SPREADING AND VANISHING IN NONLINEAR DIFFUSION PROBLEMS WITH FREE BOUNDARIES§

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Abstract. We study nonlinear diffusion problems of the form $u_t = u_{xx} + f(u)$ with free boundaries. Such problems may be used to describe the spreading of a biological or chemical species, with the free boundary representing the expanding front. For special $f(u)$ of the Fisher-KPP type, the problem was investigated by Du and Lin [8]. Here we consider much more general nonlinear terms. For any $f(u)$ which is C^1 and satisfies $f(0) = 0$, we show that the omega limit set $\omega(u)$ of every bounded positive solution is determined by a stationary solution. For monostable, bistable and combustion types of nonlinearities, we obtain a rather complete description of the long-time dynamical behavior of the problem; moreover, by introducing a parameter σ in the initial data, we reveal a threshold value σ^* such that spreading ($\lim_{t\to\infty} u = 1$) happens when $\sigma > \sigma^*$, vanishing ($\lim_{t\to\infty} u = 0$) happens when $\sigma < \sigma^*$, and at the threshold value σ^*, $\omega(u)$ is different for the three different types of nonlinearities. When spreading happens, we make use of "semi-waves" to determine the asymptotic spreading speed of the front.

1. Introduction

We consider the following problem

$$
(1.1)\qquad \begin{cases} u_t = u_{xx} + f(u), & g(t) < x < h(t),\ t > 0, \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu\, u_x(t, g(t)), & t > 0, \\ h'(t) = -\mu\, u_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0, \quad u(0, x) = u_0(x), & -h_0 \le x \le h_0, \end{cases}
$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t, x)$, μ is a given positive constant, $f : [0, \infty) \to \mathbb{R}$ is a C^1 function satisfying

$$
(1.2)\qquad f(0) = 0.
$$

The initial function u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 0$, where

$$
(1.3)\qquad \mathscr{X}(h_0) := \Big\{ \phi \in C^2([-h_0, h_0]) : \ \phi(-h_0) = \phi(h_0) = 0,\ \phi'(-h_0) > 0, \ \phi'(h_0) < 0,\ \phi(x) > 0 \text{ in } (-h_0, h_0) \Big\}.
$$

For any given $h_0 > 0$ and $u_0 \in \mathscr{X}(h_0)$, by a (classical) solution of (1.1) on the time-interval $[0, T]$ we mean a triple $(u(t, x), g(t), h(t))$ belonging to $C^{1,2}(G_T) \times C^1([0, T]) \times C^1([0, T])$, such

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that all the identities in (1.1) are satisfied pointwisely, where

$$G_T := \big\{(t,x) : t \in (0,T],\ x \in [g(t), h(t)]\big\}.$$

In the rest of the paper, the solution may also be denoted by $(u(t,x;u_0), g(t;u_0), h(t;u_0))$, or simply (u,g,h), depending on the context.

Problem (1.1) with $f(u)$ taking the particular form $f(u) = au - bu^2$ was studied recently in [8]. Such a situation arises as a population model describing the spreading of a new or invasive species, whose growth is governed by the logistic law. The free boundaries $x = g(t)$ and $x = h(t)$ represent the spreading fronts of the population whose density is represented by $u(t,x)$. The focus of [8] is on the particular logistic nonlinearity $f(u) = au - bu^2$, and many of the arguments there rely on this choice of f.

The logistic $f(u)$ mentioned above belongs to the class of "monostable" nonlinearities, and due to the pioneering works of Fisher [12] and Kolmogorov-Petrovski-Piskunov [17], it is also known as the Fisher, or KPP, or Fisher-KPP type nonlinearity. As is well-known, in population models one often needs to consider a general monostable nonlinear term ([3, 4]). Moreover, to include Allee effects, "bistable" nonlinear terms are used in many population models ([15, 18]). Bistable nonlinearity also appears in other applications including signal propagation and material science ([21, 1, 10]). Furthermore, in the study of combustion problems the typical $f(u)$ is of "combustion" type ([23, 16, 25]). A precise description of these different types of nonlinearities will be given shortly below.

The main purpose of this paper is to classify the behavior of (1.1) for all the types of nonlinearities mentioned in the last paragraph. Even restricted to the monostable type, this is an extension of [8] since we do not require the special form $f(u) = au - bu^2$, which implies that different methods have to be used.

The corresponding Cauchy problem

$$u_t = u_{xx} + f(u)\ (x \in \mathbb{R}^1,\ t > 0),\ u(0,x) = u_0(x)\ (x \in \mathbb{R}^1) \tag{1.4}$$

has been extensively studied. For example, the classical paper [3] contains a systematic investigation of this problem. Various sufficient conditions for $\lim_{t\to\infty} u(t,x) = 1$ and for $\lim_{t\to\infty} u(t,x) = 0$ are known, and when $u_0(x)$ is nonnegative and has compact support, the way $u(t,x)$ approaches 1 as $t \to \infty$ was used to describe the spreading of a (biological or chemical) species, which is characterized by certain traveling waves, and the speed of these traveling waves determines the asymptotic spreading speed of the species; see for example, [16, 10, 3, 4]. The transition between spreading ($u \to 1$) and vanishing ($u \to 0$) has not been well understood until recently. In [9], motivated by break-through results obtained in [25], a rather complete description of the sharp transition behavior was given. As we will see below, these sharp transition results of [9] for (1.4) also hold for (1.1). (Spreading and vanishing are sometimes called propagation and extinction, as in [9].) We will make use of a number of the ideas from [9], and this paper may be regarded as an extension of [9].

In most spreading processes in the natural world, a spreading front can be observed. In the one space dimension case, if the species initially occupies an interval $(-h_0, h_0)$ with density $u_0(x)$, as time t increases from 0, it is natural to expect the two end points of $(-h_0, h_0)$ to evolve into two spreading fronts, $x = g(t)$ on the left and $x = h(t)$ on the right, and the initial function $u_0(x)$ to evolve into a positive function u inside the interval $(g(t), h(t))$ governed by the equation $u_t = u_{xx} + f(u)$, with u vanishing at $x = g(t)$ and $x = h(t)$. To determine how the fronts $x = g(t)$ and $x = h(t)$ evolve with time, we assume that the fronts invade at a speed that is proportional to the spatial gradient of the density function u there, which gives rise to the free boundary conditions in (1.1). In the context of biological species, this is equivalent to saying that the front expands in a way that keeps the pressure at the front caused by the random movement of individuals at a constant level (equals μ^{-1}), balanced by a viscosity-like force representing the tendency of the individuals near the front to stay within the population range to counter the Allee effect, as the population density is close to 0 there; a detailed deduction of this fact is given in [5].

We notice that the free boundary conditions in (1.1) obtained in the above process coincide with the one-phase Stefan condition arising from the investigation of the melting of ice in contact with water ([22]). Such conditions also arise in the modeling of wound healing ([6]). For population models, [19] used such a condition for a predator-prey system over a bounded interval, showing the free boundary reaches the fixed boundary in finite time, and hence the long-time dynamical behavior of the system is the same as the well-studied fixed boundary problem; and in [20], a two phase Stefan condition was used for a competition system over a bounded interval, where the free boundary separates the two competitors from each other in the interval. A similar problem to (1.1) but with $f(u) = u^p$ $(p > 1)$ was studied in [11, 14]. Since this is a superlinear problem, its behavior is very different from (1.1) considered here as our focus is on the sublinear cases (except Theorem 1.1 and section 2). Indeed, our interests here are very different from all the previous research mentioned in this paragraph.

We now describe the main results of this paper. Firstly we assume that

$$f(u) \text{ is } C^1 \text{ and } f(0) = 0. \tag{1.5}$$

Then a simple variation of the arguments in [8] shows that, for any $h_0 > 0$ and $u_0 \in \mathscr{X}(h_0)$, (1.1) has a unique solution defined on some maximal time interval $(0, T^*)$, $T^* \in (0, \infty]$. Moreover, $g'(t) < 0$, $h'(t) > 0$ and $u(t,x) > 0$ for $t \in (0,T^*)$, $x \in (g(t), h(t))$, and if $T^* < \infty$ then $\max_{x\in[g(t),h(t)]} u(t,x) \to \infty$ as $t \to T^*$. Thus $\lim_{t\to\infty} g(t)$ and $\lim_{t\to\infty} h(t)$ always exist if $T^* = \infty$. Throughout this paper, we will use the notations

$$g_\infty := \lim_{t\to\infty} g(t), \quad h_\infty := \lim_{t\to\infty} h(t).$$

$T^* = \infty$ is guaranteed if we assume further that

$$f(u) \le Ku \text{ for all } u \ge 0 \text{ and some } K > 0. \tag{1.6}$$

A more detailed description of these statements can be found in section 2 below.

Our first main result is a general convergence theorem, which is an analogue of Theorem 1.1 in [9].

Theorem 1.1. *Suppose that* (1.5) *holds and* (u, g, h) *is a solution of* (1.1) *that is defined for all* $t > 0$, *and* $u(t,x)$ *is bounded, namely*

$$u(t,x) \le C \text{ for all } t > 0, \quad x \in [g(t), h(t)] \text{ and some } C > 0.$$

Then (g_∞, h_∞) *is either a finite interval or* $(g_\infty, h_\infty) = \mathbb{R}^1$. *Moreover, if* (g_∞, h_∞) *is a finite interval, then* $\lim_{t\to\infty} u(t,x) = 0$, *and if* $(g_\infty, h_\infty) = \mathbb{R}^1$ *then either* $\lim_{t\to\infty} u(t,x)$ *is a nonnegative constant solution of*

$$v_{xx} + f(v) = 0, \; x \in \mathbb{R}^1, \tag{1.7}$$

or

$$u(t,x) - v(x + \gamma(t)) \to 0 \text{ as } t \to \infty,$$

where v *is an evenly decreasing positive solution of* (1.7), *and* $\gamma : [0,\infty) \to [-h_0, h_0]$ *is a continuous function.*

By an evenly decreasing function we mean a function $v(x)$ satisfying $v(-x) = v(x)$ which is strictly decreasing in $[0,\infty)$. Let us note that (g_∞, h_∞) can never be a half-infinite interval. In fact, we will prove in Lemma 2.8 that

$$-2h_0 < g(t) + h(t) < 2h_0 \text{ for all } t > 0.$$

We conjecture that $\lim_{t\to\infty} \gamma(t)$ exists but were unable to prove it.

Next we focus on three types of nonlinearities:

$$(\mathrm{f}_M) \text{ monostable case,} \quad (\mathrm{f}_B) \text{ bistable case,} \quad (\mathrm{f}_C) \text{ combustion case.}$$

In the monostable case (f_M), we assume that f is C^1 and it satisfies

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad (1-u)f(u) > 0 \text{ for } u > 0, u \neq 1. \tag{1.8}$$

Clearly $f(u) = u(1-u)$ belongs to (f_M).

In the bistable case (f_B), we assume that f is C^1 and it satisfies

$$f(0) = f(\theta) = f(1) = 0, \quad f(u) \begin{cases} < 0 & \text{in } (0,\theta), \\ > 0 & \text{in } (\theta,1), \\ < 0 & \text{in } (1,\infty) \end{cases} \tag{1.9}$$

for some $\theta \in (0,1)$, $f'(0) < 0$, $f'(1) < 0$ and

$$\int_0^1 f(s)ds > 0. \tag{1.10}$$

A typical bistable $f(u)$ is $u(u-\theta)(1-u)$ with $\theta \in (0, \frac{1}{2})$.

In the combustion case (f_C), we assume that f is C^1 and it satisfies

$$f(u) = 0 \text{ in } [0,\theta], \quad f(u) > 0 \text{ in } (\theta,1), \quad f'(1) < 0, \quad f(u) < 0 \text{ in } [1,\infty) \tag{1.11}$$

for some $\theta \in (0,1)$, and there exists a small $\delta_0 > 0$ such that

$$f(u) \text{ is nondecreasing in } (\theta, \theta + \delta_0). \tag{1.12}$$

Clearly (1.5) and (1.6) are satisfied if f is of (f_M), or (f_B), or (f_C) type. Thus in these cases (1.1) always has a unique solution defined for all $t > 0$.

The next three theorems give a rather complete description of the long-time behavior of the solution, and they also reveal the related but different sharp transition natures between vanishing and spreading for these three types of nonlinearities.

Theorem 1.2. (The monostable case). *Assume that f is of (f_M) type, and $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$. Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞, h_∞) *is a finite interval with length no bigger than* $\pi/\sqrt{f'(0)}$ *and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0.$$

Moreover, if $u_0 = \sigma\phi$ with $\phi \in \mathscr{X}(h_0)$, then there exists $\sigma^* = \sigma^*(h_0,\phi) \in [0,\infty]$ such that vanishing happens when $0 < \sigma \le \sigma^*$, and spreading happens when $\sigma > \sigma^*$. In addition,

$$\sigma^* \begin{cases} = 0 & \text{if } h_0 \ge \pi/(2\sqrt{f'(0)}), \\ \in (0,\infty] & \text{if } h_0 < \pi/(2\sqrt{f'(0)}), \\ \in (0,\infty) & \text{if } h_0 < \pi/(2\sqrt{f'(0)}) \text{ and if } f \text{ is globally Lipschitz.} \end{cases}$$

Theorem 1.3. (The bistable case). *Assume that f is of* (f_B) *type, and $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$. Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0,$$

or

(iii) Transition: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and there exists a continuous function $\gamma : [0,\infty) \to [-h_0, h_0]$ such that*

$$\lim_{t\to\infty} |u(t,x) - v_\infty(x+\gamma(t))| = 0 \text{ locally uniformly in } \mathbb{R}^1,$$

where v_∞ is the unique positive solution to

$$v'' + f(v) = 0 \ (x \in \mathbb{R}^1), \ v'(0) = 0, \ v(-\infty) = v(+\infty) = 0.$$

Moreover, if $u_0 = \sigma\phi$ for some $\phi \in \mathscr{X}(h_0)$, then there exists $\sigma^* = \sigma^*(h_0,\phi) \in (0,\infty]$ such that vanishing happens when $0 < \sigma < \sigma^*$, spreading happens when $\sigma > \sigma^*$, and transition happens when $\sigma = \sigma^*$. In addition, there exists $Z_B > 0$ such that $\sigma^* < \infty$ if $h_0 \ge Z_B$, or if $h_0 < Z_B$ and f is globally Lipschitz.

Theorem 1.4. (The combustion case). *Assume that f is of* (f_C) *type, and $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$. Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0,$$

or

(iii) Transition: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = \theta \text{ locally uniformly in } \mathbb{R}^1.$$

Moreover, if $u_0 = \sigma\phi$ for some $\phi \in \mathscr{X}(h_0)$, then there exists $\sigma^* = \sigma^*(h_0,\phi) \in (0,\infty]$ such that vanishing happens when $0 < \sigma < \sigma^*$, spreading happens when $\sigma > \sigma^*$, and transition happens when $\sigma = \sigma^*$. In addition, there exists $Z_C > 0$ such that $\sigma^* < \infty$ if $h_0 \ge Z_C$, or if $h_0 < Z_C$ and f is globally Lipschitz.

Remark 1.5. The value of σ^* in the above theorems can be $+\infty$ if we drop the assumption that f is globally Lipschitz when h_0 is small. Indeed, this is the case if $f(u)$ goes to $-\infty$ fast enough as $u \to +\infty$; see Propositions 5.4, 5.8 and 5.12 for details. The values of Z_B and Z_C are determined by (4.8) and (4.9), respectively.

Remark 1.6. In [8], to determine whether spreading or vanishing happens for the special monostable nonlinearity, a threshold value of μ was established, which was shown in [8] to be always finite. Here we use σ in $u_0 = \sigma\phi$ as a varying parameter, which appears more natural especially for the bistable and combustion cases, since in these cases the dynamical behavior of (1.1) is more responsive to the change of the initial function than to the change of μ; for example, when $\|u_0\|_\infty \le \theta$, then vanishing always happens regardless of the value of μ.

Remark 1.7. Theorems 1.3 and 1.4 above are parallel to Theorems 1.3 and 1.4 in [9], where the Cauchy problem was considered. In contrast, Theorem 1.2 is very different from the Cauchy problem version, where a "hair-trigger" phenomenon appears, namely, when f is of (f_M) type, any nonnegative solution of (1.4) is either identically 0, or it converges to 1 as $t \to \infty$ (see [3, 4]).

When spreading happens, the asymptotic spreading speed is determined by the following problem

$$\begin{cases} q_{zz} - cq_z + f(q) = 0 \quad \text{for } z \in (0, \infty), \\ q(0) = 0, \ \mu q_z(0) = c, \ q(\infty) = 1, \ q(z) > 0 \text{ for } z > 0. \end{cases} \tag{1.13}$$

Proposition 1.8. *Assume that f is of (f_M), or (f_B), or (f_C) type. Then for each $\mu > 0$, (1.13) has a unique solution $(c, q) = (c^*, q^*)$.*

We call q^* a "semi-wave" with speed c^*, since the function $v(t, x) = q^*(c^*t - x)$ satisfies

$$v_t = v_{xx} + f(v) \ (t \in \mathbb{R}^1, \ x < c^*t), \ v(t, c^*t) = 0, \ v(t, -\infty) = 1,$$

and it resembles a wave moving to the right at constant speed c^*, with front at $x = c^*t$. In comparison with the normal traveling wave generated by the solution of

$$q_{zz} - cq_z + f(q) = 0 \text{ for } z \in \mathbb{R}^1, \ q(-\infty) = 0, \ q(+\infty) = 1, \tag{1.14}$$

the generator $q^*(z)$ of $v(t, x)$ here is only defined on the half line $\{z \geq 0\}$. Hence we call it a semi-wave. We notice that at the front $x = c^*t$, we have $c^* = -\mu v_x(t, x)$, namely the Stefan condition in (1.1) is satisfied by $v(t, x)$ at $x = c^*t$.

Making use of the above semi-wave, we can prove the following result.

Theorem 1.9. *Assume that f is of (f_M), or (f_B), or (f_C) type, and spreading happens. Let c^* be given by Proposition 1.8. Then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{-g(t)}{t} = c^*,$$

and for any small $\varepsilon > 0$, there exist positive constants δ, M and T_0 such that

$$\max_{|x| \leq (c^* - \varepsilon)t} |u(t, x) - 1| \leq M e^{-\delta t} \text{ for all } t \geq T_0. \tag{1.15}$$

Remark 1.10. The asymptotic spreading speed c^* depends on the parameter μ appearing in the free boundary conditions and in (1.13). Therefore we may denote c^* by c^*_μ to stress this dependence. It is well-known (see, e.g., [3, 4]) that when f is of (f_M), or (f_B), or (f_C) type, the asymptotic spreading speed determined by the Cauchy problem (1.4) is given by the speed of certain traveling wave solutions generated by a solution of (1.14). Let us denote this speed by c_0. Then we have (see Theorem 6.2): c^*_μ is increasing in μ and

$$\lim_{\mu\to\infty} c^*_\mu = c_0.$$

Remark 1.11. It is possible to show that the Cauchy problem (1.4) is the limiting problem of (1.1) as $\mu \to \infty$. This holds in much more general situations; see section 5 of [7] for the general higher space dimension case.

The rest of the paper is organized as follows. In section 2, we present some basic results which are fundamental for this research, and may have other applications. Here we only assume that f is C^1 and $f(0) = 0$, namely (1.5) holds. The proofs of some of these results are modifications of existing ones. Firstly we give two comparison principles formulated in forms that are convenient to use in this paper. Secondly we explain how the arguments in [8] can be modified to show the uniqueness and existence result for (1.1) under (1.5). Thirdly we give the proof of Theorem 1.1. This is based on a key fact proved in Lemma 2.8, which says that the solution is rather balanced in x as it evolves with time t, though it is not symmetric in x in general. The rest of the proof largely follows the approach in [9].

In section 3, for monostable, bistable and combustion nonlinearities, we give a number of sufficient conditions for vanishing (see Theorem 3.2), through the construction of suitable upper solutions.

In section 4, we obtain sufficient conditions for spreading for the three types of nonlinearities. This is achieved by constructing suitable lower solutions based on a phase plane analysis of the equation

$$q'' - cq' + f(q) = 0$$

over a bounded interval $[0, Z]$, together with suitable conditions at the ends of this interval.

Section 5 is devoted to the proofs of Theorems 1.2, 1.3 and 1.4, with the proof of each theorem constituting a subsection. The arguments here rely heavily on the results in the previous sections. The proof of the fact mentioned in Remark 1.5, namely $\sigma^* = +\infty$ when $f(u)$ goes to $-\infty$ fast enough, is rather technical, and is given in subsection 5.4.

Proposition 1.8 and Theorem 1.9 are proved in section 6, the last section of the paper. In subsection 6.1, we prove Proposition 1.8 by revisiting the well-known traveling wave solution with speed c_0 (the minimal speed for monostable type nonlinearity, and unique speed for nonlinearity of bistable or combustion type). Our phase plane analysis is related to but different from that in [3, 4]. This alternative method leads to the desired semi-wave naturally; see Remark 6.3 for further comments. Subsection 6.2 is devoted to the proof of Theorem 1.9.

2. Some Basic Results

In this section we give some basic results which will be frequently used later in the paper. The results here are for general f which is C^1 and satisfies $f(0) = 0$.

