SRB measures for non-hyperbolic systems with multidimensional expansion

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September 19, 2002

Abstract

We construct ergodic absolutely continuous invariant probability measures for an open class of non-hyperbolic surface maps introduced by [V2], who showed that they exhibit two positive Lyapunov exponents at almost every point. Our approach involves an inducing procedure, based on the notion of *hyperbolic time* that we introduce here, and contains a theorem of existence of absolutely continuous invariant measures for multidimensional piecewise expanding maps with countably many domains of smoothness.

Résumé

Nous construisons des probabilités invariantes absolument continues ergodiques pour une classe ouverte de transformations non hyperboliques dans des surfaces. Cette classe de transformations a été proposée par [V2], qui a prouvé que ces transformations ont deux exposants de Lyapunov positives presque partout. Notre approche utilise une procedure d'induction, basé sur la notion de *temps hyperboliques* que nous présentons ici, et contient un théorème d'existence des mesures invariantes absolument continues pour des tranformations dilatantes par morceaux ayant un nombre infini de domains de differentiabilité.

1 Introduction

Let $\varphi : M \to M$ be a smooth transformation on a compact manifold M. Given a map $f : M \to \mathbb{R}$, one is interested in "observing" f along the orbits of points $x \in M$. Even in cases of very simple transformations the sequences (*time-series*) $f(\varphi^j(x))$, $j \geq 0$, may have a rather complicated behaviour. Moreover, the transformations may present sensitivity on the initial conditions, i.e. a small variation on the initial point $x \in M$ gives rise to a completely different behaviour of its time-series. A more realistic task (but far from being simple) consists of studying the asymptotic time-averages of such sequences for a "large" set of points $x \in M$. In this setting, Birkhoff's ergodic theorem says that if μ is a φ-invariant finite ergodic measure, then

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(\varphi^j(x)) = \int_M f d\mu$$

for μ almost every point $x \in M$ and every integrable map $f : M \to \mathbb{R}$. However, a full μ measure set may have no special "physical" meaning, and one is particularly interested in describing the time-averages for a positive Lebesgue measure set of points in M.

The previous considerations motivate the following definition. We say that a finite measure μ in M is a *Sinai-Ruelle-Bowen measure* (*SRB measure*, for short) for the transformation $\varphi : M \to M$ if it is φ-invariant and there is a positive Lebesgue measure set $B \subset M$ such that for any continuous map $f : M \to \mathbb{R}$ one has

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(\varphi^j(x)) = \int_M f d\mu \quad \text{for every } x \in B.$$

In the light of Riesz's theorem, this in particular means that the measure μ may be "seen" by computing the time-averages for a set of points with positive Lebesgue probability. This kind of measure was introduced by Sinai for Anosov diffeomorphisms [S1], and later extended by Ruelle and Bowen for Axiom A diffeomorphisms and flows [R1, BR]. For non-hyperbolic systems the existence of such measures may be a very hard mathematical problem. Jakobson, in his celebrated paper [J], constructed SRB measures for quadratic transformations on the interval. Another important work was [BY] (based on the previous work [BC2]), where SRB measures were constructed for Hénon two dimensional maps exhibiting strange attractors. All these SRB measures are, in a sense, one-dimensional: they are absolutely continuous with respect to Lebesgue measure on the interval, in the first case, respectively with

respect to Lebesgue measure along one-dimensional local unstable manifolds, in the Hénon case. (The SRB property is a more or less direct consequence of this absolute continuity property together with ergodicity and absolute continuity of the stable foliation).

Hyperbolic systems apart, very little is known on this subject for systems with multidimensional expansion. Our goal in this work is to set up a framework for the study of statistical properties of systems with nonuniform expansion. In doing this we are primarily motivated by the (robust) examples of such systems constructed by [V2]. However, the techniques we introduce here should prove useful in much greater generality. Indeed, an application of these ideas is being given in [ABV], where partially hyperbolic systems are considered.

In order to state our first theorem, let us briefly describe the examples of [V2]. Let $\varphi_\alpha : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be a C^3 map given by $\varphi_\alpha(\theta, x) = (\hat{g}(\theta), \hat{f}(\theta, x))$, where $\hat{g}$ is a uniformly expanding map of the circle $S^1 = \mathbb{R}/\mathbb{Z}$, and $\hat{f}(\theta, x) = a(\theta) - x^2$ with $a(\theta) = a_0 + \alpha b(\theta)$. We choose $a_0 \in (1, 2)$ in such a way that $x = 0$ is pre-periodic for $q(x) = a_0 - x^2$ and take $b : S^1 \to S^1$ a Morse function. For the sake of definiteness we assume that $b(\theta) = \sin(2\pi\theta)$ and $\hat{g}$ is a linear expanding map of the form $\hat{g}(\theta) = d\theta$ (mod 1). It is easy to check that for α small enough there is an interval $I \subset (-2, 2)$ for which $\varphi_\alpha(S^1 \times I) \subset \text{int}(S^1 \times I)$.

The results in [V2] show that for $d \geq 16$ and φ in a $C^3(S^1 \times I)$ neighborhood of the map φ_α, there are two positive *Lyapunov exponents* almost everywhere. More precisely, it is proved that there is some constant $\lambda > 0$ such that

$$\liminf_{n \to +\infty} \frac{1}{n} \log \|D\varphi^n(\theta, x)v\| \geq \lambda$$

for Lebesgue almost every $(\theta, x) \in S^1 \times I$ and every non-zero $v \in T_{(\theta,x)}(S^1 \times I)$. Here we prove the following result:

Theorem A *For $d \geq 16$ and α sufficiently small, the map φ_α has a finite absolutely continuous (with respect to the bidimensional Lebesgue measure) invariant measure μ^*. Moreover, the same holds for every map φ in a sufficiently small neighborhood of φ_α in the $C^3(S^1 \times I)$ topology.*

To the best of our knowledge, these are the first examples of nonuniformly expanding systems with invariant measures absolutely continuous with respect to higher dimensional Lebesgue measure. Let us mention that [BPV] exhibits SRB measures for some multidimensional Lorenz-like attractors of flows. However, their situation is quite close to being hyperbolic, since they deal with maps which are everywhere expanding (rate of expansion bounded away from 1).

We will prove that φ has only finitely many ergodic absolutely continuous invariant probability measures – from which it follows that any absolutely continuous invariant probability measure is a linear combination of such finite number of ergodic ones. Then, as a consequence of Birkhoff's ergodic theorem we get:

Corollary *Every φ in a sufficiently small neighborhood of φ_α as in the theorem above has an SRB measure.*

In fact one can say more: in a joint work with M. Viana we show that φ is topologically mixing and ergodic with respect to Lebesgue measure. As a consequence, the SRB measure is unique, ergodic, and its support is the whole attractor.

Now we give a sketch of the proof of Theorem A. A basic idea is to introduce a new map ϕ obtained by iterating the initial map φ until we get some (uniform) expanding behaviour. This so-called *inducing* procedure goes back, at least to [J], but there is a crucial difference in our setting with respect to nonuniformly hyperbolic situations treated in previous works. Both quadratic maps and Hénon maps combine hyperbolic behaviour in large parts of the domain with non-hyperbolic behaviour in certain *critical region*, and that is also true for our systems. On the other hand, all the approaches in these previous cases rely on the existence of a well-defined *recovering period* during which the non-hyperbolic effect of the critical region is compensated for. This last ingredient does not exist in our case, which is related to the fact that the critical region $\{\det \varphi = 0\}$ intersects itself when positively iterated.

Instead, the mechanism that permits us to obtain the expanding behaviour is of a statistical type, and will be implemented by introducing the notion of *hyperbolic times* for points in $S^1 \times I$. Roughly speaking, a hyperbolic time for $(\theta, x) \in S^1 \times I$ is an iterate $h \in \mathbb{Z}^+$ for which $\|D\varphi^{-k}(\varphi^h(\theta, x))\|$ contracts for every $1 \le k \le h$ (uniformly on (θ, x) and h, and exponentially on k). The existence of positive Lyapunov exponents almost everywhere implies the existence of such hyperbolic times for most points in $S^1 \times I$. Furthermore, we are able to prove that Lebesgue almost every point in $S^1 \times I$ has infinitely many hyperbolic times.

In Section 3 we use these hyperbolic times to define a partition $\mathcal{R}$ into rectangles of $S^1 \times I$ and a map $h : \mathcal{R} \to \mathbb{Z}^+$ such that for each $R \in \mathcal{R}$ the map $\varphi^{h(R)}|\text{int}(R)$ is a diffeomorphism onto its image and exhibits uniform expanding behaviour. It is in this way that we associate to the map φ a multidimensional piecewise expanding map ϕ from $S^1 \times I$ into itself.

The ergodic properties of one-dimensional piecewise expanding maps have been studied thoroughly, starting form [LY], but the multidimensional case is much less understood. [K] proved the existence of absolutely continuous invariant mea-

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sures (a.c.i.m., for short) for piecewise analytic transformations on the unit two-dimensional square with the domains of smoothness having analytic boundaries. In general dimension, [GB1] proved the existence of a.c.i.m.'s for C^2 piecewise expanding maps with finitely many domains of smoothness having C^2 piecewise boundaries, with angles at the vertices bounded away from zero. They always assume the map to have only finitely many domains of smoothness, and so these results are not sufficient for our purposes. For this reason, we prove in Section 5 the following result on piecewise expanding maps with countably many smoothness domains, whose proof we give in general dimension.

Theorem B *Let ψ be a C^2 piecewise expanding map from the bounded region $R \subset \mathbb{R}^n$ into itself and $\{R_i\}_{i=1}^{+\infty}$ its smoothness domains. If ψ has bounded distortion and the sets $\psi(R_i)$ have C^2 piecewise boundaries with angles bounded away from zero and "large size" (see Section 5 for the precise statements), then ψ has a finite a.c.i.m..*

Recent developments on this subject include [A],[B],[S],[T]. We will use theorem B to prove that the map ϕ described above has some finite a.c.i.m. μ. Indeed, the partition $\mathcal{R}$ is constructed in such a way that the iterate $h(R)$ is a hyperbolic time for the points in R. Using this, and consequently the backward uniform contractive property, we prove in Section 4 that ϕ has a uniform bounded distortion property and, moreover, satisfies the geometric hypotheses of Theorem B.

The existence of the a.c.i.m. μ^* is then a direct consequence of the existence of the a.c.i.m. μ. From the integrability of the hyperbolic time map h and the density of μ we also deduce that μ^* is finite.

2   Hyperbolic times

The goal of this section is to recall some derivative estimates of [V2] and introduce the notion of hyperbolic times for points in $S^1 \times I$. We first assume that the map φ has the special form

$$\varphi(\theta, x) = (g(\theta), f(\theta, x)), \quad \text{with} \quad \partial_x f(\theta, x) = 0 \quad \text{if and only if} \quad x = 0, \qquad (1)$$

and prove the conclusion of Theorem A for every C^2 map φ satisfying

$$\|\varphi - \varphi_\alpha\|_{C^2} \leq \alpha \quad \text{on} \quad S^1 \times I. \tag{2}$$

In Section 7 we show how to remove assumption (1) above.

Our estimates on the derivative depend in an important way on the returns of orbits to the neighborhood $S^1 \times [-\sqrt{\alpha}, \sqrt{\alpha}\,]$ of the critical set $\{x = 0\}$. For this, we introduce a partition $\mathcal{Q}$ of I (mod 0) into the following intervals:

$$I_r = (\sqrt{\alpha}e^{-r}, \sqrt{\alpha}e^{-(r-1)}], \quad \text{for} \quad r \geq 1,$$

$$I_r = -I_{-r}, \quad \text{for} \quad r \leq -1,$$

$$I_{0+} = (\sqrt{\alpha}, e\sqrt{\alpha}\,] \quad \text{and} \quad I_{0-} = -I_{0+},$$

$$I_+ = \big(I\backslash[-e\sqrt{\alpha}, e\sqrt{\alpha}\,]\big) \cap \mathbb{R}^+ \quad \text{and} \quad I_- = \big(I\backslash[-e\sqrt{\alpha}, e\sqrt{\alpha}\,]\big) \cap \mathbb{R}^-.$$

We also introduce the following notation:

$$I_r^+ = I_{r-1} \cup I_r \cup I_{r+1}, \quad \text{for} \quad |r| \geq 1,$$

$$I_{0+}^+ = I_+ \cup I_{0+} \cup I_1 \quad \text{and} \quad I_{0-}^+ = I_- \cup I_{0-} \cup I_{-1}$$

(here we assume that $I_{r-1} = I_{0+}$ if $r = 1$, and $I_{r+1} = I_{0-}$ if $r = -1$). These families of intervals induce in a natural way analogous ones at each fiber of the type $\{\theta\} \times I$; for the sake of notational simplicity no specification will be made on which fiber they are, since this will be always clear in our settings.

In what follows we assume that $\alpha > 0$ is a sufficiently small number independent of any other constant. Furthermore, for each new constant appearing we will always specify when it depends on α. Given $(\theta, x) \in S^1 \times I$ and $j \geq 0$ we define $(\theta_j, x_j) = \varphi^j(\theta, x)$. Following [V2], for the next lemma we take η a positive constant smaller than $1/3$ depending only on the quadratic map q.

Lemma 2.1. *There are constants $C_0, C_1 > 0$ such that for every small α we have an integer $N(\alpha)$ satisfying:*

(a) If $|x| < 2\sqrt{\alpha}$, then $\prod_{j=0}^{N(\alpha)-1} |\partial_x f(\theta_j, x_j)| \geq |x|\alpha^{-1+\eta}$.

(b) If $|x| < 2\sqrt{\alpha}$, then $|x_j| > \sqrt{\alpha}$ for every $j = 1, \ldots, N(\alpha)$.

(c) $C_0 \log(1/\alpha) \leq N(\alpha) \leq C_1 \log(1/\alpha)$.

