Bifurcations in Continuous-Time Macroeconomic Systems

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Abstract

There has been increasing interest in continuous-time macroeconomic models. This research investigates bifurcation phenomena in a continuous-time model of the United Kingdom. We choose a particularly well-regarded continuous-time macroeconometric model to assure the empirical and potential policy relevance of our results. In particular, we use the Bergstrom, Nowman and Wymer continuous-time dynamic macroeconometric model of the UK economy. We find that bifurcations are important with this model for understanding the dynamic properties of the system and for determining which parameters are the most important to those dynamic properties. We have discovered that both saddle-node bifurcations and Hopf bifurcations indeed exist with this model within the model's region of plausible parameter settings.

We find that the existence of Hopf bifurcations is particularly useful since those bifurcations may provide explanations for some cyclical phenomena in the macroeconomy. We further design numerical algorithms to locate the bifurcation boundaries, which we display in three dimensional color bifurcation diagrams. A notable and perhaps surprising fact is that both types of bifurcations can coexist with this well-regarded UK model — in the same neighborhood of the parameter space.

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1 Introduction

In recent years, there has been increasing interest in using continuous-time models to describe macroeconomic systems. Continuous-time econometrics has been very important for dynamic disequilibrium modeling. The specification of econometric models in continuous-time rather than discrete-time has several advantages such as the characterization of the interaction between the variables during the unit observation period, more accurate representation of the partial adjustment processes in dynamic disequilibrium models, the independence of the unit of the observation period, and the capability of forecasting the continuous-time path of the variables. An informative discussion of the advantages is provided by Bergstrom (1996). Since the development of the first continuous-time macroeconometric model by Bergstrom and Wymer (1976), there has been a significant growth in the use of continuous-time econometric methods in macroeconomic modeling. Economy-wide continuous-time models have been developed for most of the leading industrial countries of the world, see Bergstrom et al. (1992). The idea is to model a system by a set of differential equations. An important feature of the continuous-time models is that the estimator uses a discrete model that is satisfied by the observations generated by the differential equation system irrespective of the observation interval of the sample, so that the properties of the parameters of the differential equation system can be derived from the sampling properties of the discrete model. A recent survey was given by Bergstrom (1996).

Most research on the continuous-time models focuses on estimation and model building for various economic systems. Continuous-time economic models have been built, for example, for the United Kingdom in Bergstrom and Wymer (1976) and Knight and Wymer (1978), for the United States in Donaghy (1993), for the Netherlands in Nieuwenhuis (1994), and for Italy in Tullio (1972) and Gandolfo and Padoan (1990). A complete list is provided in Bergstrom (1996). With these models available, the next stage of research is naturally performance analysis. It is important to understand the structural properties of the continuous-time economic models. There are several papers dealing with stability of continuous-time models. Particularly, Bergstrom et al. (1992) and Donaghy (1993) respectively examine the stability of the models for the United Kingdom and the United States economies. It has been noticed that for the estimated parameter values these models are slightly unstable. Bergstrom et al. (1994) analyzes the effect of monetary and fiscal feedback controls on the stability of the UK model and finds that the controls cannot stabilize the system. They further obtain a stabilizing controller, though the realizability of the controller is unclear, based on linear quadratic control theory. Nieuwenhuis and Schoonbeek (1997) investigates the relationship between the stability of the continuous-time models and the structure of the matrices appearing in the models. Their results are obtained by analyzing the dominant-diagonal structures of the matrices. Wymer (1996) suggests the study of singularities and bifurcations of continuous-time models. Barnett et al. (1996) explores, among other results, chaotic phenomena in economic systems. While these research activities represent a growing interest in understanding the continuous-time models, a comprehensive understanding of the bifurcations and the stability of continuous-time economic models are still unavailable.

This paper describes our recent effort in analyzing the continuous-time macroeconometric model of the United Kingdom as given in Bergstrom et al. (1992). It is discovered that both saddle-node bifurcations and Hopf bifurcations indeed exist. Boundaries for saddle-node bifurcations are obtained. For cases in which analytical formulas of bifurcation boundaries are not available, a numerical algorithm is provided for finding the bifurcation boundaries. The rest of the paper is organized as follows. Section 2 introduces the continuous-time macroeconometric model. Section 3 presents the linearized model and uses the gradient method to find a set of parameter values under which the system is stable. Section 4 proposes a numerical algorithm for finding bifurcation boundaries. Section 5 implements the algorithm for several special cases to locate explicit bifurcation boundaries. Finally, conclusion remarks are given and further research directions are discussed in the last section.

2 The Model

We consider the Bergstrom, Nowman and Wymer (1992) continuous-time macroeconometric model of the United Kingdom. To introduce the model, a set of variables are first defined.

