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Stability Analysis for Nonlinear Networked Control Systems: A Discrete-time Approach

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Abstract— **In this paper we develop a framework for the stabilising controller design for nonlinear Networked Control Systems (NCSs) with time-varying sampling intervals, time-varying delays and packet dropouts. As opposed to emulation-based approaches where the effects of sampling-and-hold and delays are ignored in the phase of controller design, we propose an approach in which the controller design is based on approximate discrete-time models constructed for a nominal (non-zero) sampling interval and a nominal delay while taking into account sampling-and-hold effects. Subsequently, sufficient conditions for the global exponential stability of the closed-loop NCS are provided. The results are illustrated by means of an example.**

I. Introduction

Networked control systems (NCSs) are control systems in which sensor data and control commands are being communicated over a communication network. The recent increase of interest in NCSs is motivated by many benefits they offer such as the ease of maintenance and installation, large flexibility and low cost [1]. However, still many challenges need to be faced before all the advantages of networked control systems can be exploited to their full extent. One of the major challenges is related to guaranteeing the robustness of stability and performance of the control system in the face of imperfections and constraints imposed by the communication network, see e.g. the survey papers [1], [2].

The literature on the stability analysis and controller synthesis for *linear* NCSs is extensive, see e.g. [1], [3]–[9]. Results on the stability analysis and controller design for *nonlinear* NCSs have also been obtained in the literature, though to a lesser extent than those for linear systems. In [10], [11] a continuous-time approach leading to NCS models in terms of delay-differential equations (DDEs) and stability analysis results based on the Lyapunov-Krasovskii functional method is pursued for certain classes of nonlinear systems. In [12]–[16], an emulation-based framework for the stability analysis of nonlinear NCSs has been developed. These results consider network-induced effects such as time-varying sampling intervals, delays, packet dropouts, communication constraints and quantisation; however, the results are limited to the small delay case (delays smaller than the sampling interval). Although discrete-time approaches towards the modelling and analysis of NCSs have been proven succesfull for *linear* NCSs, see e.g. [7], [17], results on discrete-time approaches for *nonlinear* NCSs are rare. Some extensions of the discrete-time approach for sampled-data systems as developed in [18], [19] towards NCS-related problem settings have been pursued in [20], [21]. In [20], an extension towards multi-rate sampled-data systems is proposed. In [21], results for NCSs with time-varying sampling intervals and delays for a specific predictive control scheme and matching protocol are presented. However, in these results the delays are always assumed to be a multiple of the sampling interval, which is in practice generally not realistic, and delays are artificially elongated to match a 'worst-case' delay, which may be detrimental to the stability and performance of the NCS.

In this paper we develop a framework for the stabilising controller design based on approximate discrete-time models for NCSs with time-varying sampling intervals, potentially large and time-varying delays, not being limited to multiples of the sampling interval, and packet dropouts. Although an emulation-based approach is powerful in its simplicity since, in the phase of controller design, one ignores sampled-data and network effects, an approach towards stability analysis and controller design based on approximate discrete-time models may exhibit several advantages over an emulation-based approach. Firstly, in the emulation approach one typically designs the controller for the case of fast sampling (and no delay) and subsequently investigates the robustness of the resulting closed-loop NCS with respect to uncertainties in the sampling intervals (and delays), see e.g. [13], [15]. In the context of networked control one generally faces the situation in which sampling intervals exhibit some level of jitter (uncertainty) around a nominal (non-zero) sampling interval and the delays exhibit some uncertainty around a nominal delay. It appeals to our intuition, which is supported by earlier results for nonlinear sampled-data systems in [18], [19], [22], that it is beneficial to design a discrete-time controller based on a nominal (non-zero) sampling interval and a nominal delay. Secondly, it has been shown in [18], [22] for the case of nonlinear sampled-data systems with fixed sampling intervals (and no delays) that controllers based on approximate discrete-time models may provide superior performance (in terms of the domain of attraction and convergence speed). Finally, we would like to note that, for the case of linear NCSs, it has been shown in [23], that the discrete-time approach may provide less conservative

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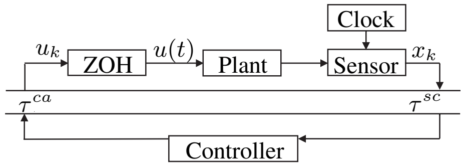

Fig. 1: Schematic overview of the networked control system.

bounds on network-induced uncertainties.

The contributions of this paper can be summarised as follows. Firstly, the results in this paper extend the results of [18], [19] on the stabilisation of nonlinear sampled-data systems based on approximate discrete-time models to the case with delays. Secondly, we develop a framework for the robustly stabilising discrete-time controller design for nonlinear NCSs with time-varying sampling intervals, large time-varying delays and packet dropouts, which extends discrete-time approaches for *linear* NCSs, as developed e.g. in [7], [17], to the realm of *nonlinear* systems.

The outline of the paper is as follows. In Section II, an (approximate) discrete-time modelling approach for nonlinear NCSs will be discussed. Based on the resulting approximate discrete-time models, parametrised by the nominal sampling interval and delay, and discrete-time controllers designed to stabilise these approximate models, we propose sufficient conditions for the global exponential stability of the closed-loop sampled-data NCS in Section III. The results are illustrated by means of an example in Section IV. Finally, concluding remarks are given in Section V.

