



Brief communication

Stability analysis for neutral delay-differential systems

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Abstract

In this paper, the problem of the stability analysis for neutral delay-differential systems is investigated. Using Lyapunov method, we present new sufficient conditions for the stability of the systems in terms of linear matrix inequality (LMI) which can be easily solved by various convex optimization algorithms. Numerical examples are given to illustrate the application of the proposed method. © 2000 The Franklin Institute. Published by Elsevier Science Ltd. All rights reserved.

Keywords: Neutral systems; Lyapunov method; Linear matrix inequality

1. Introduction

The stability analysis of neutral delay-differential systems has received considerable attention over the decades [1–9]. In the literature [5–9] Lyapunov technique, characteristic equation method, or state trajectory approach have been utilized to derive sufficient conditions for asymptotic stability of the systems. However, most of the criteria are expressed in terms of matrix norm or matrix measure of the system matrices. Unfortunately, the matrix norm operations usually make the criteria more conservative. Also the criteria in recent studies [8,9], require strong assumptions such as the matrix measures of system matrices have to be negative. These assumptions often make it difficult to apply the criteria to various systems.

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In the paper, we present two new sufficient conditions for asymptotic stability of the neutral delay-differential systems using Lyapunov's second method. One is a delay-independent criterion which investigates the stability of the system without consideration of the size of time-delay h , the other is a delay-dependent criterion which can give the maximum allowable bound of h . The derived sufficient conditions are expressed in terms of LMI to find the less conservative criteria, and can be applied under the more relaxed assumptions. The solutions of the LMIs can be easily solved by various effective optimization algorithms [10]. Three numerical examples are presented to show the application of the proposed method.

Notation: \mathcal{R}^n denotes n -dimensional Euclidean space, $\mathcal{R}^{n \times m}$ is the set of all $n \times m$ real matrices, I denotes identity matrix of appropriate order, $\|\cdot\|$ refers to either the Euclidean vector norm or induced matrix 2-norm, and $\mu(\cdot)$ denotes matrix measure of corresponding matrix. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are matrices of same dimensions, means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite).

2. Main results

Consider a neutral delay-differential system of the form

$$\dot{x}(t) = Ax(t) + Bx(t-h) + C\dot{x}(t-h) \quad (1)$$

with the initial condition function

$$x(t_0 + \theta) = \phi(\theta), \quad \forall \theta \in [-h, 0] \quad (2)$$

where $x(t) \in \mathcal{R}^n$ is the state vector, A, B and $C \in \mathcal{R}^{n \times n}$ are constant matrices, h is a positive constant time-delay, $\phi(\cdot)$ is the given continuously differentiable function on $[-h, 0]$, and the system matrix A is assumed to be a Hurwitz matrix. The system given in (1) often appears in the theory of automatic control or population dynamics.

First, we establish a delay-independent criterion, for the asymptotic stability of the neutral delay-differential system (1) using Lyapunov method in terms of LMI.

Theorem 1. *System (1) is asymptotically stable for all $h \geq 0$, if there exist positive definite matrices $P > 0$ and $R > 0$ satisfying the following LMI:*

$$\begin{bmatrix} A^T P + PA + R + A^T A & PB + A^T B & PC + A^T C \\ B^T P + B^T A & B^T B - R & B^T C \\ C^T P + C^T A & C^T B & C^T C - I \end{bmatrix} < 0. \quad (3)$$

Proof. Let the Lyapunov functional candidate be

$$V = x^T(t)Px(t) + W_1 + W_2 \quad (4)$$

where

$$W_1 = \int_{-h}^0 \dot{x}^T(t+s)\dot{x}(t+s) ds, \tag{5}$$

$$W_2 = \int_{-h}^0 x^T(t+s)Rx(t+s) ds. \tag{6}$$

The time derivative of V along the solution of (1) is given by

$$\dot{V} = x^T(A^T P + PA)x + 2x^T P B x_h + 2x^T P C \dot{x}_h + \dot{W}_1 + \dot{W}_2, \tag{7}$$

where x, x_h and \dot{x}_h denote $x(t), x(t - h)$ and $\dot{x}(t - h)$, respectively. From (5) and (6), we obtain