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Proof. Item (a) was proved in [V2]. We will follow through the ideas of its proof, and derive items (b) and (c). Throughout this proof, C will denote any large constant depending only on the map q. Take $l \geq 1$ the smallest integer for which $z = q^l(0)$ is a periodic point for q and let $k \geq 1$ be its period. Denote $\rho^k = |(q^k)'(z)|$ and note that by [S2] we must have $\rho > 1$. Fix $\rho_1, \rho_2 > 0$ with $\rho_1 < \rho < \rho_2$ and $\rho_1 > \rho_2^{1-\eta/2}$, and take $\delta_0 > 0$ small enough in order to obtain

$$\rho_1^k < \prod_{j=0}^{k-1} \left| \partial_x f(\varphi^i(\sigma, y)) \right| < \rho_2^k, \quad \text{whenever} \quad |y - z| < \delta_0$$

(and α sufficiently small). For $(\theta, x) \in S^1 \times I$ and $i \geq 0$ we denote $d_i = |x_{l+ki} - z|$. Take $\delta_1 > 0$ and α sufficiently small in such a way that

$$|x| < \delta_1 \implies d_0 \leq Cx^2 + C\alpha < \delta_0.$$

If (θ, x) and $i \geq 1$ are such that $|x| < \delta_1$ and $d_0, \ldots, d_{i-1} < \delta_0$, then $d_i \leq \rho_2^k d_{i-1} + C\alpha$ and so, inductively,

$$d_i \leq (1 + \rho_2^k + \cdots + \rho_2^{k(i-1)})C\alpha + \rho_2^{ki} d_0 \leq \rho_2^{ki}(C\alpha + Cx^2).$$

If we assume that $|x| < 2\sqrt{\alpha}$, then we have $d_i \leq \rho_2^{ki} C\alpha$. Now we take $\tilde{N}(\alpha) \geq 1$ the smallest integer for which $\rho_2^{k\tilde{N}(\alpha)} C\alpha \geq \delta_0$, and define $N(\alpha) = l + k\tilde{N}(\alpha)$. The previous considerations imply

$$d_i < \delta_0 \quad \text{for} \quad i = 0, \ldots, \tilde{N}(\alpha) - 1. \tag{3}$$

Since 0 is pre-periodic for q, there exists some constant $\epsilon > 0$ such that $|q^j(0)| > \epsilon$ for every $j > 0$. From this we deduce that

$$|x_1|, \ldots, |x_{l-1}| > \frac{\epsilon}{2}, \quad \text{whenever} \quad |x| < 2\sqrt{\alpha}, \tag{4}$$

as long as α is sufficiently small. Assume from now on that $|x| < 2\sqrt{\alpha}$.

(a) By (1) and (2) we may write $\partial_x f(\theta, x) = x\psi(\theta, x)$ with $|\psi + 2| < \alpha$ at every point $(\theta, x) \in S^1 \times I$. This, together with (4), gives

$$\prod_{j=0}^{l-1} |\partial_x f(\theta_j, x_j)| \geq \frac{1}{C}|x|. \tag{5}$$

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Taking into account that

$$\prod_{j=0}^{N(\alpha)-1} |\partial_x f(\theta_j, x_j)| = \prod_{j=0}^{l-1} |\partial_x f(\theta_j, x_j)| \prod_{i=0}^{\tilde{N}(\alpha)-1} \left(\prod_{j=0}^{k-1} |\partial_x f(\theta_{l+ki+j}, x_{l+ki+j})| \right),$$

we deduce from our previous estimates

$$\prod_{j=0}^{N(\alpha)-1} |\partial_x f(\theta_j, x_j)| \geq \frac{1}{C} |x| \rho_1^{k\tilde{N}(\alpha)} \geq \frac{1}{C} |x| \rho_2^{(1-\eta/2)k\tilde{N}(\alpha)} \geq \frac{1}{C} |x| \alpha^{-1+\eta/2} \geq |x| \alpha^{-1+\eta},$$

as long as α is sufficiently small.

(b) We know from (3) above that

$$|x_{l+ki} - z| < \delta_0 \quad \text{for} \quad i = 0, \ldots, \tilde{N}(\alpha) - 1. \tag{6}$$

Let $z_j = q^j(z)$ for $j = 1, \ldots, k$ and choose α and δ_0 small enough so that

$$|y_1 - z_1|, \ldots, |y_k - z_k| < \frac{\epsilon}{2}, \quad \text{whenever} \quad |y - z| < \delta_0. \tag{7}$$

From (6), (7) and our choice of ϵ we obtain $|x_j| > \epsilon/2$ for $j = l, \ldots, N(\alpha)$, which together with (4) gives $|x_j| > \epsilon/2$ for $j = 1, \ldots, N(\alpha)$. We conclude the proof of this item by taking $2\sqrt{\alpha} < \epsilon/2$.

(c) Recall that by our choice of $\tilde{N}(\alpha)$ we have $\rho_2^{\tilde{N}(\alpha)} C\alpha \geq \delta_0$ and $\rho_2^{k(\tilde{N}(\alpha)-1)} C\alpha < \delta_0$. Since $N(\alpha) = l + k\tilde{N}(\alpha)$ and l, k are fixed, this implies that $C_0 \log(1/\alpha) \leq N(\alpha) \leq C_1 \log(1/\alpha)$ for some constants $C_0, C_1 > 0$ not depending on α.

We have proved the three items of the lemma. $\square$

The following type of result is well-known in the literature, see e.g. Lemma 1 in [BC1]. In fact Lemma 1 in [BC1] is stated for the family $z \mapsto 1 - az^2$ but this is not important since this is affinely conjugate to $x \mapsto a - x^2$, through $x = az$. We quote the statement from Lemma 2.5 in [V2], that contains the estimates in the exact form we shall use them.

Lemma 2.2. *There are $\tau > 1$, $C_2 > 0$ and $\delta > 0$ such that for $(\theta, x) \in S^1 \times I$ and $k \geq 1$ the following holds:*

(a) If $|x_0|, \ldots, |x_{k-1}| \geq \sqrt{\alpha}$, then $\prod_{j=0}^{k-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \sqrt{\alpha} \tau^k$.

(b) If $|x_0|, \ldots, |x_{k-1}| \geq \sqrt{\alpha}$ and $|x_k| < \delta$, then $\prod_{j=0}^{k-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \tau^k$.

In the sequel we will only consider points $(\theta, x) \in S^1 \times I$ whose orbit does not hit the critical set $\{x = 0\}$. This restriction will have no special implication in our results, since the set of such points has full Lebesgue measure. For each integer $j \geq 0$ we define

$$r_j(\theta, x) = \begin{cases} |r| & \text{if } \varphi^j(\theta, x) \in I_r \text{ with } |r| \geq 1; \\ 0 & \text{if } \varphi^j(\theta, x) \in I \backslash [-\sqrt{\alpha}, \sqrt{\alpha}]. \end{cases}$$

We say that $\nu \geq 0$ is a *return situation* for (θ, x) if $r_\nu(\theta, x) \geq 1$. Let n be some positive integer and $0 \leq \nu_1 \leq \cdots \leq \nu_s \leq n - 1$ the return situations of (θ, x) from 0 to $n - 1$. It follows from Lemma 2.1 that for each $1 \leq i \leq s$

$$\prod_{j=\nu_i}^{\nu_i + N(\alpha) - 1} |\partial_x f(\theta_j, x_j)| \geq e^{-r_{\nu_i}(\theta, x)} \alpha^{-1/2 + \eta},$$

and from Lemma 2.2

$$\prod_{j=0}^{\nu_1 - 1} |\partial_x f(\theta_j, x_j)| \geq C_2 \tau^{\nu_1} \quad \text{and} \quad \prod_{j=\nu_i + N(\alpha)}^{\nu_{i+1}} |\partial_x f(\theta_j, x_j)| \geq C_2 \tau^{\nu_{i+1} - \nu_i - N(\alpha)}.$$

For the last piece of orbit (if it exists) we use again Lemma 2.2 and obtain

$$\prod_{j=\nu_s}^{n-1} |\partial_x f(\theta_j, x_j)| \geq |\partial_x f(\theta_{\nu_s}, x_{\nu_s})| C_2 \sqrt{\alpha} \tau^{n - \nu_s}.$$

Since we may write $\partial_x f(\theta, x) = x \psi(\theta, x)$ with $|\psi + 2| < \alpha$ at every point (θ, x) in $S^1 \times I$, we have

$$|\partial_x f(\theta_{\nu_s}, x_{\nu_s})| \geq (2 - \alpha) \sqrt{\alpha} e^{-r_{\nu_s}(\theta, x)}.$$

Altogether, this yields the following lower bound for $\prod_{j=0}^{n-1} |\partial_x f(\theta_j, x_j)|$:

$$\exp \left((n - (s-1)N(\alpha)) \log \tau + \sum_{i=1}^{s} \left(\left(\frac{1}{2} - \eta \right) \log \frac{1}{\alpha} - r_{\nu_i}(\theta, x) \right) - s \log C_2 - \frac{3}{2} \log \frac{1}{\alpha} \right).$$

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Considering

$$G_n(\theta, x) = \left\{ 1 \leq \nu_i \leq n - 1 : r_{\nu_i}(\theta, x) \geq \left(\frac{1}{2} - 2\eta\right) \log \frac{1}{\alpha} \right\},$$

we have

$$\sum_{i=1}^{s} \left(\left(\frac{1}{2} - \eta\right) \log \frac{1}{\alpha} - r_{\nu_i}(\theta, x)\right) \;\geq\; \eta s \log \frac{1}{\alpha} - \sum_{j \in G_n(\theta,x)} r_j(\theta, x)$$

$$\geq\; \gamma s N(\alpha) - \sum_{j \in G_n(\theta,x)} r_j(\theta, x) \qquad (8)$$

for some constant $\gamma > 0$ not depending on α or n (recall Lemma 2.1). Now we define

$$c = \frac{1}{5} \min\{\gamma, \log \tau\}$$

(note that c is independent of α and n). Taking into account that $\nu_{i+1} - \nu_i \geq N(\alpha)$ for every $1 \leq i < s$, we must have

$$s \leq \frac{n}{N(\alpha)} + 1 \leq \frac{n}{C_0 \log(1/\alpha)} + 1,$$

and so, choosing α small enough, we have $s \log C_2 \leq cn + \log C_2$ for every $n \geq 1$. Altogether, this yields

$$\prod_{j=0}^{n-1} |\partial_x f(\theta_j, x_j)| \;\geq\; \exp\left(5cn - \sum_{j \in G_n(\theta,x)} r_j(\theta, x) - cn - \log C_2 - \frac{3}{2} \log \frac{1}{\alpha}\right)$$

$$\geq\; \exp\left(4cn - \sum_{j \in G_n(\theta,x)} r_j(\theta, x) - 2 \log \frac{1}{\alpha}\right) \qquad (9)$$

for every $n \geq 1$, as long as α is sufficiently small.

Lemma 2.3. *If $(\theta, x), (\sigma, y)$ are points in $S^1 \times I$ such that $r_j(\theta, x) \leq r_j(\sigma, y) + 4$ for every $j = 0, \ldots, n - 1$, then*

$$\prod_{j=0}^{n-1} |\partial_x f(\theta_j, x_j)| \geq \exp\left(3cn - \sum_{j \in G_n(\sigma,y)} r_j(\sigma, y) - 3 \log \frac{1}{\alpha}\right).$$

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Proof. Let $0 \leq \nu_1 \leq \cdots \leq \nu_{s(\theta,x)} \leq n - 1$ (resp. $0 \leq \mu_1 \leq \cdots \leq \mu_{s(\sigma,y)} \leq n - 1$) be the returns of (θ, x) (resp. (σ, y)) from 0 to $n - 1$. From Lemma 2.1 we deduce

$$
\begin{aligned}
\sum_{i=1}^{s(\theta,x)} r_{\nu_i}(\theta, x) &= \sum_{j=0}^{n-1} r_j(\theta, x) \\
&\leq \sum_{j=0}^{n-1} r_j(\sigma, y) + \frac{4n}{N(\alpha)} + 1 \\
&= \sum_{i=1}^{s(\sigma,y)} r_{\mu_i}(\sigma, y) + \frac{4n}{N(\alpha)} + 1
\end{aligned}
$$

Using this, and estimate (8) for (σ, y), we obtain

$$
\sum_{i=1}^{s(\theta,x)} \left(\left(\frac{1}{2} - \eta \right) \log \frac{1}{\alpha} - r_{\nu_i}(\theta, x) \right) \geq \gamma s N(\alpha) - \sum_{j \in G_n(\sigma,y)} r_j(\sigma, y) - \frac{4n}{N(\alpha)} - 1,
$$

and so, as in (9), this finally yields

$$
\begin{aligned}
\prod_{j=0}^{n-1} |\partial_x f(\theta_j, x_j)| &\geq \exp \left(4cn - \sum_{j \in G_n(\sigma,y)} r_j(\sigma, y) - \frac{4n}{N(\alpha)} - 1 - 2 \log \frac{1}{\alpha} \right) \\
&\geq \exp \left(3cn - \sum_{j \in G_n(\sigma,y)} r_j(\sigma, y) - 3 \log \frac{1}{\alpha} \right)
\end{aligned}
$$

for α small enough. $\square$

Let us briefly recall how in [V2] the two Lyapunov exponents are obtained for Lebesgue almost every point in $S^1 \times I$. Since φ is close to φ_α and $\hat{g}$ is uniformly expanding, it follows that for every $(\theta, x) \in S^1 \times I$ and every non-vertical tangent vector v, $\|D\varphi^n(\theta, x)v\| \geq \mathrm{const}|(g^n)'(\theta)|$ grows exponentially fast. This provides a positive Lyapunov exponent non-collinear to the vertical direction. For obtaining the other positive Lyapunov exponent, in [V2] it was proved the following:

Lemma 2.4. *There are positive constants C and γ such that for every large n there is a set $E_n \subset S^1 \times I$ such that*

(b) $m_2(E_n) \leq C e^{-\gamma \sqrt{n}}$,

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(a) $(\theta, x) \in (S^1 \times I) \setminus E_n \Rightarrow \displaystyle\sum_{j \in G_n(\theta,x)} r_j(\theta, x) \le cn,$

where m_2 is the Lebesgue measure on $S^1 \times I$.