The following notational conventions will be used in this paper. $\mathbb{R}$ denotes the field of all real numbers and $\mathbb{N}$ denotes all nonnegative integers. By $|\cdot|$ we denote the Euclidean norm. A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and unbounded. We denote the transpose of a matrix A by A^T. For a symmetric positive definite matrix $P = P^T > 0$, $\lambda_{\max}(P)$ denotes the maximum eigenvalue of P. For a locally Lipschitz function $f(x)$, $\partial f(x)$ denotes the generalised differential of Clarke [24].

II. Discrete-time Modelling of Nonlinear NCSs

Consider a NCS as depicted schematically in Figure 1. The NCS consists of a nonlinear continuous-time plant

$$\dot{x} = f(x,u), \tag{1}$$

where $f(0,0) = 0$ and $f(x,u)$ is globally Lipschitz in x and u, $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the continuous-time control input, and a discrete-time static time-invariant controller, which are connected over a communication network. The state measurements of the plant are being sampled at the sampling instants s_k. The related sampling intervals $h_k = s_{k+1} - s_k$ are possibly time-varying and satisfy $h_k \in [\underline{h}, \overline{h}]$, $\forall k \in \mathbb{N}$, with $0 < \underline{h} \leq \overline{h}$. We write $x_k := x(s_k)$. Moreover, u_k denotes the discrete-time controller command corresponding to x_k. In the model, both the varying computation time (τ_k^c), needed to evaluate the controller, and the time-varying network-induced delays, i.e. the sensor-to-controller delay (τ_k^{sc}) and the controller-to-actuator delay (τ_k^{ca}), are taken into account. The sensor acts in a time-driven fashion and we assume that both the controller and the actuator act in an event-driven fashion (i.e. responding instantaneously to newly arrived data). Under these assumptions and given the fact that the controller is static and time-invariant, all three delays can be captured by a single delay $\tau_k := \tau_k^{sc} + \tau_k^c + \tau_k^{ca}$, see also [1]. Furthermore, we account for the fact that not all the data may be used due to message rejection, i.e. the effect that more recent control data is available before the older data is implemented and therefore the older data is neglected. We assume that the time-varying delays are bounded according to $\tau_k \in [\underline{\tau}, \overline{\tau}]$, $\forall k \in \mathbb{N}$, with $0 \leq \underline{\tau} \leq \overline{\tau}$. Note that the delays may be both smaller and larger than the sampling interval. Define $\underline{d} := \lfloor \underline{\tau}/\overline{h} \rfloor$, the largest integer smaller than or equal to $\underline{\tau}/\overline{h}$ and $\overline{d} := \lceil \frac{\overline{\tau}}{\underline{h}} \rceil$, the smallest integer larger than or equal to $\frac{\overline{\tau}}{\underline{h}}$. Finally, the zero-order-hold (ZOH) function (in Figure 1) is applied to transform the discrete-time control input u_k to a continuous-time control input $u(t) = u_{k^*(t)}$, where $k^*(t) := \max\{k \in \mathbb{N} | s_k + \tau_k \leq t\}$. More explicitly, in the sampling interval $[s_k, s_{k+1})$, $u(t)$ can be described by

$$u(t) = u_{k+j-\overline{d}} \quad \text{for} \quad t \in [s_k + t_j^k, s_k + t_{j+1}^k), \tag{2}$$

where the actuation update instants $t_j^k \in [0, h_k]$ are defined as, see [17]:

$$\begin{aligned} t_j^k = \min\Big\{ & \max\{0, \tau_{k+j-\overline{d}} - \sum_{l=k+j-\overline{d}}^{k-1} h_l\}, \\ & \max\{0, \tau_{k+j-\overline{d}+1} - \sum_{l=k+j+1-\overline{d}}^{k-1} h_l\}, \\ & \ldots, \max\{0, \tau_{k-\underline{d}} - \sum_{l=k-\underline{d}}^{k-1} h_l\}, h_k \Big\} \end{aligned} \tag{3}$$

with $t_j^k \leq t_{j+1}^k$ and $j \in \{0, 1, \ldots, \overline{d} - \underline{d}\}$. Moreover, $0 = t_0^k \leq t_1^k \leq \ldots \leq t_{\overline{d}-\underline{d}}^k \leq t_{\overline{d}-\underline{d}+1}^k := h_k$. See Figure 2 for a graphical explanation of the meaning of the control update instants t_j^k. Note that the expression for the continuous-time control input in (2), (3) accounts for possible out-of-order packet arrivals and message rejection.

Remark 1

Packet dropouts can be directly incorporated in the above model, see [17] for the appropriate expressions for t_j^k in the case of packet dropouts (replacing (3)) assuming a bound on the maximal number of subsequent packet dropouts.

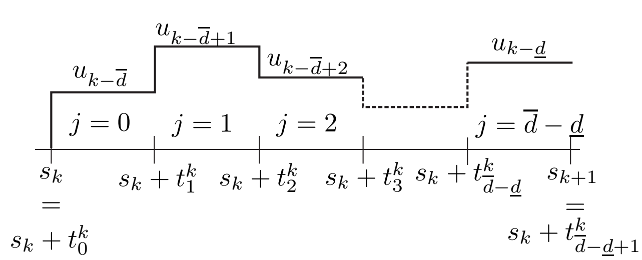

Fig. 2: Graphical interpretation of t_j^k.

Moreover, let us define the vector $\psi_j^k = \big[\tau_{k-\overline{d}+j}\ \tau_{k-\overline{d}+j+1}\ \ldots\ \tau_{k-\underline{d}}\ h_{k-\overline{d}+j}\ h_{k-\overline{d}+j+1}\ \ldots\ h_k\big]^T$ containing all past delays and sampling intervals defining t_j^k, i.e. $t_j^k = t_j^k(\psi_j^k)$. Note that $\psi_j^k \in \Psi_j := [\underline{\tau}, \overline{\tau}]^{\overline{d}-\underline{d}-j+1} \times [\underline{h}, \overline{h}]^{\overline{d}-j+1}$ for all $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, \overline{d} - \underline{d}\}$.