$$\begin{aligned} \dot{W}_1 &= \dot{x}^T \dot{x} - \dot{x}_h^T \dot{x}_h \\ &= x^T A^T A x + x_h^T B^T B x_h + \dot{x}_h^T C^T C \dot{x}_h - \dot{x}_h^T \dot{x}_h \\ &\quad + 2x^T A^T B x_h + 2x^T A^T C \dot{x}_h + 2x_h^T B^T C \dot{x}_h, \end{aligned} \tag{8}$$

$$\dot{W}_2 = x^T R x - x_h^T R x_h. \tag{9}$$

Substituting (8) and (9) into (7), we have

$$\dot{V} = \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t-h) \end{bmatrix}^T \begin{bmatrix} A^T P + PA + R + A^T A & PB + A^T B & PC + A^T C \\ B^T P + B^T A & B^T B - R & B^T C \\ C^T P + C^T A & C^T B & C^T C - I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t-h) \end{bmatrix}. \tag{10}$$

The \dot{V} is negative if inequality (3) is satisfied. This completes the proof.

Next, we develop a delay-dependent stability criterion for the neutral delay-differential systems. The following lemma, assumption and definition are necessary to derive the criterion.

Lemma 1 (Khargonekar [11]). *Let D and E be real matrices of appropriate dimensions. Then, for any scalar $\varepsilon > 0$,*

$$DE + E^T D^T \leq \varepsilon D D^T + \varepsilon^{-1} E^T E.$$

Assumption. In connection with system (1), the matrix $A + B$ has all its eigenvalues in the open left-half plane.

Definition. The matrix functions S_1 and S_2 are defined as

$$\begin{aligned} S_1(P, \varepsilon_1, \varepsilon_2, \varepsilon_3, h) &= A_0^T P + PA_0 + h P B B^T P + \varepsilon_2 P P + 2\varepsilon_3 P B B^T P \\ &\quad + (2 + \varepsilon_1^{-1}) A^T A + 3B^T B + 2\varepsilon_3^{-1} C^T C + h U^T U, \end{aligned} \tag{11}$$

$$S_2(\varepsilon_1, \varepsilon_2) = (2 + \varepsilon_1 + \varepsilon_2^{-1}) C^T C - I, \tag{12}$$

where P is a symmetric positive definite matrix, $\varepsilon_1, \varepsilon_2$, and ε_3 are positive scalars, $A_0 = A + B$, and $U = [2(A^T A + B^T B)]^{1/2}$.

Note that $S_1(P, \varepsilon_1, \varepsilon_2, \varepsilon_3, h)$ is monotonic non-decreasing with respect to h (in the sense of positive semi-definiteness).

Then the following theorem gives a delay-dependent sufficient condition for the stability of system (1).

Theorem 2. *For a given scalar h_{\max} , the system is asymptotically stable for any constant time-delay $0 < h \leq h_{\max}$, if there exist a symmetric positive definite matrix X , positive scalars $\varepsilon_1, \varepsilon_2$, and ε_3 , satisfying the following LMIs:*

$$\begin{bmatrix} Q(X) + \varepsilon_2 I + 2\varepsilon_3 B B^T & X A^T & X A^T & X B^T & X C^T & X U^T & B^T \\ A X & -\frac{1}{2} I & 0 & 0 & 0 & 0 & 0 \\ A X & 0 & -\varepsilon_1 I & 0 & 0 & 0 & 0 \\ B X & 0 & 0 & -\frac{1}{3} I & 0 & 0 & 0 \\ C X & 0 & 0 & 0 & -\frac{1}{2} \varepsilon_3 I & 0 & 0 \\ U X & 0 & 0 & 0 & 0 & -\sigma I & 0 \\ B & 0 & 0 & 0 & 0 & 0 & -\sigma I \end{bmatrix} < 0 \tag{13}$$

and

$$\begin{bmatrix} (2 + \varepsilon_1) C^T C - I & C^T \\ C & -\varepsilon_2 I \end{bmatrix} < 0 \tag{14}$$

where $Q(X) = X A_0^T + A_0 X$ and $\sigma = 1/h_{\max}$.

