Stability Advances in Robust Portfolio Optimization under Parallelepiped Uncertainty *

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Abstract

In financial markets with high uncertainties, the trade-off between maximizing expected return and minimizing the risk is one of the main challenges in modeling and decision making. Since investors mostly shape their invested amounts towards certain assets and their risk aversion level according to their returns, scientists and practitioners have done studies on that subject since the beginning of the stock markets' establishment. In this study, we model a Robust Optimization problem based on data. We found a robust optimal solution to our portfolio optimization problem. This approach includes the use of Robust Conditional Value-at-Risk under Parallelepiped Uncertainty, an evaluation and a numerical finding of the robust optimal portfolio allocation. Then, we trace back our robust linear programming model to the Standard Form of a Linear Programming model; consequently, we solve it by a well-chosen algorithm and software package. Uncertainty in parameters, based on uncertainty in the prices, and a risk-return analysis are crucial parts of this study. A numerical experiment and a comparison (back testing) application are presented, containing real-world data from stock markets as well as a simulation study. Our approach increases the stability of portfolio allocation and reduces the portfolio risk.

Keywords: Robustness and sensitivity analysis, Robust optimization, Robust conditional value-at-risk, Parallelepiped uncertainty, Risk management.

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1 Introduction

The distinction between two important notions, risk and uncertainty in modern economics has been clarified by (Knight 1921). In Knight's study, risk is for any situation when we have lack of information. Especially, we do not keep a record about the distribution of random events. On the other hand, the uncertainty refers to situations where we do not know the distribution of the randomness. This distinguishes risk and uncertainty in economic studies. This paper claims that the uncertainty concept of our research is subject to our price parameters, i.e., prices from the market. We do not assume or measure any distribution upon our price intervals (or coefficient intervals consequently) when we will construct the Parallelepiped Uncertainty set. All possible crashes, changes, and jumps in market prices are entering to our price coefficients in the Robust Optimization (RO) model. Returns could take any value in return coefficients; this is where, based on data, uncertainty enters in this study.

The classical Modern Portfolio Theory (MPT) or Mean-Variance Approach (MVA) of Markowitz's "measures" risk by standard deviation and variance (Markowitz 1952). In MPT, a trade-off is considered between risk and expected return; the objective function to be minimized represents variance (risk) and we also aim at a certain level of expected return, i.e., to satisfy a target return constraint.

The most common criticism of MVA is the usage of variance as a risk measure, especially in highly volatile markets. One of the obvious problems with variance is the existence of outliers in weight distributions at portfolio allocation results (Werner 2007). In fact, the variance shows both upside and downside movements in the market, the investor's choice (expert opinion) reveals a variety in this situation. The investor could be interested in downside portfolio movements, more than upside movements. Because of this phenomenon, the variance is not a powerful and sufficient item for measuring and presenting the risk of the market or of the portfolio. Moreover, the derivative contacts' usage for risk management gives a skewed and heavy-tailed distribution to the decision maker. This distribution reflects in an opposite way to downside and upside portfolio movements (Siu et al. 2001; Boyle et al. 2002; Fabozzi et al. 2010).

Since MVA has drawbacks like irrationality of investors, and a problem with the calculation of expected values, researchers commenced to investigate new approaches to portfolio optimization. Because of the weak specifications of the MPT (e.g., inefficiency in highly volatile markets), Black and Litterman developed a new approach to portfolio optimization. In their method, the investors can combine their views about the global look of the equities, bonds and currencies with a risk premium. Their results are intuitive and allow for diversified portfolios (Black and Litterman 1992). Recent researches, like Value-at-Risk (VaR) (RiskMetrics 1996) or Conditional Value-at-Risk (Rockafellar and Uryasev 2000), prefer to consider probability distributions. Since all returns from the market are approximated by a probability distribution, VaR or CVaR are used to obtain a risk threshold through a pre-specified confidence level.

To avoid high risk, numerous financial instruments are employed at financial markets. Moreover, to define the risk level of the investment into specific assets, various risk measurement methodologies have been developed. However, from the very beginning, addressing the term "risk" , is always connected with the phenomenon of "uncertainty". Uncertainty about asset returns and the wider uncertainty about markets, especially, of the underlying prices, are sonorous principals behind risk management literature. The investor fears from uncertainty because it affects the market decisions, structure, and future. Since the total prediction of the future is impossible due to various randomness in the markets, by the help of uncertainty quantification and related risk management methodologies, the investors try to prevent their investments from financial risk.

Nowadays, the technology of computational tools is sufficiently powerful to obtain meaningful and feasible results from optimization algorithms and software while it allows us to deal with a wide range of complex optimization problems, especially, given real-life data under uncertainty. A number of optimization methods has been used to handle this uncertainty. These methods take into account any kinds of uncertainty during modeling while computing or in the result. MPT shows us the importance of the relation between risk taking and revenue (return) from the portfolio. The main uncertainty comes from the random fluctuation of the prices of risky assets. Since returns from assets are calculated by asset prices, this type of uncertainty shows itself in the returns also. Hence, investors face uncertainty in their returns.

Our aim is, by considering the return-risk trade-off analysis under uncertain data, to obtain a higher robustness, in fact, a lower risk level under the worst-case scenario by using RCVaR. Uncertainty in parameters, based on uncertainty in the prices, and a risk-return analysis are crucial parts of this study. Hence, the trade-off (antagonism) between accuracy and risk (variance), and robustness are our main subjects. Consequently, advances on robust portfolio optimization are prominent outputs of this paper. We increase the stability of robust portfolio allocation and reduced the portfolio risk. Our achievements in dealing with uncertainty in the data, i.e., the prices, are presented. As a result, we obtain more robust and stable risk level and portfolio allocation. We can summarize our main contributions in this research as follows:

1) We apply *Parallelepiped Uncertainty* set methodology from (Özmen et al. 2011) to the portfolio optimization problem in a stock market for managing risk, and facing with the uncertainty in the data.

2) For this purpose, we apply our RO methodology to Conditional Value-at-Risk (CVaR), herewith turning it to Robust CVaR (RCVaR). The main idea is modeling of a robust portfolio optimization problem that includes our development of Robust CVaR (RCVaR) based on uncertain set-valued data.

3) We present our methodology with 40 different portfolio scenarios (different price processes). Furthermore, we compare our numerical application and back testing with nominal methodologies.

4) We introduce and discuss a more robust portfolio allocation scheme than

nominal CVaR methodology by addressing uncertainty in stock market prices. Our results show less portfolio risk level, CVaR, than nominal approaches (variance reduction) while the robust portfolio allocation is obtained, respectively.