From this and (9) above, [V2] deduces that Lebesgue almost every point (θ, x) in $S^1 \times I$ has a positive Lyapunov exponent in the x-direction. Indeed, if n is large enough, then

$$\left\| D\varphi^n(\theta, x)\frac{\partial}{\partial_x} \right\| = \prod_{j=0}^{n-1} |\partial_x f(\theta_j, x_j)| \ge e^{2cn}$$

except for $(\theta, x) \in E_n$. Thus, taking $E = \cap_{n \ge 1} \cup_{k \ge n} E_k$ we have for each n

$$m_2 \left(\bigcup_{k \ge n} E_k \right) \le \sum_{k \ge n} C e^{-\gamma\sqrt{k}} \le \text{const } e^{-\gamma\sqrt{n}},$$

which implies that $m_2(E) = 0$, and φ has another positive Lyapunov exponent in the x-direction for every point $(\theta, x) \in (S^1 \times I) \setminus E$.

For the following definition we fix $0 < \epsilon < c/2$ and recall that since we are assuming $d \ge 16$ we have in particular $e^{c+\epsilon} < d - \alpha$ for small α.

Definition 2.5. *We say that $n \ge 1$ is a hyperbolic time for $(\theta, x) \in S^1 \times I$ if for every $0 \le k < n$ we have*

$$\sum_{\substack{i \in G_n(\theta,x) \\ k \le i < n}} r_i(\theta, x) \le (c + \epsilon)(n - k).$$

Fix some large integer p (to be specified in Section 6) and let H be the set of points that has at least one hyperbolic time greater or equal to p. We decompose H into a union $H = \bigcup_{n \ge p} H_n$, where each H_n is the set of points whose first hyperbolic time greater or equal to p is n. The following result, whose proof is motivated on a lemma due to Pliss (see also [M]), gives in particular that H has full Lebesgue measure.

Proposition 2.6. *There is an integer $n_0 = n_0(p, \epsilon) \ge p$ such that for every $n \ge n_0$ one has*

$$(S^1 \times I) \setminus E_n \subset H_p \cup \cdots \cup H_n.$$

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Proof. Let n be some integer bigger than p and take $(\theta, x) \in (S^1 \times I) \setminus E_n$. We are going to prove that (θ, x) has some hyperbolic time m, with $p \leq m \leq n$. For $i = 0, \ldots n - 1$ define

$$\hat{r}_i = \begin{cases} r_i(\theta, x) & \text{if } i \in G_n(\theta, x), \\ 0 & \text{otherwise,} \end{cases}$$

and $a_i = c + \epsilon - \hat{r}_i$. Consider for $k = 1, \ldots, n$ the sums $S_k = \sum_{i=0}^{k-1} a_i$ and take $1 \leq m \leq n$ such that

$$S_m = \max\{S_k : 1 \leq k \leq n\}.$$

Using that $(\theta, x) \notin E_n$ we get

$$S_n = (c + \epsilon)n - \sum_{i=0}^{n-1} \hat{r}_i \geq (c + \epsilon)n - cn \geq \epsilon n.$$

Choose $n_0 \in \mathbb{Z}$ such that $\epsilon n_0 > (p - 1)(c + \epsilon)$. If $n \geq n_0$, then we have

$$S_n \geq \epsilon n > (p - 1)(c + \epsilon) \geq S_i$$

for $i = 0, \ldots, p - 1$. Taking into account our choice of S_m, we must have $S_m > S_i$ for $i = 0, \ldots, p - 1$, and so $m \geq p$. Now we are going to prove that m is a hyperbolic time for (θ, x). For $0 \leq k \leq m - 1$ we have

$$\begin{aligned} \sum_{\substack{i \in G_n(\theta, x) \\ k \leq i \leq m - 1}} r_i &= \sum_{i=k}^{m-1} \hat{r}_i = \sum_{i=k}^{m-1} (c + \epsilon - a_i) \\ &= (c + \epsilon)(m - k) - (S_m - S_k) \\ &\leq (c + \epsilon)(m - k) \end{aligned}$$

since by our choice of m, $S_m \geq S_k$ for all k (we assume $S_0 = 0$). So m is indeed a hyperbolic time for (θ, x). Since $p \leq m \leq n$, the proof is complete. $\square$

Remark 2.7. *From this last result we easily deduce that Lebesgue almost every point in $S^1 \times I$ has infinitely many hyperbolic times. Indeed, letting F be the set of points that have some hyperbolic time, it follows from Proposition 2.6 and the estimate on the measure of the sets E_n that F has full Lebesgue measure. On the other hand, if*

n is a hyperbolic time for (θ, x) and m is a hyperbolic time for $\varphi^n(\theta, x)$, then $n + m$ is a hyperbolic time for (θ, x). Hence, taking

$$G = \bigcup_{n \geq 1} \bigcap_{k \geq n} \varphi^{-k}(F)$$

we have that G has full Lebesgue measure and points in G have infinitely many hyperbolic times (G is precisely the set of points that fall infinitely often into F).

3 The partition

In this section we will construct a partition $\mathcal{R}$ into rectangles of $S^1 \times I$ (modulus a zero Lebesgue measure set). These rectangles will be obtained in several steps, by dividing $S^1 \times I$ according to the itineraries of points and their hyperbolic times. For this, we will consider the partition $\mathcal{Q}$ of I described in Section 2 and introduce a sequence of Markov partitions of S^1.

Assume that $S^1 = \mathbb{R}/\mathbb{Z}$ has the orientation induced by the usual order in $\mathbb{R}$ and let θ_0 be the fixed point of g close to $\theta = 0$. We define the Markov partitions $\mathcal{P}_n$, $n \geq 1$, of S^1 in the following way:

- $\mathcal{P}_1 = \{[\theta_{j-1}, \theta_j) : 1 \leq j \leq d\}$, where $\theta_0, \theta_1, \ldots, \theta_d = \theta_0$ are the pre-images of θ_0 under g (ordered according to the orientation of S^1).

- $\mathcal{P}_{n+1} = \{$ connected components of $g^{-1}(\omega) : \omega \in \mathcal{P}_n\}$ for each $n \geq 1$.

Given $\omega \in \mathcal{P}_n$ we denote by ω^- the left hand side endpoint of ω .

Before we go into the construction of the partition $\mathcal{R}$ let us make a few comments on the way $\mathcal{R}$ will be obtained, and state some basic properties that we want rectangles in $\mathcal{R}$ verify.

The partition $\mathcal{R}$ will be obtained by successive divisions of the rectangles in the initial partition $\mathcal{P}_p \times \mathcal{Q}$ of $S^1 \times I$, and will be written as a union $\mathcal{R} = \cup_{n \geq p} \mathcal{R}_n$ with the sets $\mathcal{R}_n$ defined inductively and obeying the following first condition:

(I_n) $H_n \subset \bigcup_{R \in \mathcal{R}_n} R$ and $R \cap H_n \neq \emptyset$ for every $R \in \mathcal{R}_n$.

Rectangles in $\mathcal{R}_n$ will always have the form $\omega \times J$, with ω belonging to $\mathcal{P}_n$ and J a subinterval of I_r for some $I_r \in \mathcal{Q}$. In order to obtain uniform bounded distortion for $\varphi^n | R$, $R \in \mathcal{R}_n$, we will also require the following property for all $n \geq p$:

(II_n) For every $0 \leq j < n$ and $\omega \times J \in \mathcal{R}_n$ there is $I_{r_j} \in \mathcal{Q}$ such that
  $\varphi^j(\{\omega^-\} \times J) \subset I_{r_j}^+$.

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We will be interested in that the images by φ^n of $R \in \mathcal{R}_n$ have "large size" (see Proposition 3.8) in order to apply the results of Section 5; for this it will be useful to distinguish a particular subset of $\mathcal{R}_n$

$$\mathcal{R}_n^* = \{ \, \omega \times J \in \mathcal{R}_n \ | \ \exists\, 0 \le j < n, \ \exists\, I_{r_j} \in \mathcal{Q} \, : \, I_{r_j} \subset \varphi^j(\{\omega^-\} \times J) \, \}$$

We would like $\mathcal{R}_n^*$ to coincide with the whole $\mathcal{R}_n$. This does not seem to be possible, but we are nevertheless able to show that for every $\omega \times J \in \mathcal{R}_n$ there is some $0 \le j < n$ for which $\varphi^j(\{\omega^-\} \times J)$ contains a definite fraction of some $I_{r_j} \in \mathcal{Q}$. With this in mind we introduce the following useful notion.

Definition 3.1. *We say that $\omega_n \times J_n \in \mathcal{R}_n$ is subordinate to $\omega_l \times J_l \in \mathcal{R}_l^*$ if $\omega_n \subset \omega_l$, J_n and J_l have a common endpoint, and there are $j < l$ and $I_{r_j} \in \mathcal{Q}$ for which the following holds:*
(i) $I_{r_j} \subset \varphi^j(\{\omega_l^-\} \times J_l)$;
(ii) $I_{r_j+1} \subset \varphi^j(\{\omega_l^-\} \times J_n)$ or $I_{r_j-1} \subset \varphi^j(\{\omega_l^-\} \times J_n)$.
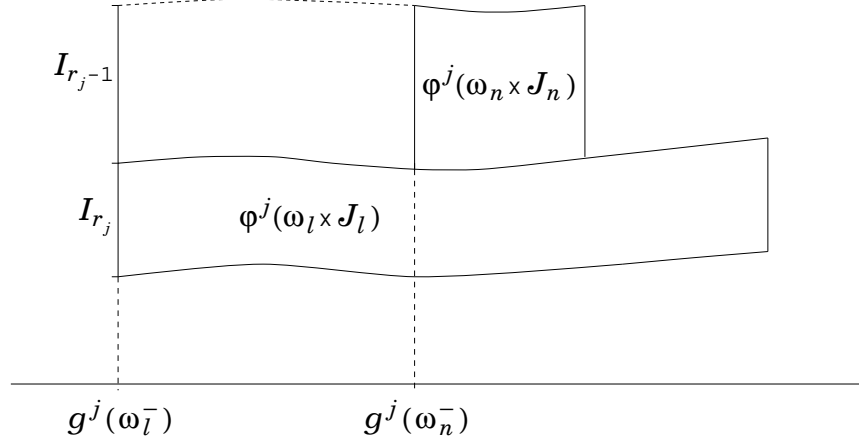

Figure 1: $\omega_n \times J_n$ is subordinate to $\omega_l \times J_l$

In order to obtain the large size property of the images of the rectangles mentioned above, we will do the construction of $\mathcal{R}$ in such a way that the following condition is satisfied for each $n \ge p$:

(III_n) For every $R \in \mathcal{R}_n$, either $R \in \mathcal{R}_n^*$ or R is subordinate to some $R^* \in \mathcal{R}_l^*$ with $l \le n$.

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At each step $n \geq p$ of our inductive process we will also obtain a partition $\mathcal{S}_n$ of the set of points that are not in the rectangles $R \in \mathcal{R}$ constructed until the moment n, i.e. $\mathcal{S}_n$ is a partition of the set

$$(S^1 \times I) \setminus \bigcup_{i=p}^{n} \bigcup_{R \in \mathcal{R}_i} R \, .$$

Rectangles in $\mathcal{S}_n$ will also have the form $\omega \times J$, with $\omega \in \mathcal{P}_n$ and J is a subinterval of some $I_{r_j} \in \mathcal{Q}$. In order to ensure property (III_j) for rectangles $R \in \mathcal{R}_j$ with $j > n$, we will require for all $n \geq p$:

> (IV_n) *For every $\omega \times J \in \mathcal{S}_n$, either $J = I_r$ for some $I_r \in \mathcal{Q}$ or $\omega \times J$ is subordinate to some $R^* \in \mathcal{R}_l^*$ with $l \leq n$.*

Let us start now the inductive construction of the partition $\mathcal{R}$.

First step.   Take an arbitrary $\omega_p \in \mathcal{P}_p$ and let $\mathcal{J}_0$ be the family of those intervals $I_r \in \mathcal{Q}$ such that $\omega_p \times I_r$ intersects H_p. Then we apply φ to $\{\omega_p^-\} \times J_0$ with $J_0 \in \mathcal{J}_0$ and consider the following two possible cases:

(a) $\varphi(\{\omega_p^-\} \times J_0)$ *contains some interval of $\mathcal{Q}$.*

   We write $J_0 = \cup_{i_1} J_{i_1}$ with the intervals J_{i_1} satisfying

$$I_{r_{i_1}} \subset \varphi(\{\omega_p^-\} \times J_{i_1}) \subset I_{r_{i_1}}^+$$

   for some $I_{r_{i_1}} \in \mathcal{Q}$. This may be done by taking $J_{i_1} = J_0 \cap \varphi^{-1}(\{g(\omega_p^-)\} \times I_{r_{i_1}})$ except for each one of the two end intervals which, if necessary, is joined to the adjacent one. We take $\mathcal{J}_1$ to be the set of those J_{i_1}'s in the union above such that $\omega_p \times J_{i_1}$ intersects H_p.

(b) $\varphi(\{\omega_p^-\} \times J_0)$ *contains no interval of $\mathcal{Q}$.*

   In this case we do not divide J_0 and say that $J_0 \in \mathcal{J}_1$.

Now take $J_1 \in \mathcal{J}_1$ and apply φ^2 to $\{\omega_p^-\} \times J_1$. If $\varphi^2(\{\omega_p^-\} \times J_1)$ contains no interval of $\mathcal{Q}$ we say that $J_1 \in \mathcal{J}_2$; otherwise we proceed as in case (a) and decompose $J_1 = \cup_{i_2} J_{i_2}$ as above. We also take $\mathcal{J}_2$ to be the set of those J_{i_2} in the union such that $\omega_p \times J_{i_2}$ intersects H_p.

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We apply this procedure until the $(p-1)^{th}$ iterate, defining in this way the family of intervals $\mathcal{J}_{p-1}$. Let $\mathcal{C}_{p-1}$ be the set of the connected components of

$$J_0 \setminus \bigcup_{J \in \mathcal{J}_{p-1}} J.$$

Finally, given $J \in \mathcal{J}_{p-1} \cup \mathcal{C}_{p-1}$ we say that $\omega_p \times J$ belongs to $\mathcal{R}_p$ if $J \in \mathcal{J}_{p-1}$, and $\omega_p \times J$ belongs to $\mathcal{S}_p$ if $J \in \mathcal{C}_{p-1}$. Doing this for every $\omega_p \in \mathcal{P}_p$ we obtain all the rectangles in $\mathcal{R}_p$ and $\mathcal{S}_p$. It is clear from our construction that $\mathcal{R}_p$ and $\mathcal{S}_p$ verify the required properties (I_p), (II_p), (III_p) and (IV_p). Actually, in this first step we have $\mathcal{R}_p^* = \mathcal{R}_p$.