Proof. Without loss of generality, it is assumed that $x(t)$ is continuously differentiable in the interval $[-2h, -h]$.

Rewrite system (1) as

$$\begin{aligned} \dot{x}(t) &= (A + B)x(t) - B \int_{t-h}^t \dot{x}(s) \, ds + C\dot{x}(t-h) \\ &= A_0 x(t) - B \int_{t-h}^t \{Ax(s) + Bx(s-h) + C\dot{x}(s-h)\} \, ds + C\dot{x}(t-h) \\ &= A_0 x(t) - B\eta - BC \int_{t-h}^t \dot{x}(s-h) \, ds + C\dot{x}(t-h) \\ &= A_0 x(t) - B\eta - BCx(t-h) + BCx(t-2h) + C\dot{x}(t-h), \end{aligned} \tag{15}$$

where

$$\eta = \int_{t-h}^t \{Ax(s) + Bx(s-h)\} ds. \tag{16}$$

Now, consider the Lyapunov functional for system (15) as

$$V = x^T(t)Px(t) + Z_1 + Z_2 + Z_3 + Z_4 \tag{17}$$

where

$$Z_1 = \int_{-h}^0 \dot{x}^T(t+s)\dot{x}(t+s) ds, \tag{18}$$

$$Z_2 = \int_{-h}^0 x^T(t+s)R_1x(t+s) ds, \tag{19}$$

$$Z_3 = \int_{-2h}^0 x^T(t+s)R_2x(t+s) ds, \tag{20}$$

$$Z_4 = 2 \int_{-h}^0 \left\{ \int_{t+\theta}^t \|Ax(s)\|^2 ds + \int_{t+\theta}^t \|Bx(s-h)\|^2 ds \right\} d\theta \tag{21}$$

and P is the matrix given in (11), and R_1 and R_2 are the positive semi-definite matrices to be found.

Then, the time derivative of V is

$$\begin{aligned} \dot{V} &= x^T(A_0^T P + PA_0)x - 2x^T PB\eta - 2x^T PBCx_h + 2x^T PBCx_{2h} \\ &\quad + 2x^T PC\dot{x}_h + \dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3 + \dot{Z}_4, \end{aligned} \tag{22}$$

where $x_{2h} = x(t-2h)$.

From (18) to (21),

$$\begin{aligned} \dot{Z}_1 &= \dot{x}^T\dot{x} - \dot{x}_h^T\dot{x}_h \\ &= (x^T A^T + x_h^T B^T + \dot{x}_h^T C^T)(Ax + Bx_h + C\dot{x}_h) - \dot{x}_h^T\dot{x}_h \\ &= x^T A^T Ax + x_h^T B^T Bx_h + \dot{x}_h^T C^T C\dot{x}_h - \dot{x}_h^T\dot{x}_h \\ &\quad + 2x^T A^T Bx_h + 2x^T A^T C\dot{x}_h + 2x_h^T B^T C\dot{x}_h, \end{aligned} \tag{23}$$

$$\dot{Z}_2 = x^T R_1 x - x_h^T R_1 x_h, \tag{24}$$

$$\dot{Z}_3 = x^T R_2 x - x_{2h}^T R_2 x_{2h}, \tag{25}$$

$$\begin{aligned} \dot{Z}_4 &= 2h\|Ax(t)\|^2 + 2h\|Bx(t-h)\|^2 - 2 \int_{t-h}^t \|Ax(s)\|^2 ds - 2 \int_{t-h}^t \|Bx(s-h)\|^2 ds. \end{aligned} \tag{26}$$