The structure of this paper is as follows. We provide the main contributions of this study in the introduction given by Section 1. In Section 2, a brief summary of literature review on CVAR under uncertainty is presented. Sections 3 and 4 reveal the methodology behind our research; these sections contain risk measures, uncertainty sets and robust optimization frameworks; the further subsections made by us give the contribution based on our robust portfolio optimization approach. Sections 5 and 6 present a numerical example and a simulation study on applying RCVaR in a robust form of portfolio optimization for illustrating the quality of our results. The last Section 7 concludes the study and indicates further investigations for the future. The core of this research is originating from (Kara 2016) and developed further.

2 A Review of Robust (Portfolio) Optimization under Uncertainty

Portfolio optimization problems in financial markets are usually outperformed under consideration of different approaches in RO framework when the data include perturbation. Soyster is one of the first researchers who studied on new approaches in RO. He focused on robust linear optimization under ellipsoidal uncertainty set where the feasible region is constrained by a set of convex inequalities instead of a set of containment definition. His suggestion, Inexact Linear Programming, means that if elements of the coefficient matrix in Linear Programming are not certainly known, coefficient could deviate, resulting in an uncertainty set (Soyster 1973).

In the study of Quarante and Zaffaroni (2008), since the optimization process leads to solutions which are likely to depend heavily on the parameters' perturbations, they proposed so-called Soyster's approach (Quaranta and Zaffaroni 2008). Huang et al. (2010) introduced a relative part of the RCVaR where the underlying probability distribution of portfolio return is only known to belong to a certain set (Huang et al. 2010). Hasuike and Katagiri (2013) studied a robust portfolio selection problem with an uncertainty set of future returns and satisfying certain levels with total returns. An ellipsoidal set of future returns is proposed by authors as an uncertainty set; then a robustness-based (worst-case objective function) selection problem is formulated as a bi-objective programming problem (Hasuike and Katagiri 2013). Natarajan et al. (2009) presented a unified theory that relates portfolio risk measures to robust optimization uncertainty sets. They identified how risk measures such as Standard Deviation, Worst-Case VaR, and CVaR can be traced back to robust optimization uncertainty sets (Natarajan et al. 2009).

Ben-Tal and Nemirovski studied convex optimization problems with lack of data, and when the data only belong to a given ellipsoidal uncertainty set (Ben-Tal and Nemirovski 1998); later, they demonstrated that LP model solutions could be infeasible because the nominal data might be perturbed (Ben-Tal and Nemirovski 2000). Bertsimas and Sim (2002) proposed a new robust LP approach which presents a new parameter to mediate robustness of the presented method against the conservativeness status (Bertsimas and Sim 2004). Bertsimas and Brown (2009) offered a new methodology to construct uncertainty sets in an RO framework for linear optimization models with uncertain parameters. Since a coherent risk measure addresses uncertainty in the data, the authors employed a convex uncertainty set in a robust optimization network (Bertsimas and Brown 2009). Zhu and Fukushima (2009) established the Worst-Case CVaR in a situation with uncertain data. They considered the minimization of the Worst-Case CVaR under mixture distribution uncertainty, Box Uncertainty and Ellipsoidal Uncertainty (Zhu and Fukushima 2009).

Tütüncü and Koenig (2004) referred to robust portfolio allocation problems with different assets, under uncertain data (unreliable portfolio asset returns). Their approach is a conservative one and also covers the Worst-Case situation (Tütüncü and Koenig 2004). Kirilyuk (2008) investigated polyhedral coherent risk measures and he applied these risk measures to risk-return optimization problems. The data of this study reveal partial uncertainty (Kirilyuk 2008). Comparisons between portfolio optimization with MVA and robust MVA are indicated by different articles. In the research of (Kuhn et al. 2009), the authors emphasized MVA's drawbacks and proposed a robust dynamic portfolio optimization methodology. Their main contribution of them is the computation of various dynamic and robust investment strategies by using uncertain input parameters. They suggested an algorithm which contains provable error bounds and a dynamic portfolio strategy. Another research, (Fonseca et al. 2011), proposed a currency investment strategy for maximization objective of a currency portfolio optimization problem. By assuming worst-case foreign currency value scenarios, they applied a novel RO methodology. In their study, the authors indicated the relationship between size of uncertainty sets and allocation of portfolio which contains foreign currencies and options. They supported their claim by numerical results and back-testing applications. The authors concluded that their hedging suggestion is more flexible than standard hedging strategies.

(Özmen, 2010) and (Özmen and Weber, 2014) used the (Conic) Multivariate Adaptive Regression Splines ((C)MARS) methodology and represented a new Robust (C)MARS (R(C)MARS) algorithm under data uncertainty. They introduced, as they say, polyhedral, in the terminology of this study, parallelepiped, and ellipsoidal uncertainties in order to get a robust optimization algorithm for (C)MARS (Özmen 2010; Özmen and Weber 2014).

3 Conditional Value-at-Risk

CVaR is a risk measure with a certain probability level α; the α-VaR of an asset (portfolio) is the lowest amount ζ such that, with a probability α, the loss will not exceed ζ, whereas the α-CVaR is the conditional expectation of losses above the amount ζ. CVaR calculates the worst α percent of returns; the average loss of a portfolio will be our α-CVaR value. In the literature, for the parameter α, three values are taken mostly: 92.5%, 95% and 99%. The definitions provided cause that the value of α-VaR cannot exceed the value of α-CVaR (Rockafellar and Uryasev 2000).

Let ξ be the vector of returns (regarded as random variables) of assets (in a portfolio) and $\boldsymbol{x}$ be the vector of the assets' (portfolio) weights; then, $\boldsymbol{x}$ is the decision vector or vector of control variables of this theory. Finally, $\boldsymbol{x}$ has to be chosen from a certain subset X of $\mathbb{R}^n$, where X is a convex set of feasible decisions, related to a given random vector ξ in $\mathbb{R}^n$ (Rockafellar and Uryasev 2000).

Definition 1 *The return of an asset or a portfolio return represents a benefit or an interest of an asset of the portfolio holder after a certain period of time. In this time interval, we assume the prices of each asset to be random and address those assets as risky assets. The mathematical definition of an asset (ξ_j) or portfolio (ξ) return is*

$$\xi := \left(\frac{p_T - p_{T-1}}{p_{T-1}}\right) = \left(\frac{p_T}{p_{T-1}} - 1\right), \tag{1}$$

where p_T is the terminal price of an asset at the end of a period and p_{T-1} is the price of an asset before the terminal period. This definition can be applied for any discrete time points like t_1 and t_2, where $0 \le t_1 < t_2 \le T$, instead of $T-1$ and T (T is the end or terminal time of maturity). The constant 1 does not lead to any change in our decision making, neither by total expected return nor in risk, as one can easily see.