Next, let us consider the exact discretisation of (1), (2), (3) at the sampling instants s_k:

$$\begin{aligned} x_{k+1} =& x_k + \int_{s_k}^{s_{k+1}} f(x(s), u(s))\,\mathrm{d}s \\ =& x_k + \sum_{j=0}^{\overline{d}-\underline{d}} \int_{s_k+t_j^k}^{s_k+t_{j+1}^k} f(x(s), u_{k+j-\overline{d}})\,\mathrm{d}s \\ =:& F_{\theta_k}^e(x_k, \bar{u}_k, u_k) \end{aligned} \tag{4}$$

with $\theta_k := \begin{bmatrix} h_k & t_1^k & t_2^k & \ldots & t_{\overline{d}-\underline{d}}^k \end{bmatrix}^T \in \mathbb{R}^{\overline{d}-\underline{d}+1}, \forall k \in \mathbb{N}$, the vector of uncertainty parameters consisting of the sampling interval h_k and the control update instants within the interval $[s_k, s_{k+1}]$. Moreover, $\bar{u}_k := \begin{bmatrix} u_{k-1}^T & u_{k-2}^T & \ldots, u_{k-\overline{d}}^T \end{bmatrix}^T$ represents a vector containing past control inputs. The uncertain parameter vector θ_k is taken from the uncertainty set Θ with

$$\begin{aligned} \Theta = \Theta(\underline{h}, \overline{h}, \underline{\tau}, \overline{\tau}) = \{&\theta \in \mathbb{R}^{\overline{d}-\underline{d}+1}\,|\, h \in [\underline{h}, \overline{h}], t_j \in [\underline{t}_j, \overline{t}_j], \\ & 1 \le j \le \overline{d} - \underline{d}, 0 \le t_1 \le \ldots \le t_{\overline{d}-\underline{d}} \le h\}, \end{aligned} \tag{5}$$

where $\underline{t}_j$ and $\overline{t}_j$ denote the minimum and maximum values of t_j^k, $j = 1, 2, \ldots, \overline{d} - \underline{d}$, respectively, given by

$$\underline{t}_j = \min_{\psi_j \in \Psi_j} t_j(\psi_j), \ \text{ and } \ \overline{t}_j = \max_{\psi_j \in \Psi_j} t_j(\psi_j), \tag{6}$$

for $1 \le j < \overline{d} - \underline{d}$. Explicit expressions for $\underline{t}_j$ and $\overline{t}_j$ are given in [25].

Let us now introduce the extended (augmented) state vector $\xi_k := \begin{bmatrix} x_k^T & u_{k-1}^T & u_{k-2}^T & \ldots & u_{k-\overline{d}}^T \end{bmatrix}^T = \begin{bmatrix} x_k^T & \bar{u}_k^T \end{bmatrix}^T \in \mathbb{R}^{n+\overline{d}m}$. Then, the exact discrete-time plant model can be written as:

$$\begin{aligned} \xi_{k+1} &= \begin{bmatrix} x_{k+1}^T & u_k^T & u_{k-1}^T & \ldots & u_{k-\overline{d}+1}^T \end{bmatrix}^T \\ &= \begin{bmatrix} F_{\theta_k}^{eT}(x_k, \bar{u}_k, u_k) & u_k^T & u_{k-1}^T & \ldots & u_{k-\overline{d}+1}^T \end{bmatrix}^T \\ &=: \bar{F}_{\theta_k}^e(\xi_k, u_k). \end{aligned} \tag{7}$$

In general the exact discrete-time model is unknown since the plant is nonlinear and, consequently, we can not explicitly compute the exact model (7). In order to design a stabilising discrete-time controller, we construct an approximate discrete-time plant model (using a discretisation scheme) based on a nominal choice θ^* for the uncertain parameters θ_k given by $\theta^* = \begin{bmatrix} h^* & t_1^* & t_2^* & \ldots & t_{\overline{d}-\underline{d}}^* \end{bmatrix}^T \in \Theta \subset \mathbb{R}^{\overline{d}-\underline{d}+1}$, where $h^* \in (\underline{h}, \overline{h}]$ is a nominal sampling interval and $t_j^* \in [\underline{t}_j, \overline{t}_j]$, $j \in \{1, 2, \ldots, \overline{d} - \underline{d}\}$, are nominal control update instants. Note that arbitrarily choosing the nominal parameter vector $\theta^* = \begin{bmatrix} h^* & t_1^* & t_2^* & \ldots & t_{\overline{d}-\underline{d}}^* \end{bmatrix}^T \in \Theta \subset \mathbb{R}^{\overline{d}-\underline{d}+1}$, such that $h^* \in (\underline{h}, \overline{h}]$ and $t_j^* \in [\underline{t}_j, \overline{t}_j]$, $j \in \{1, 2, \ldots, \overline{d} - \underline{d}\}$, may lead to sequences of control update instants that, when repeated for each sampling interval, represent unfeasible sequences of control updates for the real NCS. Therefore, we define

$$\theta^* := \begin{bmatrix} h^* & t_1^* & t_2^* & \ldots & t_{\overline{d}-\underline{d}}^* \end{bmatrix}^T \in \mathbb{R}^{\overline{d}-\underline{d}+1} \tag{8}$$