Lemma 2.1. *Suppose that* (1.5) *holds,* $T \in (0, \infty)$, $\overline{g}, \overline{h} \in C^1([0, T])$, $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T = \{(t, x) \in \mathbb{R}^2 : 0 < t \le T, \overline{g}(t) < x < \overline{h}(t)\}$, *and*

$$\begin{cases} \overline{u}_t \ge \overline{u}_{xx} + f(\overline{u}), & 0 < t \le T,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u} = 0, \quad \overline{g}'(t) \le -\mu \overline{u}_x, & 0 < t \le T,\ x = \overline{g}(t), \\ \overline{u} = 0, \quad \overline{h}'(t) \ge -\mu \overline{u}_x, & 0 < t \le T,\ x = \overline{h}(t). \end{cases}$$

If

$$[-h_0, h_0] \subseteq [\overline{g}(0), \overline{h}(0)] \quad \textit{and} \quad u_0(x) \le \overline{u}(0, x) \ \textit{in}\ [-h_0, h_0],$$

and (u, g, h) *is a solution to* (1.1), *then*

$$g(t) \ge \overline{g}(t),\ h(t) \le \overline{h}(t) \ \textit{in}\ (0, T],$$

$$u(x, t) \le \overline{u}(x, t) \ \textit{for}\ t \in (0, T] \quad \textit{and} \quad x \in (g(t), h(t)).$$

Lemma 2.2. *Suppose that* (1.5) *holds,* $T \in (0, \infty)$, $\overline{g}, \overline{h} \in C^1([0, T])$, $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T = \{(t, x) \in \mathbb{R}^2 : 0 < t \le T, \overline{g}(t) < x < \overline{h}(t)\}$, *and*

$$\begin{cases} \overline{u}_t \ge \overline{u}_{xx} + f(\overline{u}), & 0 < t \le T,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u} \ge u, & 0 < t \le T,\ x = \overline{g}(t), \\ \overline{u} = 0, \quad \overline{h}'(t) \ge -\mu \overline{u}_x, & 0 < t \le T,\ x = \overline{h}(t), \end{cases}$$

with

$$\overline{g}(t) \ge g(t) \ \textit{in}\ [0, T], \quad h_0 \le \overline{h}(0), \quad u_0(x) \le \overline{u}(0, x) \ \textit{in}\ [\overline{g}(0), h_0],$$

where (u, g, h) *is a solution to* (1.1). *Then*

$$h(t) \le \overline{h}(t) \text{ in } (0, T], \quad u(x,t) \le \overline{u}(x,t) \text{ for } t \in (0,T] \text{ and } \overline{g}(t) < x < h(t).$$

The proof of Lemma 2.1 is identical to that of Lemma 5.7 in [8], and a minor modification of this proof yields Lemma 2.2.

Remark 2.3. The function $\overline{u}$, or the triple $(\overline{u}, \overline{g}, \overline{h})$, in Lemmas 2.1 and 2.2 is often called an upper solution to (1.1). A lower solution can be defined analogously by reversing all the inequalities. There is a symmetric version of Lemma 2.2, where the conditions on the left and right boundaries are interchanged. We also have corresponding comparison results for lower solutions in each case.

The following local existence result can be proved by the same arguments as in [8] (see Theorem 2.1 and the beginning of section 5 there).

Theorem 2.4. *Suppose that* (1.5) *holds. For any given* $u_0 \in \mathscr{X}(h_0)$ *and any* $\alpha \in (0,1)$, *there is a* $T > 0$ *such that problem* (1.1) *admits a unique solution*

$$(u, g, h) \in C^{(1+\alpha)/2, 1+\alpha}(\overline{G}_T) \times C^{1+\alpha/2}([0,T]) \times C^{1+\alpha/2}([0,T]);$$

moreover,

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}(\overline{G}_T)} + \|g\|_{C^{1+\alpha/2}([0,T])} + \|h\|_{C^{1+\alpha/2}([0,T])} \le C, \tag{2.1}$$

where $G_T = \{(t,x) \in \mathbb{R}^2 : x \in [g(t), h(t)], t \in (0,T]\}$, C *and* T *only depend on* h_0, α *and* $\|u_0\|_{C^2([-h_0,h_0])}$.

Remark 2.5. As in [8], by the Schauder estimates applied to the equivalent fixed boundary problem used in the proof, we have additional regularity for u, namely, $u \in C^{1+\alpha/2, 2+\alpha}(G_T)$.

Lemma 2.6. *Suppose that* (1.5) *holds,* (u, g, h) *is a solution to* (1.1) *defined for* $t \in [0, T_0)$ *for some* $T_0 \in (0, \infty)$, *and there exists* $C_1 > 0$ *such that*

$$u(t,x) \le C_1 \text{ for } t \in [0, T_0) \text{ and } x \in [g(t), h(t)].$$

Then there exists C_2 *depending on* C_1 *but independent of* T_0 *such that*

$$-g'(t), h'(t) \in (0, C_2] \quad \text{for } t \in (0, T_0).$$

Moreover, the solution can be extended to some interval $(0, T)$ *with* $T > T_0$.

Proof. Since f is C^1 and $f(0) = 0$, there exists $K > 0$ depending on C_1 such that $f(u) \le K$ for $u \in [0, C_1]$. We may then follow the proof of Lemma 2.2 of [8] to construct an upper solution of the form

$$w(t,x) = C_1\left[2M(h(t) - x) - M^2(h(t) - x)^2\right]$$

for some suitable $M > 0$, over the region

$$\{(t,x) : 0 < t < T_0, h(t) - M^{-1} < x < h(t)\}$$

to prove that $h'(t) \le C_2$ for $t \in (0, T_0)$. The proof for $g'(t) \ge -C_2$ is parallel.

Thus, for $t \in [0, T_0)$,

$$-g(t), h(t) \in [h_0, h_0 + C_2 t], \quad -g'(t), h'(t) \in (0, C_2].$$

We now fix $\delta_0 \in (0, T_0)$. By standard L^p estimates, the Sobolev embedding theorem, and the Hölder estimates for parabolic equations, we can find $C_3 > 0$ depending only on δ_0, T_0, C_1, and C_2 such that $||u(t,\cdot)||_{C^2([g(t),h(t)])} \le C_3$ for $t \in [\delta_0, T_0)$. It then follows from the proof of Theorem 2.4 that there exists a $\tau > 0$ depending on C_3, C_2, and C_1 such that the solution of problem (1.1) with initial time $T_0 - \tau$ can be extended uniquely to the time $T_0 + \tau$. (This is similar to the proof of Theorem 2.3 in [8].) □

The above lemma implies that the solution of (1.1) can be extended as long as u remains bounded. In particular, the free boundaries never blow up when u stays bounded. We have the following result.

Theorem 2.7. *Suppose that* (1.5) *holds. Then* (1.1) *has a unique solution defined on some maximal interval* $(0, T^*)$ *with* $T^* \in (0, \infty]$. *Moreover, when* $T^* < \infty$, *we have*

$$\lim_{t \to T^*} \max_{x \in [g(t), h(t)]} u(t, x) = \infty.$$

If we further assume that (1.6) *holds, then* $T^* = \infty$.

Proof. We only need to show that $T^* = \infty$ if (1.6) holds; the other conclusions follow directly from Theorem 2.4 and Lemma 2.6.

Comparing $u(t, x)$ with the solution of the ODE

$$v_t = f(v),\ v(0) = \|u_0\|_\infty,$$

we obtain $u(t, x) \le v(t) \le \|u_0\|_\infty e^{Kt}$, since $f(v) \le Kv$. In view of Lemma 2.6, we must have $T^* = \infty$. □

The rest of this section is devoted to the proof of Theorem 1.1. We need a lemma first.

Lemma 2.8. *Suppose that* $(u(t, x), g(t), h(t))$ *is a solution of* (1.1) *as given in Theorem 1.1. Then*

$$-2h_0 < g(t) + h(t) < 2h_0 \ \text{ for all } t > 0, \tag{2.2}$$

$$u_x(t, x) > 0 > u_x(t, y) \text{ for all } t > 0,\ x \in [g(t), -h_0] \text{ and } y \in [h_0, h(t)]. \tag{2.3}$$

Proof. By continuity, $g(t) + h(t) > -2h_0$ for all small $t > 0$. Define

$$T := \sup\{s : g(t) + h(t) > -2h_0 \text{ for all } t \in (0, s)\}.$$

We show that $T = +\infty$. Otherwise T is a positive number and

$$g(t) + h(t) > -2h_0 \text{ for } t \in (0, T),\ \ g(T) + h(T) = -2h_0.$$

Hence

$$g'(T) + h'(T) \le 0. \tag{2.4}$$

We now derive a contradiction by considering

$$w(t, x) := u(t, x) - u(t, -x - 2h_0)$$

over the region

$$G := \{(t, x) : t \in [0, T],\ g(t) \le x \le -h_0\}.$$

Since $-h_0 \le -x - 2h_0 \le -g(t) - 2h_0 \le h(t)$ when $(t, x) \in G$, w is well-defined over G and it satisfies

$$w_t = w_{xx} + c(t, x)w \text{ for } 0 < t \le T,\ g(t) < x < -h_0,$$

with some $c \in L^\infty(G)$, and

$$w(t, -h_0) = 0,\ w(t, g(t)) < 0 \text{ for } 0 < t < T.$$

Moreover,

$$w(T, g(T)) = u(T, g(T)) - u(T, -g(T) - 2h_0) = u(T, g(T)) - u(T, h(T)) = 0.$$

Applying the strong maximum principle and the Hopf lemma, we deduce

$$w(t,x) < 0 \text{ for } 0 < t \le T \text{ and } g(t) < x < -h_0, \text{ and } w_x(T, g(T)) < 0.$$

But

$$w_x(T, g(T)) = u_x(T, g(T)) + u_x(T, h(T)) = -[g'(T) + h'(T)]/\mu.$$

Thus we have

$$g'(T) + h'(T) > 0,$$

a contradiction to (2.4). This proves that $g(t) + h(t) > -2h_0$ for all $t > 0$. We can similarly prove $g(t) + h(t) < 2h_0$ by considering

$$v(t,x) := u(t,x) - u(t, 2h_0 - x) \text{ over } \{(t,x) : t > 0, h_0 \le x \le h(t)\}.$$

With (2.2) proven, it is now easy to prove (2.3). For any fixed $\ell \in (g_\infty, -h_0]$, we can find a unique $T \ge 0$ such that $g(T) = \ell$. We now consider

$$z(t,x) := u(t,x) - u(t, 2\ell - x)$$

over $G_\ell := \{(t,x) : t > T, g(t) < x < \ell\}$. We have

$$z_t = z_{xx} + c(t,x)z \text{ in } G_\ell,$$

$$z(t, g(t)) < 0 \text{ and } z(t,\ell) = 0 \text{ for } t > T.$$

Hence we can apply the strong maximum principle and the Hopf lemma to deduce

$$z(t,x) < 0 \text{ in } G_\ell,\ z_x(t,\ell) > 0 \text{ for } t > T.$$

Since

$$z_x(t,\ell) = 2u_x(t,\ell),$$

we thus have

$$u_x(t, g(T)) > 0 \text{ for } t > T.$$

Now for any $t > 0$ and $x \in (g(t), -h_0]$, we can find a unique $T \in [0, t)$ such that $x = g(T)$. Hence $u_x(t,x) > 0$. This inequality is also true for $x = g(t)$, which is a consequence of the Hopf lemma applied directly to (1.1).

The proof for the other inequality in (2.3) is similar. □

Proof of Theorem 1.1: We will make use of Lemma 2.8 and then follow the ideas of [9] with suitable variations.

Let (u, g, h) be as given in Theorem 1.1. Then in view of Lemma 2.8, $I_\infty := (g_\infty, h_\infty)$ is either a finite interval or $\mathbb{R}^1$. Denote by $\omega(u)$ the ω-limit set of $u(t,\cdot)$ in the topology of $L^\infty_{loc}(I_\infty)$. Thus a function $w(x)$ belongs to $\omega(u)$ if and only if there exists a sequence $0 < t_1 < t_2 < t_3 < \cdots \to \infty$ such that

$$\lim_{n\to\infty} u(t_n, x) = w(x) \quad \text{locally uniformly in } I_\infty. \tag{2.5}$$

By local parabolic estimates, we see that the convergence (2.5) implies convergence in the $C^2_{loc}(I_\infty)$ topology. Thus the definition of $\omega(u)$ remains unchanged if the topology of $L^\infty_{loc}(I_\infty)$ is replaced by that of $C^2_{loc}(I_\infty)$.

It is well-known that $\omega(u)$ is compact and connected, and it is an invariant set. This means that for any $w \in \omega(u)$ there exists an entire orbit (namely a solution of $W_t = W_{xx} + f(W)$ defined for all $t \in \mathbb{R}^1$ and $x \in I_\infty$) passing through w. Choosing a suitable sequence $0 < t_1 < t_2 < t_3 < \cdots \to \infty$, we can find such an entire solution $W(t,x)$ with $W(0,x) = w(x)$ as follows:

$$u(t + t_n, x) \to W(t,x) \quad \text{as } n \to \infty. \tag{2.6}$$

Here the convergence is understood in the L^∞_{loc} sense in $(t,x) \in \mathbb{R}^1 \times I_\infty$, but, by parabolic regularity, it takes place in the $C^{1,2}_{loc}(\mathbb{R}^1 \times I_\infty)$ sense.

For clarity we divide the arguments below into four parts, each proving a specific claim.

Claim 1: $\omega(u)$ consists of solutions of

$$v_{xx} + f(v) = 0, \quad x \in I_\infty. \tag{2.7}$$

Let $w(x)$ be an arbitrary element of $\omega(u)$ and $W(t,x)$ the entire orbit satisfying $W(0,x) = w(x)$. Since W is a nonnegative solution of

$$W_t = W_{xx} + f(W), \quad t \in \mathbb{R}^1,\ x \in I_\infty,$$

and $f(0) = 0$, by the strong maximum principle we have either $W(t,x) > 0$ for all $t \in \mathbb{R}^1$ and $x \in I_\infty$, or $W \equiv 0$. (Note that if I_∞ is a finite interval, then it can be shown that $W(t, g_\infty) = W(t, h_\infty) = 0$ for all $t \in \mathbb{R}^1$.) In the latter case we have $w \equiv 0$, which is a solution to (2.7). In what follows we assume the former, namely $w > 0$.

By Lemma 2.8, we see that $w'(x) \geq 0$ for $x \in (g_\infty, -h_0]$ and $w'(x) \leq 0$ for $x \in [h_0, h_\infty)$. Thus there exists $x_0 \in (-h_0, h_0)$ such that $w'(x_0) = 0$, $w(x_0) = \|w\|_\infty > 0$.

Let $v(x)$ be the solution of the following initial value problem:

$$v'' + f(v) = 0, \qquad v(x_0) = w(x_0), \quad v'(x_0) = 0.$$

Then v is symmetric about $x = x_0$. Since $w(x_0) > 0$, v is either a positive solution of (2.7) in $\mathbb{R}^1$ or a solution of (2.7) with compact positive support, namely there exists $R_0 > 0$ such that

$$v(x) > 0 \text{ in } (x_0 - R_0, x_0 + R_0),\ v(x_0 \pm R_0) = 0 \text{ or } v(x_0 \pm R_0) = \infty.$$

We may now follow the argument in the proof of Lemma 3.4 in [9] (with obvious minor variations) to conclude that $w \equiv v$. This proves Claim 1.

Claim 2: If I_∞ is a finite interval, then $\omega(u) = \{0\}$.

Otherwise by Claim 1, $\omega(u)$ contains a nontrivial nonnegative solution v of the problem

$$v_{xx} + f(v) = 0 \text{ in } I_\infty,\ v(g_\infty) = v(h_\infty) = 0.$$

Due to $f(0) = 0$, by the strong maximum principle and the Hopf lemma, we have $v > 0$ in I_∞ and $v'(g_\infty) > 0 > v'(h_\infty)$. By definition, along a sequence $t_n \to +\infty$, $u(t_n, x) \to v(x)$ in $C^1_{loc}(I_\infty)$. We claim that there exists $\alpha > 0$ so that, by passing to a subsequence, $\|u(t_n,\cdot) - v(\cdot)\|_{C^{1+\alpha}([g(t_n),h(t_n)])} \to 0$ as $n \to \infty$. Indeed, if we make a change of the variable x to reduce $[g(t), h(t)]$ to the fixed finite interval $[-h_0, h_0]$ as in the proof of Theorem 2.1 of [8], so that the solution $u(t,x)$ is changed to $\tilde{u}(t,x)$, and $v(x)$ is changed to $\tilde{v}(x)$. Then we can apply the L^p estimates (and Sobolev embeddings) on the reduced equation with Dirichlet boundary conditions to conclude that $\tilde{u}(t+\cdot,\cdot)$ has a common bound in $C^{\frac{1+\nu}{2},1+\nu}([0,1]\times[-h_0,h_0])$ for all $t \geq 1$, say

$$\|\tilde{u}(t+\cdot,\cdot)\|_{C^{\frac{1+\nu}{2},1+\nu}([0,1]\times[-h_0,h_0])} \leq C_0 \ \ \forall t \geq 1. \tag{2.8}$$

Hence by extraction of a subsequence we may assume that $\tilde{u}(t_n,x) \to V(x)$ in $C^{1+\frac{\nu}{2}}([-h_0,h_0])$. But from $u(t_n,x) \to v(x)$ we know that $\tilde{u}(t_n,x) \to \tilde{v}(x)$. Thus we necessarily have $V(x) \equiv \tilde{v}(x)$, and thus $\|u(t_n,\cdot) - v(\cdot)\|_{C^{1+\frac{\nu}{2}}([g(t_n),h(t_n)])} \to 0$.

It follows that

$$h'(t_n) = -\mu u_x(t_n, h(t_n)) \to -\mu v'(h_\infty) > 0 \text{ as } n \to \infty.$$

Hence for all large n, say $n \geq n_0$,

$$h'(t_n) \geq \delta := -\mu v'(h_\infty)/2 > 0.$$

On the other hand, from (2.8), we also deduce that

$$\|u(t+\cdot,\cdot)\|_{C^{\frac{1+\nu}{2},1+\nu}(Q_t])} \leq C_1 \ \ \forall t \geq 1,$$

with $Q_t := \{(s,x) : s \in [0,1], g(t+s) \leq x \leq h(t+s)\}$. It follows that $h'(t) = -\mu u_x(t,h(t))$ is uniformly continuous in t for $t \geq 1$. Therefore $h'(t) \geq \delta/2$ for $t \in [t_n, t_n+\epsilon]$ and $n \geq n_0$ for some $\epsilon > 0$ sufficiently small but independent of n (we may assume without loss of generality that $t_{n+1} - t_n \geq 1$ for all n). Since $h'(t) > 0$ for all $t > 0$, we thus have

$$h_\infty \geq h_0 + \Sigma_{n=n_0}^{\infty} \int_{t_n}^{t_n+\epsilon} h'(t)dt = +\infty,$$

a contradiction to the assumption that I_∞ is a finite interval. The proof of Claim 2 is now complete.

Claim 3: If $I_\infty = \mathbb{R}^1$, then $\omega(u)$ is either a constant or $\omega(u) = \{v(\cdot+\alpha) : \alpha \in [\alpha_1, \alpha_2]\}$ for some interval $[\alpha_1,\alpha_2] \subset [-h_0, h_0]$, where v is an evenly decreasing positive solution of (2.7).

In view of Lemma 2.8, we only need to consider the case that $I_\infty = \mathbb{R}^1$ and $\omega(u)$ is not a singleton. Then since $\omega(u)$ is connected and compact in the topology of $C^2_{\rm loc}(\mathbb{R}^1)$, and every function $w(x)$ in $\omega(u)$ achieves its maximum at some $x_0 \in [-h_0, h_0]$, we find that there exist $0 \leq \gamma^- \leq \gamma^+$ such that $\omega(u)$ consists of solutions $v_{\alpha,\beta}$ ($\beta \in [\gamma^-, \gamma^+], \alpha \in [\alpha_1^\beta, \alpha_2^\beta]$) of (2.7) satisfying

$$v_{\alpha,\beta}(x) = v_\beta(x+\alpha) \ \ (\alpha \in [\alpha_1^\beta, \alpha_2^\beta]),$$
$$\|v_\beta\|_\infty = v_\beta(0) = \beta, \ v_\beta'(0) = 0, \ [\alpha_1^\beta, \alpha_2^\beta] \subset [-h_0, h_0] \ \ (\beta \in [\gamma^-, \gamma^+]).$$

Thus each $v_{\alpha,\beta}$ is either a constant or a symmetrically decreasing solution of (2.7). If $\gamma^- < \gamma^+$, then we may use v_β ($\beta \in [\gamma^-, \gamma^+]$) to deduce a contradiction in the same way as in section 3.3 of [9]. Thus $\gamma^- = \gamma^+$. Let $V_0(x)$ be the unique solution of (2.7) satisfying

$$V(0) = \gamma^-, \ V'(0) = 0.$$

If V_0 is a constant, then clearly $\omega(u) = \{V_0\}$. Otherwise V_0 is an evenly decreasing positive solution of (2.7), and $\omega(u) = \{V_0(\cdot+\alpha) : \alpha \in [\alpha_1, \alpha_2]\}$, $[\alpha_1, \alpha_2] \subset [-h_0, h_0]$.

