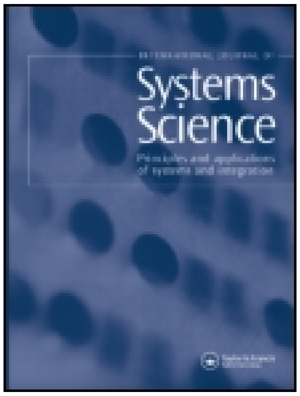

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Stabilisation of neutral systems with saturating control inputs

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Stabilisation of neutral systems with saturating control inputs

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This article focuses on the stabilisation problem of neutral systems in the presence of time-varying delays and control saturation. Based on a descriptor approach and the use of a modified sector relation, global and local stabilisation conditions are derived using Lyapunov–Krasovskii functionals. These conditions, formulated directly as linear matrix inequalities (LMIs), allow one to relate the control law to be computed to a set of admissible initial conditions, for which the asymptotic and exponential stabilities of the closed-loop system are ensured. An extension of these conditions to the particular case of retarded systems is also provided. From the theoretical conditions, optimisation problems with LMI constraints are therefore proposed to compute stabilising state feedback gains with the aim of ensuring stability for a given set of admissible initial conditions or the global stability of the closed-loop system. A numerical example illustrates the application of the proposed results.

Keywords: time delay; saturation; stabilisation; stability domains; robustness

1. Introduction

In the past years great attention has been paid to stability and control of time-delay systems (Kolmanovskii and Myshkis 1999; Niculescu 2001; Richard 2003). This is due to the fact that the behaviour of many physical systems (mechanical, chemical processes, telecommunication, etc.) can be modelled by functional differential equations. Delays can appear in the state, input or output variables (retarded systems), as well as in the state derivative (neutral systems). Furthermore, it is well known that the presence of the delays in control systems can lead to bad time-domain performances or even to the instability of the closed-loop system. Hence, we can find in the literature a great amount of techniques and methodologies dealing with the stability and stabilisation of time-delay systems (retarded and also neutral), and associated problems, such as performance, robustness and filtering.

The difficulty in controlling time-delay systems becomes even greater if the control signal is bounded. Unfortunately, this is a practical constraint, which comes from the impossibility of actuators to drive signals with unlimited amplitude or energy to the controlled plants. For retarded systems, some works addressing the stability analysis and stabilisation in the presence of saturating control signals can be found in the literature. In Oucheriah (1996) and Niculescu, Dion, and Dugard (1996) globally stabilising control laws are proposed. In Chen, Wang, and Lu (1988) and Tissir and Hmamed (1992), conditions for stability or stabilisation are proposed with state feedback and sampled-data state feedback. However, in these papers, the set of admissible initial conditions, for which the asymptotic stability is ensured (i.e. the domain of attraction) in the presence of control saturation, is not mentioned or explicitly defined. Based on invariance properties, in Dambrine, Richard, and Borne (1995) the control was computed to avoid the (input and state) saturations. In Tarbouriech and Gomes da Silva Jr (2000), Cao, Lin, and Hu (2002) and Fridman, Seuret, and Richard (2004), methods for computing stabilising state feedback control laws aiming at enlarging well defined estimates of the domain of attraction of the closed-loop system have been proposed. These methods are based on the use of polytopic differential inclusions for describing the behavior of the closed-loop system with saturating inputs. In Tarbouriech, Gomes da Silva Jr and Garcia (2003, 2004), the synthesis of stabilising static anti-windup loops is addressed for the case of retarded systems presenting fixed delays. On the other hand, considering neutral systems, we can cite only Tarbouriech and Garcia (1999). In that paper, using

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a polytopic approach for modelling saturation effects, a method for computing stabilising state feedback controls with the aim of maximising the set of admissible initial conditions is proposed. It should be pointed out that the results in Tarbouriech and Garcia (1999) are derived in the delay independent context and the obtained conditions are in the form of nonlinear matrix inequalities. Furthermore, due to the use of a polytopic approach, only local stability can be ensured.

As in Tarbouriech and Garcia (1999), this article is concerned with the asymptotic as well as the exponential stabilisation problem of neutral systems in the presence of control saturation.[1] Based on a Lyapunov–Krasosvskii functional (LKF) and on the application of a modified sector condition (Tarbouriech et al. 2004), global and local stabilisation conditions are derived in a delay dependent context. Different from Tarbouriech and Garcia (1999), these conditions allow one to consider the case of time-varying delays in a delay dependent context and they are formulated directly as linear matrix inequalities (LMIs). In addition, the extension of these conditions to the particular case of retarded systems with delays is also presented. Optimisation problems are then formulated with the aim of computing stabilising state feedback control laws. These optimisation problems allow one to search the maximal delay bound for which a global stabilising control law can be found. On the other hand, when only local stabilisation is possible (e.g. when the open-loop system is unstable), the optimisation objective consists of finding a control law that maximises an estimate of the domain of attraction or that ensures the stability for a given set of admissible initial states.

The article is organised as follows. In Section 2, the problem to be treated is formally stated. The results concerning the asymptotic stabilisation are presented in Section 3. Exponential stabilisation results are provided in Section 4. Optimisation problems to compute stabilising gains are proposed and discussed in Section 5. Finally, in Section 6, numerical examples illustrate the application of the results.

Notations. Throughout the article $\Re^n$ denotes the n dimensional Euclidean space. A_i denotes the i-th row of matrix A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A. I denotes an identity matrix of appropriate order. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote, respectively, the maximal and the minimal eigenvalues of matrix P. $\mathcal{C}_h = \mathcal{C}([-h,0],\Re^n)$ is the Banach space of continuous vector functions mapping the interval $[-h,0]$ into $\Re^n$ with the norm $\|\phi\|_c = \sup_{-h\le t\le 0}\|\phi(t)\|$. $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm. $\mathcal{C}_h^v$ is the set defined by $\mathcal{C}_h^v = \{\phi \in \mathcal{C}_h;\ \|\phi\|_c < v, v > 0\}$.

2. Problem statement

Consider the following neutral type linear system:

$$\begin{aligned} &\dot{x}(t) - F\dot{x}(t-\tau(t)) = Ax(t) + A_d x(t-\tau(t)) + Bu(t) \\ &x(t_0+\theta) = \phi(\theta),\ \ \forall\theta\in[-h,0],\ t_0\in\Re_+,\ \phi(\theta)\in\mathcal{C}_h^v, \end{aligned} \quad (1)$$

where $x(t)\in\Re^n$ and $u(t)\in\Re^m$ are, respectively, the state and the input vectors, $\tau(t)$ corresponds to a time-varying delay that satisfies

$$0 \le \tau(t) \le h, \quad \dot{\tau}(t) \le d < 1.$$

The initial function $\phi(\theta)$ is supposed to be continuously differentiable. Matrices A, A_d, B and F are real constant matrices of appropriate dimensions. To apply the Lyapunov stability theorem (Kolmanovskii and Myshkis 1999, p. 337) we assume that $\|F\| < 1$.