In our research, we prefer to specify the convex set X as follows.

Definition 2 *The convex set X of portfolios is defined by*

$$X = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i \ge 0 \; (i = 1, 2, \ldots, n) \right\}.$$

Our research comfortably permits the inclusion of further linear constraints in the definition of set X.

The vector ξ represents the n returns, but the most important matter is that it also contains uncertainty, in fact, returns; a characteristic is that information of ξ comes from a vector of market prices. With all these different variables, we can develop a loss (or alternatively, reward) function for this risk measurement technique:

$$f(\boldsymbol{x}, \xi) := -(x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n), \tag{2}$$

where ξ represents the vector of asset returns from different market scenarios and $\boldsymbol{x}$ comprises the vector of portfolio weights. By considering Equation (2), we can specify the main equation of CVAR, denoted by F_α:

$$F_\alpha(\boldsymbol{x},\zeta) = \zeta + (1-\alpha)^{-1} \int_{\xi\in\mathbb{R}^n} [f(\boldsymbol{x},\xi)-\zeta]^+ p(\xi) d\xi, \tag{3}$$

where $v^+ = \max\{v,0\}$. Here, the function f is a loss function within CVaR from Equation (3). Unlike the common acceptance, α-CVaR is not equal to an average of outcomes which are greater than α-VaR. To show this situation when the distribution is modeled by scenarios, CVaR could be determined by averaging a fractional number of scenarios (Sarykalin et al. 2008).

4 Robust Optimization

Scientists focused on RO heavily from both theoretical and practical perspectives, since it has a modeling framework for *immunizing* against parametric uncertainties in mathematical optimization. In fact, RO technique is useful when some parameters include uncertainty and are only known to belong to some uncertainty set ($\mathscr{U}$). RO finds an optimal solution that is feasible, e.g., according to *confidence intervals* due to uncertain data (Ben-Tal et al. 2009). If the uncertain coefficients within uncertainty sets are under some certain probability distribution, the probability of a feasible robust solution should be accurate. By this guarantee, the trade-off between robustness of the solution and optimal (crisp) parameters can be specified (Sim 2004).

Regarding this, first, we should consider a basic LP model and its robust counterpart:

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x} \\ &\text{subject to} \\ &\quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ &\quad \boldsymbol{x}\in X, \end{aligned} \tag{4}$$

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x} \\ &\text{subject to} \\ &\quad \breve{\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{b}, \quad \forall \breve{\boldsymbol{A}} \in \mathscr{U}, \\ &\quad \boldsymbol{x}\in X. \end{aligned} \tag{5}$$

Without loss of generality, we assume that only the matrix $\boldsymbol{A}$ implies uncertainty; in Subsection 5.1, we shall consider the uncertain $\boldsymbol{A}$ realized by different uncertainty sets. Herewith, the program of Equation (4) transforms to its *robust counterpart* given in Equation (5). In our Parallelepiped Uncertainty, the matrix $\breve{\boldsymbol{A}}$ will be included entry-wise into uncertainty intervals.

This type of optimization problem, where uncertain entries, are expressed through uncertainty sets and where we aim at an optimal solution, is called the *Robust Counterpart* of original LP problem of CVaR. In our case, that LP program will be on CVaR.

Subsequently, we will reflect on and evaluate a robustified problem of Equation (5) in every step, as we are going to show subsequently. Here, the scale and structure of such a set $\mathscr{U}$ is defined by the decision modeler. There are different structural shapes which refer to the geometry: Box, Polyhedral, Ellipsoidal, etc. In our research, we preferred to use *Parallelepiped Uncertainty*.

5 Robust Portfolio Optimization with Robust CVaR under Parallelepiped Uncertainty

5.1 Robust Conditional Value-at-Risk under Parallelepiped Uncertainty

Consider an LP model with uncertain data. Our aim consists in minimizing (or maximizing) the objective function under some constraints. Hence, our general optimization problem is the same as in Equation (4), where $\boldsymbol{x}$ is the vector of decision variables, and $\boldsymbol{A}$ is the matrix of coefficients with respect to uncertainty (Griva et al. 2009).

Herewith and by other standard arguments that we shall apply, we could move from any common representation or form of an LP model to another. Our objective function is to be minimized; hence, we could use a *height variable*, $\boldsymbol{z}$ (by an epigraph argument) and add $z - \boldsymbol{c}^T\boldsymbol{x} \geq 0$ as an inequality constraint; we include this constraint into the vector-matrix notation of the constraints $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where we employ surplus variables also.

Let us consider a row i in matrix $\boldsymbol{A}$; referring to entries, we denote by a_{ij}, $j \in J_i$, the set of columns met by row i which are subject to uncertainty (Ben-Tal and Nemirovski 2000).

We aim to solve an LP under uncertainty by a robust LP model. Since Ellipsoidal Uncertainty makes our optimization problem a Conic Quadratic Programming model, it is more robust but less tractable. Therefore, we prefer to imply Parallelepiped Uncertainty in the sense of Özmen et al. (Özmen et al. 2011). This type of uncertainty set has similarities with Box Uncertainty (Jalilvand-Nejad et al. 2016), but it generalizes the box type in a fully variable way, indeed, with independent side lengths of our parallelepipeds.

5.1.1 Parallelepiped Uncertainty

In Parallelepiped Uncertainty, the definition and construction of an uncertainty set is different than from a one with Polyhedral Uncertainty (Bertsimas and Sim 2004). Despite of similarities with the other uncertainty sets, differences of the parallelepipeds are revealed regarding their structural occurrence. Parallelepiped Uncertainty sets are products of the entry-wise intervals hosting the matrix $\breve{\boldsymbol{A}}$; we shall look at such intervals in Equation (6) soon. The benefit of parallelepipeds to our research, where important matrix entries will be return values, is that instead of a single price, we may consider multiple and flexibly varying prices of assets and, hence, take into account likewise flexible returns.

