\rm b}(\delta\omega)$ and $\delta\omega(t)$,

$$\omega_{\rm b} = \frac{16}{3\pi^2}\gamma_{\rm L}\widehat{g}(\delta\omega), \tag{50}$$

$$t = \frac{27\pi^4}{16^2}\frac{1}{\gamma_{\rm d}\gamma_{\rm L}^2}\int_0^{\delta\omega} dx\, \frac{x}{\hat{g}(x)^2}, \tag{51}$$

where $\hat{g}(x) \equiv [F(\omega_0+x)-F(\omega_0)]/F'(\omega_0)x$. When $|\delta\omega| \ll |F'/F''|$, we can set $\hat{g} \simeq 1$ to show that the frequency shift increases as the square root of time:

$$\frac{\gamma_{\rm L}}{\omega_{\rm b}} \simeq \frac{3\pi^2}{16} \quad \text{and} \quad \frac{\delta\omega}{\gamma_{\rm L}} \simeq \frac{16}{3\pi^2}\sqrt{\frac{2}{3}}\left(\gamma_{\rm d}\, t\right)^{1/2}. \tag{52}$$

Finally, note that Eq. (45) can be generalized to account for multiple resonances and species. In this case we introduce the quantity $\widetilde{\gamma}_\ell(\Gamma_\perp)$, given by

$$\widetilde{\gamma}_\ell(\Gamma_\perp) = \frac{2\pi^2\omega_0}{\operatorname{Im} G_w}|V_\ell(\Gamma_\perp)|^2\left.\frac{\partial F(\Omega,\Gamma_\perp)}{\partial\Omega}\right|_{\Omega=\omega_0}. \tag{53}$$

The generalization of Eq. (52) is found to be

$$\mathcal{S}_1 = \frac{3\pi^2}{16} \quad \text{and} \quad \delta\omega^2 = \frac{\pi^2}{8\mathcal{S}_3}\left(\gamma_{\rm d}\, t\right), \tag{54}$$

where

$$\mathcal{S}_p \equiv \sum_\ell \int d\Gamma_\perp \frac{\widetilde{\gamma_\ell}}{\omega_{\rm b}^p}. \tag{55}$$

Evidently, $\mathcal{S}_0$ is the total linear growth rate, $\gamma_{\rm L}$, and $d\Gamma_\perp$ is the phase space volume element defined earlier. The applicability of these equations requires $\alpha \ll 1$ in the phase space regions where there are significant contributions to each $\mathcal{S}_p$. However, sometimes this condition cannot be fulfilled, and we will see an example of such a case in the next section.

IV. SELF-CONSISTENT DAMPING

Up to now the background damping has been prescribed and it is assumed that it remains linear during the evolution of the wave. We now treat the damping mechanism in a self-consistent manner. We assume it is caused by particle resonances in phase space regions where $\omega_0\, F'(\omega_0) < 0$. In addition, the energetic particle population can be comprised of several species, which can be incorporated by interpreting the ℓ-summation in Eq. (17) as a sum over species. We study cases where there is no additional damping, and set $\gamma_{\rm d} = 0$.

We note that the corresponding generalization of the threshold equation, Eq. (21), leads to

$$\frac{dA}{dt} - \gamma A = - \sum_\ell \frac{\gamma_\ell r_\ell}{2} \int_0^{t/2} dz\, z^2 A(t-z) \int_0^{t-2z} dx\, K_\ell(x,z)\, A(t-z-x) A^*(t-2z-x), \quad (56)$$

where the index ℓ now runs over all resonant species. Here, A is the mode amplitude defined in such a way that $|A|$ equals the square of the trapping frequency of the destabilizing species, and $r_\ell \equiv \omega_{{\rm b}\ell}^4/|A|^2$. K_ℓ is given by Eq. (22), for each resonance.