We suppose that the input vector u is subject to amplitude limitations defined as follows:

$$|u_i| \le u_{0i}, \quad u_{0i} > 0,\ i = 1,\ldots,m. \quad (2)$$

Consider now a state feedback control law $u(t) = Kx(t)$. Due to the control bounds defined in (2), the effective control signal to be applied to the system is given by

$$u(t) = \mathrm{sat}(Kx(t))$$

where $u_i(t) = \mathrm{sat}(K_i x(t)) = \mathrm{sign}(K_i x(t))\min\{u_{0i}, K_i x(t)\}$. Hence, the closed-loop system reads

$$\dot{x}(t) - F\dot{x}(t-\tau(t)) = Ax(t) + A_d x(t-\tau(t)) + B\mathrm{sat}(Kx(t)). \quad (3)$$

System (3) is said to be *globally* asymptotically stable if for any differentiable initial condition $\phi(\theta)\in\mathcal{C}_h$, the trajectories of the system converge asymptotically to the origin (Niculescu et al. 1996; Oucheriah 1996). Similar to the case of delay-free ($\tau(t) = 0$), the determination of a global stabilising controller is only possible when some stability assumptions are verified by the open-loop system ($u(t) = 0$) (Lin and Saberi 1993). When this hypothesis is not verified, it is only possible to achieve local stabilisation. In this case, given a stabilising matrix K, we associate a *basin of attraction* to the equilibrium point $x_e(t) \equiv 0$ of system (3). The basin of attraction corresponds to all initial conditions $\phi(\theta)\in\mathcal{C}_h$ such that the corresponding trajectories of system (3) converge asymptotically to the origin. Since the determination of the exact basin of attraction is practically impossible, a problem of interest is to ensure the asymptotic stability for a set of admissible initial conditions $\phi(\theta)$ (Tarbouriech and Gomes da Silva Jr 2000; Cao et al. 2002; Fridman, Pila, and Shaked 2003). Of course, this set is included in the basin of attraction. Hence, from the above considerations, in this article we are interested in studying the stabilisation problems stated as follows.

(1) Given h and d, find K and a set of admissible initial conditions, as large as possible, for which the asymptotic (or exponential) stability of the closed-loop system is ensured.
(2) Given h, d and a set of admissible initial conditions, find K such that the asymptotic (or exponential) stability is ensured for all initial conditions belonging to the admissible set.
(3) Maximise the bound on the delay h, for which the asymptotic (or exponential) stability of the closed-loop system can be ensured for some set of admissible initial conditions and a given d.

Of course, when it is possible, the objective will be the global stabilisation of the closed-loop system. Otherwise, the set of admissible initial conditions will be defined from bounds on $\|\phi(\theta)\|_c$ and $\|\dot{\phi}(\theta)\|_c$. In the sequel, theoretical conditions that allow one to address the above stabilisation problems are proposed. Based on these conditions, optimisation problems are formulated in Section 5.

3. Asymptotic stabilisation

3.1. *Preliminaries*

Define the following function:

$$\psi(Kx(t)) = Kx(t) - \mathrm{sat}(Kx(t)). \tag{4}$$

Note that $\psi(Kx(t))$ corresponds to a decentralised deadzone nonlinearity. Considering the function $\psi(Kx(t))$, the closed-loop system can be re-written as

$$\begin{aligned}\dot{x}(t) - F\dot{x}(t-\tau(t)) = (A+BK)x(t) + A_d x(t-\tau(t))\\ - B\psi(Kx(t))\end{aligned} \tag{5}$$

Considering a matrix $G \in \Re^{m\times n}$ and defining the following polyhedral set:

$$\mathcal{S} \triangleq \{x \in \Re^n;\ |(K_i - G_i)x| \le u_{0i}, \quad i = 1,\dots,m\} \tag{6}$$

the following Lemma concerning the nonlinearity $\psi(Kx(t))$ can be stated.

Lemma 1 (Tarbouriech et al. 2004)**:** *Consider the function $\psi(Kx)$ defined in* (4). *If $x \in \mathcal{S}$ then the relation*

$$\psi(Kx)'T[\psi(Kx) - Gx] \le 0 \tag{7}$$

is verified for any matrix $T \in \Re^{m\times m}$ diagonal and positive definite.

The result in Lemma 1 can be seen as a generalised sector condition. As will be seen in the sequel, differently from the classical sector condition (used for instance in Tarbouriech et al. (2003)), this condition will allow one to obtain stability conditions directly in an LMI form.

Another instrumental result, needed in the sequel to devise the stabilisation conditions, is given by the following lemma.

Lemma 2: *Consider two scalars $a<b$ and a symmetric positive definite matrix $R \in \Re^{n\times n}$. For any continuous function $\omega : [a,b] \to \Re^n$ and any strictly positive continuous function $f : [a,b] \to \Re$, the following inequality holds*:

$$\int_a^b \omega'(s) f(s) R\omega(s)\mathrm{d}s \ge \left(\int_a^b \omega(s)\mathrm{d}s\right)' \left(\int_a^b (f(s))^{-1}\mathrm{d}s\right)^{-1} \times R\left(\int_a^b \omega(s)\mathrm{d}s\right). \tag{8}$$

Proof: Consider any $\epsilon \in [0\ 1)$. By virtue of the Schur complement, we can write that

$$\begin{bmatrix} f(s)\omega'(s)R\omega(s) & \omega'(s) \\ \omega(s) & (\epsilon f(s)R)^{-1}\end{bmatrix} \ge 0.$$

Then the proof consists of integrating the previous inequality, applying the Schur complement and taking $\epsilon \to 1$. □

Remark 1: From (8), it is simple to see that if $f(s) = 1$, the classical Jensen's inequality is obtained (Gu, Kharitonov, and Chen 2003, p. 322).