By Lemma 1, the terms on the right-hand side of (22)–(26) are satisfied by the following inequalities:

$$\begin{aligned}
 & -2x^T PB\eta \leq hx^T PBB^T Px + h^{-1}\eta^T \eta, \\
 & 2x^T A^T C\dot{x}_h \leq \varepsilon_1^{-1}x^T A^T Ax + \varepsilon_1 \dot{x}_h^T C^T C\dot{x}_h, \\
 & 2x^T PC\dot{x}_h \leq \varepsilon_2 x^T PPx + \varepsilon_2^{-1}\dot{x}_h^T C^T C\dot{x}_h, \\
 & 2x^T PBCx_{2h} \leq \varepsilon_3 x^T PBB^T Px + \varepsilon_3^{-1}x_{2h}^T C^T Cx_{2h}, \\
 & -2x^T PBCx_h \leq \varepsilon_3 x^T PBB^T Px + \varepsilon_3^{-1}x_h^T C^T Cx_h, \\
 & 2x^T A^T Bx_h \leq x^T A^T Ax + x_h^T B^T Bx_h, \\
 & 2x_h^T B^T C\dot{x}_h \leq x_h^T B^T Bx_h + \dot{x}_h^T C^T C\dot{x}_h.
 \end{aligned} \tag{27}$$

Here, η is defined in (16). Using the well-known inequality $\|u + v\|^2 \leq (1 + 1/a)\|u\|^2 + (1 + a)\|v\|^2$ for any scalar $a > 0$, $\eta^T \eta$ in (27) is satisfied:

$$\begin{aligned}
 \eta^T \eta & \leq 2 \left\| \int_{t-h}^t Ax(s) ds \right\|^2 + 2 \left\| \int_{t-h}^t Bx(s-h) ds \right\|^2 \\
 & \leq 2h \int_{t-h}^t \|Ax(s)\|^2 ds + 2h \int_{t-h}^t \|Bx(s-h)\|^2 ds
 \end{aligned} \tag{28}$$

where the second inequality is obtained using Schwartz inequality. Using (23)–(28), we have

$$\begin{aligned}
 \dot{V} & \leq x^T [A_0^T P + PA_0 + R_1 + R_2 + 2hA^T A + hPBB^T P + \varepsilon_2 PP + 2\varepsilon_3 PBB^T P \\
 & \quad + (2 + \varepsilon_1^{-1})A^T A]x + x_h^T [(3 + 2h)B^T B + \varepsilon_3^{-1}C^T C - R_1]x_h \\
 & \quad + x_{2h}^T [\varepsilon_3^{-1}C^T C - R_2]x_{2h} + \dot{x}_h^T [(2 + \varepsilon_1 + \varepsilon_2^{-1})C^T C - I]\dot{x}_h.
 \end{aligned} \tag{29}$$

Here, choosing the matrices R_1 and R_2 as $R_1 = (3 + 2h)B^T B + \varepsilon_3^{-1}C^T C$ and $R_2 = \varepsilon_3^{-1}C^T C$, respectively, (29) is simplified as

$$\begin{aligned}
 \dot{V} & \leq x^T [A_0^T P + PA_0 + hPBB^T P + \varepsilon_2 PP + 2\varepsilon_3 PBB^T P + (2 + \varepsilon_1^{-1})A^T A \\
 & \quad + 3B^T B + 2\varepsilon_3^{-1}C^T C + h(2A^T A + 2B^T B)]x + \dot{x}_h^T [(2 + \varepsilon_1 + \varepsilon_2^{-1})C^T C - I]\dot{x}_h \\
 & = x^T S_1(P, \varepsilon_1, \varepsilon_2, \varepsilon_3, h)x + \dot{x}_h^T S_2(\varepsilon_1, \varepsilon_2)\dot{x}_h \\
 & \leq x^T S_1(P, \varepsilon_1, \varepsilon_2, \varepsilon_3, h_{\max})x + \dot{x}_h^T S_2(\varepsilon_1, \varepsilon_2)\dot{x}_h,
 \end{aligned} \tag{30}$$

where $S_1(\cdot)$ and $S_2(\cdot)$ are defined in (11) and (12).

