

SQUEEZED STATES: A GEOMETRIC FRAMEWORK*

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ABSTRACT

A general definition of squeezed states is proposed and its main features are illustrated through a discussion of the standard optical coherent states represented by "Gaussian pure states" (Ref. 1).

The set-up involves representations of groups on Hilbert spaces over homogeneous spaces of the group, and relies upon the construction of a square integrable (coherent state) group representation modulo a subgroup (Ref. 2). This construction depends upon a choice of a Borel section which has a certain permissible arbitrariness in its selection; this freedom is attributable to a squeezing of the defining coherent states of the representation, and corresponds in this way to a sort of gauging.

AN EXAMPLE: GAUSSIAN PURE STATES

Gaussian pure states (GPS's), as defined by Schumaker (Ref. 3) and elaborated by Simon, Sudarshan and Mukunda (Ref. 1), are functions:

$$\psi(\vec{x}) = (\text{const}) \exp[P(\vec{x})] \quad (1)$$

where $\vec{x} \in \mathbb{R}^n$ and $P(\vec{x})$ is a quadratic

polynomial in \vec{x} with complex coefficients:

$$P(\vec{x}) = -i/2 \vec{x} Z \vec{x} + i \vec{w} \cdot \vec{x} + w_0 \quad (2)$$

with Z a symmetric ($Z^t = Z$) complex $n \times n$ matrix, $\vec{w} \in \mathbb{C}^n$, $w_0 \in \mathbb{C}$, and where, if $Z = V - iU$ is the decomposition into real and imaginary parts, then U is positive definite ($U > 0$).

Let G denote the semi-direct product of the Weyl-Heisenberg group $H(2n+1)$ with the symplectic group $Sp(2n; \mathbb{R})$ of symplectic linear maps of \mathbb{R}^{2n} .

$$G = H(2n+1) \circ Sp(2n; \mathbb{R}). \quad (3)$$

Multiplication in G is as follows:

$$g_1 g_2 = (c_1 c_2 \exp[i\Omega(Q_1, s_1 Q_2)/2], Q_1 + s_1 Q_2, s_1 s_2) \quad (4)$$

where $g = (c, Q, s)$ denotes a general element of G ; $(c, Q) \in H(2n+1)$ where $|c| = 1$, $Q \in \mathbb{R}^{2n}$; $s \in Sp(2n; \mathbb{R})$; Ω is the symplectic structure on \mathbb{R}^{2n} defined for $Q_r = (\vec{q}_r, \vec{p}_r)$, $r = 1, 2$, by

$$\Omega(Q_1, Q_2) = \vec{p}_1 \cdot \vec{q}_2 - \vec{p}_2 \cdot \vec{q}_1.$$

Let $U(g)$, $g \in G$, denote the irreducible unitary representation of G on the Hilbert

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space $\mathcal{H} = L^2(\mathbb{R}^n)$ whose restrictions to $H(2n+1)$ and $Sp(2n; \mathbb{R})$ are:

$$[U(c, Q)\psi](\vec{x}) = c \exp[i(\vec{p} \cdot \vec{x} - \vec{p} \cdot \vec{q}/2)] \psi(\vec{x} - \vec{q}) \quad (5)$$

and

$$[U(s)\psi](\vec{x}) = (2\pi)^{-n/2} (\det A)^{-1/2} \int_{\mathbb{R}^n} \exp[iS(\vec{x}, \vec{k})] \psi(\vec{k}) d\vec{k} \quad (6)$$

where

$S(\vec{x}, \vec{k}) = -1/2 \vec{x} C A^{-1} \vec{x} + \vec{k} A^{-1} \vec{x} + 1/2 \vec{k} A^{-1} B \vec{k}$, ψ is the Fourier transform of ψ , and we assume $\det A \neq 0$ if $s = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(2n; \mathbb{R})$. Actually, U defines a double-valued representation of $Sp(2n; \mathbb{R})$, i.e., a representation of the metaplectic group, but this subtlety will be ignored here.

Some facts (see Refs. 1 and 4):

(i) Let $\psi_0 = \pi^{-n/4} \exp[-\vec{x} \cdot \vec{x}/2]$ be the special GPS with $U = I_n$, $V = O_n$, $\vec{w} = \vec{0}$, $w_0 = 0$, and suppose ψ is any GPS. Then $\psi = (\text{const})U(g)\psi_0$ for some $g \in G$; moreover if $g' \in G$ also satisfies this condition then $g' = gk_0$ for some $k_0 \in K_0$:

$$K_0 = Sp(2n; \mathbb{R}) \cap O(2n; \mathbb{R}) \cong U(n) \quad (7)$$

which is a maximal compact subgroup of $Sp(2n; \mathbb{R})$, and conversely.