Notice that the value of $\omega_{\rm b}^2$ given by Eq. (12) can be quite different for different resonances. In what follows, we will illustrate the effect of this disparity for the case of two coupled resonant species – a light one and a heavy one. We will assume that there is no equilibrium field, so that the unperturbed particle moves freely, and we will discuss the electrostatic bump-on-tail-like instability resulting from the two species. For this problem we have $\omega_{{\rm b}l}/\omega_{{\rm b}h} = (M/m)^{1/2} \gg 1$, where m/M is a light-to-heavy species mass ratio, such that the charges of the two species are assumed to be equal in magnitude (the subscripts "l" and

"h" refer to the light and heavy species respectively). The inequality $\omega_{bl} \gg \omega_{bh}$ implies that the light species must exhibit nonlinear behavior at a much lower field amplitude than the heavy species when $m/M \ll 1$. It is therefore natural to first study the regime in which the light species' dynamics is nonlinear, while the response of the heavy species remains linear.

A. light species-driven instability

We expect that explosive growth of the mode amplitude should initiate frequency sweeping when γ_l is somewhat larger than the linear heavy-species stabilization rate, $|\gamma_h|$. For each species j, we have $\nu_{\text{eff},j} \ll \gamma_l + \gamma_h \equiv \gamma$. It is readily shown that the heavy-species nonlinearity in Eq. (56) is smaller than the light-species nonlinearity by a factor $(m/M)^2$. This leads to exactly the same situation as in the first part of this paper (as long as the heavy-species dynamics remains linear). The mode first grows exponentially and appears to saturate at the level $\omega_{bl} \sim \gamma^{5/4}/\gamma_l^{1/4}$, which follows straightforwardly from dimensional analysis of Eq. (21) (this scaling has also been noted by Crawford.[18]) However, after this stage the mode amplitude exhibits an oscillatory explosive growth to a much higher level, $\omega_{bl} \sim \gamma_l$. Then, the mode forms a hole and a clump in the light-species distribution and persists for a long time because the energy extraction from the light species due to frequency sweeping balances the linear energy absorption by the heavy species. The sweeping follows the scaling law $\delta\omega \propto t^{1/2}$ as long as the heavy-species response remains linear – a condition that can be written as $\dot{\delta\omega} > \omega_{bh}^2 = (m/M)\,\omega_{bl}^2$. As $\delta\omega \sim \gamma_l^{3/2}\,t^{1/2}$ and $\omega_{bl} \sim \gamma_l$, the linear approximation for the heavy species breaks down at $t_0 \sim (M/m)^2\gamma_l^{-1}$. At this moment, the mode frequency shift reaches the value $\delta\omega \sim \gamma_l^{3/2}\,t_0^{1/2} \sim \gamma_l M/m$.

It has been observed[4] that collisions restrict the lifetime of a phase space structure to $t_1 \equiv \gamma_l^2/\nu_{\text{eff},l}^3$. Thus if $t_1 \ll t_0$ [or equivalently $\nu_{\text{eff},l}^3 \gg \gamma_l^3(m/M)^2$], the linear approximation for the heavy-species response is sufficient to describe the mode frequency sweeping.

Only for extremely low collision frequency can the nonlinear dynamics of the heavy mass species enter the problem, in which case we can expect other type of responses after a time

t_0. One possibility is a stationary BGK mode where the amplitude $|A| \sim \gamma_l^2$. Such a mode is supported by a hole (or clump) in the distribution of the light species, but with the distribution of the heavy species flattened near the resonance, so that wave damping does not occur.

Another possible response involves frequency sweeping that results from the interplay of two nonlinear phase space structures – one consisting of light particles that release energy and the other consisting of heavy particles that absorb energy. Then, in the limit $F(\omega_0) - F(\omega_0 + \delta\omega) \to -\delta\omega F'(\omega_0)$, the power-transfer equation [i.e. the lower line of Eq. (45) when generalized to two species] takes the form

$$0 = \frac{16\,\delta\omega}{\pi^2}\left[c_{2h}\,\frac{\gamma_l}{\omega_{\mathrm{b}l}}\left(1-\alpha_l^2\right)^{5/4} + c_{2h}\,\frac{\gamma_h}{\omega_{\mathrm{b}l}}\left(\frac{M}{m}\right)^{3/2}\left(1-\alpha_h^2\right)^{5/4}\right]\alpha_l. \tag{57}$$