Inductive step. Assume now that we have defined the families of rectangles $\mathcal{R}_p, \ldots, \mathcal{R}_n$ and $\mathcal{S}_n$ satisfying properties (I_n), (II_n), (III_n) and (IV_n). Let us show how we define the new rectangles in $\mathcal{R}_{n+1}$ and $\mathcal{S}_{n+1}$.

Take $S \in \mathcal{S}_n$. By the inductive hypotheses we know that $S = \omega_n \times J_n$, where $\omega_n \in \mathcal{P}_n$ and $J_n \subset I_r$ for some $I_r \in \mathcal{Q}$. We write

$$S = \bigcup_{i=1}^{d} (\omega_{n+1}^i \times J_n),$$

where $\omega_{n+1}^1, \ldots, \omega_{n+1}^d$ are the intervals in the Markov partition $\mathcal{P}_{n+1}$ that cover ω_n, and distinguish the following two possible cases for each rectangle $\omega_{n+1}^i \times J_n$ in the union above:

(a) $(\omega_{n+1}^i \times J_n) \cap H_{n+1} = \emptyset$

   We say that $\omega_{n+1}^i \times J_n \in \mathcal{S}_{n+1}$. Property (IV_{n+1}) is obviously true for this new rectangle, since no division has been made on the interval J_n.

(b) $(\omega_{n+1}^i \times J_n) \cap H_{n+1} \neq \emptyset$

   Here we again distinguish two possible cases:

   (i) $\exists\, 0 \leq j \leq n\; \exists\, I_{r_j} \in \mathcal{Q}\; :\; I_{r_j} \subset \varphi^j(\{\omega_{n+1}^-\} \times J_n)$.
      We proceed as in the first step, starting our process with $\omega_{n+1}^i \times J_n$ in the place of $\omega_p \times J_0$, and define in this way the rectangles of $\mathcal{R}_{n+1}$ and $\mathcal{S}_{n+1}$ contained in $\omega_{n+1}^i \times J_n$. As before, properties (I_{n+1}), (II_{n+1}), (III_{n+1}) and (IV_{n+1}) are verified by construction.

(*ii*) Otherwise, we say that $\omega_{n+1}^i \times J_n \in \mathcal{R}_{n+1}$ and (II_{n+1}) is obviously true for this new rectangle. Property (III_{n+1}) is also true, since no division has been made on the interval J_n and property (IV_n) holds.

Our induction is complete. Since (I_n) is verified for all $n \geq p$ and H has full Lebesgue measure, we conclude that $\mathcal{R}$ is indeed a partition of $S^1 \times I$ (modulo a zero Lebesgue measure set).

The goal of the following results is to give some geometric properties of the rectangles in the partition $\mathcal{R}$ to be used later. We say that $\widehat{X} \subset S^1 \times I$ is an *admissible curve* if $\widehat{X} = \mathrm{graph}(X)$ for some $X : S^1 \longrightarrow I$ satisfying:

- X is C^2 except, possibly, for being discontinuous on the left at $\theta = \theta_0$.

- $|X'(\theta)| \leq \alpha$ and $|X''(\theta)| \leq \alpha$ at every $\theta \in S^1$.

Given $\widehat{X} = \mathrm{graph}(X)$ and $\omega \subset S^1$ we denote $\widehat{X}|\omega = \mathrm{graph}(X|\omega)$. Since we are taking the map φ with stronger expansion on the horizontal direction than in the vertical one, an application of the graph transform contruction gives rise to the following:

Lemma 3.2. *If $\widehat{X}$ is an admissible curve and $\omega \in \mathcal{P}_n$, then $\varphi^n(\widehat{X}|\omega)$ is also an admissible curve.*

Proof. See [V2], Lemma 2.1. $\square$

Corollary 3.3. *If $R \in \mathcal{R}_n$ for some $n \geq p$, then the boundary of $\varphi^n(R)$ is made by two vertical lines and two admissible curves.*

Proof. Immediate from the construction of the rectangles and Lemma 3.2. $\square$

Lemma 3.4. *There is a constant $\delta_0 > 0$ such that if α is sufficiently small, then for every $(\sigma, y) \in H_n$ and $0 \leq j < n$*

$$\left| I_{r_j(\sigma,y)+5} \right| \geq \delta_0 \alpha^{1-2\eta} e^{-(c+\epsilon)(n-j)} \geq 4\alpha(d-\alpha)^{-(n-j)}.$$

Proof. The second inequality is a direct consequence of our assumptions on d and $c + \epsilon$ and the fact that α may be taken sufficiently small. For the first inequality, we distinguish the cases $j \in G_n(\sigma, y)$ and $j \notin G_n(\sigma, y)$:

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(a) If $j \in G_n(\sigma, y)$, and since n is a hyperbolic time for (σ, y), we have in particular
$$r_j(\sigma, y) \le (c + \epsilon)(n - j).$$

Thus,
$$
\begin{aligned}
\left| I_{r_j(\sigma,y)+5} \right| &= \sqrt{\alpha} e^{-(r_j(\sigma,y)+4)} - \sqrt{\alpha} e^{-(r_j(\sigma,y)+5)} \\
&= \sqrt{\alpha}(e^{-4} - e^{-5})e^{-r_j(\sigma,y)} \\
&\ge \sqrt{\alpha}(e^{-4} - e^{-5})e^{-(c+\epsilon)(n-j)} \\
&\ge \alpha^{1-2\eta} e^{-(c+\epsilon)(n-j)},
\end{aligned}
$$

as long as α is sufficiently small (independent of $n - j$).

(b) If $j \notin G_n(\sigma, y)$, then we have
$$r_j(\sigma, y) \le \left(\frac{1}{2} - 2\eta \right) \log \frac{1}{\alpha},$$

and so
$$
\begin{aligned}
\left| I_{r_j(\sigma,y)+5} \right| &= \sqrt{\alpha} \left(e^{-4} - e^{-5} \right) e^{-r_j(\sigma,y)} \\
&\ge \alpha^{1-2\eta} \left(e^{-4} - e^{-5} \right).
\end{aligned}
$$

It suffices to take $\delta_0 = e^{-4} - e^{-5}$. $\square$

Lemma 3.5. *Let $n \ge p$ and $R \in \mathcal{R}_n$. If $(\theta, x) \in R$ and $(\sigma, y) \in R \cap H_n$, then $r_j(\theta, x) \le r_j(\sigma, y) + 4$ for every $0 \le j < n$.*

Proof. Take $0 \le j < n$ and let $(\theta_j, x_j) = \varphi^j(\theta, x)$, $(\omega_j^-, x_j^-) = \varphi^j(\omega^-, x)$. Since (θ_j, x_j) and (ω_j^-, x_j^-) lie in a same admissible curve, and
$$|\theta_j - \omega_j^-| \le (d - \alpha)^{-(n-j)},$$

it follows from Lemma 3.2 that
$$|x_j^- - x_j| \le \alpha(d - \alpha)^{-(n-j)},$$

and so
$$
\begin{aligned}
|x_j| &\ge |x_j^-| - \alpha(d - \alpha)^{-(n-j)} \\
&\ge \sqrt{\alpha} e^{-r_j(\omega^-, x)} - \alpha(d - \alpha)^{-(n-j)}.
\end{aligned}
$$

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Now, let $(\sigma_j, y_j) = \varphi^j(\sigma, y)$ and $(\omega_j^-, y_j^-) = \varphi^j(\omega^-, y)$. Using the same argument as before, we also prove that

$$
\begin{aligned}
|y_j^-| &\geq |y_j| - \alpha(d-\alpha)^{-(n-j)} \\
&\geq \sqrt{\alpha}e^{-r_j(\sigma,y)} - \alpha(d-\alpha)^{-(n-j)}.
\end{aligned}
$$

An easy application of Lemma 3.4 gives

$$
|I_{r_j(\sigma,y)+1}| > |I_{r_j(\sigma,y)+5}| > \alpha(d-\alpha)^{-(n-j)},
$$

and so

$$
|y_j^-| \geq \sqrt{\alpha}e^{-r_j(\sigma,y)} - |I_{r_j(\sigma,y)+1}|,
$$

which means that

$$
r_j(\omega^-, y) \leq r_j(\sigma, y) + 1.
$$

Taking into account (II_n), we also have

$$
r_j(\omega^-, x) \leq r_j(\omega^-, y) + 2,
$$

and so

$$
r_j(\omega^-, x) \leq r_j(\sigma, y) + 3.
$$

Altogether, this yields

$$
\begin{aligned}
|x_j| &\geq \sqrt{\alpha}e^{-r_j(\omega^-,x)} - \alpha(d-\alpha)^{-(n-j)} \\
&\geq \sqrt{\alpha}e^{-(r_j(\sigma,y)+3)} - |I_{r_j(\sigma,y)+4}| \qquad \text{(by Lemma 3.4)} \\
&\geq \sqrt{\alpha}e^{-(r_j(\sigma,y)+4)},
\end{aligned}
$$

which gives

$$
r_j(\theta, x) \leq r_j(\sigma, y) + 4,
$$

and so the proof is complete. $\square$

Corollary 3.6. *If $R \in \mathcal{R}_n$ for some $n \geq p$, then $\varphi^n|int(R)$ is a diffeomorphism onto its image.*

Proof. Take $n \geq p$ and $R \in \mathcal{R}_n$. By (I_n) we have $R \cap H_n \neq \emptyset$. Since points in $R \cap H_n$ do not hit the critical line $\{x = 0\}$ during their first $n - 1$ iterates, it follows from Lemma 3.5 that the same occurs for the other points in R. This implies that $\varphi^n|int(R)$ is a diffeomorphism onto its image. $\square$

Lemma 3.7. *Let $R \in \mathcal{R}_n$ for some $n \geq p$. If $(\theta, x) \in R$, then for $j = 0, \ldots, n-1$*

$$\prod_{i=j}^{n-1} |\partial_x f(\theta_i, x_i)| \geq \exp\left((2c - \epsilon)(n - j) - 3\log\frac{1}{\alpha}\right).$$

Proof. Take $(\theta, x) \in R$ and $0 \leq j \leq n - 1$. We have

$$\prod_{i=j}^{n-1} |\partial_x f(\theta_i, x_i)| = \prod_{i=0}^{n-j-1} |\partial_x f(\theta_{j+i}, x_{j+i})|.$$

By (I_n) and Lemma 3.5, we know that there is some $(\sigma, y) \in (\omega \times J) \cap H_n$ with

$$r_i(\theta, x) \leq r_i(\sigma, y) + 4$$

for $i = 0, \ldots, n - 1$. This in particular implies

$$r_i(\theta_j, x_j) \leq r_i(\sigma_j, y_j) + 4$$

for $i = 0, \ldots, n - j - 1$. Then, using Lemma 2.3 we obtain

$$\prod_{i=0}^{n-j-1} |\partial_x f(\theta_{j+i}, x_{j+i})| \geq \exp\left(3c(n-j) - \sum_{i \in G_{n-j}(\sigma_j, y_j)} r_i(\sigma_j, y_j) - 3\log\frac{1}{\alpha}\right).$$

We remark that since n is a hyperbolic time for (σ, y), then $(n - j)$ is a hyperbolic time for (σ_j, y_j), and so

$$\prod_{i=0}^{n-j-1} |\partial_x f(\theta_{j+i}, x_{j+i})| \geq \exp\left(3c(n-j) - (c + \epsilon)(n - j) - 3\log\frac{1}{\alpha}\right).$$

This finally gives

$$\prod_{i=j}^{n-1} |\partial_x f(\theta_i, x_i)| \geq \exp\left((2c - \epsilon)(n - j) - 3\log\frac{1}{\alpha}\right),$$

and so the proof is complete. $\square$

Proposition 3.8. *There is $\delta_1 = \delta_1(\alpha) > 0$ such that for each $n \geq p$ and $\omega \times J \in \mathcal{R}_n$ we have $|\varphi^n(\{\theta\} \times J)| \geq \delta_1$ for every $\theta \in \omega$.*

Proof. Take an arbitrary $\omega \times J \in \mathcal{R}_n$ and fix $\theta \in \omega$. We divide the proof into two parts, according to whether $\omega \times J$ belongs to $\mathcal{R}_n^*$ or not:

(a) If $\omega \times J \in \mathcal{R}_n^*$, we know that there are $0 \le j < n$ and $I_{r_j} \in \mathcal{Q}$ such that

$$I_{r_j} \subset \varphi^j(\{\omega^-\} \times J). \tag{10}$$

Fix j and r_j in these conditions. By the mean value theorem, we have some $x \in J$ for which

$$|\varphi^n(\{\theta\} \times J)| = \prod_{i=j}^{n-1} |\partial_x f(\theta_i, x_i)| \cdot |\varphi^j(\{\theta\} \times J)|,$$

and so, by Lemma 3.7

$$|\varphi^n(\{\theta\} \times J)| \ge \exp\left((2c-\epsilon)(n-j) - 3\log\frac{1}{\alpha}\right) \cdot |\varphi^j(\{\theta\} \times J)|. \tag{11}$$

Our objective now is to find a lower bound for the second factor on the right hand side of this last inequality. Let $J = [u, v]$ and consider the curves

$$\gamma_1 = \varphi^j(\omega \times \{u\}) \quad \text{and} \quad \gamma_2 = \varphi^j(\omega \times \{v\}).$$

Note that since γ_1 and γ_2 are contained in admissible curves (recall Lemma 3.2), they are graphs of maps defined on $g^j(\omega) \in \mathcal{P}_{n-j}$, whose derivatives have absolute value bounded by α. Therefore, their diameters in the x-direction are bounded from above by $\alpha(d-\alpha)^{-(n-j)}$, which together with (10) gives in particular for the points $\varphi^j(\theta, u)$ and $\varphi^j(\theta, v)$

$$|u_j| \le \sqrt{\alpha}e^{-r_j} + \alpha(d-\alpha)^{-(n-j)} \tag{12}$$

and

$$|v_j| \ge \sqrt{\alpha}e^{-(r_j-1)} - \alpha(d-\alpha)^{-(n-j)} \tag{13}$$

(we are assuming $|u_j| < |v_j|$, the other case is similar).