(ii) $Sp(2n; \mathbb{R})$ has a "block Iwasawa" decomposition:

$$Sp(2n; \mathbb{R}) = \mathbb{N} A K_0 \quad (8)$$

where

$$\mathbb{N} = \left\{ \begin{bmatrix} I_n & O \\ -V & I_n \end{bmatrix} : V^t = V \right\},$$

$$A = \left\{ \begin{bmatrix} \Delta & O_n \\ O_n & \Delta^{-1} \end{bmatrix} : \Delta^t = \Delta > 0 \right\},$$

$$K_0 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a^t a + b^t b = I_n, \quad a^t b = b^t a \right\}.$$

Moreover, defining $U = \Delta^{-2}$ and

$$s(U, V) = \begin{bmatrix} I_n & O_n \\ -V & I_n \end{bmatrix} \begin{bmatrix} U^{-1/2} & O_n \\ O_n & U^{1/2} \end{bmatrix},$$

we have for $\psi \in \mathcal{H}$:

$$[U(s(U, V))\psi](\vec{x}) = (\det U)^{1/4} \exp[i\vec{x} V \vec{x}/2] \psi(U^{1/2} \vec{x}) \quad (9)$$

and if $\psi = \psi_0$ in (i), $Z = V - iU$ then:

$$[U(s(U, V))\psi_0](\vec{x}) = \pi^{-n/4} (\det U)^{1/4} \exp[-i\vec{x} Z \vec{x}/2] \quad (10)$$

(iii) A maximal compact subgroup K of G is:

$$K = U(1) \times K_0 \cong U(1) \times U(n) \quad (11)$$

where $U(1) = \{(c, O) \in H(2n+1)\}$, and with the same meaning for $U(1)$ define also:

$$H = U(1) \times Sp(2n; \mathbb{R}). \quad (12)$$

By Y we mean the homogeneous space:

$$Y = H/K \approx Sp(2n; \mathbb{R})/K_0 \quad (13)$$

which may be regarded as the set of positive symplectic matrices; these are uniquely

expressible in the form $s(U, V)^{-1} s(U, V)^{-1}$ thereby establishing an identification of Y with the collection of GPS's centered at $Q = (\vec{0}, \vec{0}) \in \mathbb{R}^{2n}$. By X we mean the homogeneous space:

$$X = G/H \approx \mathbb{R}^{2n} \quad (14)$$

the identification being through the map sending $(c, Q, s) \in G$ to $Q \in \mathbb{R}^{2n}$. In this way X inherits

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the symplectic structure Ω which happens therefore to be G -invariant.

SQUEEZED COHERENT STATES

The theory of square integrable representations modulo a subgroup (Ref. 2) is immediately applicable in the present situation, and together with the notion of squeezing, to be outlined below, provides a convenient picture of GPS's. In what follows, a general discussion (example of which is the case of GPS's, same notation) will be given.

Let H be a closed subgroup of a Lie group G . Let $X = G/H$ and suppose $d\nu(x)$ is a left G -invariant measure on X (in general, $d\nu$ need only be quasi-invariant); let $\beta: X \rightarrow G$ denote a Borel section. Now suppose $U(\mathfrak{g})$, $\mathfrak{g} \in G$, is an irreducible unitary representation on \mathcal{H} , and suppose there exists an admissible vector $\eta \in \mathcal{H}$ such that, as a weak integral:

$$\int_X |\eta_{\beta, x}\rangle \langle \eta_{\beta, x}| d\nu(x) = A_\beta \quad (15)$$

defines a bounded, positive operator A_β with (possibly unbounded) inverse; $\eta_{\beta, x}$ denotes the vector $U(\beta(x))\eta$ (example: notation as before with $\eta = \psi_0$, $x = Q$, $\beta(x) = (1, Q, I)$, $d\nu(x) = \Omega_Q \wedge \Omega_Q \wedge \dots \wedge \Omega_Q$ (n factors), A_β is a multiple of the identity on \mathcal{H}). In this case U is square integrable mod (H, β) , and $\{\eta_{\beta, x}\}$ is a family of coherent states for U .

Let K be a closed subgroup of H of k 's:

$$U(k)\eta = \rho(k)\eta \quad (16)$$

where ρ is a 1-dimensional unitary representation of K ($|\rho(k)| = 1$). Let $\gamma: Y = H/K \rightarrow H$ denote a Borel section, and define:

$$\sigma: X \times Y \rightarrow G, \quad \sigma(x, y) = \beta(x)\gamma(y) \quad (17)$$

For fixed $y \in Y$, $\sigma(\cdot, y): X \rightarrow G$ is a Borel section and letting $\eta_{\sigma(\cdot, y), x} = U(\sigma(x, y))\eta$ it is easy to verify that U is square integrable mod $(H, \sigma(\cdot, y))$ for each $y \in Y$; in fact:

$$\int_X |\eta_{\sigma(\cdot, y), x}\rangle \langle \eta_{\sigma(\cdot, y), x}| d\nu(x) = A_{\sigma(\cdot, y)} \quad (18)$$

defines a bounded, positive invertible operator (which coincides with A_β here). The collection $\{\eta_{\sigma(\cdot, y), x}\}$, for y fixed, is a family of squeezed coherent states associated to $\{\eta_{\beta, x}\}$; one interprets $U(\gamma(y))$ as the squeezing operator defining a change of section $\beta(\cdot) \rightarrow \beta(\cdot)\gamma(y)$ (example: $k = (c, O, k_0)$, $\rho(k) = c$, $y = Z = V - iU$, $\gamma(y) = (1, O, s(U, V))$).

In this manner, squeezing in its general setting is describable in terms of changes of Borel section of the associated coherent state representation. Details of this general construction with examples will appear elsewhere.

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