3.2. *Neutral systems*

Theorem 1: *If there exist symmetric positive definite matrices Q_1, L, J, X, matrices Q_2, Q_3, Y, W and a diagonal matrix S of appropriate dimensions satisfying the LMIs* (9) *and* (10),

$$\begin{bmatrix} \tilde{\mathcal{K}} & \begin{bmatrix} J/h \\ A_dQ_1 \end{bmatrix} & \begin{bmatrix} 0 \\ FL \end{bmatrix} & \begin{bmatrix} Y' \\ -BS \end{bmatrix} & h\begin{bmatrix} Q_2' \\ Q_3' \end{bmatrix} & \begin{bmatrix} Q_2' \\ Q_3' \end{bmatrix} \\ \star & (d-1)X - J/h & 0 & 0 & 0 & 0 \\ \star & \star & (d-1)L & 0 & 0 & 0 \\ \star & \star & \star & -2S & 0 & 0 \\ \star & \star & \star & \star & -2hQ_1 + hJ & 0 \\ \star & \star & \star & \star & \star & -L \end{bmatrix} < 0$$

$$\textit{with}\ \tilde{\mathcal{K}} = \begin{bmatrix} Q_2 + Q_2' + X - J/h & Q_1A' + W'B' - Q_2' + Q_3 \\ \star & -Q_3 - Q_3' \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} Q_1 & (W-Y)_j' \\ \star & u_{0j}^2 \end{bmatrix} \ge 0, \quad j = 1,\dots,m \tag{10}$$

then for $K = WQ_1^{-1}$ *and all initial conditions satisfying*

$$\delta = (\lambda_{\max}(Q_1^{-1}) + h\lambda_{\max}(Q_1^{-1}XQ_1^{-1}))\|\phi(\theta)\|_c^2 + \left(\frac{h^2}{2}\lambda_{\max}(Q_1^{-1}JQ_1^{-1}\right) + h\lambda_{\max}(L^{-1}))\|\dot{\phi}(\theta)\|_c^2 \leq 1, \quad (11)$$

the corresponding trajectories of system (5) *converge asymptotically to the origin.*

Proof: Consider that $x(t) \in \mathcal{S}$ and the following LKF proposed in Fridman and Shaked (2002) for dealing with time-varying delays:

$$V(t) = x'(t)P_1x(t) + \int_{t-\tau(t)}^{t} x'(s)Mx(s)\mathrm{d}s + \int_{t-\tau(t)}^{t} \dot{x}'(s)U\dot{x}(s)\mathrm{d}s + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}'(s)R\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta$$

with P_1, R, M and U being symmetric positive define matrices. Noting that $(A+BK)x(t) + A_dx(t-\tau(t)) - B\psi(Kx(t)) + F\dot{x}(t-\tau(t)) - \dot{x}(t) = 0$, it follows that the derivative of the functional is given by

$$\begin{aligned}\dot{V}(t) = {} & x'(t)P_1\dot{x}(t) + \dot{x}'(t)P_1x(t) + x'(t)Mx(t) \\ & - (1-\dot{\tau}(t))x'(t-\tau(t))Mx(t-\tau(t)) \\ & + \dot{x}'(t)U\dot{x}(t) - (1-\dot{\tau}(t))\dot{x}'(t-\tau(t))U\dot{x}(t-\tau(t)) \\ & + h\dot{x}(t)'R\dot{x}(t) - \int_{-h}^{0}\dot{x}'(t+\theta)R\dot{x}(t+\theta)\mathrm{d}\theta \\ & + 2x'(t)P_2'((A+BK)x(t) + A_dx(t-\tau(t)) \\ & - B\psi(Kx(t)) + F\dot{x}(t-\tau(t)) - \dot{x}(t)) \\ & + 2\dot{x}'(t)P_3'((A+BK)x(t) + A_dx(t-\tau(t)) \\ & - B\psi(Kx(t)) + F\dot{x}(t-\tau(t)) - \dot{x}(t)).\end{aligned}$$

Introducing the vectors $\bar{x}'(t) = [x'(t)\ \dot{x}'(t)]$ and $\xi'(t) = [\bar{x}'(t)\ \bar{x}'(t-\tau(t))\ \psi'(Kx(t))]$ and the matrix $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$ we will follow the descriptor approach (Fridman and Shaked 2002). In this case, the derivative of the functional is expressed as

$$\begin{aligned}\dot{V}(t) = {} & \bar{x}'(t)\mathcal{L}\bar{x}(t) + 2\bar{x}'(t)P'\begin{bmatrix}0 & A_d'\end{bmatrix}'x(t-\tau(t)) \\ & - 2\bar{x}'(t)P'\begin{bmatrix}0 & B'\end{bmatrix}'\psi(Kx(t)) \\ & + 2\bar{x}'(t)P'\begin{bmatrix}0 & F'\end{bmatrix}'\dot{x}(t-\tau(t)) + x'(t)Mx(t) \\ & - x'(t-\tau(t))(1-\dot{\tau}(t))Mx(t-\tau(t)) \\ & + \dot{x}'(t)U\dot{x}(t) - (1-\dot{\tau}(t))\dot{x}'(t-\tau(t))U\dot{x}(t-\tau(t)) \\ & + h\dot{x}'(t)R\dot{x}(t) - \int_{t-h}^{t}\dot{x}'(s)R\dot{x}(s)\mathrm{d}s \end{aligned} \quad (12)$$

with $\mathcal{L} = \begin{bmatrix} 0 & I \\ (A+BK) & -I \end{bmatrix}' P + P'\begin{bmatrix} 0 & I \\ (A+BK) & -I \end{bmatrix}$.

Provided that $x(t) \in \mathcal{S}$, from Lemma 1, it follows that

$$\dot{V}(t) \leq \dot{V}(t) - 2\psi(Kx)'T[\psi(Kx) - Gx] \quad (13)$$

where T is a diagonal positive definite matrix.

Applying now Jensen's inequality to the last term of (12), the following inequality holds:

$$-\int_{t-h}^{t}\dot{x}'(s)R\dot{x}(s)\mathrm{d}s \leq -(x(t) - x(t-\tau(t)))'\frac{R}{h}(x(t) - x(t-\tau(t))). \quad (14)$$

Combining (13) and (14), it follows that $\dot{V}(t) \leq \xi'(t)\Gamma\xi(t)$ with

$$\Gamma = \begin{bmatrix} \mathcal{L}+\Phi & \begin{bmatrix}R/h \\ 0\end{bmatrix} + P'\begin{bmatrix}0 \\ A_d\end{bmatrix} & P'\begin{bmatrix}0 \\ F\end{bmatrix} & \begin{bmatrix}G'T \\ 0\end{bmatrix} - P'\begin{bmatrix}0 \\ B\end{bmatrix} \\ \star & (d-1)M - R/h & 0 & 0 \\ \star & \star & (d-1)U & 0 \\ \star & \star & \star & -2T \end{bmatrix}$$

where $\Phi = \begin{bmatrix} M - R/h & 0 \\ 0 & hR + U \end{bmatrix}$.