Endogenous variables

- C real private consumption
- E_n real non-oil exports
- F real current transfers abroad
- I volume of imports
- K amount of fixed capital
- K_a cumulative net real investment abroad (excluding changes in official reserve)
- L employment
- M money supply
- P real profits, interest and dividends from abroad
- p price level
- Q real net output
- q exchange rate (price of sterling in foreign currency)
- r interest rate
- w wage rate

Exogenous variables

d_x dummy variable for exchange controls ($d_x = 1$ for 1974-79, $d_x = 0$ for 1980 onwards)
E_o real oil exports
G_c real government consumption
p_f price level in leading foreign industrial countries
p_i price of imports (in foreign currency)
r_f foreign interest rate
T_1 total taxation policy variable ($(Q+P)/T_1$ is real private disposable income)
T_2 indirect taxation policy variable (Q/T_2 is real output at factor cost)
t time
Y_f real income of leading foreign industrial countries

Then the dynamic behavior of the UK economy is described by the following 14 differential equations.

Model

$$D^2 \log C = \gamma_1(\lambda_1 + \lambda_2 - D \log C) + \gamma_2 \log \left[\frac{\beta_1 e^{-\{\beta_2(r - D \log p) + \beta_3 D \log p\}}(Q + P)}{T_1 C} \right] \tag{2.1}$$

$$D^2 \log L = \gamma_3(\lambda_2 - D \log L) + \gamma_4 \log \left[\frac{\beta_4 e^{-\lambda_1 t}\{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\}^{-1/\beta_6}}{L} \right] \tag{2.2}$$

$$D^2 \log K = \gamma_3(\lambda_1 + \lambda_2 - D \log K) + \gamma_6 \log \left[\frac{\beta_5 (Q/K)^{1+\beta_6}}{r - \beta_7 D \log p + \beta_8} \right] \tag{2.3}$$

$$D^2 \log Q = \gamma_7(\lambda_1 + \lambda_2 - D \log Q) + \gamma_8 \log \left[\frac{\{1 - \beta_9 (qp/p_i)^{\beta_{10}}\}(C + G_c + DK + E_n + E_o)}{Q} \right] \tag{2.4}$$

$$D^2 \log p = \gamma_9(D \log(w/p) - \lambda_1) + \gamma_{10} \log \left[\frac{\beta_{11}\beta_4 T_2 w e^{-\lambda_1 t}\{1 - \beta_5 (Q/K)^{\beta_6}\}^{-(1+\beta_6)/\beta_6}}{p} \right] \tag{2.5}$$

$$D^2 \log w = \gamma_{11}(\lambda_1 - D \log(w/p)) + \gamma_{12} D \log(p_i/qp) + \gamma_{13} \log \left[\frac{\beta_4 e^{-\lambda_1 t}\{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\}^{-1/\beta_6}}{\beta_{12} e^{\lambda_2 t}} \right] \tag{2.6}$$

$$D^2 r = -\gamma_{14} Dr + \gamma_{15}\left[\beta_{13} + r_f - \beta_{14} D\log q + \beta_{15}\frac{p(Q+P)}{M} - r\right] \tag{2.7}$$

$$D^2 \log I = \gamma_{16}(\lambda_1 + \lambda_2 - D\log(p_i I/qp))$$

$$+\gamma_{17}\log\left[\frac{\beta_9 (qp/p_i)^{\beta_{10}}(C + G_c + DK + E_n + E_o)}{(p_i/qp)I}\right] \tag{2.8}$$

$$D^2 \log E_n = \gamma_{18}(\lambda_1 + \lambda_2 - D\log E_n) + \gamma_{19}\log\left[\frac{\beta_{16} Y_f^{\beta_{17}}(p_f/qp)^{\beta_{18}}}{E_n}\right] \tag{2.9}$$

$$D^2 F = -\gamma_{20} DF + \gamma_{21}[\beta_{19}(Q+P) - F] \tag{2.10}$$

$$D^2 P = -\gamma_{22} DP + \gamma_{23}\{[\beta_{20} + \beta_{21}(r_f - D\log p_f)]K_a - P\} \tag{2.11}$$

$$D^2 K_a = -\gamma_{24} DK_a + \gamma_{25}\{[\beta_{22} + \beta_{23}(r_f - r) - \beta_{24} D\log q - \beta_{25} d_x](Q+P) - K_a\} \tag{2.12}$$

$$D^2 \log M = \gamma_{26}(\lambda_3 - D\log M) + \gamma_{27}\log\left[\frac{\beta_{26} e^{\lambda_3 t}}{M}\right]$$

$$+\gamma_{28} D\log\left[\frac{E_n + E_o + P - F}{(p_i/qp)I}\right] + \gamma_{29}\log\left[\frac{E_n + E_o + P - F - DK_a}{(p_i/qp)I}\right] \tag{2.13}$$

$$D^2 \log q = \gamma_{30} D\log(p_f/qp) + \gamma_{31}\log\left[\frac{\beta_{27} p_f}{qp}\right] + \gamma_{32} D\log\left[\frac{E_n + E_o + P - F}{(p_i/qp)I}\right]$$

$$+\gamma_{33}\log\left[\frac{E_n + E_o + P - F - DK_a}{(p_i/qp)I}\right] \tag{2.14}$$

where D is the differential operator, $Dx = dx/dt$, $D^2 x = d^2x/dt^2$, $\beta_i, i = 1, 2, ..., 27$, $\gamma_j, j = 1, 2, ..., 33$, and $\lambda_k, k = 1, 2, 3$, are structural parameters that can be estimated from historical data. A set of their estimates using quarterly data from 1974 to 1984 are given in Table 2 of Bergstrom et al. (1992). These equations are formulated based on economic theory. The exact interpretations of these 14 equations are omitted here because they are not needed in this paper and can be found in Bergstrom et al. (1992).