Claim 4: If $\omega(u) = \{V_0(\cdot+\alpha) : \alpha \in [\alpha_1, \alpha_2]\}$ for some interval $[\alpha_1, \alpha_2] \subset [-h_0, h_0]$, then there exists a continuous function $\gamma : [0,\infty) \to [-h_0, h_0]$ such that

$$u(t,x) - V_0(x+\gamma(t)) \to 0 \text{ as } t \to \infty \text{ locally uniformly in } \mathbb{R}^1.$$

Write $w(t,x) = u_x(t,x)$. Then

$$w_t = w_{xx} + f'(u(t,x))w \text{ for } t > 0, \ x \in (g(t), h(t)),$$

and $w(t, g(t)) > 0$, $w(t, h(t)) < 0$ for all $t > 0$. Therefore by the zero number result of [2], for all large t, say $t \geq T$, $w(t,x)$ has a fixed finite number of zeros, all nondegenerate. Denote them by

$$x_1(t) < x_2(t) < ... < x_m(t) \ (m \geq 1).$$

Then each $x_i(t)$ is a continuous function of t. Due to Lemma 2.8, we must have $-h_0 \leq x_i(t) \leq h_0$ for $i = 1, ..., m$ and $t \geq T$. We show that $m = 1$. For fixed $\alpha \in [\alpha_1, \alpha_2] \subset [-h_0, h_0]$, since $V_0(\cdot+\alpha) \in \omega(u)$, there exists $t_n \to \infty$ such that $u(t_n, x) \to V_0(x+\alpha)$ in $C^2_{loc}(\mathbb{R}^1)$. Since $V_0'(x+\alpha)$ has a unique nondegenerate zero $x = -\alpha \in [-h_0, h_0]$, we find that for all large n, $w(t_n, x) = u_x(t_n, x)$ has in $[-2h_0, 2h_0]$ a unique nondegenerate zero α_n near $-\alpha$. By Lemma 2.8, we necessarily have $\alpha_n \in (-h_0, h_0)$. On the other hand, we know that $x_1(t_n), ..., x_m(t_n)$ are all the zeros of $w(t_n, x)$ in $[-h_0, h_0]$. Thus we must have $m = 1$ and $x_1(t_n) = \alpha_n$. This proves $m = 1$.

Define $\gamma(t) = -x_1(t)$ for $t \geq T$, and extend $\gamma(t)$ to a continuous function for $t \in [0, T]$ such that $\gamma(t) \in [-h_0, h_0]$ for all t. We prove that

$$u(t, x) - V_0(x + \gamma(t)) \to 0 \text{ as } t \to \infty \text{ locally uniformly in } \mathbb{R}^1.$$

Otherwise we can find $t_n \to \infty$, a bounded sequence $\{x_n\} \subset \mathbb{R}^1$ and some $\epsilon_0 > 0$ such that for all $n \geq 1$,

$$|u(t_n, x_n) - V_0(x_n + \gamma(t_n))| \geq \epsilon_0.$$

By passing to a subsequence of t_n, still denoted by itself, we may assume $u(t_n, \cdot) \to V_0(\cdot + \alpha)$ in $C^2_{loc}(\mathbb{R}^1)$ for some $\alpha \in [\alpha_1, \alpha_2]$. Hence $w(t_n, \cdot) \to V_0'(\cdot + \alpha)$ in $C^1_{loc}(\mathbb{R}^1)$. This implies that $\gamma(t_n) = -x_1(t_n) \to \alpha$, and thus, due to the boundedness of $\{x_n\}$, we have

$$V_0(x_n + \alpha) - V_0(x_n + \gamma(t_n)) \to 0 \text{ as } n \to \infty.$$

It follows that

$$\begin{aligned} \epsilon_0 &\leq |u(t_n, x_n) - V_0(x_n + \gamma(t_n))| \\ &\leq |u(t_n, x_n) - V_0(x_n + \alpha)| + |V_0(x_n + \alpha) - V_0(x_n + \gamma(t_n))| \to 0 \end{aligned}$$

as $n \to \infty$. This contradiction proves our claim.

The proof of Theorem 1.1 is now complete. □

3. Conditions for vanishing

In this section we prove some sufficient conditions that imply vanishing ($u \to 0$). The following upper bound is an easy consequence of the standard comparison principle.

Lemma 3.1. *Assume that f satisfies* (1.5) *and* (1.6). *Then, for any $h_0 > 0$ and any $\phi \in \mathscr{X}(h_0)$,*

$$u(t, x; \phi) \leq \frac{e^{Kt}}{2\sqrt{\pi t}} \int_{-h_0}^{h_0} \phi(x)dx \quad \text{for } \; g(t) \leq x \leq h(t), \; t > 0. \tag{3.1}$$

Proof. Consider the Cauchy problem

$$\begin{cases} w_t = w_{xx} + Kw, & x \in \mathbb{R}^1, \; t > 0, \\ w(0, x) = \Phi(x), & x \in \mathbb{R}^1, \end{cases} \tag{3.2}$$

where

$$\Phi(x) = \begin{cases} \phi(x), & x \in (-h_0, h_0), \\ 0, & x \notin (-h_0, h_0). \end{cases}$$

Then from the expression of w by the fundamental solution we obtain

$$w(t, x) = \frac{e^{Kt}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-\xi)^2}{4t}} w(0, \xi)d\xi \leq \frac{e^{Kt}}{2\sqrt{\pi t}} \int_{-h_0}^{h_0} \phi(\xi)d\xi.$$

By the standard comparison theorem, we have $u(t, x; \phi) \leq w(t, x)$ for $t > 0$ and $x \in [g(t), h(t)]$, and the required inequality follows. □

Theorem 3.2. *Let $h_0 > 0$ and $\phi \in \mathscr{X}(h_0)$. Then $I_\infty := (g_\infty, h_\infty)$ is a finite interval and $\lim_{t\to\infty} \|u(t, \cdot; \phi)\|_{L^\infty([g(t), h(t)])} = 0$ if one of the following conditions holds:*

(i) *f is of* (f_M) *type, $h_0 < \pi/(2\sqrt{f'(0)})$ and $\|\phi\|_{L^\infty}$ is sufficiently small;*
(ii) *f is of* (f_B) *or* (f_C) *type, and $\|\phi\|_{L^\infty} \leq \theta$;*
(iii) *f is of* (f_B) *or* (f_C) *type, and for K in* (1.6),

$$\int_{-h_0}^{h_0} \phi(x)dx \leq \theta \cdot \sqrt{\frac{2\pi}{eK}}. \tag{3.3}$$

Proof. (i) Since $h_0 < \pi/(2\sqrt{f'(0)})$, there exists a small $\delta > 0$ such that

$$\frac{\pi^2}{4(1+\delta)^2 h_0^2} - f'(0) \geq 2\delta. \tag{3.4}$$

Moreover, there exists an $s > 0$ small such that

$$\pi\mu s \leq \delta^2 h_0^2, \qquad f(u) \leq (f'(0)+\delta)u \quad \text{for } u \in [0, s].$$

Set

$$k(t) := h_0\Big(1 + \delta - \frac{\delta}{2}e^{-\delta t}\Big) \quad \text{and} \quad w(t,x) := se^{-\delta t}\cos\Big(\frac{\pi x}{2k(t)}\Big).$$

Clearly $w(t, -k(t)) = w(t, k(t)) = 0$. A direct calculation shows that, for $t > 0$ and $x \in [-k(t), k(t)]$,

$$w_t - w_{xx} - f(w) \geq \left(-\delta + \frac{\pi^2}{4k^2(t)} - f'(0) - \delta\right) w \geq 0.$$

On the other hand, by the choice of s we have

$$\mu w_x(t, -k(t)) = -\mu w_x(t, k(t)) = \frac{\pi\mu s}{2k(t)}e^{-\delta t} \leq \frac{\pi\mu s}{2h_0}e^{-\delta t} \leq \frac{\delta^2 h_0}{2}e^{-\delta t} = k'(t).$$

Therefore, $(w(t,x), -k(t), k(t))$ will be an upper solution of (1.1) if $w(0,x) \geq \phi(x)$ in $[-h_0, h_0]$.

Choose $\sigma_1 := s\cos\frac{\pi}{2+\delta}$, which depends only on μ, h_0 and f. Then when $\|\phi\|_{L^\infty} \leq \sigma_1$ we have $\phi(x) \leq \sigma_1 \leq w(0,x)$ in $[-h_0, h_0]$, since $h_0 < k(0) = h_0(1+\frac{\delta}{2})$. By Lemma 2.1 we have

$$h(t) \leq k(t) \leq h_0(1+\delta), \; h_\infty < \infty.$$

Hence I_∞ is a finite interval and by Theorem 1.1, $u \to 0$ as $t \to \infty$ locally uniformly in I_∞. In view of Lemma 2.8, this implies that $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty([g(t),h(t)])} = 0$.

(ii) (The (f_B) case) Since $u \equiv \theta$ is a stationary solution, by the strong comparison principle, there exist $\eta_1 \in (0, \theta)$ and $t_1 > 0$ such that

$$u(t_1, x; \phi) \leq \eta_1 \quad \text{for } x \in [g_1, h_1] := [g(t_1), h(t_1)].$$

Since f is of (f_B) type, there exists $M = M(\eta_1) > 0$ such that

$$f(u) \leq -Mu \quad \text{for } 0 \leq u \leq \eta_1.$$

It follows that $u(t, x; \phi) \leq \eta(t) := \eta_1 e^{-M(t-t_1)}$ for $t \geq t_1$. Choose $\rho > h_1$ such that $2M\rho^2 > \pi\mu\eta_1 e^{Mt_1}$, and then choose $0 < \delta < \min\{\frac{\rho}{2}, h_1\}$ small such that

$$u(t_1, x) < \frac{\sqrt{2}}{2}\eta_1 \quad \text{for } x \in [g_1, g_1+\delta] \cup [h_1 - \delta, h_1]. \tag{3.5}$$

For $t \geq t_1$ we define

$$\sigma(t) := \rho(2 - e^{-Mt}) \quad \text{and} \quad k(t) := h_1 - \delta + \sigma(t),$$

(so $\rho \leq \sigma(t) \leq 2\rho$, $k(t_1) > h_1$), and

$$w(t,x) := \eta(t)\cos\left[\frac{\pi(x - h_1 + \delta)}{2\sigma(t)}\right] \quad \text{for } h_1 - \delta \leq x \leq k(t), \; t \geq t_1.$$

Then, for $h_1 - \delta \leq x \leq k(t)$, $t \geq t_1$, we have

$$w_t - w_{xx} + Mw = \frac{\pi^2 w}{4\sigma^2(t)} + \eta(t)\sin\left[\frac{\pi(x-h_1+\delta)}{2\sigma(t)}\right] \cdot \frac{\pi(x-h_1+\delta)\sigma'(t)}{2\sigma^2(t)} > 0.$$

$$w(t, k(t)) = 0 \text{ and } -\mu w_x(t, k(t)) \leq \frac{\pi\mu\eta(t)}{2\rho} \leq M\rho e^{-Mt} = k'(t)$$

by the choice of ρ. Moreover, $w(t, h_1 - \delta) = \eta(t) \geq u(t, h_1 - \delta)$, and by (3.5),

$$u(t_1, x) < \frac{\sqrt{2}}{2}\eta_1 \leq w(t_1, x) \quad \text{for } h_1 - \delta \leq x \leq h_1.$$

Hence $(w(t,x), h_1 - \delta, k(t))$ is an upper solution of (1.1) for $t > t_1$ in the sense of Lemma 2.2. By the conclusion of this lemma we have $h(t) \leq k(t)$, and hence

$$h_\infty \leq \lim_{t\to\infty} k(t) = h_1 - \delta + 2\rho < \infty.$$

The rest of the proof is the same as in (i).

(ii) (The (f_C) case) In this case $u \equiv \theta$ is again a stationary solution, and by the standard comparison principle we have $u(t, x; \phi) \leq \theta$ for all $t \geq 0$. Therefore, the equation we are dealing with reduces to the heat equation $u_t = u_{xx}$. As in the proof of Lemma 3.1 we have

$$u(t, x; \phi) \leq \frac{1}{2\sqrt{\pi t}} \int_{-h_0}^{h_0} \phi(\xi) d\xi \leq \frac{\theta h_0}{\sqrt{\pi t}} \quad \text{for } g(t) \leq x \leq h(t),\ t > 0.$$

Therefore, we can find a large $t_2 > 0$ such that

$$\max_{g(t_2)\leq x\leq h(t_2)} u(t_2, x; \phi) \leq \eta_2 := \frac{1}{2}\cdot \min\Big\{\theta, \frac{\pi}{8\mu}\Big\}. \tag{3.6}$$

Take $h_2 > \max\{-g(t_2), h(t_2)\}$ such that

$$u(t_2, x; \phi) < 2\eta_2 \cos\Big(\frac{\pi x}{2h_2}\Big) \text{ for } x \in [g(t_2), h(t_2)]. \tag{3.7}$$

For this h_2 we set $\omega := \pi/(4h_2)$ and define, for $t \geq 0$,

$$k(t) := h_2(2 - e^{-\omega^2 t}), \quad w(t, x) := 2\eta_2 \cos\Big(\frac{\pi x}{2k(t)}\Big) e^{-\omega^2 t}.$$

Then, $h_2 \leq k(t) \leq 2h_2$, and for $t \geq 0$ and $-k(t) \leq x \leq k(t)$ we have

$$w_t - w_{xx} \geq \left(\frac{\pi^2}{4[k(t)]^2} - \omega^2\right) w \geq 0$$

and, by the choice of η_2,

$$k'(t) - \mu w_x(t, -k(t)) = k'(t) + \mu w_x(t, k(t)) = e^{-\omega^2 t}\left[\frac{\pi^2}{16h_2} - \frac{\pi\mu\eta_2}{k(t)}\right] \geq 0.$$

Hence $(w(t,x), -k(t), k(t))$ is an upper solution of (1.1) for $t > t_2$. It follows that $h(t + t_2) \leq k(t) < 2h_2$ for $t \geq 0$. This implies that $h_\infty < \infty$ and the rest is as before.

(iii) By (3.1), we have

$$u\Big(\frac{1}{2K}, x; \phi\Big) \leq \sqrt{\frac{eK}{2\pi}} \int_{-h_0}^{h_0} \phi(x) dx \leq \theta \quad \text{for } g\Big(\frac{1}{2K}\Big) \leq x \leq h\Big(\frac{1}{2K}\Big).$$

Then the conclusion follows from (ii). This proves the theorem. □

From Theorem 3.2 (ii), we immediately obtain

Corollary 3.3. *If f is of (f_B) or of (f_C) type, then*

$$\lim_{t\to\infty} \|u(t, \cdot)\|_{L^\infty([g(t),h(t)])} = 0$$

implies that (g_∞, h_∞) is a finite interval.

If f is of (f_M) type, this conclusion is also true. In fact, a much stronger version holds, namely

$$h_\infty - g_\infty \le \pi/\sqrt{f'(0)}.$$

This will follow from Theorem 3.2 (i) and Corollary 4.5 in the next section; see Corollary 4.6.

4. Waves of finite length and conditions for spreading

In this section, f is always assumed to be of (f_M), or (f_B), or (f_C) type. In order to obtain sufficient conditions guaranteeing spreading ($u \to 1$), we will construct suitable lower solutions to (1.1) through "waves of finite length", obtained by a phase plane analysis of the equation

$$q'' - cq' + f(q) = 0.$$

4.1. **Waves of finite length.** For $Z \in (0, \infty)$, we look for a pair $(c, q(z))$ satisfying

$$\begin{cases} q'' - cq' + f(q) = 0, & z \in [0, Z], \\ q(0) = 0, \quad q'(Z) = 0, & q(z) > 0 \text{ in } (0, Z]. \end{cases} \tag{4.1}$$

We call such a $q(z)$ a "wave of length Z with speed c", since $w(t, x) := q(ct - x)$ satisfies

$$\begin{cases} w_t = w_{xx} + f(w) \text{ for } t \in \mathbb{R}^1,\ x \in (ct - Z, ct), \\ w_x(t, ct - Z) = 0,\ w(t, ct) = 0. \end{cases}$$

Such w will be used to construct lower solutions to (1.1). We will mainly consider waves of speed $c = 0$ (stationary waves) and of speed $c > 0$ small.

Using q' to denote $\frac{dq}{dz}$, we can rewrite the first equation in (4.1) into the equivalent form

$$\begin{cases} q' = p, \\ p' = cp - f(q), \end{cases} \tag{4.2}$$

or,

$$\frac{dp}{dq} = c - \frac{f(q)}{p} \quad \text{when } p \ne 0. \tag{4.3}$$

For each $c \ge 0$ and $\omega > 0$, we use $p^c(q; \omega)$ to denote the unique solution of (4.3) with initial condition $p(q)|_{q=0} = \omega$. Such a solution is well-defined as long as it stays positive.

In the case $c = 0$, the positive solution of (4.3) with $p(q)|_{q=0} = \omega$ is given explicitly by

$$p^0(q; \omega) = \sqrt{\omega^2 - 2\int_0^q f(s)ds} \quad \text{for } q \in [0, q^\omega), \tag{4.4}$$

where q^ω is given by

$$\omega^2 = 2\int_0^{q^\omega} f(s)ds.$$

It follows that $q^\omega < 1$ if and only if $0 < \omega < \omega^0$, where

$$\omega^0 := \sqrt{2\int_0^1 f(s)ds}.$$

Moreover, it is easily seen that q^ω is strictly increasing in $\omega \in (0, \omega^0)$, and as $\omega \searrow 0$, $q^\omega \searrow 0$ in the (f_M) case, $q^\omega \searrow \bar\theta \in (\theta, 1)$ in the (f_B) case, where $\bar\theta \in (\theta, 1)$ is determined by $\int_0^{\bar\theta} f(s)ds = 0$, and $q^\omega \searrow \theta$ in the (f_C) case.

The positive solution $p^0(q; \omega)$ ($q \in [0, q^\omega)$) corresponds to a trajectory $(q_0(z), p_0(z))$ of (4.2) (with $c = 0$) that connects $(0, \omega)$ and $(q^\omega, 0)$ in the qp-plane. We may assume that it passes through $(0,\omega)$ at $z=0$ and approaches $(q^\omega,0)$ as z goes to $z^\omega\in(0,+\infty]$. Then using (4.2) with $c=0$ and (4.4) we easily deduce

$$z=\int_0^{q_0(z)}\frac{dr}{\sqrt{\omega^2-2\int_0^r f(s)ds}}=\int_0^{q_0(z)}\frac{dr}{\sqrt{2\int_r^{q^\omega}f(s)ds}}. \tag{4.5}$$

Therefore

$$z^\omega=\int_0^{q^\omega}\frac{dr}{\sqrt{\omega^2-2\int_0^r f(s)ds}}=\int_0^{q^\omega}\frac{dr}{\sqrt{2\int_r^{q^\omega}f(s)ds}}<+\infty$$

for $0<\omega<\omega^0$. We now introduce the function

$$Z(q)=\int_0^q\frac{dr}{\sqrt{2\int_r^q f(s)ds}}. \tag{4.6}$$

In the (f_M) case, define

$$Z'_M:=\inf_{0<\omega<\omega^0}z^\omega=\inf_{0<q<1}Z(q); \tag{4.7}$$

in the (f_B) case, set

$$Z_B:=\inf_{0<\omega<\omega^0}z^\omega=\inf_{\bar\theta<q<1}Z(q); \tag{4.8}$$

and in the (f_C) case, define

$$Z_C:=\inf_{0<\omega<\omega^0}z^\omega=\inf_{\theta<q<1}Z(q). \tag{4.9}$$

In the (f_M) case, as $\omega\searrow 0$, we have $q^\omega\searrow 0$ and so

$$z^\omega=\int_0^{q^\omega}\frac{(1+o(1))\;dr}{\sqrt{f'(0)}\sqrt{(q^\omega)^2-r^2}}=\frac{\pi}{2\sqrt{f'(0)}}+o(1).$$

This implies that

$$Z'_M\le Z_M:=\pi/(2\sqrt{f'(0)}). \tag{4.10}$$

It is easily seen that Z'_M, Z_B and Z_C are all positive.

As a first application of the above analysis, we have the following result.

Lemma 4.1. *If f is of (f_M) type and $Z>Z'_M$, or of (f_B) type and $Z\ge Z_B$, or of (f_C) type and $Z\ge Z_C$, then the elliptic boundary value problem*

$$v_{xx}+f(v)=0 \text{ in } (-Z,Z),\ v(-Z)=v(Z)=0 \tag{4.11}$$

has at least one positive solution v_Z. Moreover, any positive solution v_Z of (4.11) satisfies $\|v_Z\|_\infty<1$; in addition, $\|v_Z\|_\infty>\bar\theta$ if f is of (f_B) type, and $\|v_Z\|_\infty>\theta$ if f is of (f_C) type.

Proof. We only consider the case that f is of (f_B) type; the proofs of the other cases are similar.

Let $Z>Z_B$. Then from the definition of Z_B we can find $\omega_*\in(0,\omega_0)$ and correspondingly $q_*:=q^{\omega_*}\in(\bar\theta,1)$ such that $z_*:=z^{\omega_*}=Z(q_*)\in(Z_B,Z)$. Let $(q(z),p(z))$ be the trajectory of (4.2) (with $c=0$) that passes through $(0,\omega_*)$ at $z=0$ and approaches $(q_*,0)$ as z goes to z_*. Then $q(z)$ satisfies

$$q''+f(q)=0 \text{ in } (0,z_*),\ q(0)=0,\ q'(z_*)=0.$$

If we define

$$\underline{v}(x) := \begin{cases} q(x+z_*), & x \in [-z_*, 0], \\ q(-x+z_*), & x \in [0, z_*], \\ 0, & x \in [-Z, Z]\backslash[-z_*, z_*]. \end{cases}$$

Then one easily checks that $\underline{v}$ is a (weak) lower solution of (4.11). Clearly any constant $C \geq 1$ is an upper solution of (4.11). Therefore we can use the standard upper and lower solution argument to conclude that (4.11) has a maximal positive solution $\hat{v}_Z$, and $\underline{v}(x) < \hat{v}_Z(x) < 1$ in $(-Z, Z)$.