So, any of our regarded Parallelepiped Uncertainty sets $\mathscr{U}$ will be built up by entries in our uncertain matrix $\boldsymbol{A}$ where the corresponding uncertainty may, to some degree (or budget), differ from one entry to another. There are different criteria to obtain an RO result (Pachamanova 2002): (i) *Formulation flexibility*: How to allow for expressing dependencies among the uncertain coefficients along a set of constraints; (ii) *Conservativeness of optimal solution*: Associated with probabilistic guarantees, formulating the RO problem by a pre-decided level of conservativeness; (iii) *Tractability*: Preserving the RO model's computational tractability, e.g., computationally being easy to solve. Consequently, referring to our Parallelepiped Uncertainty sets, the new closed-form general Robust LP model with uncertainty implied is as in Equation (5).

In the literature, Polyhedral Uncertainty leads to entry-wise intervals of an uncertain matrix after canonical projections. We have similar, but immediate intervals in our Parallelepiped Uncertainty (without any projection). In fact, a major property of intervals involved towards our set $\mathscr{U}$ is that the uncertain coefficients $\breve{a}_{ij}$ $(j \in J_i)$ lie in intervals (Bertsimas and Brown 2009)

$$[\hat{a}_{ij} - \Delta_{ij}, \hat{a}_{ij} + \Delta_{ij}], \tag{6}$$

where $\hat{a}_{ij}$ is the arithmetic mean of the $\breve{a}_j$ and Δ_{ij} is the perturbation term for representing the margin of uncertainty in the corresponding random variable $(i = 1,2,\ldots,M;\ j = 1,2,\ldots,N)$. Any change in the value of some perturbation term Δ_{ij} could be determined by considering the standard deviations (σ_{ij}) of random variables (components ξ_j of ξ) or, in general, of uncertain matrix entries $(\breve{a}_{ij})$. This interval formula could be applied to every entry in $\boldsymbol{A}$ and turns it into an uncertain matrix $\breve{\boldsymbol{A}} \in \mathscr{U}$.

With our entries $\breve{a}_{ij}$ we address return values based on uncertain prices. Since Parallelepiped Uncertainty is based on intervals of the uncertain parameters, definitions, including positions and sizes, and relations between price and return intervals are important for the set $\mathscr{U}$ of matrices $\breve{\boldsymbol{A}}$. We shall benefit from Equation (6) with regard to both prices and returns.

5.2 Robust Portfolio Optimization with RCVaR

The earlier sections prepared us a general form for an organized program of our RO problem. In this section, we are going to imply the given but uncertain data at the place of samples, and we shall do this in the multi-valued way of intervals, according to each coordinate of the random return vector ξ. In the sense of all linear constraints, we shall address every element of those intervals. Hence, we will present and discuss the LP model, and the Robust LP will be stated in a matrix-vector form.

In order to come to this form, we use a discretized version of CVaR that we defined in Equation (3). The approximation of CVaR by a discretized form of a piecewise LP model under various constraints is given as follows:

$$\begin{aligned} &\text{minimize} \quad \zeta + \frac{1}{(1-\alpha)} \sum_{i=1}^{m} [f(\boldsymbol{x}, \xi_i) - \zeta]^{+} p(\xi_i) \\ &\text{subject to all constraints,} \end{aligned} \tag{7}$$

where f is the loss function, m is number of periods or samples (data) and p is a probability distribution function of returns. However, this type of model is not convenient by expanding the problem dimension; hence, Equation (7) is reduced to an LP form by using positive parts v^{+} and height variables $z_j \geq f(\boldsymbol{x}, \xi_j) - \zeta$. For Standard Form purposes we write $\zeta = (\zeta_1, \zeta_2)^T$ as a new decision vector; we substitute ζ by $(\zeta_1 - \zeta_2)$ with $\zeta_1, \zeta_2 \geq 0$. Consequently, the discretized version of $\min_{\boldsymbol{x} \in X}$ (CVaR) is equivalently expressed by

$$\begin{aligned} &\text{minimize}_{(\zeta, \boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^m} \quad \zeta_1 - \zeta_2 + \frac{1}{(1-\alpha) \cdot m} \sum_{i=1}^{m} z_i \\ &\text{subject to} \\ &z_i \geq f(\boldsymbol{x}, \xi_i) - (\zeta_1 - \zeta_2) \quad (i = 1, 2, \ldots, m), \\ &\sum_{j=1}^{n} x_j = 1, \\ &\boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{z} \geq \boldsymbol{0}, \zeta \geq \boldsymbol{0}. \end{aligned} \tag{8}$$

Our move from $p(\xi_i) = \pi_i$ of Stochastic Programming by a handy RO will be complete when we concentrate on particular values for each of the discrete probabilities whose sum amounts to 1, reflecting uncertainty in the form of the return data vectors $\xi_1, \xi_2, \ldots, \xi_m$, and their coordinate-wise treatment through intervals. We confine us to the case of equal (uniform) weights, but our approach works likewise well with any other (i.e., non-uniform) discrete distribution.

In the Standard Form of an LP model, the matrix representation of Equation (8) turns to become

$$\begin{aligned} &\text{minimize}_{(\zeta, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{e}) \in \mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m} \quad \zeta_1 - \zeta_2 + \frac{1}{(1-\alpha) \cdot m} \sum_{i=1}^{m} z_j \\ &\text{subject to} \\ &z_1 + \xi_1^1 x_1 + \xi_2^1 x_2 + \ldots + \xi_n^1 x_n - (\zeta_1 - \zeta_2) - e_1 = 0, \\ &z_2 + \xi_1^2 x_1 + \xi_2^2 x_2 + \ldots + \xi_n^2 x_n - (\zeta_1 - \zeta_2) - e_2 = 0, \\ &\vdots \\ &z_m + \xi_1^m x_1 + \xi_2^m x_2 + \ldots + \xi_n^m x_n - (\zeta_1 - \zeta_2) - e_m = 0, \\ &x_1 + x_2 + x_3 + \ldots + x_n = 1, \\ &\boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{z} \geq \boldsymbol{0}, \quad \boldsymbol{e} \geq \boldsymbol{0}, \quad \zeta \geq \boldsymbol{0}. \end{aligned} \tag{9}$$