The reader should also note that we assume $\gamma_l > 0$ and $\gamma_h < 0$. For the two terms in Eq. (57) to balance, we require

$$1-\alpha_h^2 \simeq \left(\frac{5}{3}\frac{\gamma_l}{|\gamma_h|}\right)^{4/5}\left(\frac{m}{M}\right)^{6/5}, \tag{58}$$

which means that $\alpha_h^2 \simeq 1$, with the physical implication that the potential well for the trapped heavy particles is shallow. According to this scaling, the correction from the heavy species to the BGK mode [i.e. the upper line in Eq. (45)] is negligible, as it is easy to verify that the light-species term dominates by $\mathcal{O}(M/m)^{8/5}$. Thus, without collisions, we find that a nonlinear frequency sweeping rate of $\dot{\delta\omega} = (m/M)\,\omega_{\mathrm{b}l}^2 \sim (m/M)\,\gamma_l^2$ can arise late in time. An interpolation formula for the sweeping rate which encapsulates both the linear and nonlinear ion responses is

$$\dot{\delta\omega} \simeq \omega_{\mathrm{b}l}^2\left(\frac{\pi^2}{16}\frac{|\gamma_h|}{\gamma_l}\frac{\omega_{\mathrm{b}l}}{\delta\omega} + \frac{m}{M}\,\mathrm{sgn}\,\delta\omega\right), \tag{59}$$

Collisionality of both species has to be low enough for Eq. (59) to be valid. It has already been noted that $\nu_{\mathrm{eff},l}^3 \ll (m/M)^2\,\gamma_l^3$ is required to achieve a nonlinear heavy-species response. For the heavy-species phase space structure to form, the collisional frequency, $\nu_{\mathrm{eff},h}$, needs to be less than the trapping frequency in the shallow well. This leads to

$$\nu_{\mathrm{eff},h}^3 < \gamma_l^3 (m/M)^{12/5}. \tag{60}$$

Then with negligible collisionality of light species, the nonlinear sweeping continues for a time $T_{\mathrm{sweep}} \sim \gamma_l^2 (m/M)^{8/5}/\nu_{\mathrm{eff},h}^3$. Afterwards, the hole in the heavy-species distribution is filled in and the distribution is flattened around the resonance region.

It can be shown that the described scenario satisfies the adiabatic condition; i.e. the frequency sweeping rate is indeed smaller than the square of the heavy-species trapping frequency.

Sweeping with a small m/M has been examined with the numerical simulation code described in Ref. 16. Figure 6 shows results for $m/M = 0.05$. In this simulation, the light species supplies the instability drive with $\gamma_l/\omega_0 = 0.05$, the heavy species supplies dissipation with $\gamma_h/\omega_0 = -0.035$, and

$$F(\omega) = \frac{\omega^2}{2\omega_0} \tag{61}$$

where ω_0 is the frequency of the linear mode. In Figs. 6a and 6b, respectively, we show the time evolution of the light-species distribution function and a plot of the mode power spectrum. Clearly there is sweeping of the upshifted spectral component, while the downshifted component does not sweep. The solid curve in Fig. 6b is the frequency shift obtained from a generalization of Eq. (59) to the case of non-constant slope of the unperturbed distribution, as used in Eq. (61). This generalization gives

$$t = \left(\frac{3\pi^2}{16}\right)^2 \int_0^{\delta\omega} \frac{x\,d\,x}{(1 + x/2\omega_0)^2\, \gamma_l^2 \left(|x|\, m/M + \frac{1}{3}|\gamma_h|\right)}. \tag{62}$$

