Spreading properties of beams radiated by partially coherent Schell-model sources

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We derive a general law for the M^2 factor of any beam generated by a partially coherent Schell-model source. The fourth power of M differs from its minimum value, attained in the coherent limit, by a term proportional to the second derivative of the modulus of the spectral degree of coherence evaluated for zero argument. Examples are given for cases of practical interest. © 1999 Optical Society of America [S0740-3232(99)01501-X]

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1. INTRODUCTION

Much research has been carried out in recent years on the development of tools for defining and measuring the quality of optical beams and resonators.[1] In particular, one of the most common parameters that characterize the quality of a light beam, the so-called M^2 factor,[2] keeps on playing a basic role in light-beam characterization.[1]

A host of general properties have been established for the coherent case,[1,3,4] and the effects of the coherence properties of a light source on the spreading features of the radiated beam have been extensively studied.[1,5–9] Nonetheless, for the partially coherent case, analytical results are restricted mainly to so-called Gaussian Schell-model (GSM) beams,[6,10,11] twisted GSM beams,[12–14] and further incoherent superpositions of Gaussian modes.[15,16]

Among classes of sources encountered in partial coherence theory, one of the most important is that of Schell-model sources.[17] One may wonder whether the analytical structure of the cross-spectral density (CSD) of a Schell-model source is sufficient for deriving some significant property of the M^2 factor of beams generated by such sources. In this paper we show that the M^2 factor of these beams has a simple expression. More precisely, the fourth power of M is obtained from the value that would pertain to the beam in the coherent limit by adding a term proportional to the second derivative of the modulus of the spectral degree of coherence at the origin.

In Section 2 a definition and some properties of the M^2 factor are given, and the analytical derivation of our result is reported in Section 3. As examples, in Section 4 two classes of Schell-model sources are considered: GSM sources and J_0-correlated sources.[18] For the latter case, an alternative derivation, based on the Wolf's modal expansion,[17] is presented in Appendix A.

2. M^2 FACTOR OF A PARTIALLY COHERENT BEAM

To simplify the formalism we consider a one-dimensional source, that is, a source whose characteristics depend only on one of the coordinates of the transverse plane. Later we will see how the results can be extended to the more general case of a two-dimensional source.

Under suitable hypotheses, the partially coherent source is characterized by the CSD function $W_0(x_1, x_2)$.[17] Since the temporal frequency is assumed to be fixed, the explicit dependence of the CSD on it has been omitted. The field generated by such a source can propagate in the form of a beam.[17] We suppose that the source plane ($z = 0$, in a suitable reference frame) need not coincide with the waist plane of the radiated beam, which we assume to be located at $z = \zeta$. In this case, in the paraxial approximation, the M^2 factor of the beam is defined as[2]

$$M^2 = 4\pi\sigma_\zeta\sigma_\infty, \tag{1}$$

where σ_ζ and σ_∞ are the square roots of the variances of the transverse intensity profiles at the waist plane, $I_\zeta(x)$, and in the far field, $I_\infty(p)$. Here p has dimensions of a spatial frequency and is related to the far-field angular coordinate θ through the relation

$$p = \theta/\lambda, \tag{2}$$

where λ is the wavelength. More precisely, we have

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$$\sigma_\zeta^2 = \frac{1}{N}\int (x - \bar{x}_\zeta)^2 I_\zeta(x)\mathrm{d}x, \tag{3}$$

$$\sigma_\infty^2 = \frac{1}{N}\int (p - \bar{p})^2 I_\infty(p)\mathrm{d}p, \tag{4}$$

where N denotes the total power carried by the beam, that is,

$$N = \int I_\zeta(x)\mathrm{d}x = \int I_\infty(p)\mathrm{d}p, \tag{5}$$

and where

$$\bar{x}_\zeta = \frac{1}{N}\int x I_\zeta(x)\mathrm{d}x, \tag{6}$$

$$\bar{p} = \frac{1}{N}\int p I_\infty(p)\mathrm{d}p \tag{7}$$

are the mean transverse coordinate at the waist and the mean propagation direction of the beam, respectively. In view of the above remarks on the physical meaning of p, we see that $\lambda\bar{p}$ is the mean propagation angle of the beam and $\lambda\sigma_\infty$ is the angular spread.