The matrix-vector representation of Equation (9) in Standard Form model will be in given terms of $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{x}$. Now, the coefficient matrix is found to be

$$\boldsymbol{A} := \left[\begin{array}{cccc|cccc|cc|cccc} \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 & 1 & 0 & \cdots & 0 & 1 & -1 & -1 & 0 & \cdots & 0 \\ \xi_1^2 & \xi_2^2 & \dots & \xi_n^2 & 0 & 1 & \dots & 0 & 1 & -1 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_1^m & \xi_2^m & \dots & \xi_n^m & 0 & 0 & \dots & 1 & 1 & -1 & 0 & \dots & 0 & -1 \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right],$$

where the entries ξ_i^j $(i = 1, \dots, m;\ j = 1, \dots, n)$ represent the uncertain return data, in the percentage returns as introduced in Equation (1). The right-hand side of Equation (9) contains the vector $\boldsymbol{b}$ defined as:

$$\boldsymbol{b} := \left[\begin{array}{ccccc} 0 & 0 & \dots & 0 & 1 \end{array}\right]^T;$$

the vectors of coefficients in the objection function, $\boldsymbol{c}$, and of decision variables, $\boldsymbol{x}$, are:

$$\boldsymbol{c} := \left[\begin{array}{ccc|ccc|cc|ccc} 0 & \dots & 0 & \dfrac{1}{m \cdot (1-\alpha)} & \cdots & \dfrac{1}{m \cdot (1-\alpha)} & 1 & -1 & 0 & \dots & 0 \end{array}\right]^T.$$

$$\boldsymbol{x} := \left[\begin{array}{cccc|cccc|cc|cccc} x_1 & x_2 & \dots & x_n & z_1 & z_2 & \dots & z_m & \zeta_1 & \zeta_2 & e_1 & e_2 & \dots & e_m \end{array}\right]^T.$$

Let us underline that this optimization modeling is applicable for any set of our dimensions, especially, for any given number of risky assets. This type of optimization problem, i.e., where uncertain entries are expressed through uncertainty sets, by intervals, and where we aim at an optimal solution, is called the Robust Counterpart of *the LP problem on CVaR*.

Those "arrays", $\boldsymbol{A}$, will be backbones of this study. From now on, the entry contents of the coefficient matrix $\boldsymbol{A}$ will be widened through products of uncertainty intervals from a set $\mathscr{U}$. This *immunizes* our problem against parameter uncertainty, in fact, against underlying data uncertainty in the prices eventually.

To obtain $\boldsymbol{A}$ with uncertainty intervals, Equation (1) will be employed. This equation is rigid if we only address nominal returns. The return formula should be extended by Parallelepiped Uncertainty based on intervals. The underlying return-interval determination formula is given in Equation (6). For the required parallelepiped setting of returns with intervals, we shall employ the following definition, herewith generalizing Equation (1).

Definition 3 *Suppose we have an asset which has prices as intervals. In that sense, our return formula is based on*

$$\frac{[a,b]}{[c,d]} = \left[\frac{a}{d}, \frac{b}{c}\right] \quad (b \geq a \geq 0;\ d \geq c > 0). \tag{10}$$

In our study, a, b, c and d will be in the role of lower and upper bounds of asset prices, respectively. These price intervals are calculated by Equation (6). Herewith, we have

$$\left[\xi^l, \xi^u\right] := \frac{\left[p_T^l, p_T^u\right]}{\left[p_{T-1}^l, p_{T-1}^u\right]} - [1,1] = \left[\frac{p_T^l}{p_{T-1}^u} - 1, \frac{p_T^u}{p_{T-1}^l} - 1\right], \tag{11}$$

where ξ^l and ξ^u are lower and upper bounds of a return interval, respectively. The nominator refers to the end and the denominator stands for the beginning of a time interval or time subinterval.

Each return in some entry of the matrix $\boldsymbol{A}$ is generalized, in fact, "randomized", with Equation (11). Either we treat all entries of the new (perturbed, uncertain) matrix $\breve{\boldsymbol{A}}$ as intervals, even if some entries are numbers (0, 1 or -1), i.e., degenerate intervals, or we could consider just the return intervals (having positive lengths) for uncertainty representation in $\breve{\boldsymbol{A}}$ by (non-degenerate) intervals. For practicability purposes, we choose the second option. Accordingly, the matrix $\breve{\boldsymbol{A}}$, located in the uncertainty set $\mathscr{U}$, is

$$\breve{\boldsymbol{A}} \in$$
$$\left[\begin{array}{ccc|cccc|cc|cccc}
\left[\xi_1^{l,1}, \xi_1^{u,1}\right] & \cdots & \left[\xi_n^{l,1}, \xi_n^{u,1}\right] & 1 & 0 & \cdots & 0 & 1 & -1 & -1 & 0 & \cdots & 0 \\
\left[\xi_1^{l,2}, \xi_1^{u,2}\right] & \cdots & \left[\xi_2^{l,2}, \xi_2^{u,2}\right] & 0 & 1 & \cdots & 0 & 1 & -1 & 0 & -1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\left[\xi_1^{l,m}, \xi_1^{u,m}\right] & \cdots & \left[\xi_n^{l,m}, \xi_n^{u,m}\right] & 0 & 0 & \cdots & 1 & 1 & -1 & 0 & \cdots & 0 & -1 \\
1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right],$$

i.e., $\breve{\boldsymbol{A}} \in \mathscr{U}$ with

$$\mathscr{U} := \operatorname{conv}(\breve{\boldsymbol{A}}^1, \breve{\boldsymbol{A}}^2, \ldots, \breve{\boldsymbol{A}}^{2^{m\cdot n}}), \tag{12}$$

being the convex hull of the canonical *vertices* $\breve{A}^l$ $(l = 1, 2, \ldots, 2^{m\cdot n})$. The matrix $\breve{\boldsymbol{A}}$ is situated in $\mathbb{R}^{M\times N}$, where $M = m+1$ and $N = n+2m+2$.

The convex hull $\mathscr{U}$ is an $m \cdot n$-dimensional (non-degenerate) polytope, embedded in a Euclidean space of $M \cdot N$. For closer information about the related convex analysis we refer to (Rockafellar 1997). In fact, $\mathscr{U}$ is a polytope given by all convex combinations of its vertices $\breve{\boldsymbol{A}}^1, \breve{\boldsymbol{A}}^2, \ldots, \breve{\boldsymbol{A}}^{2^{m\cdot n}}$. The elements of an uncertainty set $\mathscr{U}$ are founded by the Cartesian product of uncertainty intervals, and also by degenerate intervals which consist of scalar entries, 0, 1 and -1; this entire product is our Parallelepiped Uncertainty set. All the coefficient matrices represented as $\breve{\boldsymbol{A}}$ generate a lower-dimensional parallelepiped C. Then, C is

$$C = [\xi_1^{l,1}, \xi_1^{u,1}] \times [\xi_2^{l,1}, \xi_2^{u,1}] \times \ldots \times [\xi_n^{l,m}, \xi_n^{u,m}] \times \ldots =: \prod_{k=1}^{m\cdot n} C_k, \tag{13}$$

where $\xi^l \leq \xi \leq \xi^u$. We recall that there are degenerate intervals like $[a,a] = \{a\}$ inserted into the set-valued coefficient matrix, where $a \in \{-1,0,1\}$. For representing and programming the set C, these trivial or degenerate intervals of single constant values mean no complexity and no coding problem.