Let $X = P^{-1}$. Then (30) is

$$\dot{V} \leq x^T PXS_1(P, \varepsilon_1, \varepsilon_2, \varepsilon_3, h_{\max})XPx + \dot{x}_h^T S_2(\varepsilon_1, \varepsilon_2)\dot{x}_h. \tag{31}$$

Therefore, \dot{V} is negative if the following two inequalities are satisfied:

$$XS_1(P, \varepsilon_1, \varepsilon_2, \varepsilon_3, h_{\max})X < 0, \tag{32}$$

$$S_2(\varepsilon_1, \varepsilon_2) < 0. \tag{33}$$

(32) and (33) become

$$XA_0^T + A_0X + h_{\max}BB^T + \varepsilon_2I + 2\varepsilon_3BB^T + (2 + \varepsilon_1^{-1})XA^TAX + 3XB^TBX + 2\varepsilon_3^{-1}XC^TCX + h_{\max}XU^TUX < 0, \tag{34}$$

$$(2 + \varepsilon_1 + \varepsilon_2^{-1})C^TC - I < 0. \tag{35}$$

Then, by Schur complement [12], inequalities (34) and (35) are equivalent to (13) and (14), respectively. This completes the proof.

Remark 1. The LMIs given in Theorems 1 and 2 can be solved efficiently using interior point algorithm [10]. In addition, since $h_{\max} = 1/\sigma$, the maximum allowable bound of delay, h_{\max} , can be easily found by minimizing σ subjected to $X > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\sigma > 0$, and (13), (14). Note that this optimization problem has the form of an eigenvalue problem. For details, see [10].

3. Numerical examples

To illustrate the usefulness of the proposed method, we present the following three examples. Examples 1 and 2 show the application of the delay-independent criterion, and Example 3 is for the delay-dependent one.

Example 1. Consider the following system:

$$\dot{x}(t) = Ax(t) + Bx(t - h) + C\dot{x}(t - h),$$

where

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

and α is a nonzero constant.

We now determine the stability bound in terms of α . Since $\mu(A) = 0.0811 > 0$, the criteria of Li [8] and Hu [9] are not applicable. However, the system matrix A is Hurwitz, so that Theorem 1 can be applied.

Solving the LMI given in (3), we obtain the bound of α for asymptotic stability as

$$|\alpha| \leq 0.989.$$

and the solutions of the LMI as

$$P = \begin{bmatrix} 73.2481 & 72.7010 \\ 72.7010 & 74.2916 \end{bmatrix}, \quad R = \begin{bmatrix} 72.3496 & 69.7074 \\ 69.7074 & 69.2165 \end{bmatrix}.$$

Example 2. Consider the following system:

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t-h) + \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix} \dot{x}(t-h),$$

where α is a nonzero constant.

Since the system matrix A is Hurwitz and also $\mu(A)$ is negative, so that all the criteria of Li [8], Hu [9], and Theorem 1 can be applied.

By simple calculation, we obtain the bound of α for stability as

$$\text{Li [8]: } |\alpha| \leq 0.2,$$

$$\text{Hu [9]: } |\alpha| \leq 0.2,$$

$$\text{Theorem 1: } |\alpha| \leq 0.9165.$$

For $\alpha = 0.9165$, the solutions of the linear matrix inequality (3) of Theorem 1 are

$$P = \begin{bmatrix} 1.0001 & 0 \\ 0 & 1.0001 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In the example, we can see that Theorem 1 gives less conservative bound of α .

Example 3. Consider the following neutral delay-differential system:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} x(t-h) + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \dot{x}(t-h).$$

Now, in the light of Remark 1, we can compute the maximum allowable bound h_{\max} for asymptotic stability of the system by solving the LMIs (13) and (14) of Theorem 2 such that σ is minimized. Then, we have

$$\varepsilon_1 = 2.0872, \quad \varepsilon_2 = 0.0478, \quad \varepsilon_3 = 0.0315, \quad \sigma = 1.3304,$$

$$X = \begin{bmatrix} 0.1358 & -0.0485 \\ -0.0485 & 0.1174 \end{bmatrix}.$$

So, our maximum allowable bound is

$$h_{\max} = 0.7516.$$

But, Khusainov and Yun'kova's bound given in [6] found as

$$\text{Khusainov and Yun'kova: } h_{\max} = 0.1352.$$

We can see that Theorem 2 also gives less conservative bound, h_{\max} .

4. Conclusions

In this paper, we have derived new sufficient conditions for the stability of the neutral delay-differential systems. The derived sufficient conditions are expressed in terms of LMI to find the less conservative criteria, and can be applied under the more relaxed assumptions.

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