Suppose now that $\Gamma < 0$. Applying Schur's complement to the terms $\begin{bmatrix} 0 & 0 \\ 0 & hR \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & U \end{bmatrix}$, it follows that $\Gamma < 0$ is equivalent to

$$\begin{bmatrix} \tilde{\mathcal{L}} & \begin{bmatrix}R/h \\ 0\end{bmatrix} + P'\begin{bmatrix}0 \\ A_d\end{bmatrix} & P'\begin{bmatrix}0 \\ F\end{bmatrix} & \begin{bmatrix}G'T \\ 0\end{bmatrix} - P'\begin{bmatrix}0 \\ B\end{bmatrix} & h\begin{bmatrix}0 \\ I\end{bmatrix} & \begin{bmatrix}0 \\ I\end{bmatrix} \\ \star & (d-1)M - R/h & 0 & 0 & 0 & 0 \\ \star & \star & (d-1)U & 0 & 0 & 0 \\ \star & \star & \star & -2T & 0 & 0 \\ \star & \star & \star & \star & -hR^{-1} & 0 \\ \star & \star & \star & \star & \star & -U^{-1} \end{bmatrix} < 0 \quad (15)$$

where $\tilde{\mathcal{L}} = \mathcal{L} + \begin{bmatrix} M - R/h & 0 \\ 0 & 0 \end{bmatrix}$. Note now that if the previous matrix inequality is satisfied, one has $\tilde{\mathcal{L}} < 0$, which implies that $-P_3' - P_3$ is negative definite. Hence, since $P_1 > 0$, it follows that matrix P is invertible. Denote now the matrix $P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}$ and define a block diagonal matrix $\Xi = \mathrm{diag}\{Q, Q_1, U^{-1}, T^{-1}, I, I\}$. By pre- and post-multiplying (15) by Ξ_1' and Ξ_1 respectively, one obtains:

$$\begin{bmatrix} Q'\tilde{\mathcal{L}}Q & \begin{bmatrix} Q_1RQ_1/h \\ A_dQ_1 \end{bmatrix} & \begin{bmatrix} 0 \\ FU^{-1} \end{bmatrix} & \begin{bmatrix} Q_1G' \\ -BT^{-1} \end{bmatrix} & h\begin{bmatrix} Q_2' \\ Q_3' \end{bmatrix} & \begin{bmatrix} Q_2' \\ Q_3' \end{bmatrix} \\ \star & Q_1((d-1)M-R/h)Q_1 & 0 & 0 & 0 & 0 \\ \star & \star & (d-1)U^{-1} & 0 & 0 & 0 \\ \star & \star & \star & -2T^{-1} & 0 & 0 \\ \star & \star & \star & \star & -hR^{-1} & 0 \\ \star & \star & \star & \star & \star & -U^{-1} \end{bmatrix} < 0. \quad (16)$$

Consider now the change of variables: $X=Q_1MQ_1$, $J=Q_1RQ_1$, $L=U^{-1}$, $S=T^{-1}$, $Y=GQ_1$ and $W=KQ_1$. Noting that $Q'\tilde{\mathcal{L}}Q=\tilde{\mathcal{K}}$ and that the development of $(Q_1-R^{-1})R(Q_1-R^{-1})\geq 0$ implies $R^{-1}\geq 2Q_1-J$, it follows that the LMI condition (9) is obtained. Thus, we conclude that (9) implies that $\Gamma<0$, which implies that $\dot{V}(t)<0$, provided that $x(t)\in\mathcal{S}$, $t>0$.

From the definition of $V(t)$, it follows that

$$\begin{aligned} V(0) &\leq x'(0)P_1x(0)+\int_{-h}^{0}x'(s)Mx(s)\mathrm{d}s+\int_{-h}^{0}\dot{x}'(s)U\dot{x}(s)\mathrm{d}s \\ &\quad+\int_{-h}^{0}\int_{\theta}^{0}\dot{x}'(s)R\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta \\ &\leq(\lambda_{\max}(Q_1^{-1})+h\lambda_{\max}(Q_1^{-1}XQ_1^{-1}))\|\phi(\theta)\|_c^2 \\ &\quad+\Big(\frac{h^2}{2}\lambda_{\max}(Q_1^{-1}JQ_1^{-1}\Big)+h\lambda_{\max}(L^{-1}))\|\dot{\phi}(\theta)\|_c^2=\delta. \end{aligned}$$

If $\dot{V}(t)<0$, $\forall t\geq 0$, then we can conclude that

$$x(t)'P_1x(t)\leq V(t)\leq V(0)\leq\delta, \quad \forall t\geq 0. \quad (17)$$

Consider the ellipsoidal set $\mathcal{E}=\{x\in\Re^n\,;\,x'P_1x\leq 1\}$, where $P_1=Q_1^{-1}$. It is easy to see (Tarbouriech and Gomes da Silva Jr 2000) that (10) implies that $\mathcal{E}\subset\mathcal{S}$, with $\mathcal{S}$ as defined in (6). Suppose now that the initial condition $\phi(\theta)$ satisfies (11), i.e. $\delta\leq 1$, and conditions (9)–(10) hold. From (17), it follows that the state trajectory is confined in the ellipsoid $\mathcal{E}$, $\forall t\geq 0$, which ensures that $x(t)\in\mathcal{S}$, $\forall t\geq 0$. Then, $\dot{V}(t)<0$, $\forall t\geq 0$ is effectively satisfied for all initial conditions verifying (11), which concludes the proof. □

Theorem 1 considers the local (or regional) stabilisation, in the sense that the computed gain K ensures asymptotic stability just for the initial conditions satisfying (11). As pointed out in Section 2, provided the open-loop system is asymptotically stable, it can be possible to compute globally stabilising gains. The next result, which can be seen as a particularisation of Theorem 1, allows one to address this problem.

Corollary 1: *If there exist positive definite matrices Q_1, L, J, X, matrices Q_2, Q_3, W and a diagonal matrix S of appropriate dimensions satisfying* (9) *with $Y=W$, then, for $K=WQ_1^{-1}$ the origin of system* (5) *is globally asymptotically stable.*

Proof: The proof mimics the one of Theorem 1. In this case, it follows that $G=WP_1=WQ_1^{-1}=K$. Hence (7) is verified for all $x\in\Re^n$ and the global asymptotic stability follows. □

3.3. *Retarded systems*

We focus now on the stabilisation of the following retarded system:

$$\dot{x}(t)=Ax(t)+A_{\mathrm{d}}x(t-\tau(t))+B\mathrm{sat}(Kx(t)). \quad (18)$$

This system can be seen as a particular case of system (3) when $F=0$. The following theorem gives a condition to stabilise system (18).