with $h^* > 0$ chosen arbitrarily and

$$t_j^* := \begin{cases} 0, & j \in \{0, 1, \ldots, \overline{d} - \underline{d}^* - 1\} \\ \tau^* - \underline{d}^* h^*, & j = \overline{d} - \underline{d}^* \\ h^*, & j \in \{\overline{d} - \underline{d}^* + 1, \ldots, \overline{d} - \underline{d} + 1\} \end{cases}, \tag{9}$$

where $\tau^* = \eta(h^*) \in [\underline{d}h^*, \overline{d}h^*]$, in which $\eta(\cdot)$ expresses some continuous function from the nominal sampling interval h^* to the nominal delay τ^*, and $\underline{d}^* := \lfloor \tau^*/h^* \rfloor$. Note that θ^* now only depends on two nominal parameters; namely h^*, which represents the nominal sampling interval, and $\tau^* = \eta(h^*)$, which represents the nominal delay. See Figure 3 for a graphical explanation of the meaning of the resulting nominal control update instants t_j^*.

By exploiting a discretisation scheme we can now formulate the approximate discrete-time plant model as:

$$x_{k+1} = F_{\theta^*}^a(x_k, \bar{u}_k, u_k), \tag{10}$$

which leads to

$$\begin{aligned} \xi_{k+1} &= \begin{bmatrix} F_{\theta^*}^{aT}(x_k, \bar{u}_k, u_k) & u_k^T & u_{k-1}^T & \ldots & u_{k-\overline{d}+1}^T \end{bmatrix}^T \\ &=: \bar{F}_{\theta^*}^a(\xi_k, u_k) \end{aligned} \tag{11}$$

and corresponds to the nominal parameter vector θ^* defined in (8), (9). Next, we design a controller given by $u_{\theta^*}(\xi)$ for a nominal distribution of the (past) control inputs over the sampling interval $[s_k, s_{k+1})$ corresponding to the nominal parameter vector θ^* defined in (8), (9). The discrete-time controller

$$u_k = u_{\theta^*}(\xi_k) \tag{12}$$

will now be designed to stabilise this approximate discrete-time plant model (11) for a nominal parameter vector θ^*. In

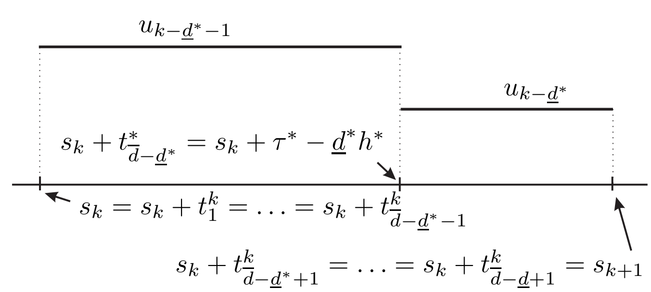

Fig. 3: Graphical interpretation of t_j^*.

fact, since θ^* only depends on h^* and τ^*, $u_{\theta^*}(\xi)$ in (12) is a controller designed to stabilise the system for the nominal sampling interval h^* and nominal delay τ^*. Let us now define the set of possible nominal parameters θ^*:

$$\begin{aligned}\Theta_0^* =&\Theta_0^*(\overline{h}^*, \underline{d}, \overline{d}, \eta(\cdot)) \\ =& \Big\{ \theta^* \in \mathbb{R}^{\overline{d}-\underline{d}+1} \,|h^* \in (0, \overline{h}^*], \\ & t_j^* := 0, \text{ for } j \in \{0, 1, \ldots, \overline{d} - \underline{d}^* - 1\}, \\ & t_j^* := \tau^* - \underline{d}^* h^*, \text{ for } j = \overline{d} - \underline{d}^*, \\ & t_j^* := h^*, \text{ for } j \in \{\overline{d} - \underline{d}^* + 1, \ldots, \overline{d} - \underline{d} + 1\}, \\ & \text{with } \tau^* = \eta(h^*)\Big\}\end{aligned} \tag{13}$$

with $\eta(h^*) \in [\underline{d}h^*, \overline{d}h^*]\ \forall h^* \in (0, \overline{h}^*]$, where $\overline{h}^*$ represents the maximal nominal sampling interval for which we aim to design stabilising controllers (stabilising the approximate discrete-time plant (11)). In Section III, we will require the approximate discrete-time plant model $\bar{F}^a_{\theta^*}(\xi, u)$, the controller $u_{\theta^*}(\xi)$ and the resulting approximate discrete-time closed-loop system $\bar{F}^a_{\theta^*}(\xi, u_{\theta^*}(\xi))$ to exhibit certain properties for $\theta^* \in \Theta^* \subseteq \Theta_0^*$ that will be used to guarantee certain stability properties for the exact discrete-time closed-loop system $\bar{F}^e_{\theta}(\xi, u_{\theta^*}(\xi))$.

The problem considered in the paper can now be formulated as follows. Given a nonlinear plant and a (family of) discrete-time controllers, parametrised by and designed for a range of nominal sampling intervals h^* and a nominal delays $\tau^* = \eta(h^*)$, we aim to provide sufficient conditions for the robust stability of the resulting sampled-data NCS in the face of (time-varying) uncertainties in the sampling interval and delays. In other words for each nominal parameter θ^* (related to a pair (h^*, τ^*)) we aim to determine the bounds $\underline{h}$, $\overline{h}$, $\underline{\tau}$ and $\overline{\tau}$ for which robust stability of the exact discrete-time closed-loop system (7), (12) (and of the sampled-data NCS (1), (2), (3), (12)) can be guaranteed.