We also show in the dashed curve the predicted frequency shift for $m/M = 0$, in which case the heavy species' response is purely linear. The simulation result agrees better with the solid curve, which is an indication of nonlinear energy absorption by a heavy-species clump. Further evidence of the clump is seen in Fig. 6c, which presents the spatial average of the perturbed heavy-species distribution $\langle \delta f_h \rangle$; the clump develops at $\omega_0 t/2\pi \sim 400$. Figure 6d shows that the mode amplitude at saturation agrees with the characteristic estimate for the sweeping solution, $|A|/\gamma_l^2 \sim 1$. Unlike the upshifted branch, the lower frequency branch does not exhibit sweeping after the heavy-species response becomes nonlinear. A likely reason for this is that, as mentioned previously, the heavy species can form a plateau at the resonance point causing dissipation and therefore frequency sweeping to stop. We also see in Fig. 6b that the lower frequency branch eventually decays, although the perturbation of the distribution function at the lowest phase velocity persists, as seen in Fig. 6c and 6d. The reason for this decay still needs to be understood. The fairly broad unshifted spectrum seen in Fig. 6b corresponds to fluctuations that persist near the original resonance after the distribution functions of both species flatten at this location. We note that frequency sweeping due to the formation of a hole and clump was also observed for $m/M = 0.25$, but sweeping did not arise for $m/M = 0.5$.

B. heavy species-driven instability

An entirely different response near marginal stability arises if there is an inverted population of heavy particles with a linear destabilization rate, $\gamma_h > 0$, and a normal population of light particles with a damping rate, $\gamma_l < 0$ (a physical example of this case is an acoustic instability where the ions have a bump-on-tail distribution and damping arises from electron Landau damping). Then the light-species nonlinearity is appreciably larger than the heavy-species nonlinearity, as is clear from Eq. (56). The effect of the light-species nonlinearity is to weaken the damping mechanism so that the instability grows faster. This leads to an explosive response $A(t) \propto 1/(t_0 - t)^{5/2}$ without any oscillations – a hard instability. This explosive behavior continues until $|A(t)| \sim (m/M)\gamma_l^2$, whereupon the light-species distribution forms a plateau near the resonance, and the further evolution of the wave is an exponential growth at a rate $\gamma \simeq \gamma_h$, until the amplitude reaches $|A| \sim \gamma_h^2$. At this stage the mode saturates as if there was no damping. The corresponding saturation level is given by $|A|^{1/2} \simeq 3.3\gamma_h$ (see Ref. 19). This level has been bracketed theoretically to within 15% by using energy conservation arguments between the wave and the destabilizing species.[20,21]

In Fig. 7 we plot the evolution of the instability for $m/M = 1/16$, with $\gamma_h = 0.025$,

$\gamma_l = -0.024$. Note that there is no frequency sweeping and the initial e-folding rate is $(\gamma_h + \gamma_l)$, as expected, but it transforms to a higher rate $\gamma \simeq .6\gamma_h$ as the damping from the light species decreases. In other simulations we find that this late time growth rate approaches γ_h as the mass ratio gets smaller. The saturated level, $|A|/\gamma_h^2 \simeq 10$ agrees with the result of Ref. 19.

A comparison at early times of this simulation and the threshold equation, Eq. (56) is shown in Fig. 8. Note that now the nonlinear term in Eq. (56) is dominated by the light species, by a factor $(M/m)^2$, and this nonlinear term has a destabilizing effect, which leads to a "hard" explosive solution. This solution can be inferred from the theory presented in Ref. 12 [we note that there is misprint in Ref. 12, and in Eqs. (31)-(33) of that paper one should replace $e^{-i\phi}$ by $-e^{-i\phi}$ to obtain the correct equation; this correction gives the hard solution now discussed, but does not alter the other explosive solutions reported in that reference]. The explicit form of the explosive solution is

$$\frac{A}{\gamma_l^2} = 42.3 \frac{1}{[\gamma_l(t_0 - t)]^{5/2}} \frac{m}{M} \tag{63}$$

where t_0 is the moment of explosion. This solution is also shown in Fig. 8.