The meaning of M^2 is that it specifies the far-field divergence properties of a light beam, once the width at the waist has been fixed. It can be proved that, for a fixed spot size at the waist, the least diverging beam is a coherent Gaussian beam of zero order. Accordingly, M^2 measures the different properties in diffractive spread of a general beam (whose M^2 is always >1) compared with a Gaussian beam ($M^2 = 1$).

In the following, for simplicity we will assume that the first-order moments of the intensity distributions (both at the source plane, $z = 0$, and in the far field) are set to zero. We obtain these values by suitably choosing the origin of the x axis and the direction of the z axis, which has to coincide with the propagation axis of the beam. In particular, it can easily be seen that these conditions imply that $\bar{x}_z = 0$ for any z. The situation is sketched in Fig. 1(a).

Since we want to derive a relationship between M^2 and the coherence properties of the source, we have to express the quantities that appear in Eq. (1) in terms of those at the plane $z = 0$, which we can do by considering that the variance of the transverse profile follows a parabolic law,[2] from which we obtain

$$\sigma_\zeta^2 = \sigma_0^2 - \lambda^2\sigma_\infty^2\zeta^2. \tag{8}$$

Moreover, the value of ζ can be expressed in terms of W_0 in the following way[9]:

$$\zeta = \frac{J}{2\pi\lambda\sigma_\infty^2}, \tag{9}$$

with

$$J = \frac{1}{N}\,\mathrm{Im}\int p\left[\frac{\partial\tilde{W}_0(-p_1, p_2)}{\partial p_1}\right]_{\substack{p_1=p\\ p_2=p}}\mathrm{d}p, \tag{10}$$

where Im stands for the imaginary part and the tilde denotes the two-dimensional Fourier transform.

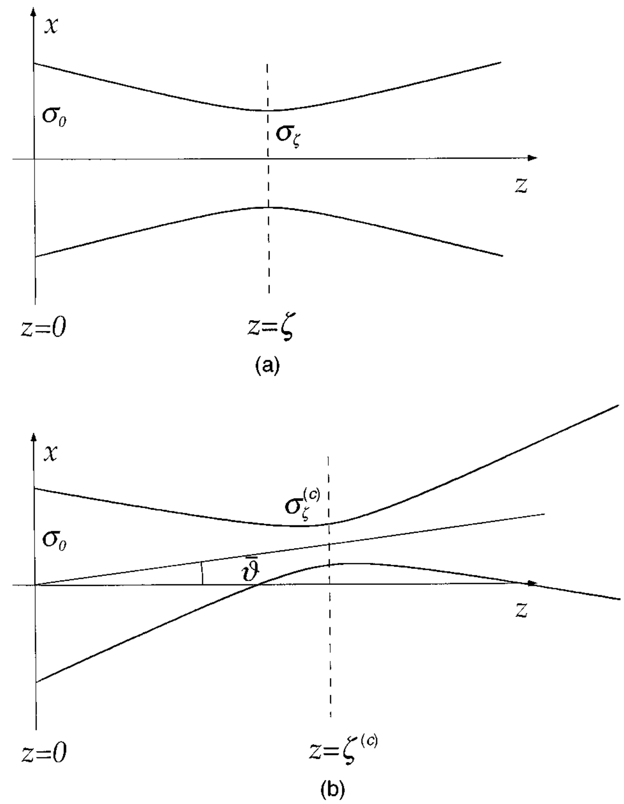

Fig. 1. Pictorial representation of the propagated beam for (a) the partially coherent source and (b) the associated coherent source.

On inserting from Eqs. (8) and (9) into Eq. (1), we obtain immediately

$$M^2 = (16\pi^2\sigma_0^2\sigma_\infty^2 - J^2)^{1/2}. \tag{11}$$

The width σ_∞ can be directly expressed in terms of the CSD of the field at the plane $z = 0$ by means of the following formula[9]:

$$\sigma_\infty^2 = \frac{1}{4\pi^2 N}\int\left[\frac{\partial^2 W_0(x_1, x_2)}{\partial x_1\partial x_2}\right]_{\substack{x_1=x\\ x_2=x}}\mathrm{d}x, \tag{12}$$

so Eq. (11) represents the generalization of Eq. (1) to the case in which the field characteristics are known only at the plane $z = 0$.