Since $(\sigma, y) \in (\omega \times J) \cap H_n$ and $r_j = r_j(\omega^-, z)$ for some $(\omega^-, z) \in \omega \times J$, we have from Lemma 3.5

$$r_j \le r_j(\sigma, y) + 4, \tag{14}$$

which together with Lemma 3.4 gives

$$|I_{r_j}| > |I_{r_j(\sigma,y)+4}| > |I_{r_j(\sigma,y)+5}| \ge 4\alpha(d-\alpha)^{-(n-j)}.$$
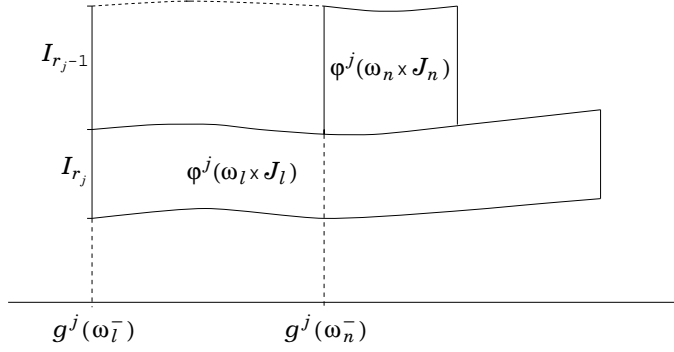
Figure 2: The length of $\varphi^j(\{\theta\} \times J)$

Hence, from (12) and (13) above and this last inequality we obtain

$$
\begin{aligned}
|v_j - u_j| &\geq \sqrt{\alpha}e^{-r_j+1} - \sqrt{\alpha}e^{-r_j} - 2\alpha(d-\alpha)^{-(n-j)} \\
&= |I_{r_j}| - 2\alpha(d-\alpha)^{-(n-j)} \\
&> \frac{|I_{r_j(\sigma,y)+4}|}{2},
\end{aligned}
$$

which by Lemma 3.4 yields

$$
|\varphi^j(\{\theta\} \times J)| > \frac{\delta_0}{2}\alpha^{1-2\eta}e^{-(c+\epsilon)(n-j)}. \tag{15}
$$

Taking into account (11) and (15) we finally get

$$
|\varphi^n(\{\theta\} \times J)| \geq \frac{\delta_0}{2}\alpha^{1-2\eta}\exp\left((c-2\epsilon)(n-j) - 3\log\frac{1}{\alpha}\right), \tag{16}
$$

which is obviously bounded from below by some constant $\delta_1(\alpha) > 0$, for every n and $j \leq n$, since we have chosen $c > 2\epsilon$.

(b) If $\omega \times J \notin \mathcal{R}_n^*$, then by (III_n) we know that there are $l \leq n$ and $\omega_l \times J_l \in \mathcal{R}_l^*$ such that $\omega \times J$ is subordinate to $\omega_l \times J_l$. This implies that for some $j \leq l-1$ and $I_{r_j} \in \mathcal{Q}$ we have

$$
I_{r_j} \subset \varphi^j(\{\omega_l^-\} \times J_l)
$$

and

$$
I_{r_j+1} \subset \varphi^j(\{\omega_l^-\} \times J) \quad \text{or} \quad I_{r_j-1} \subset \varphi^j(\{\omega_l^-\} \times J).
$$

With no loss of generality, we assume

$$I_{r_j+1} \subset \varphi^j(\{\omega_l^-\} \times J)$$

(in fact this is the worst possible case for our purpose, since we want to prove that the length of $\varphi^j(\{\theta\} \times J)$ is large). As in case (a) we have

$$|\varphi^n(\{\theta\} \times J)| \geq \exp\left((2c - \epsilon)(n - j) - 3\log\frac{1}{\alpha}\right) \cdot |\varphi^j(\{\theta\} \times J)|. \qquad (17)$$

Let $J = [u, v]$ and consider the curves

$$\gamma_1 = \varphi^j(\omega_l \times \{u\}) \quad \text{and} \quad \gamma_2 = \varphi^j(\omega_l \times \{v\}).$$

Using a similar argument to the one we used in the previous case, we deduce this time that

$$|u_j| \leq \sqrt{\alpha}e^{-(r_j+1)} + \alpha(d - \alpha)^{-(l-j)} \qquad (18)$$

and

$$|v_j| \geq \sqrt{\alpha}e^{-r_j} - \alpha(d - \alpha)^{-(l-j)}. \qquad (19)$$

It follows from (I_l) and Lemma 3.5 that there is some $(\sigma, y) \in (\omega_l \times J_l) \cap H_l$ for which

$$r_j \leq r_j(\sigma, y) + 4.$$

This, together with Lemma 3.4 gives

$$|I_{r_j+1}| > |I_{r_j(\sigma,y)+5}| \geq 4\alpha(d - \alpha)^{-(l-j)}. \qquad (20)$$

Analogously to the previous case, (18), (19), (20) lead to a similar estimate to the one obtained in (15),

$$|\varphi^j(\{\theta\} \times J)| > \frac{\delta_0}{2}\alpha^{1-2\eta}e^{-(c+\epsilon)(l-j)}.$$

Since $l - j \leq n - j$, this gives

$$|\varphi^j(\{\theta\} \times J)| \geq \frac{\delta_0}{2}\alpha^{1-2\eta}e^{-(c+\epsilon)(n-j)}. \qquad (21)$$

Hence, from (17) and (21) we obtain

$$|\varphi^n(\{\theta\} \times J)| \geq \frac{\delta_0}{2}\alpha^{1-2\eta}\exp\left((c - 2\epsilon)(n - j) - 3\log\frac{1}{\alpha}\right),$$

which is the same estimate that we obtained before. $\square$

4 Bounded distortion

We know from Corollary 3.6 that for $n \geq p$ the interiors of the rectangles $R \in \mathcal{R}_n$ are mapped diffeomorphicaly by $\varphi^n | R$ onto its image. In this section we are going to prove that there is some uniform constant bounding the distortion of such maps. It follows from assumption (1) that for each $n \geq 1$ there is a map F_n from $S^1 \times I$ to I such that for every $(\theta, x) \in S^1 \times I$ we have $\varphi^n(\theta, x) = (g^n(\theta), F_n(\theta, x))$.

Lemma 4.1. *There is a constant $C_3 > 0$ such that for every $(\theta, x) \in S^1 \times I$ and $n \geq 1$ we have*

$$\left| \frac{\partial_\theta F_n(\theta, x)}{\partial_\theta g^n(\theta)} \right| \leq C_3.$$

Proof. We start by noting that from (1), (2) and the expression of φ_α we deduce

$$|\partial_\theta g| \geq d - \alpha, \quad |\partial_\theta f| \leq \alpha |b'| + \alpha \leq 8\alpha \quad \text{and} \quad |\partial_x f| \leq |2x| + \alpha \leq 4.$$

The proof follows by induction. If $n = 1$ we have $F_1(\theta, x) = f(\theta, x)$ for every $(\theta, x) \in S^1 \times I$. Then

$$\left| \frac{\partial_\theta F_1(\theta, x)}{\partial_\theta g(\theta)} \right| < \frac{8\alpha}{d - \alpha}.$$

Assume now that the result is true for some $n \geq 1$. We have for every $(\theta, x) \in S^1 \times I$

$$F_{n+1}(\theta, x) = f(g^n(\theta), F_n(\theta, x)).$$

Hence

$$\begin{aligned}
\left| \frac{\partial_\theta F_{n+1}(\theta, x)}{\partial_\theta g^{n+1}(\theta)} \right| &= \left| \frac{\partial_\theta f(g^n(\theta), F_n(\theta, x)) \partial_\theta g^n(\theta) + \partial_x f(g^n(\theta), F_n(\theta, x)) \partial_\theta F_n(\theta, x)}{\partial_\theta g(g^n(\theta)) \partial_\theta g^n(\theta)} \right| \\
&\leq \left| \frac{\partial_\theta f(g^n(\theta), F_n(\theta, x))}{\partial_\theta g(g^n(\theta))} \right| + \left| \frac{\partial_x f(g^n(\theta), F_n(\theta, x))}{\partial_\theta g(g^n(\theta))} \right| \cdot \left| \frac{\partial_\theta F_n(\theta, x)}{\partial_\theta g^n(\theta)} \right| \\
&\leq \frac{8\alpha}{d - \alpha} + \frac{4}{d - \alpha} C_3 \leq \frac{8\alpha}{d - \alpha} + \frac{C_3}{2}
\end{aligned}$$

since $d \geq 16$. So we only have to choose $C_3 > 0$ in such a way that

$$\frac{8\alpha}{d - \alpha} \leq \frac{C_3}{2},$$

and the lemma is proved. □

Proposition 4.2. *There is a constant $C_4 = C_4(\alpha) > 0$ such that for every $n \geq p$, $R \in \mathcal{R}_n$ and $(\sigma, y) \in \varphi^n(R)$*

$$\frac{\left\| D\left(J \circ \phi^{-1}\right)(\sigma, y)\right\|}{\left|(J \circ \phi^{-1})(\sigma, y)\right|} < C_4,$$

where $\phi = \varphi^n | R : R \to \varphi^n(R)$ and J is the Jacobian of ϕ.

Proof. Taking $(\sigma, y) \in \phi(R)$ and letting $(\theta, x) = \phi^{-1}(\sigma, y)$ we write

$$\frac{\left\| D\left(J \circ \phi^{-1}\right)(\sigma, y)\right\|}{\left|(J \circ \phi^{-1})(\sigma, y)\right|} = \frac{\left\| DJ(\theta, x) \circ [D\phi(\theta, x)]^{-1}\right\|}{|J(\theta, x)|}.$$

Since $\phi(\theta, x) = (g^n(\theta), F_n(\theta, x))$, this last expression is equal to

$$\frac{1}{J^2(\theta, x)} \cdot \left\| (\partial_\theta J(\theta, x)\partial_x F_n(\theta, x) - \partial_x J(\theta, x)\partial_\theta F_n(\theta, x) , \ \partial_x J(\theta, x)\partial_\theta g^n(\theta)) \right\|.$$

Now it suffices to find uniform upper bounds for

$$A_1 = \frac{1}{J^2(\theta, x)} \cdot |\partial_\theta J(\theta, x)\partial_x F_n(\theta, x)| \quad \text{and} \quad A_2 = \frac{1}{J^2(\theta, x)} \cdot |\partial_x J(\theta, x)\partial_\theta g^n(\theta)|$$

(recall that $|\partial_\theta F_n(\theta, x)| \leq C_3 |\partial_\theta g^n(\theta)|$ by Lemma 4.1). Let us first estimate the absolute value of the partial derivatives of $J(\theta, x)$. Since

$$|J(\theta, x)| = |\partial_\theta g^n(\theta)\partial_x F_n(\theta, x)| = \exp\left(\sum_{i=0}^{n-1} \log |\partial_\theta g(\theta_i)\partial_x f(\theta_i, x_i)|\right),$$

we deduce

$$|\partial_\theta J(\theta, x)| = |J(\theta, x)| \cdot \left|\sum_{i=0}^{n-1} \frac{\partial_{\theta\theta}^2 g(\theta_i)\partial_\theta g^i(\theta)\partial_x f(\theta_i, x_i) + \partial_\theta g(\theta_i)\partial_\theta\partial_x f(\theta_i, x_i)\partial_\theta g^i(\theta)}{\partial_\theta g(\theta_i)\partial_x f(\theta_i, x_i)}\right|$$

$$= |J(\theta, x)| \cdot \left|\sum_{i=0}^{n-1} \left(\frac{\partial_{\theta\theta}^2 g(\theta_i)\partial_\theta g^i(\theta)}{\partial_\theta g(\theta_i)} + \frac{\partial_\theta\partial_x f(\theta_i, x_i)\partial_\theta g^i(\theta)}{\partial_x f(\theta_i, x_i)}\right)\right|$$

and

$$|\partial_x J(\theta, x)| = |J(\theta, x)| \cdot \left|\sum_{i=0}^{n-1} \frac{\partial_\theta g(\theta_i)\partial_{xx}^2 f(\theta_i, x_i) \prod_{j=0}^{i-1} \partial_x f(\theta_j, x_j)}{\partial_\theta g(\theta_i)\partial_x f(\theta_i, x_i)}\right|$$

$$= |J(\theta, x)| \cdot \left|\sum_{i=0}^{n-1} \frac{\partial_{xx}^2 f(\theta_i, x_i) \prod_{j=0}^{i-1} \partial_x f(\theta_j, x_j)}{\partial_x f(\theta_i, x_i)}\right|.$$

Before we go into the estimates of A_1 and A_2 let us remark that there is some constant $K > 0$ such that for every $(\theta, x) \in S^1 \times I$

$$|\partial_{\theta\theta}^2 g(\theta)|, |\partial_{xx}^2 f(\theta, x)|, |\partial_\theta \partial_x f(\theta, x)| < K.$$

Now, for the first expression we have

$$
\begin{aligned}
A_1 &= \left| \frac{\partial_x F_n(\theta, x)}{J(\theta, x)} \right| \cdot \left| \sum_{i=0}^{n-1} \left(\frac{\partial_{\theta\theta}^2 g(\theta_i) \partial_\theta g^i(\theta)}{\partial_\theta g(\theta_i)} + \frac{\partial_\theta \partial_x f(\theta_i, x_i) \partial_\theta g^i(\theta)}{\partial_x f(\theta_i, x_i)} \right) \right| \\
&\leq \left| \frac{1}{\partial_\theta g^n(\theta)} \right| \cdot \sum_{i=0}^{n-1} \left| \frac{\partial_{\theta\theta}^2 g(\theta_i) \partial_\theta g^i(\theta)}{\partial_\theta g(\theta_i)} + \frac{\partial_\theta \partial_x f(\theta_i, x_i) \partial_\theta g^i(\theta)}{\partial_x f(\theta_i, x_i)} \right| \\
&\leq \sum_{i=0}^{n-1} \left(\left| \frac{\partial_{\theta\theta}^2 g(\theta_i)}{\partial_\theta g(\theta_i) \partial_\theta g^{n-i}(\theta_i)} \right| + \left| \frac{\partial_\theta \partial_x f(\theta_i, x_i)}{\partial_x f(\theta_i, x_i) \partial_\theta g^{n-i}(\theta_i)} \right| \right) \\
&\leq \sum_{i=0}^{n-1} \frac{K}{(d - \alpha)(d - \alpha)^{n-i}} + \sum_{i=0}^{n-1} \frac{K}{|\partial_x f(\theta_i, x_i)|(d - \alpha)^{n-i}}.
\end{aligned}
$$