It is also of interest to take the mass ratio of unity (a "positron-electron" plasma). In Fig. 9 we plot the response for this case (we choose "γ_h" to denote the destabilizing species) and we see that frequency sweeping does not arise. Equation (56) cannot be applied to the equal masses case because the nonlinear terms nearly cancel in the same way as the linear terms. Therefore there is no parameter that allows the nonlinear terms to be both dominant and accurate (it can be shown that $[1 - (m/M)^2] > \gamma/\gamma_l$ is required for validity of the threshold equation). The saturation level for the equal masses case can be estimated using the same energy-momentum conservation principles, which yields $\omega_\mathrm{b} \simeq 3||\gamma_l| - |\gamma_h||$. This estimate correlates well with the mode amplitude observed in Fig. 9.

V. FREQUENCY SWEEPING WITH HEAVY SPECIES UNSTABLE

In this section we demonstrate that frequency sweeping can occur even when the instability drive comes from the heavy species if the collision frequency of the light species is sufficiently large. This frequency sweeping is easiest to demonstrate if the heavy species is collisionless ($\nu_h = 0$). In the δf particle simulation, the collisional effects are modeled by using a finite annihilation rate ν_l for the light, stabilizing species. We also use constant-slope unperturbed distribution functions for both species. We choose $\gamma_h/\omega = .025$, $\gamma_l/\omega = -.02$, $m/M = 1/16$, and scan the annihilation rate ν_l/ω (0.15, 0.0325, 0.0075). As will be discussed below, all three cases show that frequency sweeping develops. Sweeping is to be expected with the highest value of ν_l/ω, but sweeping with the lower values of ν_l/ω is surprising.

First let us consider the case ($\nu_l/\omega = 0.15$) presented in Fig. 10. We note that if $\omega_{\mathrm{b}l}/\nu_l \ll 1$, or equivalently $\omega_{\mathrm{b}h}(M/m)^{1/2} \ll \nu_l$, the nonlinear response from the light species will be a small correction to its linear response during the entire evolution of the simulation. As the saturation level for $\omega_{\mathrm{b}h}$ is of the order of γ_h, we can rewrite the condition $\omega_{\mathrm{b}h}(M/m)^{1/2} \ll \nu_l$ in the form $(\nu_l/\gamma_h)^2 m/M \equiv \delta \gg 1$. Note that this condition is satisfied for $\nu_l/\omega = 0.15$. Therefore, we are left with nearly the same problem as when the light species respond purely linearly, and the result is very similar to what we report in Sec. III.

In the limit $\delta \ll 1$, collisions are too weak to maintain the response of the light species close to the linear one. In general, the modified light species damping rate, $\gamma_{l,\mathrm{nonlinear}}$ can be described by an interpolation formula that connects their quasistationary ($dA/dt \ll \nu_l A$) response for $A \ll \nu_l^2$ to the result for $A \gg \nu_l^2$:

$$\gamma_{l,\mathrm{nonlinear}} = \gamma_l \left\{ 1 + \frac{[|A|/\delta\gamma_h^2]^2}{8\,[1 + .24(|A|/\delta\gamma_h^2)^{3/2}]} \right\}^{-1} . \tag{64}$$

Expression (64) for the limit $A \ll \nu_l^2$ was obtained by evaluating the light species nonlinearity in (56) to find $\gamma_l[1 - (|A|\gamma_h^2\delta)^2/8]$, while the result for the limit $A \gg \nu_l^2$, found in Ref. 22 is $1.9\gamma_{\mathrm{d}}(A/\delta\gamma_h^2)^{1/2}$.

Development of frequency sweeping in the case $\delta \ll 1$ is shown in Fig. 11 that presents results for $\nu_l/\omega = 0.0325$. Once a phase space structure forms and evolves adiabatically, it is straightforward to predict the sweeping rate. Equating the damping rate given by Eq. (64) to the power released by sweeping of the heavy structure [obtained from the lower equation in Eq. (45)], together with $\omega_{bh} = 16\gamma_h/3\pi^2$, gives for the frequency shift,

$$\delta\omega = .83\left[\gamma_h|\gamma_l|(m/M)^{1/2}\nu_l t\right]^{1/2}, \tag{65}$$

which agrees quite closely to the simulation results shown in Fig. 11.