Equations (2.1)-(2.14) are nonlinear. To study the steady-state behavior, it was assumed in Bergstrom et al. (1992) that the exogenous variables satisfy the following conditions.

$$d_x = 0$$

$$E_o = 0$$

$$G_c = g^*(Q+P)$$

$$p_f = p_f^* e^{\lambda_4 t}$$

$$p_i = p_i^* e^{\lambda_4 t}$$

$$r_f = r_f^*$$

$$T_1 = T_1^*$$

$$T_2 = T_2^*$$

$$Y_f = Y_f^* e^{((\lambda_1+\lambda_2)/\beta_{17})t}$$

where g^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^* and λ_4 are constants.

Under the assumption of the exogenous variables, it can be shown that $C(t)$, ..., $q(t)$ in (2.1)-(2.14) change at constant rates in equilibrium. In what follows, we study the behavior of the system of differential equations (2.1)-(2.14) near equilibria. For this purpose, let the variables $y_1(t)$, $y_2(t)$, ..., $y_{14}(t)$ be defined as follows:

$$y_1(t) = \log\{C(t)/C^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_2(t) = \log\{L(t)/L^* e^{\lambda_2 t}\}$$

$$y_3(t) = \log\{K(t)/K^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_4(t) = \log\{Q(t)/Q^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_5(t) = \log\{p(t)/p^* e^{(\lambda_3-\lambda_1-\lambda_2)t}\}$$

$$y_6(t) = \log\{w(t)/w^* e^{(\lambda_3-\lambda_2)t}\}$$

$$y_7(t) = r(t) - r^*$$

$$y_8(t) = \log\{I(t)/I^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_9(t) = \log\{E_n(t)/E_n^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{10}(t) = \log\{F(t)/F^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{11}(t) = \log\{P(t)/P^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{12}(t) = \log\{K_a(t)/K_a^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{13}(t) = \log\{M(t)/M^* e^{\lambda_3 t}\}$$

$$y_{14}(t) = \log\{q(t)/q^* e^{(\lambda_1+\lambda_2+\lambda_4-\lambda_3)t}\}$$

where $C^*, L^*, K^*, Q^*, p^*, w^*, r^*, I^*, E_n^*, F^*, P^*, K_a^*, M^*, q^*$ are functions of the vector (β, γ, λ) of 63 parameters in equations (2.1)-(2.14) and the additional parameters g^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^*, λ_4. Then the equilibrium of the system (2.1)-(2.14) corresponds to zero values of $y_i(t) = 0, i = 1, 2, ..., 14$. The set of equations satisfied by $y_i(t), i = 1, 2, ..., 14$, can be obtained from (2.1)-(2.14).

$$D^2y_1 = -\gamma_1 Dy_1 + \gamma_2\{\log(Q^*e^{y_4} + P^*e^{y_{11}}) - \log(Q^* + P^*) - \beta_2 y_7 + (\beta_2 - \beta_3)Dy_5 - y_1\} \tag{2.15}$$

$$D^2y_2 = -\gamma_3 Dy_2 + \gamma_4\left\{\frac{1}{\beta_6}\log\left[\frac{(Q^*)^{-\beta_6} - \beta_5(K^*)^{-\beta_6}}{(Q^*)^{-\beta_6}e^{-\beta_6 y_4} - \beta_5(K^*)^{-\beta_6}e^{-\beta_6 y_3}}\right] - y_2\right\} \tag{2.16}$$

$$D^2y_3 = -\gamma_5 Dy_3 + \gamma_6\Big\{(1+\beta_6)(y_4 - y_3) + \log[r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_8] - \log[y_7 + r^* - \beta_7(Dy_5 + \lambda_3 - \lambda_1 - \lambda_2) + \beta_8]\Big\} \tag{2.17}$$

$$D^2y_4 = -\gamma_7 Dy_4 + \gamma_8\Big\{\log\left[\frac{1 - \beta_9(q^*p^*/p_i^*)^{\beta_{10}}e^{\beta_{10}(y_5+y_{14})}}{1 - \beta_9(q^*p^*/p_i^*)^{\beta_{10}}}\right] + \log\left(C^*e^{y_1} + g^*(Q^*e^{y_4} + P^*e^{y_{11}}) + K^*e^{y_3}(Dy_3 + \lambda_1 + \lambda_2) + E_n^*e^{y_9}\right) - \log\left(C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*\right) - y_4\Big\} \tag{2.18}$$

$$D^2y_5 = \gamma_9(Dy_6 - Dy_5) + \gamma_{10}\left\{y_6 - y_5 - \frac{1+\beta_6}{\beta_6}\log\left[1 - \beta_5\left(\frac{Q^*}{K^*}\right)^{\beta_6}e^{\beta_6(y_4-y_3)}\right] + \frac{1+\beta_6}{\beta_6}\log\left[1 - \beta_5\left(\frac{Q^*}{K^*}\right)^{\beta_6}\right]\right\} \tag{2.19}$$