We now prove that (4.11) also has a positive solution for $Z = Z_B$. Let Z_n be a sequence decreasing to Z_B and v_n a positive solution of (4.11) with $Z = Z_n$. Setting $V_n := v_n(Z_n x)$ we find that V_n is a positive solution of

$$V'' + Z_n^2 f(V) = 0 \text{ in } (-1, 1),\ V(-1) = V(1) = 0.$$

Since $Z_n^2 f(V_n)$ is a bounded sequence in $L^\infty([-1, 1])$ it follows from standard regularity theory that by passing to a subsequence, $V_n \to V^*$ in $C^1([0, 1])$ and V^* is a weak (and hence classical) nonnegative solution of

$$V'' + Z_B^2 f(V) = 0 \text{ in } (-1, 1),\ V(-1) = V(1) = 0.$$

We claim that $V^* \not\equiv 0$. Arguing indirectly we assume that $V^* \equiv 0$, and let $\hat{V}_n := V_n/\|V_n\|_\infty$. Then

$$\hat{V}_n'' + c_n(x)\hat{V}_n = 0 \text{ in } (-1, 1),\ \hat{V}_n(-1) = \hat{V}_n(1) = 0,$$

with $c_n = Z_n^2 f(V_n)/V_n$ a bounded sequence in $L^\infty([-1, 1])$. As before, by standard elliptic regularity we have $\hat{V}_n \to \hat{V}$ in $C^1([-1, 1])$ subject to a subsequence. Moreover, since $c_n \to Z_B^2 f'(0)$, we deduce

$$\hat{V}'' + Z_B^2 f'(0)\hat{V} = 0 \text{ in } (-1, 1),\ \hat{V}(-1) = \hat{V}(1) = 0. \tag{4.12}$$

Since $\|\hat{V}\|_\infty = 1$ and $\hat{V} \geq 0$, by the strong maximum principle we conclude that $\hat{V}$ must be a positive solution of (4.12). This implies that $Z_B^2 f'(0)$ is the first eigenvalue of $(-\frac{d^2}{dx^2})$ over $(-1, 1)$ with Dirichlet boundary conditions, and hence must be positive. But this is a contradiction to $f'(0) < 0$. Thus $V^* \not\equiv 0$. By the strong maximum principle we see that it is a positive solution of (4.11) with $Z = Z_B$.

Now let v_Z be any positive solution of (4.11). Then clearly $(q(z), p(z)) := (v_Z(Z-z), -v_Z'(Z-z))$ is a trajectory for (4.2) (with $c = 0$) passing throw $(0, \omega)$ at $z = 0$ and approaching $(q^\omega, 0)$ as z goes to Z, where $\omega := -v_Z'(Z)$ and $q^\omega := v_Z(0) < 1$. Since q^ω is strictly increasing and q^ω decreases to $\overline{\theta}$ as ω decreases to 0, we find that $v_Z(0) > \overline{\theta}$. □

Next we consider (4.2) and (4.3) for small $c > 0$ as a perturbation of the case $c = 0$. It is easily seen that for small $c > 0$, (4.3) with initial data $p^c(q)|_{q=0} = \omega \in (0, \omega^0)$ has a solution $p^c(q; \omega)$ define on $q \in [0, q^{c,\omega}]$ for some $q^{c,\omega} > q^\omega$, and $p^c(q^{c,\omega}; \omega) = 0$. As before this solution corresponds to a trajectory $(q_c(z; \omega), p_c(z; \omega))$ that passes through $(0, \omega)$ at $z = 0$, and approaches $(q^{c,\omega}, 0)$ as z goes to some $z^{c,\omega} > 0$. Moreover, an elementary analysis yields the following result.

Lemma 4.2. *For any fixed $\omega \in (0, \omega_0)$ and any small $\varepsilon > 0$, there exists $\delta > 0$ small such that, if $c \in (0, \delta)$, then $q^{c,\omega} \in (q^\omega, q^\omega + \varepsilon)$, and*

$$p^0(q; \omega) \leq p^c(q; \omega) \leq p^0(q; \omega) + \varepsilon \quad \text{for } q \in [0, q^\omega];$$

moreover, $z^{c,\omega} \in (z^\omega - \varepsilon, z^\omega + \varepsilon)$ and

$$q_0(z; \omega) \leq q_c(z; \omega) \leq q_0(z; \omega) + \varepsilon \quad \text{for } z \in [0, \min\{z^\omega, z^{c,\omega}\}].$$

Let us observe that $q(z) := q_c(z, \omega)$ is a solution of (4.1) with $Z = z^{c,\omega}$. Moreover, $q'(0) = \omega$. We will use $q_c(z; \omega)$ below to construct lower solutions of (1.1).

4.2. **Conditions for spreading.**

Theorem 4.3. *Suppose that the conditions in Lemma 4.1 are satisfied and v_Z is a positive solution of (4.11). If (u, g, h) is a solution of (1.1) with $h_0 \geq Z$ and $u_0 \geq v_Z$ in $[-Z, Z]$, then*

$$(g_\infty, h_\infty) = \mathbb{R}^1 \text{ and } \lim_{t\to\infty} u(t, \cdot) = 1 \text{ locally uniformly in } \mathbb{R}^1.$$

Proof. Since v_Z is a stationary solution and $g(t) < -Z,\ h(t) > Z$ for $t > 0$, by the standard strong comparison principle we deduce

$$u(t, x) > v_Z(x) \text{ in } [-Z, Z] \text{ for all } t > 0.$$

By Theorem 1.1, $u(t, x) \to v(x)$ locally uniformly in $(g_\infty, h_\infty) = \mathbb{R}^1$ as $t \to \infty$, where v is a nonnegative solution of (1.7). v must be a positive solution since $v \geq v_Z$ in $[-Z, Z]$. Moreover, since $[-Z, Z] \subset (g_\infty, h_\infty)$, we necessarily have $v > v_Z$ in $[-Z, Z]$ due to the strong maximum principle.

Thus if we fix $t_0 > 0$ and extend v_Z by 0 outside $[-Z, Z]$, then we can find $\epsilon > 0$ small such that for all $t \geq t_0$,

$$u(t, x) > v_Z(x) + \epsilon \text{ in } [-h_0, h_0],\ u(t, x) > v_Z(0) + \epsilon \text{ in } [-\epsilon, \epsilon].$$

As in the proof of Lemma 4.1, v_Z corresponds to a trajectory $(q_0(z; \omega), p_0(z; \omega))$ of (4.2) (with $c = 0$) that passes through $(0, \omega) := (0, -v_Z'(Z))$ at $z = 0$ and approaches $(q^\omega, 0) := (v_Z(0), 0)$ as z goes to $z^\omega := Z$. By Lemma 4.2, we can find $c > 0$ sufficiently small such that the trajectory $(q_c(z; \omega), p_c(z; \omega))$ of (4.2) that passes through $(0, \omega)$ at $z = 0$ and goes to $(q^{c,\omega}, 0)$ as z approaches $z^{c,\omega}$ satisfies

$$z^{c,\omega} < z^\omega + \epsilon < h_0,\ q^{c,\omega} < q^\omega + \epsilon < 1,$$
$$q_c(z^{c,\omega} - z; \omega) < q_0(z^{c,\omega} - z; \omega) + \epsilon/2 < q_0(z^\omega - z; \omega) + \epsilon \text{ in } [\epsilon, z^{c,\omega}].$$

Since $p_c(z, \omega) = \frac{d}{dz} q_c(z; \omega) > 0$ for $z \in (0, z^{c,\omega})$, we find that

$$q_c(z^{c,\omega} - z; \omega) < q^{c,\omega} < q^\omega + \epsilon = v_Z(0) + \epsilon \text{ for } z \in (0, \epsilon].$$

Thus for such small $c > 0$, we have

$$u(t, x) > q_c(z^{c,\omega} - x; \omega) \quad \text{for } t \geq t_0,\ x \in [0, z^{c,\omega}].$$

We now fix a small $c > 0$ such that the above holds and $c < \mu\omega$. Then define, for $t \geq 0$,

$$k(t) := z^{c,\omega} + ct$$

and

$$w(t, x) := \begin{cases} q_c(k(t) - x; \omega), & x \in [ct, k(t)], \\ q_c(z^{c,\omega}; \omega), & x \in [0, ct]. \end{cases}$$

Since $q_c(z^{c,\omega}; \omega) = q^{c,\omega}$ and $f(q^{c,\omega}) > 0$, we find

$$w_t \leq w_{xx} + f(w) \text{ for } t > 0 \text{ and } x \in (0, k(t)).$$

Moreover,

$$k(0) = z^{c,\omega} < h_0 < h(t_0)$$

and

$$w(t, k(t)) = 0,\ k'(t) = c < \mu\omega = -\mu w_x(t, k(t)) \text{ for } t > 0.$$

Thus we can apply the lower solution version of Lemma 2.2 to conclude that

$$h(t + t_0) \geq k(t) \text{ and } u(t + t_0, x) \geq w(t, x) \text{ for all } t > 0 \text{ and } x \in [0, k(t)].$$

This implies that $h_\infty = \infty$ and the ω-limit of u, namely the positive solution $v(x)$ of (1.7), is defined over $\mathbb{R}^1$. Moreover, for $t > 0$ and $x \in [0, ct]$, we have

$$u(t + t_0, x) \geq w(t, x) = q^{c,\omega} > q^\omega.$$

Hence $v(x) \equiv 1$ for $x \in \mathbb{R}^1$. □

Remark 4.4. *The function $w(t, x)$ constructed above is C^1 in both variables but it is C^2 in x only for $x \in [0, ct) \cup (ct, k(t)]$; along $x = ct$, $w_{xx}(t, x)$ has a jumping discontinuity. However, as for the classical comparison principle, this does not affect the validity of Lemmas 2.1 and 2.2.*

Corollary 4.5. *If f is of* (f_M) *type and $h_0 \geq Z_M = \pi/(2\sqrt{f'(0)})$, then every positive solution (u, g, h) of* (1.1) *satisfies*

$$(g_\infty, h_\infty) = \mathbb{R}^1 \text{ and } \lim_{t\to\infty} u(t, \cdot) = 1 \text{ locally uniformly in } \mathbb{R}^1.$$

Proof. Fix $t_0 > 0$. Then $g(t_0) < -h_0$, $h(t_0) > h_0$. Set $x_0 = [g(t_0) + h(t_0)]/2$ and choose $Z_0 > h_0$ such that $[-Z_0 + x_0, Z_0 + x_0] \subset (g(t_0), h(t_0))$. Then $u(t_0, x + x_0) > 0$ in $[-Z_0, Z_0]$.

Since $q^\omega \to 0$ and $z^\omega \to \pi/(2\sqrt{f'(0)}) \leq h_0$ as ω decreases to 0, we can find $\omega > 0$ small such that $z^\omega < Z_0$ and $q^\omega < u(t_0, x + x_0)$ in $[-Z_0, Z_0]$. We now denote $Z = z^\omega$ and define

$$v_Z(x) := \begin{cases} q_0(x + Z; \omega), & x \in [-Z, 0], \\ q_0(-x + Z, \omega), & x \in [0, Z]. \end{cases}$$

Then it is easily checked that v_Z is a positive solution of (4.11), and $v_Z(x) \leq q^\omega < u(t_0, x + x_0)$ in $[-Z, Z]$.

Define

$$\tilde{u}(t, x) = u(t + t_0, x + x_0),\ \tilde{g}(t) = g(t + t_0) - x_0,\ \tilde{h}(t) = h(t + t_0) - x_0.$$

We find that $\tilde{u}_0(x) := \tilde{u}(0, x) > v_Z(x)$ in $[-Z, Z]$ and $(\tilde{u}, \tilde{g}, \tilde{h})$ solves (1.1) with initial function $\tilde{u}_0$. Applying Theorem 4.3 we deduce that

$$(\tilde{g}_\infty, \tilde{h}_\infty) = \mathbb{R}^1 \text{ and } \lim_{t\to\infty} \tilde{u}(t, \cdot) = 1 \text{ locally uniformly in } \mathbb{R}^1.$$

Clearly this implies the conclusion of the corollary. □

Corollary 4.6. *If f is of* (f_M) *type, then*

$$\lim_{t\to\infty} \|u(t, \cdot)\|_{L^\infty([g(t), h(t)])} = 0$$

implies that (g_∞, h_∞) is a finite interval with length no bigger than $\pi/\sqrt{f'(0)}$.

Proof. Otherwise we can find $t_0 > 0$ such that

$$h(t_0) - g(t_0) > \pi/\sqrt{f'(0)}.$$

Let $x_0 = [g(t_0) + h(t_0)]/2$ and define $(\tilde{u}, \tilde{g}, \tilde{h})$ by the same formulas as in the proof of Corollary 4.5; we find that the conclusion of Corollary 4.5 can be applied to $(\tilde{u}, \tilde{g}, \tilde{h})$ to deduce that $\tilde{u} \to 1$ as $t \to \infty$ locally uniformly in $\mathbb{R}^1$. In view of Lemma 2.8, this implies that

$$\lim_{t\to\infty} \|u(t, \cdot)\|_{L^\infty([g(t), h(t)])} = 1,$$

a contradiction to our assumption. □

5. Classification of dynamical behavior and sharp thresholds

In this section, based on the results of the previous sections, we obtain a complete description of the long-time dynamical behavior of (1.1) when f is of monostable, bistable or combustion type. We also reveal the related but different sharp transition behaviors between spreading and vanishing for these three types of nonlinearities.

5.1. **Monostable case.** Throughout this subsection, we assume that f is of (f_M) type.

Theorem 5.1. (Dichotomy) *Suppose that* $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$, *and* (u, g, h) *is the solution of* (1.1). *Then either spreading happens, namely,* $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1;$$

or vanishing happens, i.e., (g_∞, h_∞) *is a finite interval with length no larger than* $\pi/\sqrt{f'(0)}$ *and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0.$$

Proof. Since f is of monostable type, it is easy to see that (1.7) has no evenly decreasing positive solution, and the only nonnegative constant solutions are 0 and 1. By Theorem 1.1, we see that in this case the ω limit set of u consists of a single constant 0 or 1. Moreover, if (g_∞, h_∞) a finite interval, then $u(t,x) \to 0$ as $t \to \infty$ locally uniformly in (g_∞, h_∞). In view of Lemma 2.8, this limit implies $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty([g(t),h(t)])} = 0$. Hence we can use Corollary 4.6 to conclude that when (g_∞, h_∞) is a finite interval, its length is no larger than $\pi/\sqrt{f'(0)}$.

It remains to show that when $(g_\infty, h_\infty) = \mathbb{R}^1$, the ω limit is 1. If the limit is 0, then we can use Corollary 4.6 as above to deduce that (g_∞, h_∞) is a finite interval; hence only $\omega(u) = \{1\}$ is possible. □

Theorem 5.2. (Sharp threshold) *Suppose that* $h_0 > 0$, $\phi \in \mathscr{X}(h_0)$, *and* (u, g, h) *is a solution of* (1.1) *with* $u_0 = \sigma\phi$ *for some* $\sigma > 0$. *Then there exists* $\sigma^* = \sigma^*(h_0, \phi) \in [0, \infty]$ *such that spreading happens when* $\sigma > \sigma^*$, *and vanishing happens when* $0 < \sigma \le \sigma^*$.

Proof. By Corollary 4.5, we find that spreading happens when $h_0 \ge \pi/(2\sqrt{f'(0)})$. Hence in this case we have $\sigma^*(h_0, \phi) = 0$ for any $\phi \in \mathscr{X}(h_0)$.

In what follows we consider the remaining case $h_0 < \pi/(2\sqrt{f'(0)})$. By Theorem 3.2 (i), we see that in this case vanishing happens for all small $\sigma > 0$. Therefore

$$\sigma^* = \sigma^*(h_0, \phi) := \sup\left\{\sigma_0 : \text{ vanishing happens for } \sigma \in (0, \sigma_0]\right\} \in (0, +\infty].$$

If $\sigma^* = \infty$, then there is nothing left to prove. Suppose $\sigma^* \in (0, \infty)$. Then by definition vanishing happens when $\sigma \in (0, \sigma^*)$, and in view of Theorem 5.1, there exists a sequence σ_n decreasing to σ^* such that spreading happens when $\sigma = \sigma_n$, $n = 1, 2, \cdots$. For any $\sigma > \sigma^*$, we can find some $n \ge 1$ such that $\sigma > \sigma_n$. If we denote by (u_n, g_n, h_n) the solution of (1.1) with $u_0 = \sigma_n\phi$, then by the comparison principle, we find that $[g_n(t), h_n(t)] \subset [g(t), h(t)]$ and $u_n(t,x) \le u(t,x)$. It follows that spreading happens for such σ.

It remains to show that vanishing happens when $\sigma = \sigma^*$. Otherwise spreading must happen when $\sigma = \sigma^*$ and we can find $t_0 > 0$ such that $h(t_0) - g(t_0) > \frac{\pi}{\sqrt{f'(0)}} + 1$. By the continuous dependence of the solution of (1.1) on its initial values, we find that if $\epsilon > 0$ is sufficiently small, then the solution of (1.1) with $u_0 = (\sigma^* - \epsilon)\phi$, denoted by $(u_\epsilon, g_\epsilon, h_\epsilon)$, satisfies

$$h_\epsilon(t_0) - g_\epsilon(t_0) > \frac{\pi}{\sqrt{f'(0)}}.$$

But by Corollary 4.5, this implies that spreading happens to $(u_\epsilon, g_\epsilon, h_\epsilon)$, a contradiction to the definition of σ^*. □

From the above proof we already know that $\sigma^*(h_0, \phi) = 0$ if $h_0 \ge \frac{\pi}{2\sqrt{f'(0)}}$, regardless of the choice of $\phi \in \mathscr{X}(h_0)$. And if $h_0 < \frac{\pi}{2\sqrt{f'(0)}}$, then $\sigma^*(h_0, \phi) \in (0, +\infty]$. We now investigate when $\sigma^*(h_0, \phi)$ is finite, and when it is $+\infty$.

For a given $h_0 > 0$, since any two functions ϕ_1, $\phi_2 \in \mathscr{X}(h_0)$ can be related by

$$\sigma_1\phi_1 \le \phi_2 \le \sigma_2\phi_1$$

for some positive constants σ_1 and σ_2, we find by the comparison principle that either $\sigma^*(h_0,\phi)$ is infinite for all $\phi \in \mathscr{X}(h_0)$, or it is finite for all such ϕ. In other words, whether it is finite or not is determined by h_0 and f, but not affected by the choice of $\phi \in \mathscr{X}(h_0)$.

Proposition 5.3. *Assume that $h_0 < \pi/(2\sqrt{f'(0)})$ and $f(u) \ge -Lu$ for all $u > 0$ and some $L > 0$. Then $\sigma^*(h_0,\phi) \in (0,\infty)$ for all $\phi \in \mathscr{X}(h_0)$.*

Proof. Fix an arbitrary $\phi \in \mathscr{X}(h_0)$. By Theorem 5.2, it suffices to show that spreading happens when $u_0 = \sigma\phi$ and σ is large. We will achieve this by constructing a suitable lower solution.

We start with the following Sturm-Liouville eigenvalue problem

$$\begin{cases} \varphi''(x) + \frac{1}{2}\varphi'(x) + \lambda\varphi(x) = 0, & x \in (0,1), \\ \varphi'(0) = \varphi(1) = 0. \end{cases} \tag{5.1}$$

It is well known that the first eigenvalue λ_1 of this problem is simple and the corresponding first eigenfunction $\varphi_1(x)$ can be chosen positive in $[0,1)$. Moreover, one can easily show that $\lambda_1 > \frac{1}{16}$ and $\varphi_1'(x) < 0$ for $x \in (0,1]$. We assume further that $\|\varphi_1\|_{L^\infty([0,1])} = \varphi_1(0) = 1$.

We extend φ_1 to $[-1,1]$ as an even function. Then clearly

$$\begin{cases} \varphi_1''(x) + \frac{\mathrm{sgn}(x)}{2}\varphi_1'(x) + \lambda_1\varphi_1(x) = 0, & x \in (-1,1), \\ \varphi_1(-1) = \varphi_1(1) = 0. \end{cases} \tag{5.2}$$

We now choose constants $\varepsilon, \bar{Z}, T, \lambda, \rho$ in the following way:

$$0 < \varepsilon < \min\{1, h_0^2\},\ \bar{Z} := 1 + \pi/(2\sqrt{f'(0)}),\ T > \bar{Z}^2,$$

and

$$\lambda > \lambda_1 + L(T+1), \tag{5.3}$$

$$-2\mu\rho\varphi_1'(1) > (T+1)^\lambda. \tag{5.4}$$

Define

$$w(t,x) := \frac{\rho}{(t+\varepsilon)^\lambda}\,\varphi_1\left(\frac{x}{\sqrt{t+\varepsilon}}\right) \quad \text{for } x \in [-\sqrt{t+\varepsilon}, \sqrt{t+\varepsilon}],\ t \ge 0. \tag{5.5}$$

We show that $(w(t,x), -\sqrt{t+\varepsilon}, \sqrt{t+\varepsilon})$ is a lower solution of (1.1) on the time-interval $[0,T]$.