It is not quite clear why phase space structures forms in the first place. Their formation is likely to be associated with a remnant damping from the light species because the nonlinear suppression of the light species' damping mechanism is incomplete. With sufficient remnant damping, the initialization of holes and clumps seems to be qualitatively similar to that in the constant damping case where one observes an oscillatory transient behavior. However, this scenario needs to be revised when the light species collisionality is so low that the mode grows explosively without intermediate oscillations.

The case $\nu_l/\omega = 0.0075$ also shows frequency sweeping late in time as seen in Fig. 12. However, the initial precursor for this case is considerably different from the previous case. It is clear from Fig. 12 that in the initial stage the hard instability sets in which grows without oscillations at a rate comparable to γ_h. This phase saturates at a significantly higher level than the previously discussed case (at about 5 times the previous amplitude). Immediately after saturation the distribution of the heavy species is flattened in the trapping region, but the distribution will have a large phase space gradient at the interface between passing and trapped particles. On the other hand, for the light species a slight slope remains in its distribution function as collisions are trying to restore the distribution's equilibrium slope. Therefore, the mode amplitude will damp. As the mode amplitude damps, the light species' nonlinear dissipation rate increases, and indeed we can see that at first the rate of fall of the amplitude increases in time. However, when $|A| \sim .2\gamma_h^2$, the decay stops and two well defined frequency sidebands develop at a distance on the order of $2\gamma_h$ from the linear frequency.

The reason for sudden emergence of the frequency shifting pulses is somewhat speculative, but may be due to the following cause. We note that the heavy distribution has a sharp gradient at the interface between the trapped and passing particles when the mode amplitude was at its largest, and this gradient persists as the mode amplitude decreases. If there is no background dissipation, then the sharp gradient is sufficiently displaced from the linear resonance to allow the system to be stable (this argument has been presented in more detail in Ref. 14). However, if there is background dissipation, then a negative energy perturbation associated with the sharp positive phase space gradients may arise to become a seed for frequency sweeping.

VI. SUMMARY OF RESULTS

We have presented a description of how coherent hole-clump structures with frequency sweeping phenomena emerge in systems with weak instability. Several conditions are important for the frequency sweeping to develop. They are the following:

1. The system must be near the linear instability threshold. We find that if the system is initiated far above threshold, then there is no frequency change. Indeed, in other numerical studies we have observed that a hole and clump do not emerge when $\gamma_{\mathrm{L}} \gtrsim 2.5\gamma_{\mathrm{d}}$ where γ_{L} is the growth rate in absence of dissipation and γ_{d} is the damping rate due to background dissipation. Instead, a plateau is formed in the distribution about the resonant velocity. The wave then damps due to the remaining dissipation, with the wave frequency essentially unchanged.

2. Collisional effects need to be sufficiently weak to allow the explosive initialization of holes and clumps. For sufficiently small fluctuation levels, the condition $\nu_{\mathrm{eff}} \lesssim \gamma_{\mathrm{L}} - \gamma_{\mathrm{d}}$ is needed for explosive dynamics described in Ref. [1]. This dynamics satisfies a reduced cubic nonlinear equation, which is valid as long as particle trapping in the wave does not have a chance to develop. We find that the two sidebands formed during the explosive phase are precursors to the formation of a hole and clump in phase space. The hole and clump support a pair of BGK modes whose phase velocities slowly change, which

allows the passing particles to release the energy needed to balance the background dissipation.

3. Explosive behavior that leads to frequency chirping and coherent structure formation can also arise when the system is below the linear instability threshold if the initial perturbation is sufficiently large.