$$D^2y_6 = \gamma_{11}(Dy_5 - Dy_6) - \gamma_{12}(Dy_5 + Dy_{14}) + \gamma_{13}\left\{\frac{1}{\beta_6}\log[(Q^*)^{-\beta_6} - \beta_5(K^*)^{-\beta_6}] - \frac{1}{\beta_6}\log[(Q^*)^{-\beta_6}e^{-\beta_6 y_4} - \beta_5(K^*)^{-\beta_6}e^{-\beta_6 y_3}]\right\} \tag{2.20}$$

$$D^2y_7 = -\gamma_{14}Dy_7 + \gamma_{15}[[\beta_{15}\frac{p^*e^{y_5}(Q^*e^{y_4} + P^*e^{y_{11}})}{M^*e^{y_{13}}} - \beta_{15}\frac{p^*(Q^* + P^*)}{M^*} - \beta_{14}Dy_{14} - y_7] \tag{2.21}$$

$$D^2y_8 = \gamma_{16}(Dy_5 + Dy_{14} - Dy_8) + \gamma_{17}\Big\{(1+\beta_{10})(y_5 + y_{14}) - y_8 + \log[C^*e^{y_1} + g^*(Q^*e^{y_4} + P^*e^{y_{11}}) + K^*e^{y_3}(Dy_3 + \lambda_1 + \lambda_2) + E_n^*e^{y_9}]$$

$$-\log[C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*]\Big\} \tag{2.22}$$

$$D^2 y_9 = -\gamma_{18} D y_9 - \gamma_{19}\{\beta_{18}(y_5 + y_{14}) + y_9\} \tag{2.23}$$

$$D^2 y_{10} = -\{\gamma_{20}+2(\lambda_1+\lambda_2)\}Dy_{10}-(Dy_{10})^2+\gamma_{21}\beta_{19}\Big\{\frac{Q^* e^{y_4}+P^* e^{y_{11}}}{F^* e^{y_{10}}} - \frac{Q^*+P^*}{F^*}\Big\} \tag{2.24}$$

$$D^2 y_{11} = -\{\gamma_{22} + 2(\lambda_1 + \lambda_2)\}Dy_{11} - (Dy_{11})^2$$

$$+\gamma_{23}\{\beta_{20} + \beta_{21}(r_f^* - \lambda_4)\}\left[\frac{K_a^* e^{y_{12}}}{P^* e^{y_{11}}} - \frac{K_a^*}{P^*}\right] \tag{2.25}$$

$$D^2 y_{12} = -\{\gamma_{24} + 2(\lambda_1 + \lambda_2)\}Dy_{12} - (Dy_{12})^2 + \gamma_{25}\Big\{[\beta_{22} + \beta_{23}(r_f^* - r^* - y_7)$$

$$-\beta_{24}(Dy_{14} + \lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)]\frac{Q^* e^{y_4} + P^* e^{y_{11}}}{K_a^* e^{y_{12}}} - [\beta_{22} + \beta_{23}(r_f^* - r^*)$$

$$-\beta_{24}(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)]\frac{Q^* + P^*}{K_a^*}\Big\} \tag{2.26}$$

$$D^2 y_{13} = -\gamma_{26} Dy_{13} - \gamma_{27} y_{13} + \gamma_{28}\Big\{\frac{E_n^* e^{y_9} Dy_9 + P^* e^{y_{11}} Dy_{11} - F^* e^{y_{10}} Dy_{10}}{E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}}}$$

$$+Dy_5 + Dy_{14} - Dy_8\Big\} + \gamma_{29}\Big\{\log[E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}}$$

$$-K_a^* e^{y_{12}}(Dy_{12} + \lambda_1 + \lambda_2)] - \log[E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)]$$

$$+y_5 + y_{14} - y_8\Big\} \tag{2.27}$$

$$D^2 y_{14} = -\gamma_{30}(Dy_5 + Dy_{14}) - \gamma_{31}(y_5 + y_{14})$$

$$+\gamma_{32}\Big\{\frac{E_n^* e^{y_9} Dy_9 + P^* e^{y_{11}} Dy_{11} - F^* e^{y_{10}} Dy_{10}}{E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}}} + Dy_5 + Dy_{14} - Dy_8\Big\}$$

$$+\gamma_{33}\Big\{\log[E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}} - K_a^* e^{y_{12}}(Dy_{12} + \lambda_1 + \lambda_2)]$$

$$-\log[E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)] + y_5 + y_{14} - y_8\Big\} \tag{2.28}$$

Equations (2.15)-(2.28) form an autonomous system with equilibrium 0 for any parameter values of $\{\beta_i, \gamma_j, \lambda_k\}$. System (2.15)-(2.28) might have other equilibria. However, as a first step, we are now focusing on the properties of the trajectories of the system (2.15)-(2.28) near the equilibrium 0.