To understand this calculation and geometrical shape more clearly, the Cartesian products of the coefficient matrix with uncertain contents is represented in Figure 1. For the sake of simplicity, we do not include dimensions (entries) here where the intervals are degenerate.

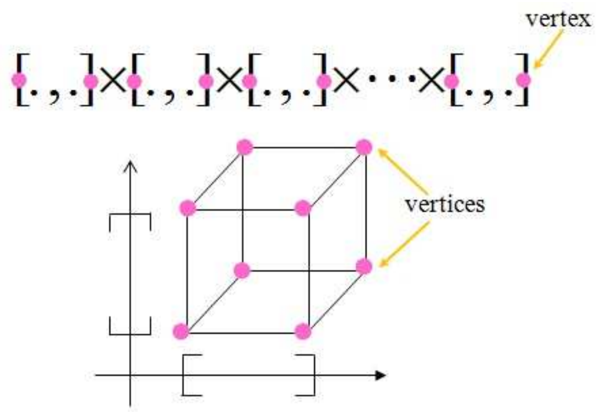


Figure 1: Parallelepiped Uncertainty Set: Cartesian product of intervals for 3 entries (Özmen et al. 2011).

Figure 1 represents the Cartesian product of intervals or, equivalently, the convex hull of vertices as an element ($\breve{\boldsymbol{A}}$) of $\mathscr{U}$ in a simplified manner. Any element of matrix $\breve{\boldsymbol{A}}$ can be represented by a vector canonically; these elements altogether generate a parallelepiped and $\mathscr{U}$ is a polytope with $2^{m \cdot n}$ vertices. By taking into account all these parallelepipeds, we have a special type of uncertainty set named *Parallelepiped Uncertainty* set. At the first view, this uncertainty set looks like a box, however, in Box Uncertainty sets, lengths of intervals are the same for every row of the uncertainty matrix $\breve{\boldsymbol{A}}$. In a Parallelepiped Uncertainty set, the lengths of intervals may vary among each other, e.g., along the columns and the rows of the regarded uncertainty matrix $\breve{\boldsymbol{A}}$.

By considering Parallelepiped Uncertainty, we trace back our robustified LP problem to an ordinary LP problem, exploiting the linearity of the robust program and the interval foundation of its uncertainty set. Here, we canonically refer to the vertex points of that set.

6 Application of CVaR and RCVaR

We used RCVaR based on historical financial data to obtain an optimal portfolio allocation. By this way, we could illustrate the implementation of RCVaR to historical financial data. Here, we refer to discussions in Subsection 5 on optimizing a portfolio by using RCVaR and satisfying a certain minimum portfolio return.

6.1 Data

There are three historical price datasets chosen for this numerical application. We created a portfolio which contains three different financial assets. The components of our portfolio are Intel, Aaon and Microsoft monthly stock prices from NASDAQ stock exchange. The data are collected from Yahoo finance. We used the historical monthly price data of these financial instruments from March 2000, to September 2016. Graphical representations of these datasets are shown in Figure 2.

All price datasets are turned into a percentage return series by addressing Equation (1). This is one of the contributions of our study, because mostly researches are done with a return calculation based on using logarithmic returns. In this research, by using percentage returns, we want to show that our return intervals (i.e., coefficient intervals in matrix $\breve{\boldsymbol{A}}$) and our RO methodology are strong enough to combat against different trends in the data. In fact, we address a finer, namely, relative (rather than absolute) viewpoint of incremental changes, which also enable the investigation of logarithmic prices that are often preferred in finance and economics.

Figure 2: Historical monthly price data of Intel, Aaon and Microsoft from NASDAQ stock market.

6.2 Algorithm

In this numerical application, we used the nominal CVaR algorithm to obtain asset allocations and risk levels of our portfolio. This application is conducted by using

Equation (9). In this sense, we only employed CVaR algorithm on MATLAB and obtained our results.

During RCVaR application, our robust algorithm is founded on some basic LP model; an Interior Point Method is chosen in this research. All optimization and CVaR codes are conducted in MATLAB Software according to the problem representation as given in Equations (8)-(9); both the nominal and the new robust optimization model are performed. We set $\alpha = 0.99$ to obtain a maximum conservatism.

To apply RCVaR, first, we created uncertainty intervals based on our price data by Equation (6). Second, we transformed the price intervals to the return interval data in the form of Equation (11). We included these uncertainty intervals into the real-world data-based matrix $\boldsymbol{A}$ in each dimension and corresponding entry; herewith, the uncertainty matrices with *Parallelepiped Uncertainty* were constructed. Consequently, we included those matrices into the problem representation of Equation (9). The part or "block" of the coefficient matrix $\breve{\boldsymbol{A}}$ with uncertainty entries showed the following form:

$$\breve{\boldsymbol{A}}_1 := \begin{bmatrix} \breve{a}_1^1 & \breve{a}_2^1 & \breve{a}_3^1 \\ \breve{a}_1^2 & \breve{a}_2^2 & \breve{a}_3^2 \\ \vdots & \vdots & \vdots \\ \breve{a}_1^{199} & \breve{a}_2^{199} & \breve{a}_3^{199} \end{bmatrix} \in \begin{bmatrix} [0.01, 1.31] & [-0.01, 0.06] & [-1.20, 0] \\ [0.01, 0.95] & [-0.01, 0.07] & [-0.23, 0.06] \\ \vdots & \vdots & \vdots \\ [-0.03, 0.91] & [0, 0.08] & [0, 0.03] \end{bmatrix}.$$

Naturally, the matrix $\breve{\boldsymbol{A}}_1$, in order to reach the full size of matrix $\breve{\boldsymbol{A}}$, should attain more row(s) and, especially, more columns. Herewith, the auxiliary variables of our LP model in Standard Form narrows down the degree of freedom as far as it came from a very small number n of risky assets. Eventually, our whole example comfortably works with portfolios whose number n of risky assets is larger than the number 3, which we chose for the ease of notation and understanding.