Results derived in this section can be obtained with different approaches, such as those based on the Wigner distribution function and on variance matrix V.[1,8,11,19–21]

3. CASE OF A SCHELL-MODEL SOURCE

In this section we specialize Eq. (11) to the case of a Schell-model source by using the pertinent CSD to evaluate J and σ_∞.

Let us recall that the CSD of a Schell-model source is

$$W_0(x_1, x_2) = T^*(x_1)T(x_2)g(x_1 - x_2), \tag{13}$$

where $T(x)$ is a generally complex function such that $|T(x)|^2 = I_0(x)$ is the optical intensity. Further, the asterisk stands for the complex conjugate, and $g(u)$ is the degree of spectral coherence ($u = x_1 - x_2$).[17]

It is convenient to write the degree of coherence by means of its modulus and phase; that is,

$$g(u) = \alpha(u)\exp[i\psi(u)], \tag{14}$$

with α and ψ real functions. Well-known properties of the CSD[17] imply that

$$\alpha(u) = \alpha(-u), \qquad \psi(u) = -\psi(-u); \tag{15}$$

i.e., α is an even function of u, whereas ψ is odd. These properties will be used in the following, where we shall assume that α and ψ and their derivatives, at least up to the second order, are continuous functions.

On inserting from Eq. (13) into Eq. (10), after some calculations we obtain

$$J = \mathrm{Im} \int T^*(x)T'(x)\mathrm{d}x, \tag{16}$$

so that J is actually independent of the degree of coherence. This important property of J will be essential for deriving the results of the present section. Note that if T is a real function, $J = 0$, regardless of the expression of g. This implies that, in such a case, the source plane always coincides with the waist plane of the propagated beam.

As regards σ_∞ [see Eq. (12)], the second derivative of the CSD in Eq. (13) turns out to be

$$\left(\frac{\partial^2 W_0}{\partial x_1 \partial x_2}\right)_{\substack{x_1=x\\x_2=x}} = |T'(x)|^2 g(0) - |T(x)|^2 g''(0) + \{T^*(x)T'(x) - [T'(x)]^*T(x)\}g'(0). \tag{17}$$

The quantities $g'(0)$ and $g''(0)$ can be profitably expressed in terms of the functions α and ψ, defined in Eqs. (15). Indeed, when the symmetry properties of these functions (and of their derivatives) and the condition $g(0) = 1$ are used, it follows that

$$g'(0) = i\psi'(0), \qquad g''(0) = \alpha''(0) - \psi'^2(0). \tag{18}$$

From Eqs. (12), (17), and (18) we see that σ_∞^2 can be written as

$$\sigma_\infty^2 = \frac{1}{4\pi^2 N}\int |T'(x)|^2\mathrm{d}x - \frac{1}{4\pi^2}[\alpha''(0) - \psi'^2(0)] + \mathrm{i}\,\frac{\psi'(0)}{2\pi^2 N}\int T^*(x)T'(x)\mathrm{d}x, \tag{19}$$

where the equality $|T(x)|^2 = I_0(x)$ and Eq. (5) have been used as well as the relation

$$\int T^*(x)T'(x)\mathrm{d}x = -\int [T'(x)]^*T(x)\mathrm{d}x, \tag{20}$$

which can be easily proved through integration by parts.

An easier interpretation of Eq. (19) can be obtained through Fourier analysis. Using well-known theorems,[22] we obtain

$$\sigma_\infty^2 = \frac{1}{N}\int p^2|\tilde{T}(p)|^2\mathrm{d}p - \frac{1}{4\pi^2}[\alpha''(0) - \psi'^2(0)] - \frac{\psi'(0)}{2\pi N}\int p|\tilde{T}(p)|^2\mathrm{d}p, \tag{21}$$

where $\tilde{T}(p)$ is the Fourier transform of the function $T(x)$.