3 Linearization of Macroeconometric Equations

Consider an ordinary differential equation

$$Dx(t) = f(x(t)) \tag{3.1}$$

where $x \in R^n$ is the state vector and the mapping $f(.) : R^n \to R^n$ is continuously differentiable (with respect to each argument). Suppose that $x^* \in R^n$ is a constant vector satisfying

$$f(x^*) = 0.$$

Then x^* is an equilibrium of the system. Let $\bar{A}$ be the Jacobian matrix of $f(x)$ evaluated at x^*

$$\bar{A} = \frac{\partial f(x)}{\partial x}|_{x=x^*}.$$

Then the following linear system

$$Dy = \bar{A}y \tag{3.2}$$

is called the linearized system of (3.1) around the equilibrium x^*. The advantage of linearization is that the (stability) behavior of trajectories of the nonlinear system (3.1) in a close neighborhood of the equilibrium x^* can be studied through that of its linearization (3.2). Briefly, if all eigenvalues of $\bar{A}$ have negative real parts, then (3.1) is stable in the neighborhood of x^*, meaning that all trajectories approach x^* as $t \to \infty$ when the initial state $x(0)$ is sufficiently close to x^*. If at least one of the eigenvalues of $\bar{A}$ has positive real part, then (3.1) is unstable in the neighborhood of x^*. In this case, there exists an initial state $x(0)$ (arbitrarily close to x^*) for which $x(t)$ does not approach x^* as $t \to \infty$. If all eigenvalues of $\bar{A}$ have nonpositive real parts and at least one has zero real part, the stability of (3.1) usually cannot be determined from the matrix $\bar{A}$. One needs to analyze higher order terms in order to determine the stability of the system. In most cases, one needs to examine the system behavior along certain manifold to determine the stability, see Khalil (1992).

Since the concept of stability adopted here is concerned with a close neighborhood of an equilibrium only, it is referred to local stability. In this paper, we only consider local stability, particularly the local stability around the equilibrium $x^* = 0$.

In many problems such as the continuous-time macroeconomic system (2.15)-(2.28), the function $f(x)$, and consequently the coefficient matrix $\bar{A}$ of the corresponding linearized system (3.2), depend on some parameters. In this case, write (3.2) in the following form

$$Dy = \bar{A}(\theta)y, \tag{3.3}$$

where $\theta \in \Theta$ is the vector of parameters taking values in the parameter space Θ. Since θ may change eigenvalues of $\bar{A}(\theta)$, the stability of (3.2) might depend on θ.

In systems theory, a bifurcation is said to occur if a system exhibits different structural properties such as stability when some parameter values are crossed. Bifurcation phenomena have been a subject of intensive research in many disciplines, see Guckenheimer and Holmes (1983). To study possible bifurcation phenomena in the continuous-time macroeconomic system (2.15)-(2.28), we consider its linearization. The parameter θ is chosen to be those that were estimated from real data:

$$\theta = [\beta_1, ..., \beta_{27}, \gamma_1, ..., \gamma_{33}, \lambda_1, \lambda_2, \lambda_3]'.$$

So $\theta \in R^{63}$ is a 63-dimensional column vector. The feasible region Θ is specified by the bounds of the parameters (see, Table 2 of Bergstrom et al. (1992). It is a bounded region.

The linearized system of (2.15)-(2.28) is

$$D^2 y_1 = -\gamma_1 D y_1 + \gamma_2 \left\{ \frac{Q^* y_4 + P^* y_{11}}{Q^* + P^*} - \beta_2 y_7 + (\beta_2 - \beta_3) D y_5 - y_1 \right\} \tag{3.4}$$

$$D^2 y_2 = -\gamma_3 D y_2 + \gamma_4 \left\{ \frac{(Q^*)^{-\beta_6} y_4 - \beta_5 (K^*)^{-\beta_6} y_3}{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}} - y_2 \right\} \tag{3.5}$$

$$D^2 y_3 = -\gamma_5 D y_3 + \gamma_6 \left\{ (1 + \beta_6)(y_4 - y_3) - \frac{y_7 - \beta_7 D y_5}{r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_8} \right\} \tag{3.6}$$

$$\begin{aligned} D^2 y_4 = -\lambda_7 D y_4 + \gamma_8 \Big\{ & - y_4 - \frac{\beta_9 (q^* p^* / p_i^*)^{\beta_{10}}}{1 - \beta_9 (q^* p^* / p_i^*)^{\beta_{10}}} \beta_{10} (y_5 + y_{14}) \\ & + \frac{C^* y_1 + g^*(Q^* y_4 + P^* y_{11}) + K^* D y_3 + K^*(\lambda_1 + \lambda_2) y_3 + E_n^* y_9}{C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*} \Big\} \end{aligned} \tag{3.7}$$

$$D^2 y_5 = \gamma_9 (D y_6 - D y_5) + \gamma_{10} \left\{ (1 + \beta_6) \frac{\beta_5 (Q^*/K^*)^{\beta_6}}{1 - \beta_5 (Q^*/K^*)^{\beta_6}} (y_4 - y_3) + y_6 - y_5 \right\} \tag{3.8}$$

$$D^2 y_6 = \gamma_{11} (D y_5 - D y_6) - \gamma_{12} (D y_5 + D y_{14}) + \gamma_{13} \frac{(Q^*)^{-\beta_6} y_4 - \beta_5 (K^*)^{-\beta_6} y_3}{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}} \tag{3.9}$$