Theorem 2: *If there exist positive definite matrices Q_1, X, J, matrices Q_2, Q_3, Y, W and a diagonal matrix S of appropriate dimensions satisfying the LMIs* (10) *and* (19)

$$\begin{bmatrix} \tilde{\mathcal{K}} & \begin{bmatrix} J/h \\ A_dQ_1 \end{bmatrix} & \begin{bmatrix} Y' \\ -BS \end{bmatrix} & h\begin{bmatrix} Q_2' \\ Q_3' \end{bmatrix} \\ \star & (d-1)X-J/h & 0 & 0 \\ \star & \star & -2S & 0 \\ \star & \star & \star & -2hQ_1+hJ \end{bmatrix}<0 \quad (19)$$

then for $K=WQ_1^{-1}$ and all initial conditions satisfying

$$\begin{aligned} \delta_r &= (\lambda_{\max}(Q_1^{-1})+h\lambda_{\max}(Q_1^{-1}XQ_1^{-1}))\|\phi(\theta)\|_c^2 \\ &\quad+\frac{h^2}{2}\lambda_{\max}(Q_1^{-1}JQ_1^{-1})\|\dot{\phi}(\theta)\|_c^2\leq 1 \end{aligned}$$

the corresponding trajectories of system (18) *converge asymptotically to the origin.*

Proof: Considering the following LKF:

$$\begin{aligned} V(t) &= x'(t)P_1x(t)+\int_{t-\tau(t)}^{t}x'(s)Mx(s)\mathrm{d}s \\ &\quad+\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}'(s)R\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta \end{aligned}$$

with P_1, R and M being symmetric positive define matrices, it suffices to follow the same steps of the proof of Theorem 1 considering $U=0$. □

Concerning the global stabilisation, the following result follows in this case.

Corollary 2: *If there exist positive definite matrices Q_1, X, J, matrices Q_2, Q_3, W and a diagonal matrix S of appropriate dimensions satisfying the LMI* (19) *with $Y = W$, then the state feedback gain $K = WQ_1^{-1}$ ensures that the origin of system* (18) *is globally asymptotically stable.*

Remark 2: The result presented in Theorem 2 can be easily adapted to consider the case of delays that can vary arbitrarily fast (i.e. $\dot{\tau}(t)$ is not bounded by d). This can be done by setting the matrix X equal to zero in (19), which corresponds to set $M = 0$ in the LKF.

4. Exponential stabilisation

Exponential stability properties can be an interesting way to characterise the convergence rate of the system. As usual (Niculescu, de Souza, Dion, and Dugard 1998; Sun, Zhao and Hill 2006; Xu, Lam, and Zhong 2006), given some rate $\alpha > 0$, a system (5) is said to be α-stable, or 'exponentially stable with the rate α', if there exists a scalar $\beta \geq 1$ such that its solution $x(t; t_0, \phi(\theta))$, with any initial continuously differentiable function $\phi(\theta)$, satisfies

$$\|x(t, t_0; \phi(\theta))\| \leq \beta[\|\phi(\theta)\|_c + \|\dot{\phi}(\theta)\|_c] e^{-\alpha(t-t_0)}. \quad (20)$$

The following theorem provides a sufficient condition to ensure the exponential stabilisation of system (5), with a decay rate α.

Theorem 3: *If, for a positive number α, there exist positive definite matrices Q_1, X, L, J, matrices Q_2, Q_3, Y, W and a diagonal matrix S of appropriate dimensions satisfying the LMIs* (10) *and* (21),

$$\begin{bmatrix} \tilde{\mathcal{K}}_\alpha & \begin{bmatrix} Je^{-2\alpha h}/\eta_1 \\ A_d Q_1 \end{bmatrix} & \begin{bmatrix} 0 \\ FL \end{bmatrix} & \begin{bmatrix} Y' \\ -BS \end{bmatrix} & h\begin{bmatrix} Q_2' \\ Q_3' \end{bmatrix} & \begin{bmatrix} Q_2' \\ Q_3' \end{bmatrix} \\ \star & ((d-1)X - J/\eta_1)e^{-2\alpha h} & 0 & 0 & 0 & 0 \\ \star & \star & (d-1)Le^{-2\alpha h} & 0 & 0 & 0 \\ \star & \star & \star & -2S & 0 & 0 \\ \star & \star & \star & \star & -2hQ_1 + hJ & 0 \\ \star & \star & \star & \star & \star & -L \end{bmatrix} < 0 \quad (21)$$

$$\text{with } \tilde{\mathcal{K}}_\alpha = \begin{bmatrix} Q_2 + Q_2' + X + 2\alpha Q_1 - Je^{-2\alpha h}/\eta_1 & Q_1 A' + W'B' - Q_2' + Q_3 \\ \star & -Q_3 - Q_3' \end{bmatrix}$$

then, for $K = WQ_1^{-1}$ and all initial conditions satisfying

$$\delta_e = (\lambda_{\max}(Q_1^{-1}) + \eta_1 \lambda_{\max}(Q_1^{-1} X Q_1^{-1}))\|\phi(\theta)\|_c^2 + (\eta_1 \lambda_{\max}(L^{-1}) + \eta_2 \lambda_{\max}(Q_1^{-1} J Q_1^{-1}))\|\dot{\phi}(\theta)\|_c^2 \leq 1 \quad (22)$$

with

$$\eta_1 = \frac{1 - e^{-2\alpha h}}{2\alpha} \quad \text{and} \quad \eta_2 = \frac{e^{-2\alpha h} - 1 + 2\alpha h}{4\alpha^2}, \quad (23)$$

the corresponding trajectories of system (5) *converge exponentially to the origin, with a decay rate α.*