The integrals in Eq. (21) can be interpreted as follows: Let us consider the coherent source that has a field distribution equal to $T(x)$ at the plane $z = 0$, that is, the source specified by the CSD of Eq. (13), when the degree of spectral coherence equals 1 for any value of $(x_1 - x_2)$. Such source will be referred to as the coherent source associated with the partially coherent one. Of course, the intensity profile at a typical cross section of the beam radiated by the associated coherent source differs, in general, from that of the partially coherent beam, except for $z = 0$. In particular, $\tilde{T}(p)$ can be taken as the far-zone field distribution of the coherent beam, and the corresponding far-zone intensity will be given by

$$I_\infty^{(c)}(p) = |\tilde{T}(p)|^2. \tag{22}$$

This means that the integrals in Eq. (21) can be interpreted as moments of the intensity profile of the coherent beam in the far zone, which suggests that the M^2 of the partially coherent beam can be somehow related to that of the associated coherent beam. To this end, we introduce the mean propagation direction of the associated coherent beam, namely, $\bar{p}^{(c)}$, and the square root of the variance of the intensity profile in the far zone, $\sigma_\infty^{(c)}$, defined through expressions similar to those given in Eqs. (7) and (4).

It is worth noting that not only the variance in the far zone but also the mean propagation direction of the coherent beam is generally different from that of the partially coherent beam, $\bar{p}$. This means that, even if we choose the z axis to be coincident with the propagation direction of the partially coherent beam, this coincidence will not necessarily occur for the associated coherent beam. The situation that corresponds to the propagated coherent beam is sketched in Fig. 1(b), where the longitudinal coordinate of the waist plane, $\zeta^{(c)}$, is also shown. It should be noted that, although the parameter J is independent of the coherence state of the source, this is not the case for the position of the waist plane, because of the relation in Eq. (9), where J is divided by σ_∞^2, which depends on g, as will be clear below.

On inserting from Eqs. (7) and (4) into Eq. (21) we obtain the following expression for σ_∞^2:

$$\sigma_\infty^2 = {\sigma_\infty^{(c)}}^2 - \frac{\alpha''(0)}{4\pi^2} + \left[\bar{p}^{(c)} - \frac{1}{2\pi}\psi'(0)\right]^2. \tag{23}$$

As we are going to show in a moment, the term in square brackets in Eq. (23) gives the mean propagation direction of the partially coherent beam. Indeed, the expression of $\bar{p}$ in terms of the CSD of the source is (see, for example, Ref. 23)

$$\bar{p} = \frac{1}{N}\int p\tilde{W}_0(-p, p)\mathrm{d}p. \tag{24}$$

On inserting a CSD of the form of Eq. (13) into Eq. (24), we obtain after some calculations

$$\bar{p} = \frac{i}{2\pi}\left[g'(0) - \frac{1}{N}\int T^*(x)T'(x)\mathrm{d}x\right], \tag{25}$$

which, using Eqs. (18), we can easily transform into the form

$$\bar{p} = \bar{p}^{(c)} - \frac{1}{2\pi}\psi'(0). \tag{26}$$

The right-hand side of the Eq. (26) just coincides with the term in square brackets in Eq. (23). Since we have assumed that the z axis coincides with the mean propagation direction of the partially coherent beam, that term vanishes and we have

$$\sigma_\infty^2 = \sigma_\infty^{(c)2} - \frac{\alpha''(0)}{4\pi^2}. \tag{27}$$

Since, as can be easily seen, $\alpha''(0) \leq 0$, Eq. (27) confirms the intuitive feeling that a reduction of the coherence features of the source leads to a widening of the angular spread of the radiated beam. Consequences of this fact [from Eqs. (9) and (8)] are that, in the case of the coherent source, the width at the waist is smaller and the distance between the source and the waist plane is greater than for the partially coherent case.

By use of Eq. (27), the expression for the M^2 factor of the beam radiated by the partially coherent source turns out to be, from Eq. (11),

$$M^2 = [16\pi^2\sigma_0^2\sigma_\infty^2 - 4\sigma_0^2\alpha''(0) - J^2]^{1/2}. \tag{28}$$

Now, if we recall that the quantity J^2 is independent of the coherence state of the source [see Eq. (16)], we can join it with the first term in square brackets in Eq. (28) to obtain, through Eq. (11), the M^2 factor, say, M_c^2, of the beam radiated by the associated coherent source. We thus obtain

$$M^2 = [M_c^4 - 4\sigma_0^2\alpha''(0)]^{1/2}. \tag{29}$$

Equation (29) represents the main result of this study and shows how the coherence properties of a Schell-model source affect the spreading features of the radiated beam. It is remarkable that the quality factor depends on the degree of coherence only through the second-order derivative of its modulus, evaluated at $x_1 - x_2 = 0$. This means, in particular, that two sources with the same intensity profile but different coherence functions will show the same spreading properties (in the sense of the second-order moments), provided that the quantity $\alpha''(0)$ is the same for both the sources.