Since $d - \alpha > 1$, it suffices to find a uniform upper bound for the second term in the sum above. Taking into account (1) and (2) we have $|\partial_x f(\theta_i, x_i)| \geq (2 - \alpha)\sqrt{\alpha} e^{-r_i(\theta, x)}$, for $i = 0, \ldots, n - 1$. We also have from (I_n) and Lemma 3.5 that there is some $(\sigma, y) \in R \cap H_n$ for which $r_i(\theta, x) \leq r_i(\sigma, y) + 4$, for $i = 0, \ldots, n - 1$. Hence

$$\sum_{i=0}^{n-1} \frac{K}{|\partial_x f(\theta_i, x_i)|(d - \alpha)^{n-i}} \leq \sum_{i=0}^{n-1} \frac{K}{(2 - \alpha)\sqrt{\alpha} e^{-r_i(\sigma,y)-4}(d - \alpha)^{n-i}},$$

and taking $C(\alpha) = K/((2 - \alpha)\sqrt{\alpha} e^{-4})$ this last sum is equal to

$$C(\alpha) \left(\sum_{i \in G_n(\sigma,y)} \frac{1}{e^{-r_i(\sigma,y)}(d - \alpha)^{n-i}} + \sum_{i \notin G_n(\sigma,y)} \frac{1}{e^{-r_i(\sigma,y)}(d - \alpha)^{n-i}} \right),$$

which is uniformly bounded from above, since for $i \in G_n(\sigma, y)$ we have $r_i(\sigma, y) \leq (c + \epsilon)(n - i)$ (recall that n is a hyperbolic time for (σ, y), and $e^{c+\epsilon} < d - \alpha$), and for $i \notin G_n(\sigma, y)$ we have $r_i(\sigma, y) \leq (1/2 - 2\eta) \log 1/\alpha$.

For the second expression we have

$$
\begin{aligned}
A_2 &= \left| \frac{\partial_\theta g^n(\theta)}{J(\theta, x)} \right| \cdot \left| \sum_{i=0}^{n-1} \frac{\partial_{xx}^2 f(\theta_i, x_i) \prod_{j=0}^{i-1} \partial_x f(\theta_j, x_j)}{\partial_x f(\theta_i, x_i)} \right| \\
&\leq \sum_{i=0}^{n-1} \left| \frac{\partial_{xx}^2 f(\theta_i, x_i)}{\partial_x f(\theta_i, x_i) \prod_{j=i}^{n-1} \partial_x f(\theta_j, x_j)} \right| \\
&\leq \sum_{i=0}^{n-1} \frac{K}{\left| \partial_x f(\theta_i, x_i) \prod_{j=i}^{n-1} \partial_x f(\theta_j, x_j) \right|}.
\end{aligned}
$$

By (I_n) and Lemma 3.5 there is some $(\sigma, y) \in R \cap H_n$ with $r_i(\theta, x) \leq r_i(\sigma, y) + 4$ for $i = 0, \dots, n-1$. Thus, from Lemma 2.3 we have

$$
\prod_{j=i}^{n-1} |\partial_x f(\theta_j, x_j)| = \exp \left(3c(n-i) - \sum_{j \in G_{n-i}(\sigma_i, y_i)} r_j(\sigma_i, y_i) - 3\log \frac{1}{\alpha} \right).
$$

Since n is a hyperbolic time for (σ, y), then $n-i$ is a hyperbolic time for (σ_i, y_i), and so

$$
\prod_{j=i}^{n-1} |\partial_x f(\theta_j, x_j)| \geq \exp \left(3c(n-i) - (c+\epsilon)(n-i) - 3\log \frac{1}{\alpha} \right).
$$

On the other hand, from (1) and (2)

$$
|\partial_x f(\theta_i, x_i)| \geq (2-\alpha)\sqrt{\alpha} e^{-r_i(\theta, x)} \geq (2-\alpha)\sqrt{\alpha} e^{-r_i(\sigma, y)-4}.
$$

Thus, taking $C'(\alpha) = e^4 / ((2-\alpha)\alpha^{7/2})$, we obtain

$$
A_2 \leq C'(\alpha) \sum_{i=0}^{n-1} \frac{1}{\exp\left((2c-\epsilon)(n-i) - r_i(\sigma, y)\right)},
$$

which again by splitting into sums over $i \in G_n(\sigma, y)$ and $i \notin G_n(\sigma, y)$ we obtain a uniform upper bound, similarly to what we have done for A_1. $\square$

5   Multidimensional piecewise expanding maps

This section is independent of the previous ones both in content and in notations. Here we will give sufficient conditions for the existence of absolutely continuous invariant probabilities for higher dimensional C^2 piecewise expanding maps with infinitely many domains of smoothness. In [GB1] sufficient conditions are given for the existence of such probabilities, in the case of only a finite number of domains of smoothness. We will follow through the approach in [GB1] and prove that under certain general conditions the result may be extended for countably many domains.

Let R be some bounded region in $\mathbb{R}^n$ and ϕ a map from R to R. We say that ϕ is a C^2 *piecewise expanding map* if the following conditions hold:

(E_1) There is a partition $\{R_i\}_{i=1}^{\infty}$ of R, such that each R_i is a closed domain with piecewise C^2 boundary of finite $(n-1)$-dimensional measure.

(E_2) Each $\phi_i = \phi|R_i$ is a C^2 bijection from the interior of R_i onto its image and has a C^2 extension to R_i.

(E_3) There is $0 < \sigma < 1$ such that $\|D\phi_i^{-1}\| < \sigma$ for every $i \geq 1$.

The piecewise expanding map ϕ is said to have *bounded distortion* if:

(D) There is some $K > 0$ such that for every $i \geq 1$

$$\frac{\left\| D\left(J \circ \phi_i^{-1}\right)\right\|}{\left|J \circ \phi_i^{-1}\right|} < K,$$

where J is the Jacobian of ϕ.

Let S be some closed region in $\mathbb{R}^n$ with piecewise C^2 boundary of finite $(n-1)$-dimensional measure and U a neighborhood of ∂S in S. We say that U is a *regular collar* for S if there are a C^1 unitary vector field H in ∂S and $\beta(S), \rho(S) > 0$ with the following properties:

(C_1) U may be written as the disjoint union of the segments joining $x \in \partial S$ to $x + \rho(S)H(x)$.

(C_2) For every $x \in \partial S$ and $v \in T_x\partial S$ the angle between $H(x)$ and v is bounded away from zero, with $|\sin \angle(v, H(x))| \geq \beta(S)$.
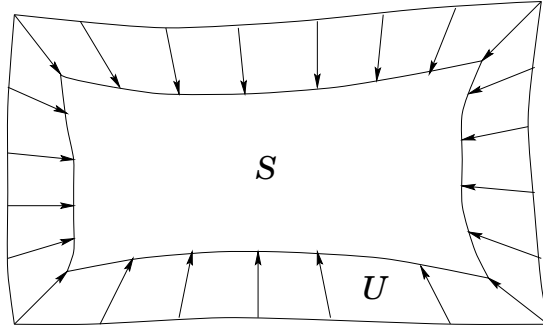
Figure 3: The regular collar U

Remark 5.1. *Here we assume that at the points $x \in \partial S$ where ∂S is not smooth the vector $H(x)$ is a common C^1 extension of H restricted to each $(n-1)$-dimensional smooth component of ∂S having x in its boundary. We also assume that the tangent space of any such singular point x is the union of the tangent spaces to the $(n-1)$-dimensional smooth components it belongs to.*

Theorem 5.2. *Let $\phi : R \to R$ be a C^2 piecewise expanding map with bounded distortion and $\{R_i\}_{i=1}^{\infty}$ its domains of smoothness. Assume that there are $\beta, \rho > 0$ such that each $\phi(R_i)$ has a regular collar with $\beta(\phi(R_i)) > \beta$ and $\rho(\phi(R_i)) > \rho$. If $\sigma(1 + 1/\beta) < 1$, then ϕ has an absolutely continuous invariant probability.*

The main ingredient for the proof of this theorem is the notion of variation for maps in multidimensional spaces. We adopt the definition given in [G]. In all that follows we denote by m_n the Lebesgue measure of $\mathbb{R}^n$ and m_{n-1} the induced Lebesgue measure in any piecewise smooth $(n-1)$-dimensional submanifold. For $f \in L^1(\mathbb{R}^n)$ with compact support we define the *variation* of f as

$$V(f) = \sup \left\{ \int_{\mathbb{R}^n} f \text{div}(g) dm_n : g \in C_0^1(\mathbb{R}^n, \mathbb{R}^n), \|g\| \leq 1 \right\},$$

where $C_0^1(\mathbb{R}^n, \mathbb{R}^n)$ is the set of C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ with compact support and $\| \cdot \|$ is the supremum norm in $C_0^1(\mathbb{R}^n, \mathbb{R}^n)$. We will make use of the following properties, whose proofs may be found in [G] or [EG]:

(V1) If $f \in C^1(\mathbb{R}^n)$, then $V(f) = \int_{\mathbb{R}^n} \|Df\| dm_n$.

(V2) If $A \subset \mathbb{R}^n$ is a closed domain with piecewise smooth $(n-1)$-dimensional boundary and $f \in L^1(\mathbb{R}^n)$ is such that $f|(\mathbb{R}^n \backslash A) = 0$, $f|A$ is continuous and $f|\text{int}(A)$ is C^1, then

$$V(f) = \int_{\text{int}(A)} \|Df\| dm_n + \int_{\partial A} |f| dm_{n-1}.$$

Let R be the domain of the C^2 piecewise expanding map ϕ. In the sequel we will consider the space

$$BV(R) = \left\{ f \in L^1(R) : V(f) < +\infty \right\},$$

which is called the space of *bounded variation* functions in $L^1(R)$. The proof of the following results may also be found in [G] or [EG]:

(B1) $BV(R)$ is dense in $(L^1(R), \| \cdot \|_1)$.

(B2) Given $f \in BV(R)$, there is a sequence $(f_k)_k$ of functions in $C^1(R)$ such that $\lim_k \|f_k - f\|_1 = 0$ and $\lim_k V(f_k) = V(f)$.

(B3) If $(f_k)_k$ is a sequence of functions in $BV(R)$ converging to f in $L^1_{loc}(R)$, then $V(f) \leq \liminf_k V(f_k)$.

(B4) If $(f_k)_k$ is a sequence in $BV(R)$ such that $\left(\|f_k\|_1 \right)_k$ and $\left(V(f_k) \right)_k$ are bounded in $\mathbb{R}$, then $(f_k)_k$ has some subsequence converging in the L^1-norm to some function in $BV(R)$.

Now we introduce the linear Perron-Frobenius operator

$$\mathcal{L} : L^1(R) \longrightarrow L^1(R)$$

defined as

$$\mathcal{L}f = \sum_{i=1}^{\infty} \frac{f \circ \phi_i^{-1}}{|J \circ \phi_i^{-1}|} \chi_{\phi(R_i)}.$$

It is well-known that $\mathcal{L}$ has the following two properties:

(L1) $\|\mathcal{L}f\|_1 \leq \|f\|_1$ for every $f \in L^1(R)$.

(L2) $\mathcal{L}f = f$ if and only if f is the density of an absolutely continuous invariant probability measure for ϕ.

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We will prove Theorem 5.2 by showing that there is some fixed point of the Perron-Frobenius operator in $BV(R)$. For this we need some auxiliary lemmas.

Lemma 5.3. *Let S be some closed domain having a regular collar. If $f \in C^1(S)$, then*

$$\int_{\partial S} f \, dm_{n-1} \le \frac{1}{\beta(S)} \left(\frac{1}{\rho(S)} \int_S f \, dm_n + \int_S \|Df\| \, dm_n \right).$$

Proof. See [GB1], Lemma 3. $\square$

From now on we assume that the C^2 piecewise expanding map $\phi : R \to R$ and its domains of smoothness $\{R_i\}_{i=1}^{+\infty}$ satisfy the hypotheses of Theorem 5.2.

Lemma 5.4. *There is a constant $K_0 > 0$ such that for every $f \in BV(R)$*

$$V(\mathcal{L}f) \le \sigma(1 + 1/\beta)V(f) + K_0 \|f\|_1.$$

Proof. We start by proving this in the case $f \in C^1(R)$. We have

$$\mathcal{L}f = \sum_{i=1}^{\infty} F_i \chi_{\phi(R_i)} \quad \text{where} \quad F_i = (f \circ \phi_i^{-1})/(J \circ \phi_i^{-1}).$$

Hence, using the subaditivity of variation and (V2)

$$
\begin{aligned}
V(\mathcal{L}f) &\le \sum_{i=1}^{\infty} V(F_i \chi_{\phi(R_i)}) \\
&= \sum_{i=1}^{\infty} \left(\int_{\phi(R_i)} \|DF_i\| \, dm_n + \int_{\partial\phi(R_i)} |F_i| \, dm_{n-1} \right).
\end{aligned}
$$

Let us estimate each one of the terms involved in this last sum. For the first one we have

$$
\begin{aligned}
&\int_{\phi(R_i)} \|DF_i\| \, dm_n \\
&\le \int_{\phi(R_i)} \frac{\|D(f \circ \phi_i^{-1})\|}{|J \circ \phi_i^{-1}|} dm_n + \int_{\phi(R_i)} \left\| (f \circ \phi_i^{-1}) \cdot D\left(\frac{1}{J \circ \phi_i^{-1}} \right) \right\| dm_n \\
&\le \sigma \int_{\phi(R_i)} \frac{\|Df(\phi_i^{-1})\|}{|J \circ \phi_i^{-1}|} dm_n + \int_{\phi(R_i)} K \left| \frac{f \circ \phi_i^{-1}}{J \circ \phi_i^{-1}} \right| dm_n,
\end{aligned}
$$

where $K > 0$ is the constant given by the bounded distortion property of ϕ. By a change of variables induced by ϕ in these last two integrals we obtain

$$\int_{\phi(R_i)} \|DF_i\| dm_n \le \sigma \int_{R_i} \|Df\| dm_n + K \int_{R_i} |f| dm_n.$$

For the second term in the sum above, we have by Lemma 5.3

$$\begin{aligned}
\int_{\partial\phi(R_i)} |F_i| dm_{n-1} &\le \frac{1}{\beta} \left(\frac{1}{\rho} \int_{\phi(R_i)} |F_i| dm_n + \int_{\phi(R_i)} \|DF_i\| dm_n \right) \\
&\le \frac{1}{\beta\rho} \int_{R_i} |f| dm_n + \frac{1}{\beta} \int_{\phi(R_i)} \|DF_i\| dm_n \\
&\le \left(\frac{1}{\beta\rho} + \frac{K}{\beta} \right) \int_{R_i} |f| dm_n + \frac{\sigma}{\beta} \int_{R_i} \|Df\| dm_n.
\end{aligned}$$

Altogether, this yields

$$\begin{aligned}
V(\mathcal{L}f) &\le \sum_{i=1}^{\infty} \left(\left(\sigma + \frac{\sigma}{\beta} \right) \int_{R_i} \|Df\| dm_n + \left(K + \frac{1}{\beta\rho} + \frac{K}{\beta} \right) \int_{R_i} |f| dm_n \right) \\
&\le \sigma \left(1 + \frac{1}{\beta} \right) V(f) + \left(K + \frac{1}{\beta\rho} + \frac{K}{\beta} \right) \int_R |f| dm_n,
\end{aligned}$$

from which we deduce the result for the special case $f \in C^1(R)$, simply by taking $K_0 = K + 1/(\beta\rho) + K/\beta$.