III. Global Exponential Stability of the NCS

In this section we aim to formulate conditions under which the closed-loop sampled-data system (1), (2), (3), (12) is globally exponentially stable (GES).

A. Sufficient conditions for GES

Let us adopt the following assumptions for a set of nominal parameters Θ^* satisfying $\Theta^* \subseteq \Theta_0^*(\overline{h}^*, \underline{d}, \overline{d}, \eta(\cdot))$ with $\Theta_0^*(\overline{h}^*, \underline{d}, \overline{d}, \eta(\cdot))$ as in (13) for some given $\overline{h}^*$, $\underline{d}$, $\overline{d}$ and $\eta(\cdot)$.

Assumption 1
There exist a parametrised family of functions $V_{\theta^*}(\xi)$, a parametrised family of controllers $u_{\theta^*}(\xi)$, $a_i > 0$, $i = 1, 2, 3$, such that the following inequalities hold for some $1 \le p < \infty$:

$$\begin{aligned}&\frac{V_{\theta^*}(\bar{F}^a_{\theta^*}(\xi, u_{\theta^*}(\xi))) - V_{\theta^*}(\xi)}{h^*} \le -a_3|\xi|^p, \\ &a_1|\xi|^p \le V_{\theta^*}(\xi) \le a_2|\xi|^p,\ \forall \xi \in \mathbb{R}^{n+\overline{d}m},\ \forall \theta^* \in \Theta^*.\end{aligned} \tag{14}$$

This assumption requires that the control law $u_{\theta^*}(\xi)$ globally exponentially stabilises the *approximate* discrete-time plant (formulated for the nominal parameter set θ^*). Note that this assumption does not guarantee the stability of the *exact* closed-loop plant model for time-varying $\theta_k \in \Theta$.

Assumption 2
The parametrised family of functions $V_{\theta^*}(\xi)$ is locally Lipschitz and satisfies the following condition uniformly over $\theta^* \in \Theta^*$: there exists an $L_v > 0$, such that $\sup_{\zeta \in \partial V_{\theta^*}(\xi)} |\zeta| \le L_v|\xi|^{p-1}$, $\forall \xi \in \mathbb{R}^{n+\overline{d}m}$, and $\forall \theta^* \in \Theta^*$, with p in accordance with Assumption 1.

Note that Assumption 2 is a reasonable assumption that holds for a broad class of (possibly non-smooth) Lyapunov functions (e.g. for $p = 1$, L_v reflects a global Lipschitz constant and, for the case of quadratic Lyapunov functions $V = \frac{1}{2}\xi^T P \xi$, with $P = P^T > 0$, we have that $p = 2$ and $L_v = \lambda_{\max}(P)$).

Assumption 3
The parametrised family of approximate nominal discrete-time plant models $\bar{F}^a_{\theta^*}(\xi, u)$ is one-step consistent with the parametrised family of exact nominal discrete-time plant models $\bar{F}^e_{\theta^*}(\xi, u)$ uniformly over $\theta^* \in \Theta^*$, i.e. there exists $\hat{\rho} \in \mathcal{K}_\infty$ such that $|\bar{F}^a_{\theta^*}(\xi, u) - \bar{F}^e_{\theta^*}(\xi, u)| \le h^*\hat{\rho}(h^*)\left(|\xi| + |u|\right)$, $\forall \xi \in \mathbb{R}^{n+\overline{d}m}, u \in \mathbb{R}^m$ and $\forall \theta^* \in \Theta^*$.

The notion of consistency is commonly used in the numerical analysis literature, see e.g. [26], to address the closeness of solutions of families of models (obtained by numerical integration). Moreover, the notion of one-step consistency has been used before in the scope of the stabilisation of nonlinear sampled-data systems based on approximate discrete-time models [18], [19]. In [27], a one-step consistent integration scheme is presented with which approximate discrete-time plant models satisfying Assumption 3 can be constructed.

Assumption 4
The right-hand side $f(x, u)$ of the continuous-time plant model is globally Lipschitz, i.e. there exists $L_f > 0$ such that $|f(x_1, u_1) - f(x_2, u_2)| \le L_f\left(|x_1 - x_2| + |u_1 - u_2|\right)$, $\forall x_1, x_2 \in \mathbb{R}^n$, $u_1, u_2 \in \mathbb{R}^m$.

Assumption 5
The parametrised family of discrete-time control laws $u_{\theta^*}(\xi)$ is linearly bounded uniformly over $\theta^* \in \Theta^*$, i.e. there exists $L_u > 0$ such that $|u_{\theta^*}(\xi)| \le L_u|\xi|$, $\forall \xi \in \mathbb{R}^{n+\overline{d}m}$, and $\forall \theta^* \in \Theta^*$.