Although we obtained Eq. (29) by starting from a one-dimensional source, it could be easily shown, following an analogous procedure, that Eq. (29) retains its validity when the M^2 factor of the beam radiated by a two-dimensional source has to be evaluated along one of the transverse coordinates.

Moreover, for cases in which intensity distribution is axially symmetric, Eq. (29) can be extended to the radial quality factor, M_r^2, which is defined as[2]

$$M_r^2 = 2\pi\sigma_\zeta\sigma_\infty, \tag{30}$$

where now σ_ζ^2 and σ_∞^2 are the radial second-order moments of the intensity distributions.[2] Such an extension reads as

$$M_r^2 = [M_{\mathrm{cr}}^4 - \sigma_0^2\nabla^2\alpha(0, 0)]^{1/2}, \tag{31}$$

where σ_0^2 is the radial variance of the source intensity distribution, ∇^2 denotes the Laplace operator, and M_{cr}^2 is the radial quality factor in the coherent limit.

The main virtue of Eqs. (29) and (31) is that they allow us to evaluate the M^2 factor of the beam produced by any Schell-model source by solving the propagation problem for the coherent limit instead of for the more demanding partially coherent case. This is particularly useful whenever the coherent solution is already known, as we shall see through the examples given in Section 4. It should be noted, however, that this does not apply to sources, such as those of the twisted Gaussian Schell-model type, for which the spectral degree of coherence $\mu(\mathbf{r}_1, \mathbf{r}_2)$ does not depend only on the vectorial difference $\mathbf{r}_1 - \mathbf{r}_2$.[12]

4. EXAMPLES

A. Gaussian Schell-Model Source

The CSD of a one-dimensional GSM source can be written in the form of Eq. (13), with the functions T and g given by[24]

$$T(x) = T_0 \exp\left(-\frac{x^2}{4\sigma_0^2}\right), \tag{32}$$

$$g(u) = \exp\left(-\frac{u^2}{2\sigma_\mu^2}\right). \tag{33}$$

Here, T_0 is a positive parameter and σ_0^2 and σ_μ^2 are the variances of the intensity profile and of the degree of coherence, respectively.

Because of the Gaussian form of the function T, the M^2 factor of the beam radiated by the associated coherent source is unitary, and we can write

$$M^2 = [1 - 4\sigma_0^2\alpha''(0)]^{1/2}. \tag{34}$$

Since g is real, it also coincides with the function α defined in Eq. (14). By performing the second derivative of α we obtain

$$\alpha''(u) = -\frac{1}{\sigma_\mu^2}\left(1 - \frac{u^2}{\sigma_\mu^2}\right)\exp\left(-\frac{u^2}{2\sigma_\mu^2}\right), \tag{35}$$

so, from Eq. (34), the M^2 factor of the GSM beam turns out to be

$$M^2 = \left(1 + 4\frac{\sigma_0^2}{\sigma_\mu^2}\right)^{1/2}, \tag{36}$$

in agreement with the propagation law derived in Ref. 10. It is worth recalling that for this kind of beam a global coherence parameter can be defined as

$$\eta = \sigma_\mu/\sigma_0. \tag{37}$$

From Eq. (36) we see that the M^2 factor depends only on such a parameter and that the beam quality is an increasing function of it, as expected.

Starting from Eq. (31), it can be easily shown that Eq. (36) also gives the radial beam quality factor for a two-dimensional GSM beam, provided that the quantities σ_0^2 and σ_μ^2 now represent the two-dimensional variances of the intensity and the degree, respectively, of the coherence profiles.

B. Gaussian J_0-Correlated Schell-Model Source

In this section we study the spreading properties of another class of partially coherent fields originated by a Schell-model planar source, namely, the J_0-correlated Schell-model (JSM) beams.[18] Sources of this kind were first synthesized starting from an annular incoherent source,[25] and later it was shown that the radiated beams constitute a class of partially coherent modes inside a Fabry–Perot resonator.[26] The degree of coherence of such sources is a Bessel function of the first kind and zero order, whereas, as for the previous example, the intensity profile has been chosen as Gaussian. Thus we have

$$T(\mathbf{r}) = T_0 \exp\left(-\frac{r^2}{2\sigma_0^2}\right), \tag{38}$$

$$g(\mathbf{u}) = J_0(\beta|\mathbf{u}|), \tag{39}$$

where β is a positive parameter.