$$\begin{aligned} D^2 y_7 = -\gamma_{14} D y_7 + \gamma_{15} \Big\{ & - \beta_{14} D y_{14} - y_7 \\ & + \frac{\beta_{15}}{M^*} [(Q^* + P^*) p^* (y_5 - y_{13}) + p^* (Q^* y_4 + P^* y_{11})] \Big\} \end{aligned} \tag{3.10}$$

$$\begin{aligned} D^2 y_8 = \gamma_{16} (D y_5 + D y_{14} - D y_8) + \gamma_{17} \Big\{ & (1 + \beta_{10})(y_5 + y_{14}) - y_8 \\ & + \frac{C^* y_1 + g^*(Q^* y_4 + P^* y_{11}) + K^*(\lambda_1 + \lambda_2) y_3 + K^* D y_3 + E_n^* y_9}{C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*} \Big\} \end{aligned} \tag{3.11}$$

$$D^2 y_9 = -\gamma_{18} D y_9 - \gamma_{19} \{ \beta_{18} (y_5 + y_{14}) + y_9 \} \tag{3.12}$$

$$D^2 y_{10} = -[\gamma_{20} + 2(\lambda_1 + \lambda_2)]Dy_{10} + \frac{\gamma_{21}\beta_{19}}{F^*}[Q^*(y_4 - y_{10}) + P^*(y_{11} - y_{10})] \tag{3.13}$$

$$D^2 y_{11} = -[\gamma_{22} + 2(\lambda_1 + \lambda_2)]Dy_{11} + \gamma_{23}[\beta_{20} + \beta_{21}(r_f^* - \lambda_4)]\frac{K_a^*}{P^*}(y_{12} - y_{11}) \tag{3.14}$$

$$D^2 y_{12} = -[\gamma_{24} + 2(\lambda_1 + \lambda_2)]Dy_{12} + \gamma_{25}\Big\{ -\beta_{24}\frac{Q^*+P^*}{K_a^*}Dy_{14} - \beta_{23}\frac{Q^*+P^*}{K_a^*}y_7$$

$$+[\beta_{22}+\beta_{23}(r_f^*-r^*)-\beta_{24}(\lambda_1+\lambda_2+\lambda_4-\lambda_3)]\frac{Q^*(y_4-y_{12})+P^*(y_{11}-y_{12})}{K_a^*}\Big\} \tag{3.15}$$

$$D^2 y_{13} = -\gamma_{26}Dy_{13} - \gamma_{27}y_{13}$$

$$+\gamma_{28}\Big\{\frac{E_n^*Dy_9 + P^*Dy_{11} - F^*Dy_{10}}{E_n^* + P^* - F^*} + Dy_5 + Dy_{14} - Dy_8\Big\}$$

$$+\gamma_{29}\Big\{\frac{E_n^*y_9+P^*y_{11}-F^*y_{10}-K_a^*(\lambda_1+\lambda_2)y_{12}-K_a^*Dy_{12}}{E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)}+y_5+y_{14}-y_8\Big\} \tag{3.16}$$

$$D^2 y_{14} = -\gamma_{30}(Dy_5 + Dy_{14}) - \gamma_{31}(y_5 + y_{14})$$

$$+\gamma_{32}\Big\{\frac{E_n^*Dy_9 + P^*Dy_{11} - F^*Dy_{10}}{E_n^* + P^* - F^*} + Dy_5 + Dy_{14} - Dy_8\Big\}$$

$$+\gamma_{33}\Big\{\frac{E_n^*y_9 + P^*y_{11} - F^*y_{10} - K_a^*(\lambda_1 + \lambda_2)y_{12} - K_a^*Dy_{12}}{E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)} + y_5 + y_{14} - y_8\Big\} \tag{3.17}$$

or in matrix form

$$\dot{x} = A(\theta)x \tag{3.18}$$

where

$$x = [y_1 \;\; Dy_1 \;\; y_2 \;\; Dy_2 \;\; \ldots \;\; y_{14} \;\; Dy_{14}]' \in R^{28}$$

and $A(\theta) \in R^{28\times 28}$ is the coefficient matrix. For the set of estimated values of $\{\beta_i\}$, $\{\gamma_j\}$, and $\{\lambda_k\}$ given in Table 2 of Bergstrom et al. (1992), all the eigenvalues of $A(\theta)$ are stable (having negative real parts) except three

$$s_1 = 0.0033, \; s_2 = 0.0090 + 0.0453i, \; s_3 = 0.0090 - 0.0453i,$$

where $i = \sqrt{-1}$ is the imaginary unit. However, the real parts of the unstable eigenvalues are so small that it is unclear whether they are caused by errors in estimation or by the structural properties of the system itself.

Note that the system (2.15)-(2.28), or the linearized system (3.4)-(3.17), operates in locally unstable region. We are interested in locating the stable region and the boundary. Our approach is to first find a stable sub-region of Θ and then expand the sub-region to find its boundary. Such boundary is the bifurcation boundary. To this end, we next find a parameter vector $\theta^* \in \Theta$ such that (3.18) is stable. From this θ^* we will find the stable region of θ and the boundary of bifurcations. We use the gradient method to find a θ^* such that all eigenvalues of $A(\theta^*)$ have strictly negative real parts.