Proof: Consider the following LKF:

$$\begin{aligned} V_\alpha(t) = & x'(t)P_1 x(t) + \int_{t-\tau(t)}^{t} x'(s)e^{2\alpha(s-t)} M x(s) \mathrm{d}s \\ & + \int_{t-\tau(t)}^{t} \dot{x}'(s) e^{2\alpha(s-t)} U \dot{x}(s) \mathrm{d}s \\ & + \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}'(s) e^{2\alpha(s-t)} R \dot{x}(s) \mathrm{d}s\, \mathrm{d}\theta \end{aligned}$$

with P_1, R, M and $U > 0$ being symmetric positive define matrices. Following the proof of Theorem 1, the differentiation of the LKF along the trajectories of system (1) leads to

$$\begin{aligned} \dot{V}_\alpha(t) = & \bar{x}'(t)\mathcal{L}\bar{x}(t) + 2\bar{x}'(t)P'\begin{bmatrix} 0 & A_d' \end{bmatrix}' x(t - \tau(t)) \\ & - 2\bar{x}'(t)P'\begin{bmatrix} 0 & B' \end{bmatrix}' \psi(Kx(t)) \\ & + 2\bar{x}'(t)P'\begin{bmatrix} 0 & F' \end{bmatrix}' \dot{x}(t - \tau(t)) \\ & + x'(t)Mx(t) + \dot{x}'(t)U\dot{x}(t) \\ & - (1 - \dot{\tau}(t))x'(t - \tau(t))e^{-2\alpha\tau(t)} Mx(t - \tau(t)) \\ & - (1 - \dot{\tau}(t))\dot{x}'(t - \tau(t))e^{-2\alpha\tau(t)} U\dot{x}(t - \tau(t)) \\ & + h\dot{x}'(t)R\dot{x}(t) - \int_{t-h}^{t} \dot{x}'(s)e^{2\alpha(s-t)} R\dot{x}(s)\mathrm{d}s \\ & + 2\alpha x'(t)P_1 x(t) - 2\alpha V_\alpha(t). \end{aligned}$$

Applying Lemma 2 to the integral term of the previous expression, the following inequality is obtained:

$$\begin{aligned} & - \int_{t-h}^{t} \dot{x}'(s)e^{2\alpha(s-t)} R\dot{x}(s)\mathrm{d}s \\ & \leq -\frac{2\alpha}{e^{2\alpha h} - 1}(x(t) - x(t - \tau(t)))' R(x(t) - x(t - \tau(t))). \end{aligned}$$

Noting that $\frac{2\alpha}{e^{2\alpha h}-1} = e^{-2\alpha h}/\eta_1$ and that for all delay $\tau(t) \in [0\ h]$, $e^{-2\alpha\tau(t)} \geq e^{-2\alpha h}$, then the following inequality holds:

$$\begin{aligned}\dot{V}_\alpha(t) + 2\alpha V_\alpha(t) \leq{} & \bar{x}'(t)\mathcal{L}\bar{x}(t) + 2\bar{x}'(t)P'\begin{bmatrix}0 & A_d'\end{bmatrix}' x(t-\tau(t)) \\ & - 2\bar{x}'(t)P'\begin{bmatrix}0 & B'\end{bmatrix}'\psi(Kx(t)) \\ & + 2\bar{x}'(t)P'\begin{bmatrix}0 & F'\end{bmatrix}'\dot{x}(t-\tau(t)) \\ & + x'(t)Mx(t) + \dot{x}'(t)U\dot{x}(t) \\ & - (1-\dot{\tau}(t))x'(t-\tau(t))e^{-2\alpha h}Mx(t-\tau(t)) \\ & - (1-\dot{\tau}(t))\dot{x}'(t-\tau(t))e^{-2\alpha h}U\dot{x}(t-\tau(t)) \\ & + h\dot{x}'(t)R\dot{x}(t) - e^{-2\alpha h}/\eta_1(x(t) \\ & - x(t-\tau(t)))'R(x(t)-x(t-\tau(t))) \\ & + 2\alpha x(t)'P_1x(t).\end{aligned}$$

The end of the proof strictly follows the line of Theorem 1. Thus if LMI (21) is satisfied, it follows that $\dot{V}_\alpha(t) + 2\alpha V_\alpha(t) < 0$ for all $x(t) \in \mathcal{S}$ and consequently, by integration, that $V_\alpha(t)$ exponentially decreases with the decay rate 2α. This implies that the condition (20), for the exponential stability of the solution of system (1), holds (Sun et al. 2006).

On the other hand, one has

$$\begin{aligned}V_\alpha(0) \leq{} & \left(\lambda_{\max}(P_1) + \left(\int_{-h}^{0} e^{2\alpha s}\mathrm{d}s\right)\lambda_{\max}(M)\right)\|\phi(\theta)\|_c^2 \\ & + \left(\left(\int_{-h}^{0} e^{2\alpha s}\mathrm{d}s\right)\lambda_{\max}(U)\right. \\ & \left. + \left(\int_{-h}^{0}\int_{\theta}^{0} e^{2\alpha s}\mathrm{d}s\,\mathrm{d}\theta\right)\lambda_{\max}(R)\right)\|\dot{\phi}(\theta)\|_c^2 \\ \leq{} & (\lambda_{\max}(P_1) + \eta_1\lambda_{\max}(M))\|\phi(\theta)\|_c^2 \\ & + (\eta_1\lambda_{\max}(U) + \eta_2\lambda_{\max}(R))\|\dot{\phi}(\theta)\|_c^2. \end{aligned} \tag{24}$$

From the definition of Q_1, X, J and L as in Theorem 1, it follows that $\lambda_{\max}(P_1) = \lambda_{\max}(Q_1^{-1})$, $\lambda_{\max}(M) = \lambda_{\max}(Q_1^{-1}XQ_1^{-1})$, $\lambda_{\max}(R) = \lambda_{\max}(Q_1^{-1}JQ_1^{-1})$ and $\lambda_{\max}(U) = \lambda_{\max}(L^{-1})$. Hence, if $\phi(\theta)$ verifies (22), we can conclude that

$$x'(t)P_1x(t) \leq V_\alpha(t) \leq e^{-2\alpha t}V_\alpha(0) \leq \delta_e \leq 1, \quad \forall t \geq 0,$$

which implies that $x(t) \in \mathcal{E}$, $\forall t \geq 0$. Then, since (10) implies that $\mathcal{E} \subset \mathcal{S}$, as in Theorem 1, we can effectively conclude that (21) implies $\dot{V}_\alpha(t) < -2\alpha V_\alpha(t) < 0$, $\forall t \geq 0$, for all initial conditions verifying (22). □

Remark 3: Since the exponential function is convex, from (23) it follows that $\eta_1 \geq h$ and $\eta_2 \geq \frac{h^2}{2}$. This ensures that the set of initial conditions for the asymptotic case is greater than the one for the exponential case. Moreover, when $\alpha \to 0$, $\eta_1 \to h$ and $\eta_2 \to \frac{h^2}{2}$, which ensures the continuity of the set with respect to α. Thus the set of admissible initial conditions of Theorem 1 is recovered when $\alpha \to 0$.