We note that these assumptions are natural extensions of the assumptions used in the scope of the stabilisation of nonlinear sampled-data systems (with constant sampling intervals and no delays), see [18]. Assumption 3 bounds the difference between the approximate and exact nominal discrete-time plant models. Assumption 4 is typically needed to bound the intersample behaviour, which, in turn, is needed to bound the difference between the nominal and uncertain exact discrete-time plant models. Moreover, the satisfaction of

$$\frac{L_v\,(L_a)^{p-1}}{h^*}\left(h^*\hat{\rho}(h^*)\,(1+L_u)+\rho_\theta\left(h^*,M_h,M_{t_1},\ldots,M_{t_{\overline{d}-\underline{d}}}\right)\right)\leq(1-\beta)a_3 \tag{15}$$

$$\rho_\theta\left(h^*,M_h,M_{t_1},\ldots,M_{t_{\overline{d}-\underline{d}}}\right):=e^{L_f h^*}\left((1+\max(1,L_u))\left(e^{L_f M_h}-1\right)+2L_f\max(1,L_u)\sum_{j=1}^{\overline{d}-\underline{d}}M_{t_j}\right) \tag{16}$$

Assumption 1 guarantees GES of the approximate discrete-time plant model, for any fixed $\theta^*\in\Theta^*$, and avoids non-uniform bounds on the overshoot and non-uniform convergence rates for the solutions of the approximate nominal discrete-time plant model, whereas Assumption 5 avoids non-uniform bounds on the controls. Finally, Assumption 2 implies continuity of the Lyapunov function. It has been shown in [18], [19] that if Assumptions 1, 2 and 5 are not satisfied then the approximate closed-loop discrete-time system does not exhibit sufficient robustness to account for the mismatch between the approximate and exact discrete-time models.

Based on these assumptions we can formulate a result that provides sufficient conditions under which the closed-loop uncertain exact discrete-time system (7), (12) is GES. Hereto, consider the following definition: $L_a := \left(2+L_u+(1+\max(1,L_u))\left(e^{L_f\overline{h}}-1\right)\right) + h^*\hat{\rho}(h^*)\,(1+L_u)$.

Theorem 1

Consider the exact discrete-time plant model (7) with $\theta_k\in\Theta(\underline{h},\overline{h},\underline{\tau},\overline{\tau})$, $\forall k\in\mathbb{N}$ and $\Theta(\underline{h},\overline{h},\underline{\tau},\overline{\tau})$ as in (5). Moreover, consider the set $\Theta_0^*(\overline{h}^*,\underline{d},\overline{d},\eta(\cdot))$ of nominal parameter vectors as in (13) for given $\overline{h}^*$, $\underline{d}$, $\overline{d}$ and $\eta(\cdot)$. Furthermore, consider lower and upper bounds on the sampling interval and delay such that $0<\underline{h}<h^*\leq\overline{h}$ and $0\leq\underline{\tau}\leq\tau^*\leq\overline{\tau}$. The following two statements hold:

- *If Assumptions 1-5 are satisfied for $\Theta^*=\{\theta^*\}$, for some $\theta^*\in\Theta_0^*(\overline{h}^*,\underline{d},\overline{d},\eta(\cdot))$, and if there exists $0<\beta<1$ such that the inequality (15), on top of this page, is satisfied where the function $\hat{\rho}$ follows from Assumption 3 and ρ_θ is defined in (16) with $M_h:=\max_{h\in[\underline{h},\overline{h}]}|h-h^*|$, $M_{t_j}:=\max_{t_j\in[\underline{t}_j,\overline{t}_j]}|t_j-t_j^*|$, $j=1,2,\ldots,\overline{d}-\underline{d}$, and $\underline{t}_j$ and $\overline{t}_j$ defined in (6), then the closed-loop uncertain exact discrete-time system (7), (12) is globally exponentially stable for $\theta_k\in\Theta(\underline{h},\overline{h},\underline{\tau},\overline{\tau})$, $\forall k\in\mathbb{N}$;*
- *If Assumptions 1-5 are satisfied for $\Theta^*=\Theta_0^*(\overline{h}^*,\underline{d},\overline{d},\eta(\cdot))$, then there exists an $h^*_{max}\leq\overline{h}^*$ such that for all $h^*\in(0,h^*_{max}]$, there exist $\underline{h}(\theta^*)$, $\overline{h}(\theta^*)$, $\underline{\tau}(\theta^*)$, $\overline{\tau}(\theta^*)$, with $\underline{h}(\theta^*)<\overline{h}(\theta^*)$, $\underline{\tau}(\theta^*)<\overline{\tau}(\theta^*)$, and $0<\beta<1$ satisfying (15). Consequently, the family of closed-loop uncertain exact discrete-time systems (7), (12) is globally exponentially stable for all $\theta^*\in\Theta_0^*(h^*_{max},\underline{d},\overline{d},\eta(\cdot))$ and for $\theta_k\in\Theta(\underline{h}(\theta^*),\overline{h}(\theta^*),\underline{\tau}(\theta^*),\overline{\tau}(\theta^*))$, $\forall k\in\mathbb{N}$.*

Proof: The proof is omitted for the sake of brevity and can be found in [27]. ■

The first statement of the theorem can be interpreted as follows. If Assumptions 1-5 hold for a fixed $\theta^*\in\Theta^*$ (i.e. for a fixed nominal sampling interval h^* and nominal delay τ^*) and condition in (15) is satisfied for that fixed θ^*, then system (7), (12) is GES for $\theta_k\in\Theta(\underline{h},\overline{h},\underline{\tau},\overline{\tau})$, $\forall k\in\mathbb{N}$ (i.e. for $h_k\in[\underline{h},\overline{h}]$ and $\tau_k\in[\underline{\tau},\overline{\tau}]$, $\forall k\in\mathbb{N}$). Note that the condition in (15) involves two distinct terms:

1) $L_v\,(L_a)^{p-1}\,\hat{\rho}(h^*)\,(1+L_u)$, which reflects the effect of approximately discretising the nonlinear plant using a nominal parameter vector θ^* (i.e. corresponding to a nominal sampling interval h^* and a nominal delay τ^*);
2) $\frac{L_v(L_a)^{p-1}}{h^*}\rho_\theta\left(h^*,M_h,M_{t_1},\ldots,M_{t_{\overline{d}-\underline{d}}}\right)$, which reflects the effect of the uncertainty in the sampling interval and delay.