To find such a θ^*, we consider the following problem of minimizing the maximum real parts of matrix $A(\theta)$:

$$\min_{\theta \in \Theta} R_{\max}(A(\theta)) \tag{3.19}$$

where

$$R_{\max}(A(\theta)) = \max_i \{\text{real}(\lambda_i) : \lambda_1, \lambda_2, ..., \lambda_{28} \text{ are eigenvalues of } A(\theta)\}.$$

Since the dimension of A is 28 which is relatively high, it is infeasible to have a closed-form expression for $R_{\max}(A(\theta))$. We use the gradient method to solve the minimization problem (3.19).

Consider the following recursive algorithm. Let θ_0 be the estimated set of parameter values given in Table 2 of Bergstrom et al. (1992). At step n, $n \geq 0$, with θ_n, let

$$\theta_{n+1} = \theta_n - a_n \frac{\partial R_{\max}(A(\theta))}{\partial \theta}|_{\theta=\theta_n}$$

where $\{a_n, n = 0, 1, 2, ...\}$ is the sequence of (positive) step sizes. After several iterations (20 iterations in this case), the algorithm arrived at the following θ^*:

$$\begin{aligned}\theta^* = [&0.9400, 0.2256, 2.3894, 0.2030, 0.2603, 0.1936, 0.1829, 0.0183, 0.2470,\\ &-0.2997, 1.0000, 23.5000, -0.0100, 0.1260, 0.0082, 13.5460, 0.4562,\\ &1.0002, 0.0097, 0.0049, 0.2812, -0.1000, 44.9030, 0.1431, 0.0004,\\ &71.4241, 0.8213, 3.9998, 0.8973, 0.6698, 0.0697, 0.1064, 0.0010,\\ &3.9901, 0.3652, 1.0818, 0.0081, 3.5988, 0.6626, 0.1172, 0.8452,\\ &0.0421, 1.4280, 0.3001, 3.9969, 3.6512, 3.9995, 4.0000, 3.9995,\\ &3.9410, 0.5861, 0.0040, 0.7684, 0.0427, 0.1183, 0.0708, 2.3187,\\ &0.1659, 0.0017, 0.000, 0.0100, 0.0100, 0.0067].\end{aligned}$$

The corresponding $R_{\max}(A(\theta^*)) = -0.0039$. Therefore, all eigenvalues of $A(\theta^*)$ have strictly negative real parts and the system (3.18) is stable at θ^*. Starting from this stable point, in the next section we will find the stable region of the parameter space and the bifurcation boundaries.

4 Determination of Bifurcation Boundaries

The goal of this section is to find bifurcation boundaries. Since the linearized system (3.18) only determines the local stability of (2.15)-(2.28), we are dealing with local bifurcations, as opposed to global bifurcations.

The system (2.15)-(2.28) can be written as

$$Dx = A(\theta)x + F(x, \theta) \tag{4.1}$$

where $F(x, \theta) = O(x^2)$ includes terms of higher orders.

On one hand, we have seen in the previous section that $A(\theta)$ has three eigenvalues with strictly positive real parts for the set of parameter values given in Table 2 of Bergstrom et al. (1992). On the other hand, all eigenvalues of (3.18) have strictly negative real parts for $\theta = \theta^*$. Since eigenvalues are continuous functions of entries of $A(\theta)$, there must exist at least one eigenvalue of $A(\theta)$ with zero real part on the bifurcation boundary. Different types of bifurcations may occur according to the way unstable eigenvalues are created. Two main types of bifurcations are considered in this paper: the saddle-node bifurcations and Hopf bifurcations.

Saddle-node bifurcations

A saddle-node bifurcation occurs when a system has a nonhyperbolic equilibrium with a geometrically simple zero eigenvalue at the bifurcation point and additional transversality conditions are satisfied (given by the Sotomayor's Theorem, see Sotomayor (1973)).

When $det(A(\theta)) = 0$, $A(\theta)$ has at least one zero eigenvalue. If $A(\theta)$ has exactly one simple zero eigenvalue (with multiplicity one), under additional technical transversality conditions, this point corresponds to a saddle-node bifurcation. So the first condition we are going to use to find the bifurcation boundary is

$$det(A(\theta)) = 0. \tag{4.2}$$

Note that $A(\theta)$ is a sparse matrix. Analytical forms of bifurcation boundaries can be obtained for most parameters. To demonstrate the feasibility of this approach, we consider finding the bifurcation boundaries for β_2 and β_5.