For the general case, we observe that by (B2), given $f \in BV(R)$ we may choose a sequence $(f_k)_k$ of functions in $C^1(R)$ such that

$$\lim_k \|f_k - f\|_1 = 0 \quad \text{and} \quad \lim_k V(f_k) = V(f).$$

As a consequence of what we have seen for the case $f \in C^1(R)$, we have in particular that $\mathcal{L}(C^1(R)) \subset BV(R)$. By (L1), the sequence $(\mathcal{L}f_k)_k$ also converges in $L^1(R)$ to $\mathcal{L}f$, and so we may apply (B3) and deduce

$$\begin{aligned}
V(\mathcal{L}f) &\le \liminf_{k\to+\infty} V(\mathcal{L}f_k) \\
&\le \liminf_{k\to+\infty} \left(\sigma(1 + 1/\beta) V(f_k) + K_0 \|f_k\|_1 \right) \\
&= \sigma(1 + 1/\beta) V(f) + K_0 \|f\|_1.
\end{aligned}$$

This proves the general case. $\square$

Lemma 5.5. *There are constants $0 < \lambda < 1$ and $K_1 > 0$ such that for every $f \in BV(R)$ and $j \geq 1$*

$$V(\mathcal{L}^j f) \leq \lambda^j V(f) + K_1 \|f\|_1.$$

Proof. Let $\lambda = \sigma(1 + 1/\beta)$ and take $f \in BV(R)$. It follows from (L1) and Lemma 5.4 that

$$
\begin{aligned}
V(\mathcal{L}^j f) &\leq \lambda V(\mathcal{L}^{j-1} f) + K_0 \|f\|_1 \\
&\leq \lambda^2 V(\mathcal{L}^{j-2} f) + (\lambda + 1) K_0 \|f\|_1 \\
&\cdots \\
&\leq \lambda^j V(f) + (\lambda^{j-1} + \cdots + 1) K_0 \|f\|_1.
\end{aligned}
$$

It suffices to take $K_1 = K_0 \sum_{j=0}^{\infty} \lambda^j$. $\square$

Proof of Theorem 5.2. As we said before, we are going to prove that the Perron-Frobenius operator associated to ϕ has a fixed point in $BV(R)$. Consider for $k \geq 1$

$$f_k = \frac{1}{k} \sum_{j=0}^{k-1} \mathcal{L}^j 1.$$

It follows from Lemma 5.5 that for each $k \geq 1$

$$V(f_k) \leq \frac{1}{k} \sum_{j=0}^{k-1} V(\mathcal{L}^j 1) \leq \frac{1}{k} \sum_{j=0}^{k-1} (\lambda^j V(1) + K_1 \|1\|_1) \leq K_1.$$

Since we also have $\|f_k\|_1 \leq \|1\|_1$ for every $k \geq 1$, it follows from (B4) that $(f_k)_k$ has some accumulation point f_0 in the L^1-norm, which is obviously invariant by $\mathcal{L}$. Thus, $\mu = f_0 m_n$ is an absolutely continuous ϕ-invariant finite measure. $\square$

Remark 5.6. *It follows from the proof above that the accumulation point f_0 belongs to $BV(R)$. In higher dimensions a bounded variation function need not to be bounded (see [GB2]). However, by Sobolev's inequality (see Theorem 1.28 in [G] or Theorem 1 in Section 5.6 of [EG]), there is some constant $K(n) > 0$ only depending on the dimension n such that for any $f \in BV(R)$*

$$\left(\int |f|^r dm_n \right)^{1/r} \leq K(n)\, V(f), \quad \text{where} \quad r = \frac{n}{n-1}.$$

This in particular gives $BV(R) \subset L^r(R)$.

Now we are going to derive some ergodic properties of the map ϕ. For this, we start by proving that given any $f \in L^1(R)$ the sequence $1/k \sum_{j=0}^{k-1} \mathcal{L}^j f$ has accumulation points in $BV(R)$ (in the L^1-norm). We follow some well-known arguments, e.g. those used in [LY], where they also prove that every fixed point of $\mathcal{L}$ is in $BV(R)$. We could also prove such result by using the same arguments, but we do not need it for our purposes.

Let $f \in L^1(R)$ and take a sequence $(f_l)_l$ in $BV(R)$ converging to f in the L^1-norm. It is no restriction to assume that $\|f_l\|_1 \leq 2\|f\|_1$ for every $l \geq 1$ and do it. For each $l \geq 1$ we have

$$V(\mathcal{L}^j f_l) \leq \lambda^j V(f_l) + K_1 \|f_l\|_1 \leq 3K_1 \|f\|_1$$

for every large j. So, increasing k if necessary, we have

$$V\left(\frac{1}{k} \sum_{j=0}^{k-1} \mathcal{L}^j f_l\right) \leq 4K_1 \|f\|_1.$$

It follows from (B4) that there exists some $\hat{f}_l \in BV(R)$ and a sequence $(k_i)_i$ such that $1/k_i \sum_{j=0}^{k_i-1} \mathcal{L}^j f_l$ converges in the L^1-norm to $\hat{f}_l$ as i goes to $+\infty$. Moreover, by (B3) we have $V(\hat{f}_l) \leq 4K_1 \|f\|_1$. Hence, we may apply the same argument to the sequence $(\hat{f}_l)_l$ in order to obtain a subsequence $(l_i)_i$ such that $(\hat{f}_{l_i})_i$ converges in the L^1-norm to some $\hat{f}$ with $V(\hat{f}) \leq 4K_1 \|f\|_1$. We deduce by a triangular inequality argument that there is some sequence $(k_m)_m \to +\infty$ for which $1/k_m \sum_{j=0}^{k_m-1} \mathcal{L}^j f_{l_m}$ converges to $\hat{f}$ in the L^1-norm as $m \to +\infty$. On the other hand,

$$\|\frac{1}{k_m} \sum_{j=0}^{k_m-1} \left(\mathcal{L}^j f_{l_m} - \mathcal{L}^j f\right)\|_1 \leq \frac{1}{k_m} \sum_{j=0}^{k_m-1} \|f_{l_m} - f\|_1 = \|f_{l_m} - f\|_1$$

and this last term goes to 0 as $m \to +\infty$. This finally implies that $1/k_m \sum_{j=0}^{k_m-1} \mathcal{L}^j f$ converges to $\hat{f}$ in the L^1-norm.

We claim that given any ϕ-invariant set $A \subset R$ with positive Lebesgue measure, there exists some absolutely continuous ϕ-invariant probability measure μ_A whose density has variation less than $4K_1$ and $\mu_A(A) = 1$. Indeed, for some sequence $(k_i)_i$ the sequence $1/k_i \sum_{j=0}^{k_i-1} \mathcal{L}^j \chi_A$ converges in the L^1-norm to some f_A with $V(f_A) \leq 4K_1 \|\chi_A\|_1 \leq 4K_1$ and $\|f_A\|_1 > 0$. Thus, taking μ_A equal to the normalization of the measure $f_A m_n$, we have that μ_A is an absolutely continuous ϕ-invariant probability

measure. Up to multiplying by the total mass of μ_A and letting $A^c = R \backslash A$ we have

$$\mu_A(A^c) = \lim_i \frac{1}{k_i} \sum_{j=0}^{k_i-1} \int_{A^c} \mathcal{L}^j \chi_A \, dm_n = \lim_i \frac{1}{k_i} \sum_{j=0}^{k_i-1} \int_R \chi_{A^c} \circ \phi^j \cdot \chi_A \, dm_n = 0.$$

Now let $A \subset R$ be some ϕ-invariant set with positive Lebesgue measure, and $\mu_A = f_A m_n$ an absolutely continuous ϕ-invariant probability measure giving full weight to A with $V(f_A) \leq 4K_1$. Let $r = n/(n-1)$ and $q = 1/n$. Combining Remark 5.6 with Hölder's inequality we obtain

$$1 = \|f_A\|_1 \leq \|f_A\|_{L^r} m_n(A)^{1/q} \leq K(n) 4K_1 m_n(A)^n. \tag{22}$$

Taking $\widehat{K}(n) = (K(n)4K_1)^{-1/n}$ we have that $m_n(A) \geq \widehat{K}(n)$.

It immediately follows that R can be decomposed into finitely many minimal ϕ-invariant sets $A_1, \ldots, A_p$ with positive Lebesgue measure. By minimality, for each $i = 1, \ldots p$, the absolutely continuous ϕ-invariant measure μ_{A_i} giving full weight to A_i is ergodic. Moreover, any absolutely continuous ϕ-invariant probability measure μ can be written as $\mu = \sum_{i=1}^p \mu(A_i) \, \mu_{A_i}$.

6   The measure

Here we will use the results of the previous sections to prove that the map φ has a finite absolutely continuous invariant measure. Let $\mathcal{R} = \cup_{n \geq p} \mathcal{R}_n$ be the partition of $S^1 \times I \pmod 0$ constructed in Section 3. We define the map $\phi : S^1 \times I \longrightarrow S^1 \times I$ by taking

$$\phi|\text{int}(R) = \varphi^n|\text{int}(R) \quad \text{if} \quad R \in \mathcal{R}_n,$$

and extending it arbitrarily to the boundaries of the rectangles. In order to use the results of Section 5, we may view ϕ as a C^2 piecewise expanding map from $[0,1] \times I \subset \mathbb{R}^2$ into itself, since the interiors of the rectangles in $\mathcal{R}$ and their images by ϕ do not intersect the set $\{\theta = 0\}$.

Our objective now is to show that the map ϕ defined above satisfies the hypotheses of Theorem 5.2.

- (E_1) and (E_2) are easily verified.

- For (E_3), note that if (θ, x) belongs to $R \in \mathcal{R}_n$, then

$$D\phi(\theta, x) = \begin{pmatrix} \partial_\theta g^n(\theta) & 0 \\ \partial_\theta F_n(\theta, x) & \partial_x F_n(\theta, x) \end{pmatrix},$$

and so

$$(D\phi(\theta, x))^{-1} = \frac{1}{J(\theta, x)} \begin{pmatrix} \partial_x F_n(\theta, x) & 0 \\ -\partial_\theta F_n(\theta, x) & \partial_\theta g^n(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_\theta g^n(\theta))^{-1} & 0 \\ -\partial_\theta F_n(\theta, x)(\partial_\theta g^n(\theta)\partial_x F_n(\theta, x))^{-1} & (\partial_x F_n(\theta, x))^{-1} \end{pmatrix}.$$

Taking into account Lemma 4.1 this implies

$$\|D\phi^{-1}(\phi(\theta, x))\| \le \max\left\{ |\partial_\theta g^n(\theta)|^{-1} + C_3 |\partial_x F_n(\theta, x)|^{-1}, |\partial_x F_n(\theta, x)|^{-1} \right\}.$$

We have

$$|\partial_\theta g^n(\theta)|^{-1} \le (d - \alpha)^{-n}.$$

Using (I_n) and Lemmas 2.3 and 3.5 we also have

$$|\partial_x F_n(\theta, x)|^{-1} \le \exp\left(-(2c - \epsilon)n + 3\log\frac{1}{\alpha} \right).$$

Hence,

$$\|D\phi^{-1}(\phi(\theta, x))\| \le (d - \alpha)^{-n} + (1 + C_3)\exp\left(-(2c - \epsilon)n + 3\log\frac{1}{\alpha} \right), \quad (23)$$

which can be made smaller than one, by taking p large enough.

- (D) is a direct consequence of Proposition 4.2.

- For (C_1) and (C_2), we observe that by Corollary 3.3 the boundary of $\phi(R)$, $R \in \mathcal{R}$, is made by two vertical lines and two admissible curves. Hence, the C^2 components of the boundaries of $\phi(R)$ meet at angles uniformly bounded away from zero. This, together with Proposition 3.8, provides regular collars for $\phi(R)$ with constants $\beta(\phi(R))$ and $\rho(\phi(R))$ uniformly bounded away from zero.

Now we are in conditions to specify our choice of p: having in mind Theorem 5.2 and estimate (23), let $p \ge 1$ be an integer such that

$$\left((d - \alpha)^{-p} + (1 + C_3)\exp\left(-(2c - \epsilon)p + 3\log\frac{1}{\alpha} \right) \right) \left(1 + \frac{1}{\beta} \right) < 1.$$

Thus, we may apply Theorem 5.2 and conclude that ϕ has an absolutely continuous invariant probability μ. Finally, defining the sequence of sets

$$R_1 = \cdots = R_{p-1} = \emptyset, \quad \text{and} \quad R_n = \bigcup_{R \in \mathcal{R}_n} R \quad \text{for} \quad n \geq p,$$

we take

$$\mu^* = \sum_{j=0}^{\infty} \varphi_*^j \left(\mu \mid \bigcup_{n>j} R_n \right).$$

We will show that μ^* is a φ-invariant absolutely continuous finite measure, and then derive the existence of a finite number of measures with the same properties of μ^* and moreover being ergodic.