In this case, only a single Lyapunov function $V_{\theta^*}(\xi)$ and a single controller $u_{\theta^*}(\xi)$ need to be found, which is a relatively simple task. Note, however, that for a priori fixed θ^* there is no guarantee that condition (15) will be satisfied, because the discretisation error (expressed by the term under point 1) above) may be too large. If condition (15) is not satisfied one has to resort to designing a Lyapunov function $V_{\theta^*}(\xi)$ and a controller $u_{\theta^*}(\xi)$ for a smaller nominal sampling interval h^* (and corresponding θ^*) and, subsequently, checking whether condition (15) is satisfied. Although this approach is beneficial in the sense that one only needs the existence of a Lyapunov function and controller for a fixed θ^*, it may lead to an iterative design procedure for Lyapunov functions and controllers. Therefore, we formulated the second statement of Theorem 1, which makes explicit that we can always choose the nominal sampling interval h^*, the uncertainty on the sampling interval $\overline{h}-\underline{h}$ and the uncertainty on the delay $\overline{\tau}-\underline{\tau}$ sufficiently small such that (15) is satisfied. Note that the definition of Θ_0^* in (13) allows h^* to be taken arbitrarily close to zero. To validate such a statement, we required in the second statement of Theorem 1 that Assumptions 1, 2, 3 and 5 hold for all $\theta^*\in\Theta_0^*$. Hereto, in turn, we need to design a parametrised family of controllers $u_{\theta^*}(\xi)$ and construct a parametrised family of Lyapunov functions $V_{\theta^*}(\xi)$. In order to design (families of) control laws and Lyapunov functions satisfying such an assumption, one may exploit, for instance, (extensions of) the results presented in [28] on backstepping designs for Euler approximate discrete-time models. When exploiting the second statement of Theorem 1, one typically computes $\underline{h}(\theta^*)$, $\overline{h}(\theta^*)$, $\underline{\tau}(\theta^*)$, $\overline{\tau}(\theta^*)$ using (15) for each fixed $\theta^* \in \Theta_0^*(h^*_{max}, \underline{d}, \overline{d}, \eta(\cdot))$. Note that, even for each fixed θ^*, different combinations of $\underline{h}(\theta^*)$, $\overline{h}(\theta^*)$, $\underline{\tau}(\theta^*)$, $\overline{\tau}(\theta^*)$ may satisfy (15), which may be used to investigate trade-offs between time-varying delays and time-varying sampling intervals.

Remark 2

Based on the results on the global exponential stability of the exact uncertain discrete-time model, we can also conclude that the sampled-data NCS (1), (2), (3), (12) *is GES. Namely, it can be shown that the intersample behaviour is linearly globally uniformly bounded over the maximum sampling interval* $\overline{h}$, *see [27]. Next, we can use the results in [29] to conclude that the closed-loop sampled-data NCS* (1), (2), (3), (12) *is globally exponentially stable.*

IV. Illustrative Example

Let us consider a NCS as depicted in Figure 1 with a class of scalar nonlinear continuous-time plants of the form $\dot{x} = f(x) + u$, where $x \in \mathbb{R}$, $u \in \mathbb{R}$, and $f(x)$ is globally Lipschitz with Lipschitz constant L_{fx}. Consequently, the right-hand side $f(x) + u$ satisfies Assumption 4 with $L_f = \max(1, L_{fx})$. Let us consider the case without delays, but with uncertain time-varying sampling intervals. For an example with time-varying delays we refer to [27]. We use an Euler discretisation scheme to construct the following family of approximate discrete-time plant models: $x_{k+1} = x_k + h^*(f(x_k)+u_k) =: F^a_{h^*}(x_k, u_k)$. It is straightforward to show that this family of approximate discrete-time models satisfies Assumption 3 with $h^*\hat{\rho}(h^*) = \frac{L_{fx}}{L_f}\left(e^{L_f h^*} - 1 - L_f h^*\right)$. Moreover, consider the following controllers

$$u_k = -f(x_k) - x_k \tag{17}$$

$$u_k = -f(x_k) - x_k - h^* x_k, \tag{18}$$

where the first controller is independent of h^* and could be regarded as an example of an emulation-based controller, whereas the second controller is clearly parametrised by the nominal sampling interval h^*. Below, we will exploit the candidate Lyapunov function $V(x) = |x|$, which is independent of θ^* and which clearly satisfies Assumption 2 with $L_v = 1$ and $p = 1$. Note that both controllers approach each other for $h^* \downarrow 0$.

Let us first consider controller (17). This controller clearly satisfies Assumption 5 with $L_u = L_{fx} + 1$. In order to assess the satisfaction of Assumption 1, we note that we can take $a_1 = a_2 = 1$ and we evaluate

$$\begin{aligned}\frac{V(F^a_{h^*}(x_k,u_k)) - V(x_k)}{h^*} &= \frac{|x_k + h^*(f(x_k)+u_k)| - |x_k|}{h^*} \\ &= \begin{cases} -|x_k| & \text{if } 0 < h^* \leq 1 \\ (1 - \frac{2}{h^*})|x_k| & \text{if } 1 \leq h^* \leq 2 \end{cases},\end{aligned} \tag{19}$$

which can be used to conclude that the approximate closed-loop discrete-time system, with controller (17), satisfies Assumption 1 with $p = 1$ for $0 < h^* \leq 2 - \varepsilon$, with $\varepsilon > 0$.