In fact, for $x \in (-\sqrt{t+\varepsilon}, \sqrt{t+\varepsilon})$ and $t \in [0,T]$ we have

$$\begin{aligned} w_t - w_{xx} - f(w) &\le w_t - w_{xx} + Lw \\ &= \frac{-\rho}{(t+\varepsilon)^{\lambda+1}}\left[\varphi_1'' + \frac{x}{2\sqrt{t+\varepsilon}}\varphi_1' + (\lambda - L(t+\varepsilon))\varphi_1\right] \\ &\le \frac{-\rho}{(t+\varepsilon)^{\lambda+1}}\left[\varphi_1'' + \frac{\mathrm{sgn}(x)}{2}\varphi_1' + (\lambda - L(t+\varepsilon))\varphi_1\right] \\ &\le \frac{-\rho}{(t+\varepsilon)^{\lambda+1}}\left[\varphi_1'' + \frac{\mathrm{sgn}(x)}{2}\varphi_1' + \lambda_1\varphi_1\right] = 0. \end{aligned}$$

Clearly $w(t,\pm\sqrt{t+\varepsilon}) = 0$, and by (5.4) we have

$$(\sqrt{t+\varepsilon})' \pm \mu w_x(t,\pm\sqrt{t+\varepsilon}) \le \frac{1}{2\sqrt{t+\varepsilon}}\left(1 + \frac{2\mu\rho}{(T+1)^\lambda}\varphi_1'(1)\right) < 0.$$

Finally, since $\varepsilon < h_0^2$ we can choose $\hat{\sigma} > 0$ large such that

$$w(0,x) = \frac{\rho}{\varepsilon^\lambda}\varphi_1\Big(\frac{x}{\sqrt{\varepsilon}}\Big) < \hat{\sigma}\phi(x) \quad \text{for } x \in [-\sqrt{\varepsilon}, \sqrt{\varepsilon}] \subset [-h_0, h_0].$$

Hence $(w(t,x), -\sqrt{t+\varepsilon}, \sqrt{t+\varepsilon})$ is a lower solution of (1.1) over the time interval $[0,T]$ if in (1.1) we take $u_0(x) = \sigma\phi(x)$ with $\sigma \geq \hat{\sigma}$. It follows that if (u,g,h) is the solution of (1.1) with $u_0 = \sigma\phi$ and $\sigma \geq \hat{\sigma}$, then

$$g(t) \leq -\sqrt{t+\varepsilon},\ h(t) \geq \sqrt{t+\varepsilon} \quad \text{for } t \in [0,T].$$

In particular, $h(T) - g(T) > 2\sqrt{T} > 2\bar{Z} > \pi/\sqrt{f'(0)}$. So spreading happens by Corollary 4.5 for such (u,g,h). □

Proposition 5.4. *Assume that*

$$\lim_{s\to\infty} \frac{-f(s)}{s^{1+2\beta}} = \infty, \tag{5.6}$$

for some

$$\beta > \frac{3+\sqrt{13}}{2}. \tag{5.7}$$

Then there exists $Z_M^0 \in (0, \pi/(2\sqrt{f'(0)}))$ *such that for every* $\phi \in \mathscr{X}(h_0)$,

(i) $\sigma^*(h_0,\phi) = \infty$ *if* $h_0 \in (0, Z_M^0]$, *and*

(ii) $\sigma^*(h_0,\phi) \in (0,\infty)$ *if* $h_0 \in \big(Z_M^0, \pi/(2\sqrt{f'(0)})\big)$.

The proof of this result is rather technical and is postponed to the end of this section. Clearly Theorem 1.2 follows from Theorems 5.1, 5.2 and Proposition 5.3 above.

5.2. **Bistable case.** Throughout this subsection, we assume that f is of (f_B) type.

Theorem 5.5. (Trichotomy) *Suppose that* $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$ *and* (u,g,h) *is the solution of* (1.1). *Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty} \max_{g(t)\leq x\leq h(t)} u(t,x) = 0,$$

or

(iii) Transition: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and there exists a continuous function* $\gamma : [0,\infty) \to [-h_0, h_0]$ *such that*

$$\lim_{t\to\infty} |u(t,x) - v_\infty(x+\gamma(t))| = 0 \text{ locally uniformly in } \mathbb{R}^1,$$

where v_∞ *is the unique positive solution to*

$$v'' + f(v) = 0\ (x \in \mathbb{R}^1),\ v'(0) = 0,\ v(-\infty) = v(+\infty) = 0.$$

Proof. By Theorem 1.1, we have either (g_∞, h_∞) is a finite interval or $(g_\infty, h_\infty) = \mathbb{R}^1$. In the former case, $\lim_{t\to\infty} u(t,x) = 0$ locally uniformly in (g_∞, h_∞), which, together with Lemma 2.8, implies that (ii) holds.

Suppose now $(g_\infty, h_\infty) = \mathbb{R}^1$; then either $\lim_{t\to\infty} u(t,x)$ is a nonnegative constant solution of

$$v_{xx} + f(v) = 0 \text{ in } \mathbb{R}^1, \tag{5.8}$$

or

$$u(t,x) - v(x+\gamma(t)) \to 0 \text{ as } t \to \infty \text{ locally uniformly in } \mathbb{R}^1,$$

where v is an evenly decreasing positive solution of (5.8), and $\gamma : [0,\infty) \to [-h_0, h_0]$ is a continuous function.

Since f is of bistable type, it is well-known (see [9]) that bounded nonnegative solutions of (5.8) consist of the following:

(1) constant solutions: 0, θ, 1;
(2) a family of periodic solutions satisfying $0 < \min v < \theta < \max v < \overline{\theta}$;
(3) a family of symmetrically decreasing solutions $v_\infty(\cdot - a)$, $a \in \mathbb{R}^1$, where v_∞ is uniquely determined by

$$v_\infty'' + f(v_\infty) = 0 \text{ in } \mathbb{R}^1, \quad v_\infty(0) = \overline{\theta}, \quad v_\infty'(0) = 0,$$

which necessarily satisfies $\lim_{|x|\to\infty} v_\infty(x) = 0$.

From this list, clearly only 0, θ, 1 and $v_\infty(\cdot - a)$ are possible members of $\omega(u)$.

By Corollary 3.3, $\omega(u) = \{0\}$ is impossible. It remains to show that $\omega(u) \neq \{\theta\}$. We argue indirectly by assuming that $u(t,x) \to \theta$ as $t \to \infty$ locally uniformly in $\mathbb{R}^1$. Let $v_0(x)$ be a periodic solution of (5.8) as given in (2) above. We now consider the number of zeros of the function

$$w(t,x) := u(t,x) - v_0(x)$$

in the interval $[g(t), h(t)]$, and denote this number by $\mathcal{Z}(t)$. Clearly $w(t, g(t)) < 0$ and $w(t, h(t)) < 0$ for all $t > 0$. Therefore we can use the zero number result of [2] to the equation satisfied by w to conclude that $\mathcal{Z}(t)$ is finite and non-increasing in t for $t > 0$. (We could use a change of variable to change the varying interval $[g(t), h(t)]$ into a fixed one, and then use [2] to the reduced equation.) On the other hand, since $u(t,x) \to \theta$ and $v_0(x)$ oscillates around θ, we find that $\mathcal{Z}(t) \to \infty$ as $t \to \infty$. This contradiction shows that $v = \theta$ is impossible. The proof is complete. □

Theorem 5.6. (Sharp threshold) *Suppose that $h_0 > 0$, $\phi \in \mathscr{X}(h_0)$, and (u, g, h) is a solution of* (1.1) *with $u_0 = \sigma\phi$ for some $\sigma > 0$. Then there exists $\sigma^* = \sigma^*(h_0, \phi) \in (0, \infty]$ such that spreading happens when $\sigma > \sigma^*$, vanishing happens when $0 < \sigma < \sigma^*$, and transition happens when $\sigma = \sigma^*$.*

Proof. By Theorem 3.2 (ii) we find that vanishing happens if $\sigma < \theta/\|\phi\|$. Hence

$$\sigma^* = \sigma^*(h_0, \phi) := \sup\left\{\sigma_0 : \text{ vanishing happens for } \sigma \in (0, \sigma_0]\right\} \in (0, +\infty].$$

If $\sigma^* = +\infty$, then there is nothing left to prove. So we assume that σ^* is a finite positive number.

By definition, vanishing happens for all $\sigma \in (0, \sigma^*)$. We now consider the case $\sigma = \sigma^*$. In this case, we cannot have vanishing, for otherwise we have, for some large $t_0 > 0$, $u(t_0, x) < \theta$ in $[g(t_0), h(t_0)]$, and due to the continuous dependence of the solution on the initial values, we can find $\epsilon > 0$ sufficiently small such that the solution $(u_\epsilon, g_\epsilon, h_\epsilon)$ of (1.1) with $u_0 = (\sigma^* + \epsilon)\phi$ satisfies

$$u_\epsilon(t_0, x) < \theta \text{ in } [g_\epsilon(t_0), h_\epsilon(t_0)].$$

Hence we can apply Theorem 3.2 (ii) to conclude that vanishing happens to $(u_\epsilon, g_\epsilon, h_\epsilon)$, a contradiction to the definition of σ^*. Thus at $\sigma = \sigma^*$ either spreading or transition happens.

We show next that spreading cannot happen at $\sigma = \sigma^*$. Suppose this happens. Let v_Z be a stationary solution as given in Lemma 4.1. Then we can find $t_0 > 0$ large such that

$$[-Z, Z] \subset (g(t_0), h(t_0)), \; u(t_0, x) > v_Z(x) \text{ in } [-Z, Z]. \tag{5.9}$$

By the continuous dependence of the solution on initial values, we can find a small $\epsilon > 0$ such that the solution $(u^\epsilon, g^\epsilon, h^\epsilon)$ of (1.1) with $u_0 = (\sigma^* - \epsilon)\phi$ satisfies (5.9), and by Theorem 4.3, spreading happens for $(u^\epsilon, g^\epsilon, h^\epsilon)$. But this is a contradiction to the definition of σ^*.

Hence transition must happen when $\sigma = \sigma^*$. We show next that spreading happens when $\sigma > \sigma^*$. Let (u, g, h) be a solution of (1.1) with some $\sigma > \sigma^*$, and denote the solution of (1.1) with $\sigma = \sigma^*$ by (u_*, g_*, h_*). By the comparison theorem we know that

$$[g_*(1), h_*(1)] \subset (g(1), h(1)),\ u_*(1, x) < u(1, x) \text{ in } [g_*(1), h_*(1)].$$

Hence we can find $\epsilon_0 > 0$ small such that for all $\epsilon \in [0, \epsilon_0]$,

$$[g_*(1) - \epsilon, h_*(1) - \epsilon] \subset (g(1), h(1)),\ u_*(1, x + \epsilon) < u(1, x) \text{ in } [g_*(1) - \epsilon, h_*(1) - \epsilon].$$

Now define

$$u_\epsilon(t, x) = u_*(t + 1, x + \epsilon),\ g_\epsilon(t) = g_*(t + 1) - \epsilon,\ h_\epsilon(t) = h_*(t + 1) - \epsilon.$$

Clearly $(u_\epsilon, g_\epsilon, h_\epsilon)$ is a solution of (1.1) with $u_0(x) = u_*(1, x + \epsilon)$. By the comparison principle we have, for all $t > 0$ and $\epsilon \in (0, \epsilon_0]$,

$$[g_\epsilon(t), h_\epsilon(t)] \subset (g(t + 1), h(t + 1)),\ u_\epsilon(t, x) \le u(t + 1, x) \text{ in } [g_\epsilon(t), h_\epsilon(t)].$$

If $u_*(t, x) - v_\infty(x + \gamma(t)) \to 0$ as $t \to \infty$ and $\omega(u) \ne \{1\}$, then necessarily $u(t, x) - v_\infty(x + \tilde{\gamma}(t)) \to 0$ as $t \to \infty$. Here both limits are locally uniform in $\mathbb{R}^1$, and $\gamma, \tilde{\gamma}$ are continuous functions from $[0, \infty)$ to $[-h_0, h_0]$.

On the other hand, the above inequalities imply that

$$\limsup_{t\to\infty} [v_\infty(x + \epsilon + \gamma(t)) - v_\infty(x + \tilde{\gamma}(t))] \le 0$$

for all $x \in \mathbb{R}^1$ and $\epsilon \in (0, \epsilon_0]$. Since v_∞ is an evenly decreasing function, this implies that $\lim_{t\to\infty}[\epsilon + \gamma(t) - \tilde{\gamma}(t)] = 0$ for every $\epsilon \in (0, \epsilon_0]$. Clearly this is impossible. Thus we must have $\omega(u) = \{1\}$. This proves that spreading happens for $\sigma > \sigma^*$. □

Next we determine when $\sigma^*(h_0, \phi)$ is finite and when it is infinite.

Proposition 5.7. *Let Z_B be given by (4.8). Then $\sigma^*(h_0, \phi) < \infty$ for all $\phi \in \mathscr{X}(h_0)$ if $h_0 \ge Z_B$, or if $h_0 \in (0, Z_B)$ and $f(u) \ge -Lu$ for all $u > 0$ and some $L > 0$.*

Proof. Let $h_0 > 0$, $\phi \in \mathscr{X}(h_0)$ and (u, g, h) be a solution of (1.1) with $u_0 = \sigma\phi$. It suffices to show that spreading happens for all large σ under the given conditions.

First we suppose that $h_0 \ge Z_B$. By Lemma 4.1, (4.11) has a positive solution v_Z with $Z = h_0$. For sufficiently large $\sigma > 0$ clearly $\sigma\phi \ge v_Z$. Thus we can apply Theorem 4.3 to conclude that spreading happens for (u, g, h) with such σ, as we wanted.

Next we consider the case that $h_0 \in (0, Z_B)$ and $f(u) \ge -Lu$ for all $u > 0$ and some $L > 0$. In this case we construct a lower solution as in the proof of Proposition 5.3 with the following changes: $\bar{Z} \ge 1 + \pi/(2\sqrt{f'(0)})$ is replaced by $\bar{Z} \ge 1 + Z_B$, and we add a further restriction for ρ, namely

$$\frac{\rho}{(T + \varepsilon)^\lambda} \varphi_1\Big(\frac{x}{\sqrt{T + \varepsilon}}\Big) \ge v_{Z_B}(x) \text{ in } [-Z_B, Z_B].$$

We deduce as in the proof of Proposition 5.3 that, for $\sigma \ge \hat{\sigma}$,

$$h(T) - g(T) > 2Z_B,\ u(T, x) \ge w(T, x) \ge v_{Z_B} \text{ in } [-Z_B, Z_B].$$

Then by Theorem 4.3, we deduce that spreading happens for (u, g, h) with $\sigma \ge \hat{\sigma}$. The proof is complete. □

Clearly Theorem 1.3 is a consequence of Theorems 5.5, 5.6 and Proposition 5.7. The following result gives conditions for $\sigma^*(h_0,\phi)=\infty$, whose proof will be given in the last subsection of this section.

Proposition 5.8. *Assume that*

$$\lim_{s\to\infty}\frac{-f(s)}{s^{1+2\beta}}=\infty \quad \text{for some } \beta>2. \tag{5.10}$$

Then there exists $Z_B^0\in(0,Z_B)$ such that, for every $\phi\in\mathscr{X}(h_0)$,

(i) $\sigma^*(h_0,\phi)=\infty$ *if* $h_0\le Z_B^0$, *and*

(ii) $\sigma^*(h_0,\phi)\in(0,\infty)$ *if* $h_0>Z_B^0$.

5.3. **Combustion case.** Throughout this subsection, we assume that f is of (f_C) type.

Theorem 5.9. (Trichotomy) *Suppose that $h_0>0$, $u_0\in\mathscr{X}(h_0)$ and (u,g,h) is the solution of* (1.1). *Then either*

(i) Spreading: $(g_\infty,h_\infty)=\mathbb{R}^1$ *and*

$$\lim_{t\to\infty}u(t,x)=1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞,h_∞) *is a finite interval and*

$$\lim_{t\to\infty}\max_{g(t)\le x\le h(t)}u(t,x)=0,$$

or

(iii) Transition: $(g_\infty,h_\infty)=\mathbb{R}^1$ *and*

$$\lim_{t\to\infty}u(t,x)=\theta.$$

Proof. One easily sees that bounded nonnegative solutions of

$$v_{xx}+f(v)=0 \text{ in } \mathbb{R}^1, \tag{5.11}$$

with a combustion type f, consists of the following constant solutions only: 0, every $c\in(0,\theta)$, θ, 1.

Therefore, by Theorem 1.1, we have either (g_∞,h_∞) is a finite interval and $\lim_{t\to\infty}u(t,x)=0$ locally uniformly in (g_∞,h_∞), or $(g_\infty,h_\infty)=\mathbb{R}^1$ and $\lim_{t\to\infty}u(t,x)=v$ locally uniformly in $\mathbb{R}^1$, with v a constant nonnegative solution of (5.11). As before we can use Lemma 2.8 to conclude that when (g_∞,h_∞) is a finite interval, then vanishing happens.

It remains to show that when $(g_\infty,h_\infty)=\mathbb{R}^1$, then $v\equiv1$ or $v\equiv\theta$. As before Corollary 3.3 shows that $v=0$ is impossible when $(g_\infty,h_\infty)=\mathbb{R}^1$. We show next that $v\ne c$ for any $c\in(0,\theta)$. Suppose by way of contradiction that $v\equiv c\in(0,\theta)$. Then in view of Lemma 2.8, for some large $t_0>0$ we have $\|u(t_0,\cdot)\|_{L^\infty}<\theta$. Thus we can apply Theorem 3.2 to conclude that vanishing happens to (u,g,h), a contradiction to $\omega(u)=\{c\}$.

The proof is complete. □

Theorem 5.10. (Sharp threshold) *Suppose that $h_0>0$, $\phi\in\mathscr{X}(h_0)$, and (u,g,h) is a solution of* (1.1) *with $u_0=\sigma\phi$ for some $\sigma>0$. Then there exists $\sigma^*=\sigma^*(h_0,\phi)\in(0,\infty]$ such that spreading happens when $\sigma>\sigma^*$, vanishing happens when $0<\sigma<\sigma^*$, and transition happens when $\sigma=\sigma^*$.*

Proof. The proof is identical to that of Theorem 5.6 except the last part, where it shows that spreading happens when $\sigma>\sigma^*$. This part has to be proved differently.

Let (u_*, g_*, h_*) be the solution of (1.1) with $u_0 = \sigma^*\phi$, and (u, g, h) a solution with $u_0 = \sigma\phi$ and $\sigma > \sigma^*$. Since $u_*(t,x) \to \theta$ locally uniformly in $\mathbb{R}^1$ as $t \to \infty$, in view of Lemma 2.8, we can find $T > 0$ large such that

$$u_*(t,x) < \theta + \delta_0/2 \text{ for } t \geq T/2,\ x \in [g_*(t), h_*(t)], \tag{5.12}$$

where δ_0 is given in (1.12). By the comparison principle we have

$$[g_*(T), h_*(T)] \subset (g(T), h(T)),\ u_*(T,x) < u(T,x) \text{ in } [g_*(T), h_*(T)]. \tag{5.13}$$

Now, for $\xi \in (0,1)$, we define

$$v^\xi(t,x) := \xi^{-1} u_*(\xi t, \sqrt{\xi}x),\ g^\xi(t) = \xi^{-1/2} g_*(\xi t),\ h^\xi(t) = \xi^{-1/2} h_*(\xi t).$$

Then, by (5.12) and (5.13), we can choose $\xi_0 \in (0,1)$ close enough to 1 so that, for every $\xi \in [\xi_0, 1)$,

$$v^\xi(t,x) \leq \theta + \delta_0 \quad \text{for all } t \geq T,\ x \in [g^\xi(t), h^\xi(t)], \tag{5.14}$$

and

$$[g^\xi(T), h^\xi(T)] \subset (g(T), h(T)),\ v^\xi(T,x) \leq u(T,x) \text{ in } [g^\xi(T), h^\xi(T)]. \tag{5.15}$$

Observe that v^ξ satisfies the equation

$$v^\xi_t = v^\xi_{xx} + f(\xi v^\xi) \text{ for } t > T,\ x \in [g^\xi(t), h^\xi(t)].$$

By (5.14) and (1.12), we have $f(\xi v^\xi) \leq f(v^\xi)$. Therefore in view of (5.15), we find that (v^ξ, g^ξ, h^ξ) is a lower solution of (1.1) for $t \geq T$. It follows that

$$u(t,x) \geq v^\xi(t,x) \text{ and } v \geq \lim_{t\to\infty} v^\xi(t,x) = \theta/\xi,$$

where v is the ω-limit of u. Thus we must have $v \equiv 1$, as we wanted. □

Proposition 5.11. *Let Z_C be given by* (4.9). *Then $\sigma^*(h_0, \phi) < \infty$ for all $\phi \in \mathscr{X}(h_0)$ if $h_0 \geq Z_C$, or if $h_0 \in (0, Z_C)$ and $f(u) \geq -Lu$ for all $u > 0$ and some $L > 0$.*

Proposition 5.12. *Assume that f satisfies* (5.10). *Then there exists $Z_C^0 \in (0, Z_C)$ such that, for every $\phi \in \mathscr{X}(h_0)$,*

(i) *$\sigma^*(h_0, \phi) = \infty$ if $h_0 \leq Z_C^0$, and*
(ii) *$\sigma^*(h_0, \phi) \in (0, \infty)$ if $h_0 > Z_C^0$.*

The proof of Proposition 5.11 is identical to that of Proposition 5.7; all we need is to replace Z_B by Z_C in the proof. The proof of Proposition 5.12 is given in the next subsection. Evidently, Theorem 1.4 is a consequence of Theorems 5.9, 5.10 and Proposition 5.11.

5.4. **Proof of Propositions 5.4, 5.8 and 5.12.** In this subsection we always assume that f is of (f_M), or (f_B), or (f_C) type. We will prove Propositions 5.8 and 5.12 first, and then prove Proposition 5.4.