Remark 1 *If Parallelepiped Uncertainty sets are used, then there can be a computational drawback in the numerical experiment. The number of vertices might be too large, and to handle them computationally causes a high complexity. Additionally, the matrix* $\breve{\boldsymbol{A}}$ *has a very big dimension in our numerical practice, and the capacity at some computer might not be enough for such a size of the coefficient matrix. Hence, we employed a Weak Parallelepiped Robustification to solve that practical problem. Weak robustification means an entry-wise robustification with respect to the matrix* $\breve{\boldsymbol{A}}$. *This finite robustification process goes row by row, and it represents all the other data according to interval midpoints (ceteris paribus). Eventually, our "weak" version of RO approach addresses the worst- (robust) case with respect to all entry-wise robustifications (Özmen 2016).*

Remark 2 *One major assumption during the calculation of returns series from prices series is that the first-period returns for each asset have zero value. The* *logic behind the assumption is that an investor does not get a return when he/she invested in a stock on the market in the first period. Then, the investor only invests money. To calculate a return, we need to have different prices at two different, usually neighboring, time points.*

6.3 Nominal Conditional Value-at-Risk Application

This application provides a numerical result for portfolio optimization. The considered model here is the objective given in Equation (3), under all its constraints. Additionally, we employed a constraint on a minimum required amount of return which is discussed in Section 5; such a constraint can be inserted into the model and its given constraints to improve the entire return figure.

Two different confidence levels are implied into the CVaR model. First, since $\alpha = 0.95$ is considered as a standard value, we employed it for a first experiment. Second, we optimized our portfolio regarding the conservation level $\alpha = 0.99$. During the RO process, our research is interested in worst-case situations. The second confidence-level value is employed for various difficult situations when the decision should be more conservative than usually.

Table 1: Nominal portfolio weights and CVaR results.

	Intel (Asset 1)	**Aaon (Asset 2)**	**Microsoft (Asset 3)**	**CVaR**
α**=0.95**	0.0840	0.4760	0.4400	0.1217
α**=0.99**	0.1991	0.5030	0.2979	0.1717

According to the nominal portfolio optimization results in Table 1, the decision vector put particular weights on Asset 2 (Aaon) and Asset 3 (Microsoft). Furthermore, the confidence interval changed the amount of risk (objective value) which our portfolio faces and the distribution of our portfolio weights (decision vector).

6.4 Robust Conditional Value-at-Risk Application

Given our main aim on robust portfolio allocation, we used our optimization model and method on real-time data. Herewith, by addressing the robustified version of Equation (9), we obtained robust solutions.

Table 2: Robust portfolio weights and CVaR results.

	Intel (Asset 1)	**Aaon (Asset 2)**	**Microsoft (Asset 3)**	**CVaR**
Nominal	0.1991	0.5030	0.2979	0.1717
Robust	0.1482	0.4650	0.3868	0.1683

Based on the results in Table 2, while our portfolio risk is minimized as we aimed, the asset weights are allocated differently than in the nominal case.

7 Simulation Study and Stability Evaluation (Back Testing)

After we conducted a nominal and an RO-supported portfolio optimization by using CVaR and RCVaR, respectively. In this section, we compared CVaR and RCVaR algorithms using different price datasets generated by Monte-Carlo (MC) simulation, according to our real-data supported descriptive statistics. The comparison is based on statistical properties, namely, the variance of obtained results, i.e., of our risk values. Moreover, we generated 3 different asset prices for 199 months between March 2000 and September 2016. In this work, to avoid from negative prices and, we generated uniformly distributed prices. According to (Huang et al. 2010) some of the important researches in RO literature could be accepted for absolute RO. The absolute RO means parameters in RO models follow a uniform distribution in the uncertain set. Moreover, these RO models emphasize the extreme worst-case situations in the uncertain parameters of the optimization model. Hence, we prepared our random asset prices under a *uniform distribution*.

MC simulation provided uncertain model scenarios and allowed us to use them for our purpose. Since the perturbed prices by MC simulation are generated under a specific probability distribution, they included uncertainty. This situation permitted us to apply our RO technique on new datasets.

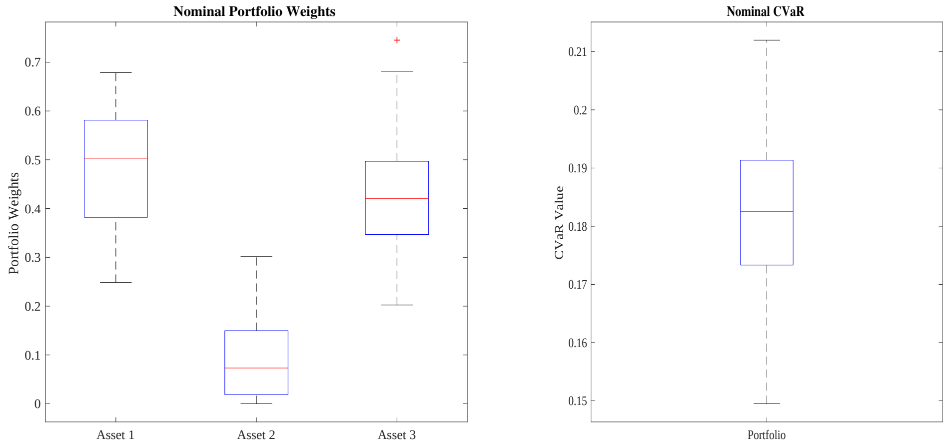
In this context, 199 *uniformly distributed asset prices* were generated under 40 different price scenarios, actually, for 3 assets, we generated 40 different price series. We made a portfolio optimization with a Robust Portfolio Optimization conducted for each price scenario. Optimal asset weights and CVaR results for each scenario are presented in Figure 3, respectively. The algorithm of this scenario application is based upon (Werner 2007; Schöttle and Werner 2006).

These results explained us the variations of weights among similar portfolio prices and their returns. Here, RO aimed to reduce these variations to obtain a more stable objective function result.

For our simulation study of RCVaR, we obtained 40 different interval values under Parallelepiped Uncertainty. According to those intervals, 40 different uncertainty scenarios were generated. Hereby, the weights of all the reproduced uniformly distributed portfolios are shown in Figure 4a. Furthermore, the RCVaR results are displayed in Figure 4b. All these calculations and optimization codes have been constructed by MATLAB Software.

The robustified version of Equation (9) aims to reduce the risk in the portfolio and to obtain a robust portfolio allocation. From Figure 4b, we may claim that while our portfolio risk is tending to be minimized, asset weights of the portfolio converge to their robust values in the sense of Parallelepiped Uncertainty set.