Remark 4: Since the LMIs in Theorems 1–3, as well as in Corollaries 1 and 2, are affine in the system matrices A, A_d, B and F, the extension of the conditions to consider uncertain systems described by polytopic uncertainties is straightforward. Note that if these matrices can be computed as a convex combination of the vertices of a polytope of matrices, given by (A^i, A_d^i, B^i, F^i), $i = 1, \ldots, N$, then, by convexity, it suffices to verify the LMIs at each vertex of the polytope simultaneously. Furthermore, for each vertex, different matrices Q_3^i and Q_2^i can be considered.

5. Optimisation problems

In this section, we show how the proposed theoretical conditions can be cast into LMI-based optimisation problems to determine a suitable stabilisation gain K. In particular, three criteria are considered: the maximisation of the delay bound h for which global stability can be ensured; the maximisation of the set of admissible initial conditions, which indirectly corresponds to determine K in order to maximise the region of attraction of the closed-loop system and the maximisation of the delay bound h or a quadratic performance criteria, while ensuring the stability for a given set of admissible initial conditions.

5.1. *Maximisation of the delay for which global stability is ensured*

In the case where the system can be globally asymptotically stabilised in the absence of the delays, an interesting problem consists of finding the maximal bound h^* on the time-varying delay $\tau(t)$, for which system (5) can be globally stabilised, considering a given bound d on $\dot{\tau}(t)$. This can be accomplished by solving the following optimisation problem:

$$\begin{gathered}\max h \\ \text{subject to} \\ (9) \text{ with } Y = W.\end{gathered} \tag{25}$$

Note that, due to the product between h and the variables Q_2, Q_3 and J, the solution of this problem can be obtained by iteratively increasing h and testing the feasibility of (9), which is an LMI for a fixed h.

5.2. *Maximisation of the set of admissible initial conditions*

Consider given h and d. In order to ensure the stability of system (5) by using Theorem 1, the admissible initial conditions must verify condition (11). Assume that $\|\phi(\theta)\|_c^2 = \delta_1$ and $\|\dot{\phi}(\theta)\|_c^2 = \delta_2$. Note that the smaller the maximal eigenvalues of $Q_1^{-1}, Q_1^{-1}JQ_1^{-1}$,

$Q_1^{-1}XQ_1^{-1}$, and L^{-1}, the larger δ_1 and δ_2 for which (11) is verified. Hence, the problem of finding K leading to the maximisation of the region of stability of the closed-loop system can be achieved by minimising these maximal eigenvalues. With this aim, consider the following auxiliary LMIs:

$$\begin{bmatrix} \lambda_{Q_1} I & I \\ \star & Q_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \lambda_X I & I \\ \star & 2Q_1 - X \end{bmatrix} \geq 0,$$
$$\begin{bmatrix} \lambda_J I & I \\ \star & 2Q_1 - J \end{bmatrix} \geq 0, \quad \begin{bmatrix} \lambda_L I & I \\ \star & L \end{bmatrix} \geq 0. \tag{26}$$

From the fact that $R^{-1} \geq 2Q_1 - J$ and $M^{-1} \geq 2Q_1 - X$ it follows that these LMIs are, respectively, equivalent to $\lambda_{Q_1} \geq \lambda_{\max}(Q_1^{-1})$, $\lambda_X \geq \lambda_{\max}(Q_1^{-1}XQ_1^{-1})$, $\lambda_J \geq \lambda_{\max}(Q_1^{-1}JQ_1^{-1})$ and $\lambda_L \geq \lambda_{\max}(L^{-1})$.

Hence, the following optimisation problem can be considered:

$$\begin{gathered} \min \beta_1\lambda_{Q1} + \beta_2\lambda_X + \beta_3\lambda_J + \beta_4\lambda_L \\ \text{subject to} \\ (9),\ (10) \text{ and } (26) \end{gathered} \tag{27}$$

where β_1, β_2, β_3 and β_4 are weights that should be tuned in order to satisfy some trade-off between δ_1 and δ_2. The choice of these weighting parameters are performed in an *ad hoc* way. In general, the minimisation of one of the eigenvalues is more critical to obtain larger values of δ_1 and/or δ_2. In this case, the weight associated with the appropriate eigenvalue should be increased.

5.3. *Maximisations for a given set of admissible initial conditions*

Consider now $\delta_1 > 0$ and $\delta_2 > 0$. The idea is then to compute K in order to guarantee the stability for all initial conditions satisfying $\|\phi(\theta)\|_c^2 \leq \delta_1$ and $\|\dot{\phi}(\theta)\|_c^2 \leq \delta_2$. This case can be addressed considering the auxiliary LMIs (26) and the following additional constraint:

$$(\lambda_{Q1} + h\lambda_X)\delta_1 + (0.5h^2\lambda_J + h\lambda_L)\delta_2 - 1 \leq 0. \tag{28}$$

Note that if $\|\phi(\theta)\|_c^2 \leq \delta_1$ and $\|\dot{\phi}(\theta)\|_c^2 \leq \delta_2$, (28) implies that (11) is verified. In this case, for instance, the following optimisation criteria can be considered.

5.3.1. *Maximisation of the bound h for which is possible to find a stabilising gain*

In this case, a problem analogous to (25) can be formulated as follows:

$$\begin{gathered} \max h \\ \text{subject to} \\ (9),\ (10),\ (26) \text{ and } (28). \end{gathered} \tag{29}$$

5.3.2. *Minimisation of an upper bound to a given cost function (guaranteed cost problem)*

A natural performance measure is given by the following quadratic criterion on plant states:

$$\mathcal{J} = \int_0^\infty x'(t)C'Cx(t)\mathrm{d}t \quad \text{where } C'C \geq 0,\ C'C \in \Re^{n\times n}.$$

If we are now able to show that

$$\dot{V}(t) + \frac{1}{\gamma}\bar{x}'\begin{bmatrix} C' \\ 0 \end{bmatrix}\begin{bmatrix} C & 0 \end{bmatrix}\bar{x} < 0, \tag{30}$$

it follows that $\mathcal{J} < \gamma V(0) < \gamma$, $\forall\phi(\theta)$ satisfying (11).