If f satisfies

$$\lim_{u\to\infty} \frac{-f(u)}{u^{1+2\beta}} = \infty \quad \text{for some } \beta > 2, \tag{5.16}$$

then taking

$$L = L(\beta) = \frac{1+\beta}{\beta^2} \cdot 2^{1+2\beta}, \tag{5.17}$$

we can find $s = s(\beta) > 1$ such that

$$-f(u) \geq Lu^{1+2\beta} \quad \text{for } u \geq s. \tag{5.18}$$

Let $h_0 > 0$, and (u, g, h) be the solution of (1.1) with $u_0 = \phi \in \mathscr{X}(h_0)$. We show that $\int_{g(t)}^{h(t)} u(t, x; \phi)$ can be made as small as we want if h_0 is small enough and t is chosen suitably, regardless of the choice of ϕ.

Lemma 5.13. *Suppose that* (5.16) *or* (5.18) *holds. Then given any* $\varepsilon > 0$ *we can find* $h_0^* = h_0^*(\varepsilon) > 0$ *such that, for each* $h_0 \in (0, h_0^*]$ *there exists* $t_0 = t_0(h_0) > 0$ *so that*

$$\int_{g(t_0)}^{h(t_0)} u(t_0, x; \phi)dx < \varepsilon \text{ for all } \phi \in \mathscr{X}(h_0).$$

Proof. For any $h_0 \in (0, 1)$, set

$$\psi(x) = (x - h_0)^{-\frac{1}{\beta}} - h_0^{-\frac{1}{\beta}} \quad \text{for } h_0 < x \le 2h_0.$$

For $h_0 + ct < x \le 2h_0 + ct,\ t > 0$, we define

$$w(t, x) := \psi(x - ct), \quad k(t) = 2h_0 + ct \quad \text{with } c = \frac{\mu}{\beta h_0^{\frac{1+\beta}{\beta}}}. \tag{5.19}$$

With s given by (5.18), we set

$$\varepsilon_1 := h_0(1 + s)^{-\beta}\ (< h_0) \quad (\text{or equivalently, } \psi(h_0 + \varepsilon_1) = s h_0^{-\frac{1}{\beta}}). \tag{5.20}$$

We now consider w for $h_0 + ct + \varepsilon_1 \le x \le 2h_0 + ct,\ t > 0$. It is easily seen that

$$w_t(t, x) = -c\psi'(x - ct) = \frac{c}{\beta(x - h_0 - ct)^{\frac{1+\beta}{\beta}}} \ge \frac{c}{\beta h_0^{\frac{1+\beta}{\beta}}},$$

$$w_{xx}(t, x) = \frac{1 + \beta}{\beta^2(x - h_0 - ct)^{\frac{1+2\beta}{\beta}}} \le \frac{1 + \beta}{\beta^2} \cdot \frac{(1 + s)^{1+2\beta}}{h_0^{\frac{1+2\beta}{\beta}}}.$$

Hence, with $F := \sup_{0 \le \xi < \infty} f(\xi)$,

$$\begin{aligned} w_t - w_{xx} - f(w) \ge & \frac{1}{\beta h_0^{\frac{1+2\beta}{\beta}}} \left[c h_0 - \frac{1 + \beta}{\beta}(1 + s)^{1+2\beta} - F\beta h_0^{\frac{1+2\beta}{\beta}} \right] \\ = & \frac{1}{\beta h_0^{\frac{1+2\beta}{\beta}}} \left[\frac{\mu}{\beta} h_0^{-1/\beta} - \frac{1 + \beta}{\beta}(1 + s)^{1+2\beta} - F\beta h_0^{\frac{1+2\beta}{\beta}} \right] \ge 0, \end{aligned}$$

provided h_0 is sufficiently small.

Next we consider w for $h_0 + ct < x \le h_0 + ct + \varepsilon_1,\ t > 0$. In this range, we have

$$\begin{aligned} w(t, x) \ & \ge \ \frac{1}{(x - h_0 - ct)^{\frac{1}{\beta}}} \left(1 - \left(\frac{\varepsilon_1}{h_0} \right)^{\frac{1}{\beta}} \right) \\ & = \ \frac{1}{(x - h_0 - ct)^{\frac{1}{\beta}}} \left(\frac{s}{1 + s} \right) \\ & \ge \ \frac{s}{\varepsilon_1^{1/\beta}(1 + s)} = \frac{s}{h_0^{\frac{1}{\beta}}} > s. \end{aligned}$$

Thus,

$$\begin{aligned} w_t - w_{xx} - f(w) &\geq -c\psi'(x-ct) - \psi''(x-ct) + L[\psi(x-ct)]^{1+2\beta} \\ &\geq L\left[\frac{1}{(x-h_0-ct)^{\frac{1}{\beta}}}\Big(\frac{s}{1+s}\Big)\right]^{1+2\beta} - \frac{1+\beta}{\beta^2(x-h_0-ct)^{\frac{1+2\beta}{\beta}}} \\ &= \frac{1}{(x-h_0-ct)^{\frac{1+2\beta}{\beta}}}\left[L\Big(\frac{s}{1+s}\Big)^{1+2\beta} - \frac{1+\beta}{\beta^2}\right] \\ &> \frac{1}{(x-h_0-ct)^{\frac{1+2\beta}{\beta}}}\left[L\Big(\frac{1}{2}\Big)^{1+2\beta} - \frac{1+\beta}{\beta^2}\right] = 0. \end{aligned}$$

Clearly,

$$k'(t) + \mu w_x(t, k(t)) = c - \frac{\mu}{\beta h_0^{\frac{1+\beta}{\beta}}} = 0.$$

We now compare (u, h) with (w, k) over the region

$$\Omega := \{(t, x) : h_0 + ct \leq x \leq k(t)\} \cap \{(t, x) : 0 \leq x \leq h(t)\}.$$

By definition,

$$u(t, x) = 0 \text{ for } x = h(t),\ w(t, x) = +\infty \text{ for } x = h_0 + ct.$$

Thus we can apply Lemma 2.2 to deduce that whenever $J(t) := \{x : (t, x) \in \Omega\}$ is nonempty, we have $h(t) \leq k(t)$ and $u(t, x) \leq w(t, x)$ in $J(t)$. Thus we have $h(t) \leq k(t)$ for all $t > 0$.

By (5.18) and the definition of c,

$$\tau_1 := \int_{\frac{s}{h_0^{\frac{1}{\beta}}}}^{\infty} \frac{dr}{-f(r)} \leq \frac{h_0^2}{2\beta L s^{2\beta}}, \quad c\tau_1 \leq \frac{\mu}{2\beta^2 L s^{2\beta}} h_0^{\frac{\beta-1}{\beta}}.$$

Let $\zeta(t)$ be the solution of

$$\zeta'(t) = f(\zeta), \quad \zeta(0) = \|\phi\|_{L^\infty} + 1.$$

Then $u(t, x; \phi) \leq \zeta(t)$ for $t \geq 0$. We claim that $u(t, x; \phi) \leq s h_0^{-\frac{1}{\beta}}$ for $t \geq \tau_1$.

Indeed, since $f(1) = 0$ and $f(\xi) < 0$ for $\xi > 1$, we find that $\zeta(t) > 1$ and is decreasing for $t > 0$. Moreover,

$$\begin{aligned} \tau_1 &= \int_{\|\phi\|_\infty+1}^{\zeta(\tau_1)} \frac{d\zeta}{f(\zeta)} \\ &= \int_{\|\phi\|_\infty+1}^{\infty} \frac{d\zeta}{f(\zeta)} - \int_{\zeta(\tau_1)}^{\infty} \frac{d\zeta}{f(\zeta)} \\ &< \int_{\zeta(\tau_1)}^{\infty} \frac{d\zeta}{-f(\zeta)}. \end{aligned}$$

Thus

$$\int_{s/h_0^{1/\beta}}^{\infty} \frac{d\zeta}{-f(\zeta)} < \int_{\zeta(\tau_1)}^{\infty} \frac{d\zeta}{-f(\zeta)}.$$

It follows that $\zeta(\tau_1) < s/h_0^{1/\beta}$ and hence $\zeta(t) < s/h_0^{1/\beta}$ for all $t \geq \tau_1$, which implies the claim.

By this estimate of $u(t, x; \phi)$ we obtain, for $t = \tau_1$,

$$\int_0^{h(\tau_1)} u(\tau_1, x; \phi)dx \leq (2h_0 + c\tau_1) s h_0^{-\frac{1}{\beta}} \leq h_0^{\frac{\beta-2}{\beta}} \left[2s h_0^{\frac{1}{\beta}} + \frac{\mu}{2\beta^2 L s^{2\beta-1}}\right].$$

Since $\beta > 2$, $\int_0^{h(\tau_1)} u(\tau_1, x; \phi)dx$ can be as small as possible when $h_0 \to 0$. By a parallel consideration, the same is true for $\int_{g(\tau_1)}^0 u(\tau_1, x; \phi)dx$. This completes the proof. $\square$

We also need the following technical lemma.

Lemma 5.14. *Suppose $h_0 > 0$ and $\phi \in \mathscr{X}(h_0)$. Then there exists $\varepsilon_0 \in (0, h_0)$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, any $\phi_\varepsilon \in \mathscr{X}(h_0 - \varepsilon)$, and any sufficiently large $\sigma > 0$,*

$$g_\varepsilon(t_1) \le -h_0,\ h_\varepsilon(t_1) \ge h_0,\ u_\varepsilon(t_1, x) \ge \phi(x) \quad \text{for } x \in [-h_0, h_0]$$

at some $t_1 > 0$, where $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$ denotes the solution of (1.1) with $u_0 = \sigma\phi_\varepsilon$.

Proof. We prove the conclusion by constructing a suitable lower solution. Let $\varphi_1(x)$ be the positive function satisfying (5.2) and $\|\varphi_1\|_{L^\infty([-1,1])} = \varphi_1(0) = 1$.

Choose $\rho_0 > 0$ such that

$$\rho_0 > \frac{1}{-2\mu\varphi_1'(1)}, \qquad \rho_0\varphi_1\Big(\frac{x}{h_0}\Big) \ge \phi(x) \quad \text{for } x \in [-h_0, h_0]. \tag{5.21}$$

This is possible since $\varphi_1'(1) < 0$ and $\phi \in C^1([-h_0, h_0])$. Fix such a ρ_0; then there exists $M = M(\rho_0) > 0$ such that

$$f(s) \ge -Ms \quad \text{for } s \in [0, 2\rho_0]. \tag{5.22}$$

Set

$$\lambda := \lambda_1 + Mh_0^2, \quad \varepsilon_0 := (1 - 2^{-\frac{1}{2\lambda}})h_0. \tag{5.23}$$

For any $\varepsilon \in (0, \varepsilon_0)$, any $\phi_\varepsilon \in \mathscr{X}(h_0 - \varepsilon)$, we will show that, when

$$\sigma\phi_\varepsilon \ge \rho_0\Big(\frac{h_0}{h_0 - \varepsilon}\Big)^{2\lambda} \cdot \varphi_1\Big(\frac{x}{h_0 - \varepsilon}\Big),$$

we have

$$u(t_1, x; \sigma\phi_\varepsilon) \ge \phi(x) \quad \text{on } [-h_0, h_0],$$

at time $t_1 := 2\varepsilon h_0 - \varepsilon^2 > 0$. To prove this result, we first show that $(w, -k, k)$ given by

$$w(t, x) := \rho_0\Big(\frac{h_0}{k(t)}\Big)^{2\lambda} \cdot \varphi_1\Big(\frac{x}{k(t)}\Big), \quad k(t) := \sqrt{(h_0 - \varepsilon)^2 + t}$$

forms a lower solution of (1.1) on the time interval $t \in [0, t_1]$.

When $t \in [0, t_1]$, we have $h_0 - \varepsilon \le k(t) \le h_0$ and

$$\|w(t, x)\|_{L^\infty([-k(t), k(t)])} = w(t, 0) \le \rho_0\Big(\frac{h_0}{h_0 - \varepsilon}\Big)^{2\lambda} \le 2\rho_0$$

by the definition of ε_0. Hence, for $-k(t) \le x \le k(t)$, $0 \le t \le t_1$,

$$\begin{aligned} w_t - w_{xx} - f(w) &\le w_t - w_{xx} + Mw \\ &= \frac{-\rho_0 h_0^{2\lambda}}{[k(t)]^{2\lambda+2}}\left[\varphi_1'' + \frac{x}{2k(t)}\varphi_1' + [\lambda - M(k(t))^2]\varphi_1\right] \\ &\le \frac{-\rho_0 h_0^{2\lambda}}{[k(t)]^{2\lambda+2}}\left[\varphi_1'' + \frac{\operatorname{sgn}(x)}{2}\varphi_1' + \lambda_1\varphi_1\right] = 0, \end{aligned}$$

$$k'(t) + \mu w_x(t, k(t)) \le \frac{1}{2k(t)}\left[1 + 2\mu\rho_0\Big(\frac{h_0}{k(t)}\Big)^{2\lambda}\varphi_1'(1)\right] < 0,$$

and

$$-k'(t) + \mu w_x(t, -k(t)) \ge -\frac{1}{2k(t)}\left[1 + 2\mu\rho_0\Big(\frac{h_0}{k(t)}\Big)^{2\lambda}\varphi_1'(1)\right] > 0.$$

If σ is chosen such that $w(0,x) \le \sigma\phi_\varepsilon$, we find that $(w(t,x), -k(t), k(t))$ is a lower solution of (1.1) with $u_0 = \sigma\phi_\epsilon$. Now it is clear that the required inequalities follow from this and (5.21). □

Proof of Propositions 5.8 and 5.12: We only consider the (f_B) case; the proof of the (f_C) case is identical.

By Lemma 5.13 and Theorem 3.2 (iii), we see that for sufficiently small $h_0 > 0$, vanishing happens for any $\phi \in \mathscr{X}(h_0)$ and any $\sigma > 0$. So $\sigma^*(h_0) = \infty$ for small h_0. Here and in what follows we write $\sigma^*(h_0) = \infty$ instead of $\sigma^*(h_0, \phi) = \infty$, since $\phi \in \mathscr{X}(h_0)$ plays no role for the validity of this identity.

Define

$$Z_B^0 := \sup \Pi \qquad \text{where } \ \Pi := \{h_0 > 0 \mid \sigma^*(h_0) = \infty\}. \tag{5.24}$$

In view of the above fact and Proposition 5.7, we have $0 < Z_B^0 \le Z_B$. By the comparison principle, we see that $\sigma^*(h_0) = \infty$ when $h_0 \in (0, Z_B^0)$, that is, $(0, Z_B^0) \subset \Pi$.

We claim that the set $(0,\infty)\backslash\Pi$ is open, and so Π is closed. To see this, suppose h_0 belongs to this set and so $\sigma^*(h_0,\phi) < \infty$ for every $\phi \in \mathscr{X}(h_0)$. Hence there exists $\sigma_1 > 0$ so that spreading happens when $u_0 = \sigma\phi$ and $\sigma \ge \sigma_1$. By Lemma 5.14, for sufficiently small $\epsilon > 0$ and any $\phi_\epsilon \in \mathscr{X}(h_0 - \epsilon)$, there exists $\sigma_2 > 0$ and $t_1 > 0$ such that

$$u(t_1, x; \sigma_2\phi_\epsilon) \ge \sigma_1\phi(x) \text{ in } [-h_0, h_0].$$

It follows that

$$u(t + t_1, x; \sigma_2\phi_\epsilon) \ge u(t, x; \sigma_1\phi) \text{ for all } t > 0.$$

This implies that $\sigma^*(h_0 - \epsilon, \phi_\epsilon) < \infty$ and hence $h_0 - \epsilon \in (0,\infty)\backslash\Pi$. By the comparison principle, clearly any $h > h_0$ belongs to this set. Thus it is an open set.

Hence Π is relatively closed in $(0,+\infty)$ and $\Pi = (0, Z_B^0]$. By Proposition 5.7, $\sigma^*(Z_B) < +\infty$. Therefore $Z_B^0 < Z_B$, and for $h_0 > Z_B^0$, $\sigma^*(h_0) < +\infty$.

The proof is complete. □

The proof of Proposition 5.4 needs the following result.

Lemma 5.15. *Under the assumptions of Proposition 5.4, there exists $h_0^* > 0$ small such that, for any $h_0 \in (0, h_0^*)$ and any $\phi \in \mathscr{X}(h_0)$, we have $\|u(t,\cdot;\phi)\|_\infty \to 0$ as $t \to \infty$.*

Proof. By (5.7) we have

$$\beta < \frac{2\beta^2 - 2\beta - 1}{1+\beta}.$$

Fix a constant α between them, we may assume without loss of generality that

$$\limsup_{s\to\infty} \frac{-f(s)}{s^{1+2\alpha}} < \infty. \tag{5.25}$$

Indeed, if (5.25) does not hold, we can modify f to be $f_1 \in C^1$ such that $f(u) \le f_1(u)$ for $u \ge 0$ and that f_1 satisfies (5.25). Replace f by f_1 in (1.1) and denote the problem by $(1.1)_1$. It is easily seen that when a solution $u_1(t,x;\phi)$ of $(1.1)_1$ vanishes, the solution $u(t,x;\phi)$ of (1.1) also vanishes. Hence we may prove the lemma under the additional condition (5.25).

In what follows we always choose $h_0 \in (0,1)$. Conditions (5.6) and (5.25) imply that there exist $s > 1$, $K_\beta > 0$ and $K_\alpha > 0$ such that

$$K_\beta u^{1+2\beta} \le -f(u) \le K_\alpha u^{1+2\alpha} \qquad \text{for } u \ge s. \tag{5.26}$$

Moreover, we could have chosen $K_\beta = L$ given by (5.17). Therefore we can define c, $w(t,x)$ and $k(t)$ as in the proof of Lemma 5.13 to deduce

$$h(t) \le 2h_0 + ct \text{ for all } t > 0.$$

Similarly

$$g(t) \ge -2h_0 - ct \text{ for all } t > 0.$$

Step 1. A bound from the proof of Lemma 5.13. We denote $\omega_1 := \frac{1+2\beta}{2\beta^2} < 1$; then as in the proof of Lemma 5.13 we have

$$\tau_1 := \int_{\frac{s}{h_0^{\omega_1}}}^{\infty} \frac{dr}{-f(r)} \le \frac{h_0^{2\beta\omega_1}}{2\beta K_\beta s^{2\beta}},$$

$$c\tau_1 \le \frac{\mu}{2\beta^2 K_\beta s^{2\beta}} h_0, \qquad \max\{-g(\tau_1), h(\tau_1)\} \le 2h_0 + c\tau_1 \le \Big(2 + \frac{\mu}{2\beta^2 K_\beta s^{2\beta}}\Big) h_0,$$

and

$$\int_{g(\tau_1)}^{h(\tau_1)} u(\tau_1, x; \phi)dx \le \Big(4s + \frac{\mu}{\beta^2 K_\beta s^{2\beta-1}}\Big) h_0^{1-\omega_1}. \tag{5.27}$$

Step 2. A bound for g and h at a later time τ_2. By condition (5.7), there exists $0 < \delta < \beta$ such that

$$\alpha < \frac{2\beta^2 - 2\beta - 1}{1 + \beta + \delta}.$$

Hence

$$1 - \omega_1 = \frac{2\beta^2 - 2\beta - 1}{2\beta^2} > \alpha\omega_2 := \alpha \cdot \frac{1 + \beta + \delta}{2\beta^2}.$$

Note that $\omega_2 < \omega_1$ and so $h_0^{-\omega_2} < h_0^{-\omega_1}$.

Define

$$\tau_2 := \int_{\frac{s}{h_0^{\omega_2}}}^{\infty} \frac{dr}{-f(r)} \le \frac{h_0^{2\beta\omega_2}}{2\beta K_\beta s^{2\beta}} < \tau_2^0 := \frac{1}{2\beta K_\beta s^{2\beta}}.$$

Then

$$\max\{-g(\tau_2), h(\tau_2)\} \le 2h_0 + c\tau_2 \le 2h_0 + \frac{\mu}{2\beta^2 K_\beta s^{2\beta}} h_0^{\frac{\delta}{\beta}} \le \frac{\pi}{3\sqrt{f'(0)}} \tag{5.28}$$

provided h_0 is sufficiently small.

Step 3. A key bound for u. Direct calculation shows that

$$\tau_2 - \tau_1 \ge \int_{\frac{s}{h_0^{\omega_2}}}^{\frac{s}{h_0^{\omega_1}}} \frac{dr}{K_\alpha r^{1+2\alpha}} = \frac{h_0^{2\alpha\omega_2}(1 - h_0^{2\alpha(\omega_1-\omega_2)})}{2\alpha K_\alpha s^{2\alpha}} \ge \frac{h_0^{2\alpha\omega_2}}{4\alpha K_\alpha s^{2\alpha}}$$

provided that h_0 is sufficiently small such that

$$1 - h_0^{2\alpha(\omega_1-\omega_2)} \ge \frac{1}{2}. \tag{5.29}$$

Therefore, for $g(\tau_2) \le x \le h(\tau_2)$, we have by (5.27)

$$\frac{e^{K(\tau_2-\tau_1)}}{2\sqrt{\pi(\tau_2-\tau_1)}} \int_{g(\tau_1)}^{h(\tau_1)} u(\tau_1, x; \phi)dx \le \widetilde{M} h_0^{1-\omega_1-\alpha\omega_2} < \sigma_1, \tag{5.30}$$

provided $h_0 > 0$ is sufficiently small, where $K > 0$ is chosen such that (1.6) holds, $\sigma_1 > 0$ is small so that the conclusion in Theorem 3.2 (i) holds when $\|\phi\|_\infty \le \sigma_1$, and

$$\widetilde{M} := \frac{e^{K\tau_2^0}}{\sqrt{\pi}} \sqrt{4\alpha K_\alpha s^{2\alpha}} \Big(2s + \frac{\mu}{2\beta^2 K_\beta s^{2\beta-1}} \Big).$$

Step 4. Completion of the proof. For the above chosen $h_0 > 0$,

$$h(\tau_1) < h(\tau_2) < \frac{\pi}{3\sqrt{f'(0)}},\ g(\tau_1) > g(\tau_2) > -\frac{\pi}{3\sqrt{f'(0)}}.$$

By the proof of Lemma 3.1 we know that

$$u(\tau_1 + t, x; \phi) \le \frac{e^{Kt}}{2\sqrt{\pi t}} \int_{g(\tau_1)}^{h(\tau_1)} u(\tau_1, x; \phi) dx \quad \text{for all } t \ge 0.$$

Hence for $g(\tau_2) \le x \le h(\tau_2)$,

$$u(\tau_2, x; \phi) \le \frac{e^{K(\tau_2 - \tau_1)}}{2\sqrt{\pi(\tau_2 - \tau_1)}} \int_{g(\tau_1)}^{h(\tau_1)} u(\tau_1, x; \phi) dx < \sigma_1.$$

Consequently, $u(\tau_2 + t, x; \phi) \to 0$ by Theorem 3.2 (i). □

Proof of Proposition 5.4: With the help of the above lemma, one can proceed as in the proof of Propositions 5.8 and 5.12. □

6. Semi-waves and spreading speed

Throughout this section we assume that f is of type (f_M), or (f_B), or (f_C), and (u, g, h) is a solution of (1.1) for which spreading happens. To determine the spreading speed, we will construct suitable upper and lower solutions based on semi-waves and waves of finite length with speed close to that of the semi-waves.