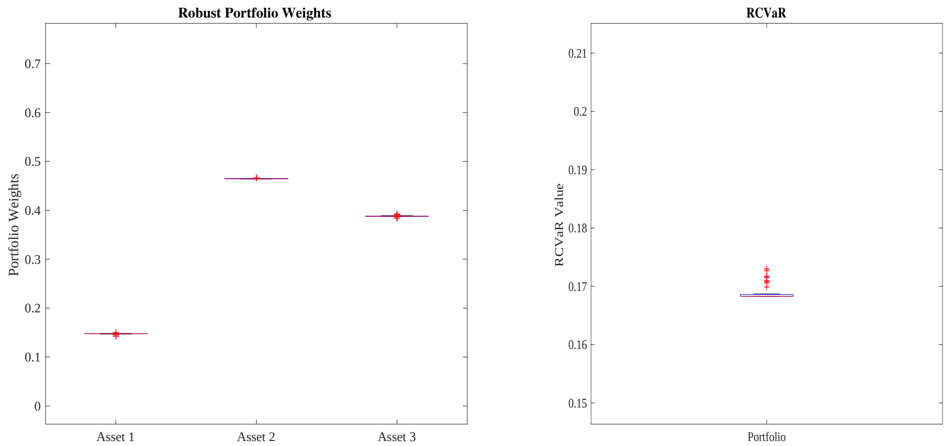
From Figures 3 and 4 we observe that the variability of the portfolio weights has decreased. For more clear visualizations, nominal and robust portfolio optimization weights are presented in Figure 5, related to our numerical results, respectively.



(a) Variety of portfolio weights per simulation.

(b) Variety of CVaR values per simulation.

Figure 3: Nominal simulation results.



(a) Variety of robust portfolio weights per simulation.

(b) Variety of RCVaR values per simulation.

Figure 4: Robust simulation results.

A sample of 40 nominal and robust simulated portfolio optimization results are included in Table 3.

One can understand that the portfolio optimization with RCVaR under Parallelepiped Uncertainty provides a stable portfolio asset allocation, ensuring also a

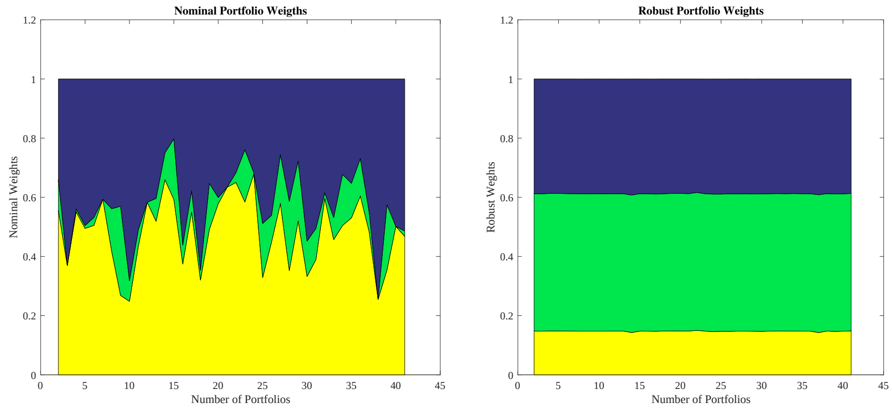


Figure 5: Area plots of nominal and robust portfolio weights.

Table 3: A sample of simulation results for nominal and robust models.

Portfolio	Nominal Weights			CVaR	Robust Weights			RCVaR
1	0.5554	0.1034	0.3411	0.1494	0.1479	0.4643	0.3878	0.1683
2	0.3691	0	0.6308	0.1885	0.1475	0.4645	0.3881	0.1683
3	0.5509	0.0097	0.4392	0.1988	0.1482	0.4650	0.3868	0.1683
4	0.4948	0.0096	0.4954	0.1890	0.1482	0.4650	0.3868	0.1683
5	0.5052	0.0266	0.4681	0.1876	0.1480	0.4644	0.3876	0.1715
6	0.5935	0	0.4064	0.1768	0.1477	0.4645	0.3878	0.1706
7	0.4170	0.1443	0.4386	0.1742	0.1477	0.4643	0.3879	0.1699
8	0.2681	0.3015	0.4304	0.1928	0.1476	0.4645	0.3879	0.1687
9	0.2484	0.0701	0.6815	0.1967	0.1477	0.4644	0.3879	0.1685
10	0.4341	0.0562	0.5097	0.2045	0.1477	0.4643	0.3879	0.1730

good *Diversification*, and a reduced risk level. In addition, our robust approach *did not mask* any of our assets in the diversification sense that assets are kept (investing all assets) rather than dropping off (not investing one or more assets) from the portfolio. We apply the notion of "masking" from statistical learning in out context; cf. (Friedman et al. 2001). This result is a very good achievement for risk management and portfolio optimization.

8 Conclusion and Future Outlook

The trade-off between maximizing the expected return and minimizing the risk under uncertainty is a great challenge to the decision making process for all quantitative investors. Uncertainty in the given information as given, i.e., by the data, affects the amount of variety and distribution, risk and return of the investments.

In this research, we prepared and conducted a robust decision making model, algorithm and methodology, applied on given real-time market data according to Parallelepiped Uncertainty in our setting.

Our aim has been to find robust portfolio optimization results (the selected quantities of assets) by using a well-developed mathematical approach. Indeed, here we introduced Robust Conditional Value-at-Risk methodology under Parallelepiped Uncertainty by considering the amount of uncertainty in the real-world data from a stock market. Moreover, based on real-time data, we generated 40 new portfolios by MC simulation and we obtained 40 different scenarios to apply our RCVaR approach. We compared both the variety of portfolio weights and the portfolio risk values of CVaR and RCVaR. As a result, our RCVaR methodology under Parallelepiped Uncertainty provided us a more stable portfolio allocation and a reduced the portfolio risk. Diversification Effect on portfolio assets became more stable after our RCVaR application.

Furthermore, the topological dependence and continuity properties under noise and perturbation of our uncertainty set with respect to the data, and of the optimal portfolios with respect to the uncertainty sets as well, could be investigated. The two directions of study might be merged, compared with the nominal case and cases of other kinds of uncertainty sets. Both could be elaborated by researchers in the context of (Jongen and Weber 1991) and (Jongen et al. 1992). Regarding the future work on this new model and methodology, robust portfolio optimization in multi-periods with given initial and, then, gradually rebalanced and reallocated wealth (rather than equating the budget constraint to 1) could become beneficial for the practice of risk management. This multi-period model might even permit for Regime Switches (Savku and Weber 2017). Additionally, time consistency is a rising topic in the field of applied mathematics, economics and Operations Research. Herewith, time consistent robustly optimized portfolios should be an another interesting research topic at a new horizon of Operations Research.

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