By using Eq. (31) and taking the fact that g is real into account (so $\alpha = g$), we have

$$\nabla^2\alpha = \nabla^2 g = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}}{\mathrm{d}r}J_0(\beta r)\right] = -\frac{\beta}{r}\frac{\mathrm{d}}{\mathrm{d}r}[rJ_1(\beta r)], \tag{40}$$

and, using the well-known relation[27]

$$\frac{\mathrm{d}}{\mathrm{d}x}[xJ_1(x)] = xJ_0(x), \tag{41}$$

after some calculation we obtain

$$\nabla^2\alpha(0, 0) = -\beta^2. \tag{42}$$

Since in this case [see Eq. (38)] $M_{\mathrm{cr}}^2 = 1$, we obtain the following expression for the radial quality factor for JSM beams:

$$M_r^2 = (1 + \beta^2\sigma_0^2)^{1/2}. \tag{43}$$

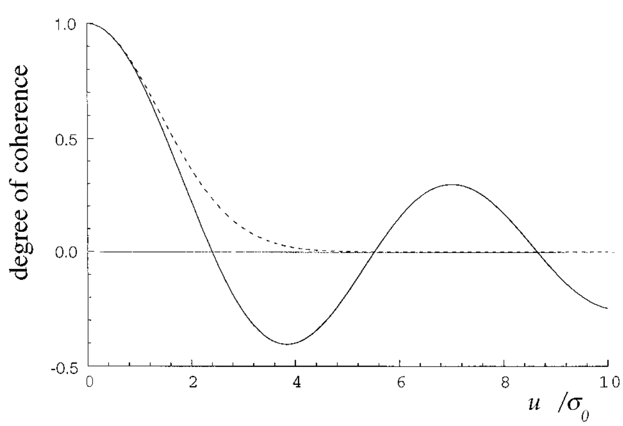

Fig. 2. Degree of coherence, as a function of the normalized variable u/σ_0, for a GSM source (dashed curve) and a JSM source (solid curve) with the same global coherence parameter $\eta = 2$.