6.1. **Semi-waves.** We call $q(z)$ a semi-wave with speed c if $(c, q(z))$ satisfies

$$\begin{cases} q'' - cq' + f(q) = 0 & \text{for } z \in (0, \infty), \\ q(0) = 0,\ q(\infty) = 1,\ q(z) > 0 & \text{for } z \in (0, \infty). \end{cases} \tag{6.1}$$

As before, the first equation in (6.1) can be written in the equivalent form

$$q' = p,\ p' = cp - f(q). \tag{6.2}$$

So a solution $q(z)$ of (6.1) corresponds to a trajectory $(q(z), p(z))$ of (6.2) that starts from the point $(0, \omega)$ $(\omega = q'(0) > 0)$ in the qp-plane and ends at the point $(1, 0)$ as $z \to +\infty$.

If $p(z) = q'(z) > 0$ for all $z > 0$, then the trajectory can be expressed as a function $p = P(q)$, $q \in [0, 1]$, which satisfies

$$\frac{dP}{dq} \equiv P' = c - \frac{f(q)}{P} \ \text{ for } q \in (0, 1),\ P(0) = \omega,\ P(1) = 0. \tag{6.3}$$

It is easily checked that

$$P_0(q) := \sqrt{2 \int_q^1 f(s) ds}, \quad q \in [0, 1],$$

solves (6.3) with $c = 0$ and $\omega = \omega^0 := \sqrt{2 \int_0^1 f(s) ds} > 0$. Moreover,

$$P_0'(1) = -\sqrt{-f'(1)}.$$

Suppose $c \geq 0$ and consider the equilibrium point $(1,0)$ of (6.2). A simple calculation shows that $(1,0)$ is a saddle point, and hence by the theory of ODE (cf. [24]) there are exactly two trajectories of (6.2) that approach $(1,0)$ from $q<1$; one of them, denoted by T_c, has slope $\frac{c-\sqrt{c^2-4f'(1)}}{2}<0$ at $(1,0)$, and the other has slope $\frac{c+\sqrt{c^2-4f'(1)}}{2}>0$ at $(1,0)$. A part of T_c that lies in the set $S:=\{(q,p): 0\leq q\leq 1, p\geq 0\}$ and contains $(1,0)$ is a curve which can be expressed as $p=P_c(q)$, $q\in[q_c,1]$, where $q_c\in[0,1)$, $P_c(q)>0$ in $(q_c,1)$ and the point $(q_c,P_c(q_c))$ lies on the boundary of S. Thus $P_c(q)$ satisfies

$$P'=c-\frac{f(q)}{P} \text{ in } (q_c,1),\ P(1)=0,\ P'(1)=\frac{c-\sqrt{c^2-4f'(1)}}{2}. \tag{6.4}$$

Clearly, when $q_c>0$ we have $P_c(q_c)=0$.

If $q_c>0$, then as q is decreased from 1, $P_c(q)$ stays positive and approaches 0 from above as q decreases to q_c. Checking the sign of $P_c'(q)$ by the differential equation we easily see that this cannot happen before q reaches $\tilde{\theta}$, where

$$\tilde{\theta}=\begin{cases} 0, & \text{in } (\mathrm{f}_M) \text{ case},\\ \theta, & \text{in } (\mathrm{f}_B) \text{ and } (\mathrm{f}_C) \text{ case}.\end{cases}$$

Thus we always have $q_c\leq\tilde{\theta}$. For convenience of notation, we assume that $P_c(q)=0$ for $q\in[0,q_c)$ when $q_c>0$, so that $P_c(q)$ is always defined for $q\in[0,1]$. Denote $T_c^1:=\{(q,p): p=P_c(q), q\in[0,1]\}$.

Since

$$0>\frac{c-\sqrt{c^2-4f'(1)}}{2}>-\sqrt{-f'(1)}=P_0'(1),$$

we have $P_c'(1)>P_0'(1)$, and by comparing the differential equations of $P_c(q)$ and $P_0(q)$ we easily see that $P_c(q)$ never touches $P_0(q)$ from below as q decreases from 1 to q_c. Thus

$$0<P_c(q)<P_0(q) \quad \text{for } q\in(q_c,1),$$

which implies $P_c(q)<P_0(q)$ for $q\in(0,1)$.

In $(q_c,1)$, we have

$$(P_c^2-P_0^2)'=2cP_c\leq 2cP_0\leq 2cM:=2c\|P_0\|_{L^\infty([0,1])}.$$

Integrating this inequality over $[q,1]\subset[q_c,1]$ we obtain

$$P_0(q)\geq P_c(q)\geq\sqrt{P_0^2(q)-2cM(1-q)} \quad \text{for } q\in(q_c,1].$$

This means that for sufficiently small $c>0$ we have $q_c=0$ and $P_c(q)>0$ in $[0,1)$.

Define

$$c_0:=\sup\Lambda,\ \Lambda:=\{\xi>0:\ P_c(q)>0 \text{ in } [0,1) \text{ for all } c\in(0,\xi]\}. \tag{6.5}$$

Then the above observation implies that $c_0\in(0,\infty]$. We claim that

$$c_0\leq 2\sqrt{K}, \quad \text{where } K:=\sup_{s>0}\frac{f(s)}{s}. \tag{6.6}$$

Since $f(u)\leq Ku$ for $u\geq 0$, we have

$$P_c'\geq c-\frac{Kq}{P_c} \text{ in } (q_c,1).$$

If $c\geq 2\sqrt{K}$, then the linear function $L(q)=\frac{c+\sqrt{c^2-4K}}{2}q$ satisfies

$$L'=c-\frac{Kq}{L} \text{ for } q\in\mathbb{R}^1.$$

It follows that $P_c(q) \le L(q)$ in $(q_c, 1)$, which implies that $P_c(q_c) = 0$ and $c \notin \Lambda$. Therefore $c_0 \le c$ for any such c, and hence $c_0 \le 2\sqrt{K}$.

Lemma 6.1. *For any* $0 \le c_1 < c_2 \le c_0$ *and* $\bar{c} \ge 0$,

$$P_{c_1}(q) > P_{c_2}(q) \text{ in } [0,1), \ \lim_{c\to\bar{c}} P_c(q) = P_{\bar{c}}(q) \ \text{ uniformly in } [0,1].$$

Moreover, $P_{c_0}(0) = 0$ *and* $P_{c_0}(q) > 0$ *in* $(0,1)$. *Furthermore, when* f *is of* (f_B) *or of* (f_C) *type,* $q_c > 0$ *for* $c > c_0$, *and when* f *is of* (f_M) *type,* $P_c(0) = 0$, $P_c(q) > 0$ *in* $(0,1)$ *for all* $c \ge c_0$.

Proof. When $0 \le c_1 < c_2$, from the formula for $P_c'(1)$ we find $P_{c_1}'(1) < P_{c_2}'(1)$. Since

$$P_{c_1}' < c_2 - \frac{f(q)}{P_{c_1}},$$

we find that as q decreases from $q = 1$, the curve $p = P_{c_2}(q)$ remains below the curve $p = P_{c_1}(q)$. Therefore, $q_{c_2} \ge q_{c_1}$ and for $q \in (q_{c_2}, 1)$, $P_{c_1}(q) > P_{c_2}(q)$. It follows that $P_c(q)$ is non-increasing in c for $q \in [0,1]$. Therefore for any $\bar{c} \ge 0$, as c increases to $\bar{c}$, $P_c(q)$ converges monotonically to some $R(q)$ in $[0,1]$ uniformly. $R(q)$ represents a trajectory of (6.2) with $c = \bar{c}$ that approaches $(1,0)$ from $q < 1$, and its slope at $(1,0)$ is negative. Therefore by the uniqueness of $T_{\bar{c}}$, $R(q)$ must coincide with $P_{\bar{c}}(q)$. We can similarly show that $P_c(q)$ converges to $P_{\bar{c}}(q)$ when c decreases to $\bar{c}$. Thus the curve T_c^1 varies continuously in c for $c \ge 0$.

If we assume further that $c_2 < c_0$, then by the definition of c_0 we know that $P_{c_i}(q) > 0$ in $[0,1)$ for $i = 1, 2$. Thus in this case the above argument yields $P_{c_1}(q) > P_{c_2}(q)$ for $q \in [0,1)$.

We now consider $P_{c_0}(q)$. We must have $P_{c_0}(0) = 0$, for otherwise $P_{c_0}(0) > 0$ which implies that $P_{c_1}(0) > 0$ for $c_1 > c_0$ but close to c_0. However, this implies that $P_c(q) > 0$ in $[0,1)$ for all $c \in (0, c_1]$ and thus $(0, c_1] \subset \Lambda$. But this implies $c_0 \ge c_1$, a contradiction. Thus we always have $P_{c_0}(0) = 0$.

To show that $P_{c_0}(q) > 0$ in $(0,1)$, it suffices to prove that $q_{c_0} = 0$. Suppose by way of contradiction that $q_{c_0} > 0$. Since $q_{c_0} \le \tilde{\theta}$, and $\tilde{\theta} = 0$ when f is monostable, we find that $q_{c_0} > 0$ cannot happen if f is of (f_M) type.

Suppose that f is of bistable type. Choose $\eta \in (0, q_{c_0}) \subset (0, \theta)$. Since $(\eta, 0)$ is a regular point for (6.2), there is a unique trajectory $T_{c,\eta}$ passing through $(\eta, 0)$. Since $f(\eta) < 0$, $T_{c,\eta}$ has a part in S that is a curve that can be expressed by $p = V_c(q)$, $q \in [\eta, q^c]$ for some $q^c \in (\eta, 1]$, and $(q^c, V_c(q^c))$ lies on the boundary of S, $V_c(q) > 0$ in (η, q^c),

$$V_c' = c - \frac{f(q)}{V_c} \text{ in } (\eta, q^c).$$

The curve $p = V_{c_0}(q)$, $q \in (\eta, q^{c_0})$, is increasing for $q \in (0, \theta)$ and it cannot intersect $T_{c_0}^1$. Hence it remains above $T_{c_0}^1$. This implies that $q^{c_0} = 1$. It cannot join $(1,0)$ since T_{c_0} is the only trajectory approaching this point with a non-positive slope there. Therefore necessarily $V_{c_0}(1) > 0$. Thus this curve is a piece of trajectory of (6.2) with $c = c_0$ that stays away from any equilibrium point. Hence for all c close to c_0, $V_c(q)$ stays close to $V_{c_0}(q)$ in $[\eta, 1]$. In particular, for all $c < c_0$ close to c_0, $V_c(q) > 0$ in $(\eta, 1]$. This implies that for such c, T_c^1 must lie below the curve $p = V_c(q) (\eta \le q \le 1)$. This is impossible since by the definition of c_0, for such c, $P_c(q) > 0$ in $[0,1)$, which leads to $0 = V_c(\eta) \ge P_c(\eta) > 0$.

For the case that f is of combustion type, the arguments need to be modified, since now $(\eta, 0)$ is an equilibrium point of (6.2). Choose $\epsilon > 0$ small so that $\eta - \epsilon c^{-1} > 0$. Then (η, ϵ) is a regular point of (6.2), and hence there is a unique trajectory $T_{\eta,c,\epsilon}$ passing through it. Since $f(u) = 0$ in $(0, \theta]$, we see that the trajectory is a straight line with slope c near (η, ϵ), and it intersects the q-axis at $(\eta - \epsilon c^{-1}, 0)$. Much as in the bistable case above a piece of $T_{\eta,c,\epsilon}$ in S can be expressed as $p = \hat{V}_c(q)$, and $p = \hat{V}_{c_0}(q)$ lies above $T^1_{c_0}$ with $\hat{V}_{c_0}(1) > 0$. We can now derive a contradiction in the same way as in the bistable case. Thus we must have $q_{c_0} = 0$.

If f is of (f_M) type, then for $c \geq c_0$, $P_c(0) \leq P_{c_0}(0) = 0$. On the other hand, since $q_c = 0$ we know that $P_c(q) > 0$ for $q \in (0,1)$. Thus for such c, $P_c(0) = 0$ and $P_c(q) > 0$ in $(0,1)$.

If f is of (f_B) type, then $(0,0)$ is a saddle equilibrium point of (6.2) for all $c \geq 0$, and from the ODE theory we find that there are exactly two trajectories of (6.2) that approach $(0,0)$ from $q > 0$, one denoted by T^0_c has slope $\frac{c+\sqrt{c^2-4f'(0)}}{2} > 0$ at $(0,0)$, the other has slope $\frac{c-\sqrt{c^2-4f'(0)}}{2} < 0$ at (0,0). For such f, if there exists $c > c_0$ such that $q_c = 0$, then we must have $P_c(0) = 0$ for otherwise, $P_c(0) > 0$ and by the monotonicity of $P_c(q)$ on c, we deduce $c_0 \geq c$, contradicting to the choice of c. Thus $P_c(0) = 0$, and $p = P_c(q)$, $q \in [0,1]$, represents a trajectory of (6.2) that connects $(0,0)$ and $(1,0)$. So it must coincide with T^0_c and hence $P'_c(0) = \frac{c+\sqrt{c^2-4f'(0)}}{2} > 0$. For the same reason we have $P'_{c_0}(0) = \frac{c_0+\sqrt{c_0^2-4f'(0)}}{2} > 0$. It follows that $P'_c(0) > P'_{c_0}(0)$. On the other hand, from

$$P'_c(1) = \frac{c - \sqrt{c^2 - 4f'(1)}}{2} \text{ and } P'_{c_0}(1) = \frac{c_0 - \sqrt{c_0^2 - 4f'(1)}}{2}$$

we deduce $P'_c(1) > P'_{c_0}(1)$. Thus there exists $q_* \in (0,1)$ such that $P_c(q) > P_{c_0}(q)$ in $(0, q_*)$ and $P_c(q_*) > P_{c_0}(q_*)$. It follows that $P'_c(q_*) \leq P'_{c_0}(q_*)$. However, from the differential equations we deduce $P'_c(q_*) - P'_{c_0}(q_*) = c - c_0 > 0$. This contradiction shows that we must have $q_c > 0$ for $c > c_0$ in the (f_B) case.

If f is of (f_C) type, and if $q_c = 0$ for some $c > c_0$, then we have $0 \leq P_c(0) \leq P_{c_0}(0) = 0$, and thus $P_c(0) = 0$. We notice from the differential equation for $P_c(q)$ that $P'_c(q) = c$ in $(0,\theta]$ and hence $P_c(q) = cq$ in this range. For the same reason $P_{c_0}(q) = c_0 q$ in $(0,\theta]$. Thus we again have $P'_c(0) > P'_{c_0}(0)$. We can now derive a contradiction as in the (f_B) case above.

The proof of the lemma is now complete. □

Theorem 6.2. *Let c_0 and $P_{c_0}(q)$ be defined as above. Then the trajectory represented by $p = P_{c_0}(q)$, $q \in (0,1)$, gives rise to a solution $q_0(z)$ of the problem*

$$\begin{cases} q'' - cq' + f(q) = 0 & \text{for } z \in \mathbb{R}^1, \\ q(-\infty) = 0,\ q(\infty) = 1,\ q(z) > 0 & \text{for } z \in \mathbb{R}^1, \end{cases} \tag{6.7}$$

with $c = c_0$. Moreover, $q_0(z)$ is unique up to translation of the variable z. This problem has no solution for any other nonnegative value of c if f is of (f_B) or of (f_C) type, and when f is of (f_M) type, it has a unique solution (up to translation) for every $c \geq c_0$, and has no solution for $c \in [0, c_0)$.

For each $\mu > 0$, there exists a unique $c^* = c^*_\mu \in (0, c_0)$ such that $P_{c^*}(0) = \frac{c^*}{\mu}$. Moreover, (6.1) has a unique solution $(c,q) = (c^*, q^*)$ satisfying $q'(0) = c/\mu$, and c^*_μ is increasing in μ with

$$\lim_{\mu\to\infty} c^*_\mu = c_0.$$

Proof. Let $(q_0(z), p_0(z))$, $z \in \mathbb{R}^1$, be the trajectory of (6.2) corresponding to $p = P_{c_0}(q)$, $q \in (0,1)$. Then clearly $q_0(z)$ satisfies (6.7) with $c = c_0$. Conversely, a solution of (6.7) gives rise to a function $P(q)$ satisfying (6.3). Thus $P(q) \equiv P_c(q)$. The conclusions about the existence and nonexistence of solutions to (6.7) now follow directly from Lemma 6.1. The solution is unique up to translation because the trajectory $T^1_{c_0}$ is the only one that approaches $(1,0)$ from $q < 1$ that has a negative slope there.

By Lemma 6.1 and the definition of c_0, we find that for each $c \in [0, c_0)$, $P_c(0) > 0$ and it decreases continuously as c increases in $[0, c_0]$. Moreover, $P_0(0) > 0$ and $P_{c_0}(0) = 0$. We now consider the continuous function

$$\xi(c) = \xi_\mu(c) := P_c(0) - \frac{c}{\mu},\ c \in [0, c_0].$$

By the above discussion we know that $\xi(c)$ is strictly decreasing in $[0, c_0]$. Moreover, $\xi(0) = P_0(0) > 0$ and $\xi(c_0) = -c_0/\mu < 0$. Thus there exists a unique $c^* = c^*_\mu \in (0, c_0)$ such that $\xi(c^*) = 0$.

If we view $(c^*_\mu, c^*_\mu/\mu)$ as the unique intersection point of the decreasing curve $y = P_c(0)$ with the increasing line $y = c/\mu$ in the cy-plane, then it is clear that c^*_μ increases to c_0 as μ increases to ∞.

Finally the curve $p = P_{c^*}(q)$, $q \in [0, 1)$, corresponds to a trajectory of (6.2), say $(q^*(z), p^*(z))$, $z \in [0, \infty)$, that connects the regular point $(0, P_{c^*}(0))$ with the equilibrium $(1, 0)$. It follows from (6.2) with $c = c^*$ that (c^*, q^*) solves (6.1) with $(q^*)'(0) = c^*/\mu$. If (c, q) is another solution of (6.1) satisfying $q'(0) = c/\mu$, then it corresponds to a trajectory of (6.2) connecting $(0, c/\mu)$ and $(1, 0)$ in the set S. Since for each $c \geq 0$ there is only one such trajectory joining $(1, 0)$, it coincides with $p = P_c(q)$, $q \in [0, 1)$. Thus we necessarily have $P_c(0) = c/\mu$ and hence $c = c^*$. It follows that $q = q^*$.

The proof is complete. □

Remark 6.3. *The function $q_0(z)$ is usually called a traveling wave with speed c_0. Its existence is well known. Our proof of the existence of $(c_0, q_0(z))$ is somewhat different from* [3, 4]*, so that our version of the proof can be easily used to obtain the semi-wave $q^*(z)$ and to reveal the relationship between c^* and c_0. Since $(1, 0)$ is always a saddle equilibrium point of* (6.2)*, our construction of the connecting orbit between $(0, 0)$ and $(1, 0)$, based on the latter point, is slightly simpler, compared with that in* [3, 4]*, where the construction is based on $(0, 0)$ instead.*

Proposition 1.8 clearly follows from Theorem 6.2.

Next we show how a suitable perturbation of the above setting can be used to produce a semi-wave that can be used to construct upper solutions for (1.1). Let $\hat{\theta} \in (0, 1)$ be the biggest maximum point of f in $(0, 1)$. For small $\varepsilon > 0$, let $f_\varepsilon(u)$ be a C^1 function obtained by modifying $f(u)$ over $[\hat{\theta}, 2]$ such that $f(u) \leq f_\varepsilon(u)$ for $u \in \mathbb{R}^1$, $f_\varepsilon(u)$ has a unique zero $1 + \varepsilon$ in $[\hat{\theta}, 2]$, $f'_\varepsilon(1 + \varepsilon) < 0$, and f_ε decreases to f in the C^1 norm over $[\hat{\theta}, 2]$ as ε decreases to 0.

Replacing f by f_ε, we have a parallel version of Theorem 6.2. We denote the corresponding wave and semi-wave by $(c_0^\varepsilon, q_0^\varepsilon(z))$ and $(c^*_\varepsilon, q^*_\varepsilon(z))$ respectively. We have the following result.