Invariance. Let A be an arbitrary Borelean in $S^1 \times I$. We have

$$
\begin{aligned}
\mu^*(\varphi^{-1}(A)) &= \sum_{j=0}^{\infty} \mu \left(\varphi^{-j}(\varphi^{-1}(A)) \cap (\cup_{n>j} R_n) \right) \\
&= \sum_{j=0}^{\infty} \mu \left(\varphi^{-(j+1)}(A) \cap (R_{j+1} \cup (\cup_{n>j+1} R_n)) \right) \\
&= \sum_{j=0}^{\infty} \mu \left(\varphi^{-(j+1)}(A) \cap R_{j+1} \right) + \sum_{j=0}^{\infty} \mu \left(\varphi^{-(j+1)}(A) \cap (\cup_{n>j+1} R_n) \right).
\end{aligned}
$$

In this last equality we used that $(R_n)_n$ is a sequence of disjoint sets (mod 0). Now we have

$$
\begin{aligned}
\sum_{j=0}^{\infty} \mu \left(\varphi^{-(j+1)}(A) \cap R_{j+1} \right) &= \mu \left(\cup_{j \geq 1} \left(\varphi^{-j}(A) \cap R_j \right) \right) \\
&= \mu(\phi^{-1}(A)) = \mu(A)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=0}^{\infty} \mu \left(\varphi^{-(j+1)}(A) \cap (\cup_{n>j+1} R_n) \right) &= \mu^*(A) - \mu(A \cap (\cup_{n>0} R_n)) \\
&= \mu^*(A) - \mu(A),
\end{aligned}
$$

which altogether gives

$$\mu^*(\varphi^{-1}(A)) = \mu^*(A),$$

and so the measure μ^* is φ-invariant.

Absolute continuity. This is a direct consequence of the absolute continuity of the measure μ. Note that if A is a Borelean in $S^1 \times I$ with $m_2(A) = 0$, then $m_2(\varphi^{-j}(A)) = 0$ for every $j \geq 0$. Then, by the absolute continuity of μ, we have $\mu(\varphi^{-j}(A)) = 0$ for every $j \geq 0$. Hence

$$\mu^*(A) = \sum_{j=0}^{\infty} \mu\left(\varphi^{-j}(A) \cap \left(\bigcup_{n>j} R_n\right)\right) = 0,$$

which shows that μ^* is absolutely continuous with respect to m_2.

Finitness. Let f_0 be the density of μ with respect to the Lebesgue measure m_2. We have

$$
\begin{aligned}
\mu^*\left(S^1 \times I\right) &= \sum_{j=0}^{\infty} \mu\left(\cup_{n>j} R_n\right) \\
&= \sum_{j=0}^{\infty} \int_{\cup_{n>j} R_n} f_0 \, dm_2 \\
&= \sum_{j=0}^{\infty} \int_{S^1 \times I} f_0 \cdot \chi_{\cup_{n>j} R_n} \, dm_2.
\end{aligned}
$$

Now, recall that by Remark 5.6 we have $f_0 \in L^2(S^1 \times I)$. On the other hand, it follows from Proposition 2.6 that for every $j \geq p$

$$\bigcup_{n>j} R_n \subset \left(\left(S^1 \times I\right) \setminus \left(H_p \cup \cdots \cup H_j\right)\right) \subset E_j,$$

and so, by Lemma 2.4

$$m_2\left(\bigcup_{n>j} R_n\right) \leq m_2(E_j) \leq Ce^{-\gamma\sqrt{j}}.$$

Altogether, this yields

$$\mu^*(S^1 \times I) \leq \sum_{j=0}^{\infty} \|f_0\|_{L^2} \cdot \|\chi_{\cup_{n>j} R_n}\|_{L^2}$$

$$\leq p\|f_0\|_{L^2} + \|f_0\|_{L^2} \sum_{j=p}^{\infty} \|\chi_{\cup_{n>j} R_n}\|_{L^2}$$

$$\leq p\|f_0\|_{L^2} + \|f_0\|_{L^2} \sum_{j=p}^{\infty} C^{1/2} e^{-\gamma\sqrt{j}/2}$$

which is obviously finite.

Ergodicity. For proving that φ has some ergodic absolutely continuous invariant probability measure, we will show that for any φ-invariant set $A \subset S^1 \times I$ with positive Lebesgue measure:

1. A has Lebesgue measure uniformly bounded away from zero;

2. there is some absolutely continuous φ-invariant probability measure giving full weight to A.

This allows us to decompose $S^1 \times I$ into a finite number of minimal positive Lebesgue measure φ-invariant sets. Then everything follows exactly in the same way as for the piecewise expanding map ϕ in Section 5.

Let $A \subset S^1 \times I$ be any φ-invariant set with $m_2(A) > 0$. We have

$$\phi^{-1}(A) = \left\{ x \in \bigcup_{i \geq p} R_i : \phi(x) \in A \right\} = \bigcup_{i \geq p} \left(\varphi^{-i}(A) \cap R_i \right) = A,$$

and so the set A is also ϕ-invariant. It follows from what we have seen in Section 5 that $m_2(A) \geq \widehat{K}(2)$, and so we have proved 1. above.

Now let μ_A be a measure as in Section 5 and take

$$\mu_A^* = \sum_{j=0}^{\infty} \varphi_*^j \left(\mu_A \mid \bigcup_{n>j} R_n \right).$$

In order to obtain 2. it suffices to show that $\mu_A^*(A^c) = 0$, where $A^c = (S^1 \times I)\backslash A$. Since A is φ-invariant, A^c is also φ-invariant, and so

$$\mu_A^*(A^c) = \sum_{j=0}^{\infty} \mu_A \left(\varphi^{-j}(A^c) \cap \left(\bigcup_{n>j} R_n \right) \right) = \sum_{j=0}^{\infty} \mu_A \left(A^c \cap \left(\bigcup_{n>j} R_n \right) \right) = 0$$

Hence, the normalized μ_A^* is an absolutely continuous φ-invariant probability measure giving full weight to A.

7 Conclusion of the proof

Here we show how to remove assumption (1) from our hypotheses and conclude that any C^3 map φ satisfying $\|\varphi - \varphi_\alpha\| < \epsilon$ for sufficiently small ϵ (with respect to α) has a finite absolutely continuous invariant measure.

The main step is to show that such a φ admits an invariant foliation $\mathcal{F}^c$ by C^1 leaves C^1 close to vertical lines in $S^1 \times I$. This is a consequence of the fact that the set of vertical lines constitutes a normally expanding invariant foliation for the map φ_α.

Let $\mathcal{H}$ be the space of continuous maps $\xi : S^1 \times I \to [-1, 1]$ endowed with the sup norm, and define the map $F : \mathcal{H} \to \mathcal{H}$ by

$$F\xi(z) = \frac{\partial_x f(z)\xi(\varphi(z)) - \partial_x g(z)}{-\partial_\theta f(z)\xi(\varphi(z)) + \partial_\theta g(z)}, \quad z = (\theta, x) \in S^1 \times I.$$

Note that F is well-defined, since

$$|F\xi(z)| \leq \frac{(4 + \epsilon) + \epsilon}{-(\text{const } \alpha + \epsilon) + (d - \epsilon)} < 1$$

for small $\alpha > 0$ and $\epsilon > 0$. Moreover, F is a contraction on $\mathcal{H}$: given $\xi, \eta \in \mathcal{H}$ and $z \in S^1 \times I$

$$
\begin{aligned}
|F\xi(z) - F\eta(z)| &\leq \frac{|\det D\varphi(z)||\xi(z) - \eta(z)|}{|(-\partial_\theta f(z)\xi(\varphi(z)) + \partial_\theta g(z))(-\partial_\theta f(z)\eta(\varphi(z)) + \partial_\theta g(z))|} \\
&\leq \frac{((d + \epsilon)(4 + \epsilon) + \epsilon)|\xi(z) - \eta(z)|}{(d - \text{const}\alpha)^2}.
\end{aligned}
$$

This last quantity can be made smaller than $1/2|\xi(z) - \eta(z)|$, as long as α and ϵ are chosen sufficiently small. This shows that F is a contraction of the Banach space $\mathcal{H}$, and so has a fixed point $\xi^c \in \mathcal{H}$. Note that since the map F depends continuously on the dynamics φ, and for φ_α the fixed point coincides with the constant map equal to zero, then for φ close to φ_α the fixed point ξ^c of F is close to zero. We have defined F in such a way that if we take $E^c(z) = \text{span}\{(\xi^c(z), 1)\}$, then for every $z \in S^1 \times I$

$$D\varphi(z)E^c(z) \subset E^c(\varphi(z)). \tag{24}$$

Now we take $\mathcal{F}^c$ to be the set of integral curves of the vector field $z \to (\xi^c(z), 1)$. Since the vector field is taken of class C^0, it does not follow immediately that through each point in $S^1 \times I$ passes only one integral curve. However, we will prove uniqueness of solutions by using the fact that the map φ has a big expansion in the horizontal direction.

Assume by contradiction that there are two distinct integral curves $Y_1, Y_2 \in \mathcal{F}^c$ with a common point. So we may take three distinct nearby points $z_0, z_1, z_2 \in S^1 \times I$ such that $z_0 \in Y_1 \cap Y_2$, $z_1 \in Y_1$, $z_2 \in Y_2$ and z_1, z_2 have the same x-coordinate. Consider the horizontal curve X joining z_1 to z_2. If we iterate X by φ, such iterates are admissible curves (nearly horizontal) and locally grow in the horizontal direction by a factor al least $d - \mathrm{const}\, \alpha$ in each iterate. Hence, after a certain number of iterates the images of X wrap many times around the cylinder $S^1 \times I$. On the other hand, since the iterates of Y_1 and Y_2 are always tangent to the vector field $z \to (\xi^c(z), 1)$, it follows that all the iterates of Y_1 and Y_2 have small amplitude in the θ-direction. This gives a contradiction, since all the iterates by φ of the homotopically zero closed curve made by Y_1, Y_2 and X is always homotopic to zero in $S^1 \times I$. Thus, we have uniqueness of solutions of the vector field $z \to (\xi^c(z), 1)$, and from (24) it follows that $\mathcal{F}^c$ is a φ-invariant foliation of $S^1 \times I$ by C^1 leaves C^1 close to vertical lines.

The existence of this invariant foliation $\mathcal{F}^c$ replaces the assumption of the skew-product form of φ in (1). One also needs an analog of the second part of assumption (1). Since (24) holds and $E^c(z), E^c(\varphi(z))$ are unidimensional spaces, there must be some scalar $\Delta(z)$ satisfying

$$D\varphi(z)(\xi^c(z), 1) = \Delta(z)(\xi^c(\varphi(z)), 1)$$

for every $z \in S^1 \times I$. We define the *critical set* of φ by $\mathcal{C} = \{z \in S^1 \times I : \Delta(z) = 0\}$. By an easy implicit function argument it is shown in Section 2.5 of [V2] that $\mathcal{C}$ is the graph of some C^2 map $\eta : S^1 \to I$ arbitrarily C^2-close to zero if ϵ is small. This means that up to a change of coordinates C^2-close to the identity we may suppose that $\eta \equiv 0$ and, hence, write $\Delta(\theta, x) = x\psi(\theta, x)$ with $|\psi + 2|$ close to zero if ϵ and α are small. This provides an analog to the second part of assumption (1). At this point, the arguments of [V2] apply with $\partial_x f(\theta, x)$ replaced by $\Delta(\theta, x)$, to show that $\prod_{i=0}^{n-1} \Delta(\theta_i, x_i)$ grows exponentially fast almost surely.

Now the proof of Theorem A follows in just the same way as before, with the leaves of $\mathcal{F}^c$ replacing the vertical lines. For the sake of completeness a few words are required, concerning the construction of the partition $\mathcal{R}$. In this case the boundary of the rectangles will be made by two horizontal segments (as before) and two segments

of leaves in $\mathcal{F}^c$. For each $n \geq 1$ we define a partition $\mathcal{P}_n$ of $S^1 \times \{0\}$ in the following way: consider the map

$$\begin{aligned} \widehat{X}_n : \quad S^1 \times \{0\} &\longrightarrow S^1 \times I \\ (\theta, 0) &\longmapsto \varphi^n(\theta, 0) \end{aligned}$$

and let F_0 be the leaf of $\mathcal{F}^c$ close to $\{\theta = 0\}$ that is fixed under φ. Then we define

$$\mathcal{P}_n = \left\{ [\theta', \theta''] : (\theta', \theta'') \text{ is a connected component of } \widehat{X}_n^{-1}((S^1 \times I) \backslash F_0) \right\}.$$

This partition easily induce a partition of $S^1 \times I$ into nearly vertical strips

$$\widehat{\mathcal{P}}_n = \left\{ \Omega = \bigcup_{\theta \in \omega} F_\theta : \omega \in \mathcal{P}_n \right\},$$

where each F_θ is the leaf of $\mathcal{F}^c$ that contains the point $(\theta, 0) \in S^1 \times I$. Note that for each $x \in I$ and $\Omega \in \widehat{\mathcal{P}}_n$ the length of $\omega = (S^1 \times \{x\}) \cap \Omega$ depends on x in an unimportant way. In fact we have

$$(d + \mathrm{const}\alpha)^{-n} \leq |\omega| \leq (d - \mathrm{const}\alpha)^{-n}$$

for every $x \in I$ and $n \geq 1$. Now the construction of the partition $\mathcal{R}$ follows as in Section 3, starting our inductive process with the nearly vertical strips of the partition $\widehat{\mathcal{P}}_p$. Here we use the left hand side fibers of the strips in $\widehat{\mathcal{P}}_n$ ($n \geq p$) to determine the itineraries of points. Having defined the elements of $\mathcal{R}_n$ and $\mathcal{S}_n$ for some $n \geq p$, we use the partition $\widehat{\mathcal{P}}_{n+1}$ to divide the remaining rectangles in $\mathcal{S}_n$.

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