Note that (30) is satisfied if the following matrix inequality is verified:

$$\begin{bmatrix} \tilde{\mathcal{K}} & \begin{bmatrix} J/h \\ A_dQ_1 \end{bmatrix} & \begin{bmatrix} 0 \\ FL \end{bmatrix} & \begin{bmatrix} Y' \\ -BS \end{bmatrix} & h\begin{bmatrix} Q_2' \\ Q_3' \end{bmatrix} & \begin{bmatrix} Q_2' \\ Q_3' \end{bmatrix} & \begin{bmatrix} Q_1'C' \\ 0 \end{bmatrix} \\ \star & (d-1)X - J/h & 0 & 0 & 0 & 0 & 0 \\ \star & \star & (d-1)L & 0 & 0 & 0 & 0 \\ \star & \star & \star & -2S & 0 & 0 & 0 \\ \star & \star & \star & \star & -2hQ_1 + hJ & 0 & 0 \\ \star & \star & \star & \star & \star & -L & 0 \\ \star & \star & \star & \star & \star & 0 & -\gamma I \end{bmatrix} < 0 \tag{31}$$

Hence, the following optimisation problem can be formulated in order to minimise the bound γ (guaranteed cost) on the performance quadratic criterion:

$$\begin{gathered} \min \gamma \\ \text{subject to} \\ (31),\ (10),\ (26) \text{ and } (28). \end{gathered} \tag{32}$$

Remark 5: The optimisation problems above can be straightforwardly adapted to the problem of retarded systems and to the exponential stabilisation. It suffices to consider the conditions stated in Theorems 2 and 3. In particular, for the case of exponential stabilisation, another problem of interest is the maximisation of the decay rate α, for which it is possible to ensure the stability for a given set of admissible initial states or the global stability.

Remark 6: It should be noticed that the derived results apply also to the analysis problem. In this case, the results and the optimisation problems can be straightforwardly adapted to consider a given gain K.

6. Numerical examples

Example 1: Consider system (1) with

$$A = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 1 \end{bmatrix},$$
$$F = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad u_0 = 15.$$

Considering the optimisation problem (27) with $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$, the gain $K = [-0.1325\ 0.0153]$ is obtained for $h = 1$ and $d = 0.1$. This gain ensures the asymptotic stability for any $\phi(\theta)$ satisfying

$$11.4\|\phi(\theta)\|_c^2 + 8.57\|\dot{\phi}(\theta)\|_c^2 \leq 10^5. \tag{33}$$

It is worth noticing that in our previous work (Gomes da Silva Jr et al. 2005), where a direct descriptor approach was adopted, considering the same problem, the asymptotic stability was ensured for $\phi(\theta)$ satisfying $51\|\phi(\theta)\|_c^2 + 9.34\|\dot{\phi}(\theta)\|_c^2 \leq 10^4$. This shows that the set of admissible initial conditions obtained from the application of Theorem 1 are significantly less conservative.

Note that the set of admissible $\phi(\theta)$ given by (33) denotes a trade-off between the amplitude and the derivative of the initial conditions. Hence, for instance, if we consider that $\|\phi(\theta)\|_c = \|\dot{\phi}(\theta)\|_c$ the stability is ensured for $\phi(\theta)$ such that $\|\phi(\theta)\|_c = \|\dot{\phi}(\theta)\|_c < 70.74$. On the other hand, if we consider that the initial states are constant over the interval $[-h, 0]$, that is $\|\dot{\phi}(\theta)\|_c = 0$, it follows that all $\phi(\theta)$ such that $\|\phi(\theta)\|_c \leq 93.65$ are admissible.

Consider now $d = 0.1$ and the optimisation problem (27) with $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$. In Table 1, considering the initial conditions such that $\|\phi(\theta)\|_c = \|\dot{\phi}(\theta)\|_c \leq \bar{\delta}$, the maximal value of $\bar{\delta}$ and the respective gain, obtained from the solution of (27), are shown for different values of h. As expected, the set of admissible initial conditions reduces as the upperbound on the delay increases. The value of $h = 3.53$ corresponds to the maximum upperbound on the delay for which the LMIs are feasible. The same behaviour appears when h is fixed and the parameter d varies. For instance, for $h = 1$, the set of initial conditions reduces as d increases and it is almost empty for $d = 0.947$.

Table 1. $h \times$ region of stability.

h	$\bar{\delta}$	K
1	70.74	[−0.1325 0.0153]
2	56.17	[−0.1201 −0.0421]
3	18.17	[−0.1681 −0.0137]
3.53	62.8×10^{-3}	[−1.2062 11.1614]

Concerning the performance analysis with Theorem 3 together with the optimisation problem (27), one can see that the set of admissible initial conditions reduces as the exponential decay rate α increases. This fact is illustrated in Table 2, where the maximal value of $\bar{\delta}$ obtained considering $\|\phi(\theta)\|_c = \|\dot{\phi}(\theta)\|_c \leq \bar{\delta}$, $h = 1$ and $d = 0.1$ is shown for different values of the decay rate α. For $h = 1$ and $d = 0.1$, the maximum exponential decay rate, for which the LMIs are feasible, is $\alpha = 1.85$.

Example 2: Consider a retarded system given by (18), with the matrices A, A_d and B defined in the Example 1, $u_0 = 15$, $h = 1$ and $d = 0.1$. It is possible to ensure the asymptotic stability of initial conditions satisfying $\|\phi(\theta)\|_c = \|\dot{\phi}(\theta)\|_c < 83.55$ with the gain $K = [-0.1950\ 0.0649]$. Note that the bound on the admissible conditions is larger than the ones obtained in Fridman et al. (2003) (79.43) and in Gomes da Silva Jr et al. (2005) (79.54). This indicates that the proposed method is less conservative than the previous approaches.

7. Concluding remarks

The synthesis of stabilising gains for linear neutral systems in the presence of saturating inputs and time-varying delays has been addressed. First, conditions that allow the computation of a state feedback matrix associated with a set of initial conditions, for which the asymptotic closed-loop stability can be ensured, have been derived. Considering the case of open-loop asymptotically stable systems, this condition can be slightly modified to address the problem of computing globally stabilising gains. It has also been shown that the conditions can be particularised to consider retarded systems. Following the same approach, exponential stabilisation conditions have been derived.

Based on the theoretical conditions, convex optimisation problems (with LMI constraints) have been proposed in order to compute the stabilising gains aiming at: maximising the delay for which global

Table 2. $\alpha \times$ region of stability.

α	0.0	0.2	0.4	0.6	0.8	1.0	1.2
$\bar{\delta}$	70.74	44.38	35.50	26.38	21.33	18.13	9.58

stability can be ensured; maximising the set of admissible initial conditions, which indirectly corresponds to determine K in order to maximise the region of attraction of the closed-loop system; or maximising the delay or a quadratic performance criterion, while ensuring the stability for a given set of admissible initial conditions.

The extension of the results to uncertain polytopic systems is straightforward. Another interesting possible extension regards the problem of static anti-windup design.

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Note

1. A preliminary version of the present work has been presented in Gomes da Silva Jr, Fridman, Seuret, and Richard (2005).

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