Proposition 6.4.

$$c_0^\varepsilon \geq c_0,\ c^*_\varepsilon > c^*,\ \lim_{\varepsilon \to 0} c_0^\varepsilon = c_0,\ \lim_{\varepsilon \to 0} c^*_\varepsilon = c^*.$$

Proof. Let $P_c^\varepsilon(q)$ denote the correspondent of $P_c(q)$. Since $f_\varepsilon \geq f$, as q decreases from $1 + \varepsilon$, the curve $p = P_c^\varepsilon(q)$ cannot touch the curve $p = P_c(q)$ from above. As before, it cannot touch $p = 0$ before q reaches $\tilde{\theta}$. Therefore $P_c^\varepsilon(q)$ is positive over $(\tilde{\theta}, 1 + \varepsilon)$ and $P_c^\varepsilon(q) > P_c(q)$ in $[q_c, 1)$. Thus for $c \in (0, c_0)$, $P_c^\varepsilon(0) > P_c(0) > 0$. This implies that $c_0^\varepsilon \geq c_0$.

Using the monotonicity of f_ε on ε, we easily deduce that $P_c^\varepsilon(q)$ is non-decreasing in ε. Thus $P_c^\varepsilon(q)$ converges to some $R(q)$ as $\varepsilon \to 0$ uniformly in $[0, 1]$. Since $p = R(q)$ represents a trajectory of (6.2) that approaches $(1, 0)$ with a non-positive slope at $(1, 0)$, and there is only one such trajectory, $R(q)$ must coincide with $P_c(q)$. In particular, we have $P_c^\varepsilon(0) \to P_c(0)$ as $\varepsilon \to 0$.

In view of the definition of c_0^ε, the monotonicity of $P_c^\varepsilon(q)$ on ε implies that c_0^ε is nondecreasing in ε. Therefore $\hat{c}_0 := \lim_{\varepsilon \to 0} c_0^\varepsilon$ exists and $\hat{c}_0 \geq c_0$.

Suppose $\hat{c}_0 > c_0$, we are going to deduce a contradiction. Choose $c \in (c_0, \hat{c}_0)$ and consider $P_c^\varepsilon(q)$. Since $c < c_0^\varepsilon$, we have $P_c^\varepsilon(q) > 0$ in $[0, 1+\varepsilon)$ for all ε.

Then in the case that f is of (f_B) type or of (f_C) type, we have $(P_c^\varepsilon(q))' \geq c$ in $(0, \theta]$ and hence $P_c^\varepsilon(q) \geq cq$ in $[0, \theta]$. Letting $\varepsilon \to 0$, we deduce $P_c(q) \geq cq$ in $[0, \theta]$. We already know from the proof of Lemma 6.1 that $P_c(q) > 0$ in $(q_c, 1) \supset (\theta, 1)$. Thus $P_c(q) > 0$ in $(0, 1)$. If $P_c(0) > 0$ then by the monotonicity in c we have $P_{c'}(0) > 0$ for all $c' \in (0, c]$ and hence $c_0 \geq c$, a contradiction to our choice of c. If $P_c(0) = 0$, then $p = P_c(q)$, $q \in (0, 1)$, represents a trajectory of (6.2) connecting $(0, 0)$ and $(1, 0)$. By Theorem 6.2 such a trajectory exists only if $c = c_0$, so we again reach a contradiction.

Thus we have proved that $\lim_{\varepsilon \to 0} c_0^\varepsilon = c_0$ when f is of (f_B) type or of (f_C) type.

We now consider the case that f is of (f_M) type. Suppose that $\hat{c}_0 > c_0$ and fix $c \in (c_0, \hat{c}_0)$. Note that from the monotonicity of c_0^ε in ε, we always have $c_0^\varepsilon \geq \hat{c}_0 > c$. Moreover, from the differential equation we easily see that as ε decreases to 0, $P_c^\varepsilon(q)$ decreases to $P_c(q)$ uniformly in $[0, 1]$, and $P_c(q) < P_{c_1}(q)$ in $(0, 1)$ if $c > c_1 > c_0$. We fix such a c_1. Thus for sufficiently small $\varepsilon > 0$, $P_c^\varepsilon(\hat{\theta}) < P_{c_1}(\hat{\theta})$. We now consider $P_c^\varepsilon(q)$ for $q \in [0, \hat{\theta}]$. We notice that in this range $f_\varepsilon(q) = f(q)$, and thus $P_c^\varepsilon(q)$ satisfies

$$P' = c - \frac{f(q)}{P}$$

for $q \in (0, \hat{\theta}]$. Since $P_c^\varepsilon(\hat{\theta}) < P_{c_1}(\hat{\theta})$, the curve $p = P_c^\varepsilon(q)$ remains below the curve $p = P_{c_1}(q)$ as q is decreased from $q = \hat{\theta}$. Thus, due to $P_{c_1}(0) = 0$ (because $c_1 > c_0$), we necessarily have $P_c^\varepsilon(0) = 0$. On the other hand, due to $c < c_0^\varepsilon$, we must have $P_c^\varepsilon(0) > 0$. This contradiction shows that $\hat{c}_0 = c_0$ in the monostable case as well.

For $c \in (0, c_0)$, since $P_c^\varepsilon(0) > P_c(0)$, we have $\xi_\varepsilon(c) := P_c^\varepsilon(0) - c/\mu > \xi(c) := P_c(0) - c/\mu$. It follows that $\xi(c^*) = 0 < \xi_\varepsilon(c^*)$, which implies $c_\varepsilon^* > c^*$ since $\xi_\varepsilon(c)$ is strictly decreasing in c. Since $P_c^\varepsilon(0)$ is non-decreasing in ε, we deduce that c_ε^* is non-decreasing in ε. The fact that $c_\varepsilon^* \to c^*$ as $\varepsilon \to 0$ now follows easily from the uniqueness of c^* as a solution of $\xi(c) = 0$.

The proof is now complete. □

Finally we show how a semi-wave can be perturbed to give a wave of finite length which is more convenient to use in the construction of lower solutions for (1.1). So let (c^*, q^*) be the unique solution to (6.1). Denote $\omega^* := c^*/\mu$ and for each $c \in (0, c^*)$ consider the problem

$$P' = c - \frac{f(q)}{P}, \; P(0) = \omega^*. \tag{6.8}$$

Since $c < c^*$, we easily see that the unique solution $P^c(q)$ of this problem stays below $P_{c^*}(q)$ as q increases from 0. Therefore there exists some $Q^c \in (0, 1]$ such that $P^c(q) > 0$ in $[0, Q^c)$ and $P^c(Q^c) = 0$. We must have $Q^c < 1$ because otherwise we would have $P^c(q) \equiv P_c(q)$ due to the uniqueness of the trajectory of (6.2) that approaches $(1, 0)$ from $q < 1$ with a non-positive slope there, but this is impossible since $P_c(0) > P_{c^*}(0) = \omega^* = P^c(0)$. It is also easily seen that, as c increases to c^*, Q^c increases to 1 and $P^c(q) \to P_{c^*}(q)$ uniformly, in the sense that $\|P^c - P_{c^*}\|_{L^\infty([0,Q^c])} \to 0$. Let $(q^c(z), p^c(z))$ denote the trajectory of (6.2) represented by the curve $p = P^c(q)$, $q \in [0, Q^c]$, with $(q^c(0), p^c(0)) = (0, \omega^*)$ and $(q^c(z^c), p^c(z^c)) = (Q^c, 0)$, then clearly $q^c(z)$ solves (4.1) with $Z = z^c$. Moreover, we have

$$c < c^* = \mu\omega^* = \mu\,(q^c)'(0) \tag{6.9}$$

and

$$\lim_{c \nearrow c^*} z^c = +\infty, \; \lim_{c \nearrow c^*} \|q^c - q^*\|_{L^\infty([0,z^c])} = 0. \tag{6.10}$$

6.2. **Asymptotic spreading speed.** Let $(c^*, q^*(z))$ be given as in Theorem 6.2. For $c \in (0, c^*)$, let $q^c(z)$, Q^c and z^c be as in the previous subsection. For $t \geq 0$ we define

$$k(t) = k_c(t) := z^c + ct$$

and

$$w(t,x) = w_c(t,x) := \begin{cases} q^c(k(t)-x), & x \in [ct, k(t)], \\ q^c(z^c), & x \in [-ct, ct], \\ q^c(k(t)+x), & x \in [-k(t), -ct]. \end{cases}$$

We will use $(w, -k, k)$ as a lower solution to (1.1) in the proof of the following result.

Lemma 6.5. *Let (u, g, h) be a solution of (1.1) for which spreading happens. Then for any $c \in (0, c^*)$ and any $\delta \in (0, -f'(1))$, there exist positive numbers T_* and M such that for $t \geq T_*$,*

(i) $[g(t), h(t)] \supset [-ct, ct]$;

(ii) $u(t,x) \geq 1 - Me^{-\tilde{\delta}t}$ *for $x \in [-ct, ct]$ and some $\tilde{\delta} = \tilde{\delta}(c) \in (0, \delta)$;*

(iii) $u(t,x) \leq 1 + Me^{-\delta t}$ *for $x \in [g(t), h(t)]$.*

Proof. (i) Fix $\hat{c} \in (c, c^*)$. Since spreading happens we can find $T_1 > 0$ such that

$$[g(T_1), h(T_1)] \supset [-k_{\hat{c}}(0), k_{\hat{c}}(0)] \text{ and } u(T_1, x) > w_{\hat{c}}(0, x) \text{ in } [-k_{\hat{c}}(0), k_{\hat{c}}(0)].$$

One then easily checks that $(w_{\hat{c}}(t-T_1, x), -k_{\hat{c}}(t-T_1), k_{\hat{c}}(t-T_1))$ is a lower solution of (1.1) for $t \geq T_1$. Hence for $t \geq T_2$ with some $T_2 > T_1$,

$$g(t) \leq -k_{\hat{c}}(t-T_1) < -\hat{c}(t-T_1) < -ct, \;\; h(t) \geq k_{\hat{c}}(t-T_1) > \hat{c}(t-T_1) > ct$$

and

$$u(t,x) \geq w_{\hat{c}}(t-T_1, x) \text{ for } x \in [-k_{\hat{c}}(t-T_1), k_{\hat{c}}(t-T_1)] \supset [-ct, ct].$$

(ii) Since $w_{\hat{c}}(t-T_1, x) \equiv q^{\hat{c}}(z^{\hat{c}}) = Q^{\hat{c}} > Q^c$ for $|x| \leq ct < \hat{c}(t-T_1)$ for all $t \geq T_2$, we find from the above estimate for u that

$$u(t,x) \geq Q^c \quad \text{for } -ct \leq x \leq ct, \; t \geq T_2.$$

Since $f'(1) < 0$, for any $\delta \in (0, -f'(1))$ we can find $\rho = \rho(\delta) \in (0,1)$ such that

$$f(u) \geq \delta(1-u) \;\; (u \in [1-\rho, 1]), \qquad f(u) \leq \delta(1-u) \;\; (u \in [1, 1+\rho]). \tag{6.11}$$

Recall that $Q^c \to 1$ as c increases to c^*. Without loss of generality we may assume that c has been chosen so that $Q^c > 1 - \rho$.

Fix $T \geq T_2$ and let ψ be the solution of

$$\begin{cases} \psi_t = \psi_{xx} - \delta(\psi - 1), & -cT < x < cT, \; t > 0, \\ \psi(t, \pm cT) \equiv Q^c, & t > 0, \\ \psi(0, x) \equiv Q^c, & -cT \leq x \leq cT. \end{cases} \tag{6.12}$$

Since $\underline{\psi} \equiv Q^c$ is a lower solution of the corresponding elliptic problem of (6.12), and $\overline{\psi} \equiv 1$ is an upper solution, $\psi(t,x)$ increases in t and $\psi \in [Q^c, 1]$. Moreover, ψ is a lower solution for the equation satisfied by $u(t+T, x)$ in the region $(t,x) \in [0, \infty) \times [-cT, cT]$, and so

$$\psi(t,x) \leq u(t+T, x) \quad \text{for } -cT \leq x \leq cT, \; t \geq 0. \tag{6.13}$$

Set $\Psi := (\psi - Q^c)e^{\delta t}$, then

$$\begin{cases} \Psi_t = \Psi_{xx} + \delta(1 - Q^c)e^{\delta t}, & -cT < x < cT, \; t > 0, \\ \Psi(t, \pm cT) \equiv 0, & t > 0, \\ \Psi(0, x) \equiv 0, & -cT \leq x \leq cT, \end{cases} \tag{6.14}$$

The Green function of this problem can be expressed in the form (see page 84 of [13])

$$\widetilde{G}(t,x)=\sum_{n\in\mathbb{Z}}(-1)^n G(t,x-2ncT),$$

which yields

$$\widetilde{G}(t,x)\geq \widehat{G}(t,x):=G(t,x)-G(t,x-2cT)-G(t,x+2cT),$$

where G is the fundamental solution of the heat equation:

$$G(t,x)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}.$$

Hence

$$\begin{aligned}\Psi(t,x) &= \int_0^t d\tau\int_{-cT}^{cT}\widetilde{G}(t-\tau,x-\xi)\delta(1-Q^c)e^{\delta(t-\tau)}d\xi\\ &\geq \delta(1-Q^c)\int_0^t e^{\delta(t-\tau)}d\tau\int_{-cT}^{cT}\widehat{G}(t-\tau,x-\xi)d\xi\end{aligned}$$

For any $\epsilon\in(0,1)$, consider (t,x) satisfying

$$|x|\leq(1-\epsilon)cT,\quad 0<t\leq\frac{\epsilon^2c^2T}{4}. \tag{6.15}$$

For such (t,x) and any $\tau\in(0,t)$, we have

$$\frac{cT\pm x}{2\sqrt{t-\tau}}\geq\frac{\epsilon cT}{2\sqrt{t-\tau}}\geq\frac{\epsilon cT}{2\sqrt{t}}=\sqrt{T}\cdot\frac{\epsilon c\sqrt{T}}{2\sqrt{t}}\geq\sqrt{T}\geq 1. \tag{6.16}$$

So for (t,x) satisfying (6.15) we have

$$\int_{-cT}^{cT}G(t-\tau,x-\xi)d\xi=\left(\int_{-\infty}^{\infty}-\int_{-\infty}^{-cT}-\int_{cT}^{\infty}\right)G(t-\tau,x-\xi)d\xi=1-I_1-I_2$$

where

$$I_1:=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{-\frac{cT+x}{2\sqrt{t-\tau}}}e^{-r^2}dr,\ I_2:=\frac{1}{\sqrt{\pi}}\int_{\frac{cT-x}{2\sqrt{t-\tau}}}^{\infty}e^{-r^2}dr.$$

Using the elementary inequality

$$\int_y^\infty e^{-r^2}dr\leq\int_y^\infty re^{-r^2/2}dr=e^{-y^2/2}\text{ for all }y\geq 1,$$

$(cT\pm x)\geq\epsilon cT$, and (6.16), we deduce

$$I_1,\ I_2\leq\frac{1}{\sqrt{\pi}}e^{-\frac{(\epsilon cT)^2}{8(t-\tau)}}.$$

But (6.16) also infers

$$\frac{(\epsilon cT)^2}{8(t-\tau)}\geq T/2.$$

Thus

$$I_1,\ I_2\leq\frac{1}{\sqrt{\pi}}e^{-T/2},$$

and

$$\int_{-cT}^{cT}G(t-\tau,x-\xi)d\xi\geq 1-\frac{2}{\sqrt{\pi}}e^{-T/2}.$$

Similarly,

$$\begin{aligned}\int_{-cT}^{cT} G(t-\tau, x-\xi-2cT)d\xi &= \frac{1}{\sqrt{\pi}}\int_{-cT}^{cT}\frac{1}{2\sqrt{t-\tau}}e^{-\frac{(x-\xi-2cT)^2}{4(t-\tau)}}d\xi \\ &\le \frac{1}{\sqrt{\pi}}\int_{-cT}^{\infty}\frac{1}{2\sqrt{t-\tau}}e^{-\frac{(x-\xi-2cT)^2}{4(t-\tau)}}d\xi \\ &= \frac{1}{\sqrt{\pi}}\int_{\frac{cT-x}{2\sqrt{t-\tau}}}^{\infty} e^{-r^2}dr \le \frac{1}{\sqrt{\pi}}e^{-T/2}.\end{aligned}$$

Consequently, for (t,x) satisfying (6.15), we have

$$\Psi(t,x) \ge \delta(1-Q^c)\int_0^t e^{\delta(t-\tau)}\left(1-\frac{4}{\sqrt{\pi}}e^{-T/2}\right)d\tau = (1-Q^c)\left[1-\frac{4}{\sqrt{\pi}}e^{-T/2}\right](e^{\delta t}-1).$$

This implies that, for such (t,x),

$$\psi(t,x) \ge 1-\frac{4}{\sqrt{\pi}}e^{-T/2}-e^{-\delta t}.$$

Taking $t=\frac{\epsilon^2c^2}{4}T$ we obtain

$$\psi\Big(\frac{\epsilon^2c^2}{4}T,x\Big) \ge 1-\frac{4}{\sqrt{\pi}}e^{-T/2}-e^{-\epsilon^2c^2\delta T/4}.$$

We only focus on small $\epsilon>0$ such that $\epsilon^2c^2\delta<2$, so

$$\psi\Big(\frac{\epsilon^2c^2}{4}T,x\Big) \ge 1-M_0e^{-\epsilon^2c^2\delta T/4} \quad \text{with } M_0 := \frac{4}{\sqrt{\pi}}+1$$

for $|x|\le(1-\epsilon)cT$ and $T\ge T_2$.

By (6.13), for such T and x, we have

$$u\Big(\frac{\epsilon^2c^2}{4}T+T,x\Big) \ge 1-M_0e^{-\epsilon^2c^2\delta T/4}.$$

Finally, if we rewrite

$$t=\frac{\epsilon^2c^2}{4}T+T,$$

then

$$T=\left(1+\frac{\epsilon^2c^2}{4}\right)^{-1}t.$$

Thus

$$u(t,x)\ge 1-M_0e^{-\tilde{\delta}t} \quad \text{for } |x|\le(1-\epsilon)\left(1+\frac{\epsilon^2c^2}{4}\right)^{-1}ct,\ t\ge T_3,$$

where $\tilde{\delta}:=\frac{\epsilon^2c^2}{4}\left(1+\frac{\epsilon^2c^2}{4}\right)^{-1}\delta$ and $T_3:=\frac{\epsilon^2c^2}{4}T_2+T_2$. Since this is true for any $c\in(0,c^*)$ close to c^*, and any small $\epsilon>0$, the above estimate implies the conclusion in (ii).

(iii) Consider the equation $\eta'(t)=f(\eta)$ with initial value $\eta(0)=\|u_0\|_{L^\infty}+1$. Then η is an upper solution of (1.1). So $u(t,x)\le\eta(t)$ for all $t\ge0$. Since $f(u)<0$ for $u>1$, $\eta(t)$ is a decreasing function converging to 1 as $t\to\infty$. Hence there exists $T_4>0$ such that $\eta(t)<1+\rho$ for $t\ge T_4$. Now, for $t\ge T_4$, $\eta'(t)=f(\eta)\le\delta(1-\eta)$, and so

$$u(t,x)\le\eta(t)\le 1+\rho e^{-\delta(t-T_4)} \quad \text{for } g(t)\le x\le h(t),\ t\ge T_4.$$

This completes the proof. □

Proof of Theorem 1.9. Assume that spreading happens. Then for any given small $\varepsilon > 0$, we can apply Lemma 6.5 to obtain some $T_1 > 0$ large such that for $t \geq T_1$,

$$[g(t), h(t)] \supset [-(c^* - \varepsilon)t, (c^* - \varepsilon)t] \text{ and } |u(t,x) - 1| \leq Me^{-\delta t} \text{ for } |x| \leq (c^* - \varepsilon)t. \tag{6.17}$$

We now make use of the perturbation method introduced in the previous subsection. For small $\varepsilon_1 > 0$, we modify f to obtain f_{ε_1} and $(c^*_{\varepsilon_1}, q^*_{\varepsilon_1})$ as described there. Since $c^*_{\varepsilon_1} \to c^*$ as $\varepsilon_1 \to 0$, we can choose $\varepsilon_1 > 0$ small enough such that $c^*_{\varepsilon_1} < c^* + \varepsilon$.

By Lemma 6.5 we see that for some large $T_2 > 0$, $u(t,x) < 1 + \varepsilon_1/2$ for $t \geq T_2$. We then choose $M' > 0$ large enough such that

$$-c^*_{\varepsilon_1} T_2 - M' < g(T_2), \;\; c^*_{\varepsilon_1} T_2 + M' > h(T_2),$$

and

$$1 + \varepsilon_1/2 < q^*_{\varepsilon_1}(c^*_{\varepsilon_1} T_2 + M' - x) \quad \text{for } x \in [g(T_2), h(T_2)].$$

Therefore if we define

$$k(t) := c^*_{\varepsilon_1} t + M', \; w(t,x) := q^*_{\varepsilon_1}(k(t) - x),$$

then (w, g, k) is an upper solution of (1.1) for $t \geq T_2$, and we can use Lemma 2.2 to deduce that

$$h(t) \leq c^*_{\varepsilon_1} t + M' < (c^* + \varepsilon)t + M' \quad \text{for } t \geq T_2.$$

We can similarly show that

$$g(t) \geq -(c^* + \varepsilon)t - M' \quad \text{for } t \geq T_2.$$

These estimates and (6.17) clearly imply

$$\lim_{t\to\infty} \frac{-g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = c^*.$$

This completes the proof of Theorem 1.9. □

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