Let us next consider controller (18). This controller clearly satisfies Assumption 5 with $L_u = L_{fx} + 1 + \overline{h}^*$. In order to assess the satisfaction of Assumption 1, we evaluate

$$\begin{aligned}\frac{V(F^a_{h^*}(x_k,u_k)) - V(x_k)}{h^*} &= \frac{|x_k + h^*(f(x_k)+u_k)| - |x_k|}{h^*} \\ &= \begin{cases} (-1 - h^*)|x_k| & \text{if } 0 < h^* \leq \frac{1}{2}(\sqrt{5}-1) \\ \frac{-2+h^*+h^{*2}}{h^*}|x_k| & \text{if } \frac{1}{2}(\sqrt{5}-1) \leq h^* \leq 1 \end{cases},\end{aligned} \tag{20}$$

which can be used to conclude that the approximate closed-loop discrete-time system, with controller (18), satisfies Assumption 1 with $p = 1$ for $0 < h^* \leq 1 - \varepsilon$, with $\varepsilon > 0$.

For both controllers, the second statement of Theorem 1 can now be exploited to conclude that there always exists a sufficiently small nominal sampling interval and a sufficiently small level of uncertainty on the sampling interval such that the exact closed-loop sampled-data networked control system can be guaranteed to be globally exponentially stable. This example also shows that the results proposed in this paper can be used to study both emulation-based controllers as well as discrete-time controllers parametrised by the nominal sampling interval.

Next, let us use condition (15) in Theorem 1 to compute (estimates of the) uncertainty bounds $\underline{h}$, $\overline{h}$ on the sampling interval, depending on h^*, that still guarantee GES of the closed-loop system. Here, we consider the case of symmetric uncertainty intervals for h around h^*, i.e. $h^* = (\underline{h} + \overline{h})/2$ represents the middle of the uncertainty interval $[\underline{h}, \overline{h}]$. We note that condition (15) in Theorem 1 also allows to compute asymmetric uncertainty intervals around h^* (i.e. $h^* \neq (\underline{h} + \overline{h})/2$). Bounds for $\underline{h}$, $\overline{h}$ are depicted, depending on the choice for h^*, in Figure 4 for the case that $L_{fx} = 0.82$. Figure 4 indicates that, for this particular example, the controller that explicitly takes into account the nominal sampling interval may allow for a larger uncertainty in the sampling interval than the emulation-based controllers. However, we stress here that this is by no means a generic fact and we note that, firstly, the bounds given here only represent sufficient conditions, which may exhibit a certain level of conservatism and, secondly, that these bounds on the allowable jitter depend on many factors such as the particular controller designed, the particular integration scheme used to obtain

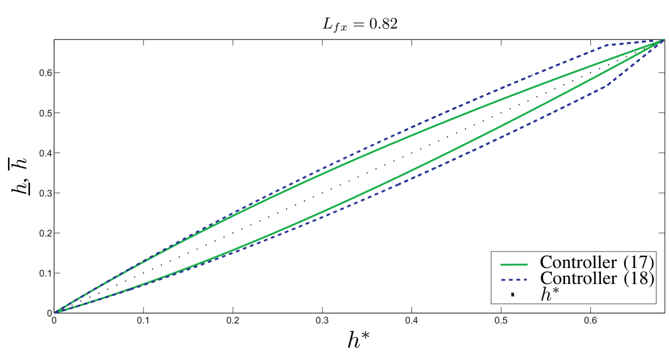

Fig. 4: Bounds $\underline{h}$, $\overline{h}$ on the uncertainty of the sampling interval for controllers (17), (18) for $L_{fx} = 0.82$.

the approximate discrete-time plant model, the particular Lyapunov function used to study stability etc. To assess the possible conservatism of these results to some extent, we consider the case in which $f(x) = L_{fx}x$, with $L_{fx} = 0.82$, and the sampling interval is constant. In this case we can straightforwardly compute an upperbound on the sampling interval (because the discrete-time closed-loop system for fixed h is linear), which is $h \approx 1.184$ for controller (17) and $h \approx 0.792$ for controller (18). Considering the fact that we consider an entire class of nonlinear systems and time-varying sampling intervals, the bounds depicted in Figure 4 are not extremely conservative.

V. Conclusions

This paper presents a framework for the stabilising controller design for nonlinear Networked Control Systems (NCSs) with time-varying sampling intervals and time-varying delays (that may be larger than the sampling interval). We have developed a framework for the controller design based on approximate discrete-time plant models. As opposed to emulation-based approaches where the effects of sampling-and-hold and delays are ignored in the phase of controller design, we propose an approach in which the controller design is based on approximate discrete-time models constructed for a nominal (non-zero) sampling interval and a nominal delay. Subsequently, sufficient conditions for the global exponential stability of the closed-loop NCS with time-varying sampling intervals and delays are provided.

The results in this paper represent extensions to the existing literature in several ways. Firstly, the results presented in this paper extend the results in [18], [19] on the controller design for nonlinear sampled-data systems (with *constant* sampling intervals and *no* delays) based on approximate discrete-time models to the case of nonlinear sampled-data systems with delays (this represents an extension even for the case with constant delays). Moreover, the results in this paper further extend these works in the sense that we allow for *time-varying* uncertain sampling intervals and delays. From a different perspective, the results in this paper extend the results on discrete-time modelling and stability analysis for *linear* NCSs with time-varying sampling intervals, delays and packet dropouts in [7], [17] to the realm of *nonlinear* systems.

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