4. In addition to the case where linear dissipation is imposed, we have studied physical cases where dissipation arises self-consistently from a second resonant species. When both species are collisionless, we find the frequency sweeping effects similar to the case of a fixed linear dissipation. Such a response requires the trapping frequency of the stabilizing species be much less than the trapping frequency of the destabilizing species. In the opposite limit, the stabilizing effect quickly disappears, and the instability proceeds as if there is no background damping. When the trapping frequencies of the stabilizing and destabilizing species are equal, frequency sweeping effects are not observed, and the level of saturation is proportional to the increment above linear stability.

5. Frequency sweeping can be restored in the case of a collisionless heavy destabilizing component competing with linear dissipation supplied by the light species when the collision frequency for the light species is high enough. Then the response of the light species will remain purely linear, allowing phase space structure formation in the heavy species in accordance with the theory presented in Sec. III. However, frequency sweeping is observed even if the light species is moderately affected by collisions. In this case, the rate of sweeping is determined by the collisions that attempt to offset the plateau formation. The surprise is that conditions can be established in the first place for phase space structure formation. At very low collision frequencies the sweeping mechanism is perhaps caused by collisionally induced dissipation that destabilizes negative energy perturbations from the steep gradients of the destabilizing

distribution function.

6. Though our study here concentrates on idealized systems for which there are just one or two resonances, our approach is applicable to more complicated physical systems. For example, Eq. (54) allows the calculation of the frequency sweeping rate for a TAE mode in the presence of many simultaneous but nonoverlapping resonances.[21] Work is in progress to attempt to apply this theory to frequency sweeping phenomena observed in experimental situations.

REFERENCES

[1] H.L. Berk, B.N. Breizman, and M. Pekker, Phys. Rev. Lett. **76** 1256 (1996).

[2] I.B. Bernstein, S.K. Trehan, and M.P.H. Weenik, Nucl. Fusion **4** 61 (1964).

[3] M.N. Rosenbluth and P.H. Rutherford, Phys. Rev. Lett. 34 1428 (1975).

[4] H.L. Berk, B.N. Breizman, and N.V. Petviashvili, Phys. Lett. A **234** 213 (1997).

[5] H.L. Berk, B.N. Breizman, and N.V. Petviashvili, Phys. Lett. A **238** 408 (1998).

[6] H.L. Berk, C.E. Nielsen, and K.V. Roberts, Phys. Fluids **13** 980 (1970).

[7] P.W. Terry, P.H. Diamond, and T.S. Hahm, Phys. Fluids **2** 2048 (1990).

[8] M.E. Mauel, Journal de Physique IV **7** C4–307 (1997).

[9] T.H. Dupree, Phys. Fluids **15** 334 (1972).

[10] R.H. Berman, D.J. Tetreault, and T.H. Dupree, Phys. Fluids **26** 2437 (1983).

[11] B.N. Breizman, H.L. Berk, M.S. Pekker, F. Porcelli, G.V. Stupakov, and K.L. Wong, Phys. Plasmas **4** 1559 (1997).

[12] H.L. Berk, B.N. Breizman, and M.S. Pekker, Plasma Phys. Rep. **9** 778 (1997).

[13] A.N. Kaufman, Phys. Fluids **15** 1063 (1972).

[14] H.L. Berk, B.N. Breizman, and H. Ye, Phys. Rev. Lett. **68** 3563 (1992).

[15] I.B. Bernstein, J.M. Greene, and M.D. Kruskal, Phys. Rev. Lett. **108** 546 (1957).

[16] J. Candy, J. Comput. Phys. **129** 160 (1996).

[17] H.L. Berk and B.N. Breizman, Phys. Fluids B **2** 2246 (1991).

[18] J.D. Crawford, Bull. Am. Phys. Soc. **42** 12057 (1997).

[19] M.B. Levin, M.G. Lyubarskii, I.N. Onishchenko, V.D. Shapiro, and V.I. Schevchenko, Zh.

Eksp. Teor. Fiz. **62** 1725 (1972); [Translation: JETP **35** 898 (1972)].