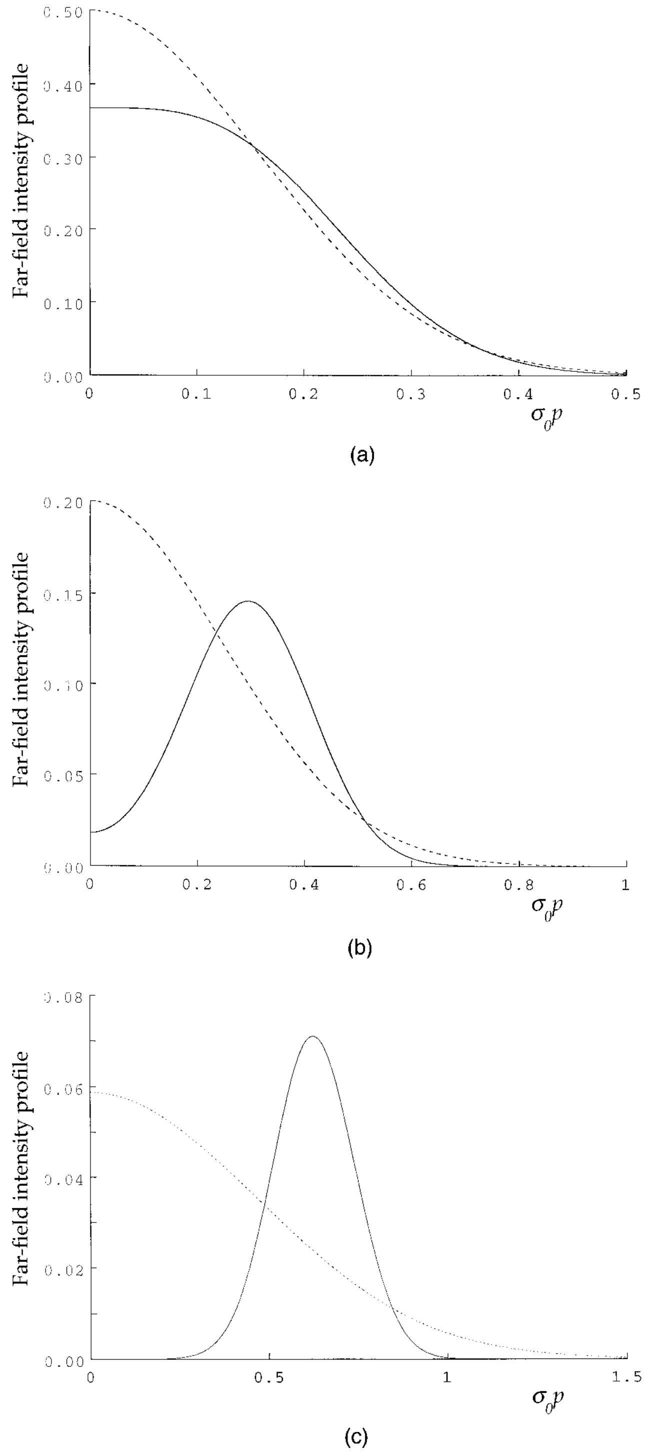

Fig. 3. Far-field intensity profiles for a GSM source (dashed curves) and a JSM source (solid curves) with the same global coherence parameter, as functions of the normalized spatial frequency $\sigma_0 p$. Values of η are (a) 2, (b) 1, and (c) 0.5.

An alternative procedure, which leads to the same result starting from knowledge of the modal expansion of the J_0-correlated source,[18] is reported in Appendix A.

Notice that, similarly to the case of GSM sources, a global coherence parameter can be defined as

$$\eta = \frac{2}{\sigma_0\beta}, \tag{44}$$

which is sufficient to completely characterize a JSM beam with respect to its divergence properties. In particular, Eq. (43) coincides with Eq. (36) if the η parameters of the two sources coincide, i.e., if

$$\sigma_\mu = 2/\beta. \tag{45}$$

As an example, in Fig. 2 a plot of the degree of coherence of a JSM source is shown (solid curve), together with that of the associated GSM source (dashed curve), for $\eta = 2$.

The two sources exhibit exactly the same intensity profile but, because of the different coherence properties with which they are endowed, the propagated beams behave in different ways, as we can understood by exploiting the results of Wolf's modal theory of coherence.[17] Indeed, since the coherent modes of a GSM source[24,28] are shape invariant, through paraxial propagation the beam obtained as an incoherent superposition of such modes keeps this property.[29] This is not true for a JSM beam, because it is obtained by superimposing Bessel–Gauss beams,[18,30,31] which present approximately the same transverse profile only within a certain distance from the waist plane. Despite the difference in shape of the propagated profiles, the coincidence between the M^2 factors of the beams ensures that their propagated widths are equal to each other at any transverse plane.

In Fig. 3 we show far-field intensity profiles of a GSM beam (dashed curves) and a JSM beam (solid curves) with identical source intensity profiles and the same global coherence parameter [see Eq. (45)], as a function of the normalized far-field coordinate $\sigma_0 p$, for some values of the global coherence parameter η.

5. CONCLUSIONS

A simple rule for determining the M^2 factor of the beam radiated by any Schell-model source has been derived. It stems from the knowledge of the M^2 factor of the beam radiated by the associated coherent source, which is characterized by the same CSD function as the partially coherent source but with the degree of coherence set to 1 for all pairs of points at the source plane. Thus the rule turns out to be particularly useful whenever the solution for the coherent case is already known, as for a Gaussian-shaped source. Examples have been given for GSM and J_0-correlated sources.

APPENDIX A: ALTERNATIVE EVALUATION OF M^2 FOR A J_0-CORRELATED BEAM

We start from the CSD in Eq. (13), with T and g given by Eqs. (38) and (39), respectively. For such a source the waist coincides with the plane $z = 0$, so the radial M^2 factor is given by Eq. (30) with $\sigma_\zeta = \sigma_0$.