[20] J. Candy, D. Borba, H.L. Berk, G.T.A. Huysmans, and W. Kerner, Phys. Plasmas **4** 2597 (1997).

[21] H.V. Wong and H.L. Berk, Phys. Plasmas **5** 2781 (1998).

[22] H.L. Berk and B.N. Breizman, Phys. Fluids B **2**, 2226 (1991).

FIGURE CAPTIONS

FIG. 1. Time evolution of normalized mode amplitude for $\gamma_{\mathrm{d}}/\gamma_{\mathrm{L}} = 0.7$ and $\nu_{\mathrm{eff}}/\gamma_{\mathrm{L}} = 0.1$ (these parameters also apply to Figs. 2 and 3).

FIG. 2. Contour plots of the evolving spectral intensity $|A_\omega|^2$ vs. time using a Gaussian time window $\exp(-(t-t_0)^2/\Delta^2))$, with $\Delta = 30\gamma_L^{-1}$. Dotted line is the theory prediction for the collisionless case.

FIG. 3. Coherent hole-clump structures with time-dependent frequencies. (a) The spatially averaged particle distribution as a function of time and the distance from the linear resonance $\Omega - \omega_0$ with $\Omega \equiv kv$ for the bump-on-tail instability. (b) Spectral intensity $|A_\omega|^2$ as a function of time and $\omega - \omega_0$.

FIG. 4. Oscillatory mode growth in weak nonlinear regime. Solid curve shows the result of full-scale numerical simulation, dashed curve shows the solution of the reduced equation, Eq. (21), with $\gamma = 0.1\gamma_{\mathrm{L}}$.

FIG. 5. Formation of frequency sweeping solution in a linearly stable system. (a) Evolution of the mode amplitude. (b) Plot of the evolving Fourier spectrum.

FIG. 6. Evolution of bump-on-tail instability resulting from two resonant species with $m/M = 0.05$, $\gamma_l/\omega_0 = 0.05$, $\gamma_h/\omega_0 = -0.035$, and collision frequency zero for both species. (a) Spatial average of the light species distribution function; (b) plot of the mode spectral intensity. Solid curve shows the analytical prediction for the frequency shift (Eq. (62) with $m/M = 0.05$); dashed curve shows the predicted shift for $m/M = 0$; (c) Perturbed heavy species distribution function; (d) Time dependence of the mode amplitude.

FIG. 7. Evolution of bump-on-tail instability for two resonant species with $M/m = 16$, $\gamma_h/\omega_0 = 0.025$, $\gamma_l/\omega_0 = -0.024$, and with zero collision frequency. (a) Time de-

pendence of the mode amplitude; (b) The heavy species distribution function; note plateau formation near the resonant velocity ($kv/\omega_0 = 1$).

FIG. 8. Comparison of numerical simulations (solid curve) with the solution of the reduced nonlinear equation, Eq. (56), (dashed curve) for a heavy-species driven instability in a two-species system. Dots plot the self-similar explosive solution, Eq. (63). The straight line is the exponential growth given by linear theory.

FIG. 9. Response of two species system for $m/M = 1$ with $\gamma_h/\omega_0 = 0.05$, $\gamma_l/\omega_0 = -0.035$ and with zero collision frequency for both species. The mode amplitude vs. time is presented in (a), the spatially-averaged distribution for the destabilizing species is shown in (b).

FIG. 10. Response of two species system for $m/M = 1/16$, $\gamma_h/\omega_0 = 0.025$, $\gamma_l/\omega_0 = -0.02$, $\nu/\omega_0 = 0.15$ for the light species, and with zero collision frequency for the heavy species. The amplitude vs. time is shown in (a), the wave spectrum vs. time is shown in (b).

FIG. 11. Same as Fig. 10 except that $\nu_l/\omega_0 = 0.0325$ for the light species. Solid curve in Fig. 10b shows the theoretical prediction given by Eq. (65).

FIG. 12. Same as Fig. 10 except that $\nu_l/\omega_0 = 0.0075$ for the light species.