To evaluate the beam width in the far field we use the standard techniques of the modal theory of coherence in the spatial-frequency domain.[32] To this end we introduce the modal decomposition of the CSD of a Gaussian J_0-correlated source, i.e.,

$$W_0(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n=-\infty}^{+\infty} \lambda_n \phi_n^*(\mathbf{r}_1)\phi_n(\mathbf{r}_2), \tag{A1}$$

where the eigenvalue λ_n and the eigenfunctions $\phi_n(\mathbf{r})$ are given by[18,31]

$$\lambda_n = \pi\sigma_0^2 T_0^2 \exp\left(-\frac{\beta^2\sigma_0^2}{2}\right) I_n\left(\frac{\beta^2\sigma_0^2}{2}\right), \tag{A2}$$

$$\phi_n(\mathbf{r}) = \phi_{0n} \exp\left(-\frac{r^2}{2\sigma_0^2}\right) J_n(\beta r)\exp(-in\theta), \tag{A3}$$

where J_n and I_n are the nth-order Bessel function and a modified Bessel function, respectively, of the first kind, (r, θ) are polar coordinates of $\mathbf{r}$, and

$$\phi_{0n} = T_0/\sqrt{\lambda_n}. \tag{A4}$$

Equation (A3) states that the modes of a JSM source are the so-called Bessel–Gauss beams of nth order.[30]

Now, it is known[17] that the intensity distribution on the spatial-frequency plane, say, $I^{(\infty)}(\mathbf{p})$, is given by

$$I^{(\infty)}(\mathbf{p}) = \tilde{W}_0(\mathbf{p}, -\mathbf{p}). \tag{A5}$$

By using modal expansion (A1), after some algebra we obtain

$$\tilde{W}_0(\mathbf{p}_1, \mathbf{p}_2) = \sum_{n=-\infty}^{+\infty} \lambda_n \tilde{\phi}_n^*(-\mathbf{p}_1)\tilde{\phi}_n(\mathbf{p}_2) \tag{A6}$$

and thus

$$I^{(\infty)}(\mathbf{p}) = \sum_{n=-\infty}^{+\infty} \lambda_n |\tilde{\phi}_n(\mathbf{p})|^2. \tag{A7}$$

The explicit expression of Eq. (A7) can be given a closed form. Indeed, starting from the Fourier transform of the modes ϕ_n (Ref. 33),

$$\tilde{\phi}_n(\mathbf{p}) = \pi\sigma_0^2(-i)^n\phi_{0n}\exp\left(-\frac{\beta^2\sigma_0^2}{2}\right)\exp(-2\pi^2\sigma_0^2p^2) \times I_n(2\pi\beta\sigma_0^2 p)\exp(-in\varphi), \tag{A8}$$

with (p, φ) the polar coordinates of $\mathbf{p}$, we obtain

$$\begin{aligned} I^{(\infty)}(\mathbf{p}) &= 4\pi^2\sigma_0^4 \exp(-\beta^2\sigma_0^2)\exp(-4\pi^2\sigma_0^2p^2) \times \sum_{n=-\infty}^{+\infty} I_n^2(2\pi\beta\sigma_0^2 p) \\ &= 4\pi^2\sigma_0^4 \exp(-\beta^2\sigma_0^2) \times \exp(-4\pi^2\sigma_0^2p^2) I_0(4\pi\beta\sigma_0^2 p), \end{aligned} \tag{A9}$$

where use has been made of formula (5.8.6.1) of Ref. 34. From Eq. (A9) it can be seen that the intensity profile on the spatial-frequency plane is a radial function, so its variance is

$$\sigma_\infty^2 = \frac{\int_0^\infty I_0(4\pi\beta\sigma_0^2 p)\exp(-4\pi^2\sigma_0^2p^2)p^3\mathrm{d}p}{\int_0^\infty I_0(4\pi\beta\sigma_0^2 p)\exp(-4\pi^2\sigma_0^2p^2)p\,\mathrm{d}p}. \tag{A10}$$

The evaluation of this quantity can be performed with formula (6.631.10) of Ref. 35 and the relation $I_0(x) = J_0(ix)$, yielding

$$\sigma_\infty^2 = \frac{1}{4\pi^2\sigma_0^2}(1 + \beta^2\sigma_0^2). \tag{A11}$$

Finally, the radial quality factor M_r^2 of the J_0-correlated beam turns out to be

$$M_r^2 = (1 + \beta^2\sigma_0^2)^{1/2}, \tag{A12}$$

which coincides with Eq. (43).

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