Theorem 1. The bifurcation boundary for β_2 and β_5 is determined by

$$1.36\beta_2\beta_5 + 21.78\beta_5 - 2.05\beta_2 - 10.05 = 0. \tag{4.3}$$

Proof. Denote

$$A(\theta) = [a_{i,j}].$$

We know from (3.4)-(3.17) that only the following entries of $A(\theta)$ are functions of β_2 and β_5. All other entries do not depend on β_2 and β_5.

$$a_{2,10} = \gamma_2(\beta_2 - \beta_3), \quad a_{2,13} = -\gamma_2\beta_2,$$

$$a_{4,5} = -\gamma_4 \frac{(K^*)^{-\beta_6}\beta_5}{(Q^*)^{-\beta_6} - (K^*)^{-\beta_6}\beta_5}, \quad a_{4,7} = \gamma_4 \frac{(Q^*)^{-\beta_6}}{(Q^*)^{-\beta_6} - (K^*)^{-\beta_6}\beta_5},$$

$$a_{10,5} = -\gamma_{10}(1+\beta_6)\frac{\beta_5(Q^*/K^*)^{\beta_6}}{1-\beta_5(Q^*/K^*)^{\beta_6}}, \quad a_{10,7} = \gamma_{10}(1+\beta_6)\frac{\beta_5(Q^*/K^*)^{\beta_6}}{1-\beta_5(Q^*/K^*)^{\beta_6}},$$

$$a_{12,5} = -\gamma_{13} \frac{(K^*)^{-\beta_6}\beta_5}{(Q^*)^{-\beta_6} - (K^*)^{-\beta_6}\beta_5}, \quad a_{12,7} = \gamma_{13} \frac{(Q^*)^{-\beta_6}}{(Q^*)^{-\beta_6} - (K^*)^{-\beta_6}\beta_5}.$$

Setting parameter values at θ^* except for β_2 and β_5, we obtain from direct calculation that

$$det(A) = -4.63 \frac{-1.005\times 10^{-15} - 2.05\times 10^{-16}\beta_2 + 2.178\times 10^{-15}\beta_5 + 1.36\times 10^{-16}\beta_2\beta_5}{0.48 - 0.32\beta_5}.$$

Hence, (4.3) immediately follows from setting $det(A(\theta)) = 0$. □

The boundary (4.3) is illustrated as the dashed line in Figure 1.

Hopf bifurcations

A Hopf bifurcation occurs at points where the system has a nonhyperbolic equilibrium connected with a pair of purely imaginary eigenvalues, but non-zero eigenvalues, and additional transversality conditions are satisfied, according to the Hopf Theorem, see Guckenheimer and Holmes (1983).

Consider the case of $det(A(\theta)) \neq 0$ but $A(\theta)$ has at least one pair of pure imaginary eigenvalues (with zero real parts and non-zero imaginary parts.) If $A(\theta)$ has exactly one such pair, and under some additional transversality conditions, this point corresponds to the Hopf bifurcation.

To find Hopf bifurcation points, let $p(s) = det(sI - A(\theta))$ be the characteristic polynomial of $A(\theta)$ and express it as

$$p(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3 + \ldots + c_{n-1}s^{n-1} + s^n$$

where $n = 28$ for the system (3.18). Construct the following $(n-1)$ by $(n-1)$ matrix

$$S = \begin{bmatrix} c_0 & c_2 & \ldots & c_{n-2} & 1 & 0 & 0 & \ldots & 0 \\ 0 & c_0 & c_2 & \ldots & c_{n-2} & 1 & 0 & \ldots & 0 \\ & \ldots & & & & & & \ldots & \\ 0 & 0 & \ldots & 0 & c_0 & c_2 & c_4 & \ldots & 1 \\ c_1 & c_3 & \ldots & c_{n-1} & 0 & 0 & & \ldots & 0 \\ 0 & c_1 & c_3 & \ldots & c_{n-1} & 0 & 0 & \ldots & 0 \\ & \ldots & & & & & & \ldots & \\ 0 & 0 & & \ldots & 0 & c_1 & c_3 & \ldots & c_{n-1} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} \frac{n-2}{2} \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} \frac{n}{2} \text{ rows} \end{matrix}$$

Let S_0 be obtained by deleting rows 1 and n/2 and columns 1 and 2, and let S_1 be obtained by deleting rows 1 and n/2 and columns 1 and 3. Then the following theorem of Guckenheimer et al. (1997) gives a condition for $A(\theta)$ to have exactly one pair of pure imaginary eigenvalues.

Theorem 2. The matrix $A(\theta)$ has exact one pair of pure imaginary eigenvalues if

$$det(S) = 0, \quad det(S_0) * det(S_1) > 0.$$

If $det(S) \neq 0$ or if $det(S_0) * det(S_1) < 0$, $A(\theta)$ has no pure imaginary eigenvalues. □

Therefore, the second condition of bifurcation boundary is

$$det(S) = 0, \quad det(S_0) * det(S_1) > 0. \tag{4.4}$$

We will use (4.4) to find candidates of bifurcation boundaries and then check which segments are the true boundaries.

In principle, the approach outlined in the proof of Theorem 1 can also be applied to find boundaries for Hopf bifurcations. However, in most cases, direct calculation of $det(S)$ is prohibitive. The following numerical procedure could be used to find the bifurcation boundaries. For the sake of simplicity, we only consider two parameters here, say θ_1 and θ_2.

Procedure (P1)

(1) For any fixed θ_1, we treat θ_2 as a function of θ_1 and find the θ_2 satisfying the condition (4.2), i.e., $h(\theta_2) = det(A(\theta)) = 0$. First find the number of zeros of $h(\theta_2)$. Starting with approximations of zeros, use the following gradient algorithm to find all zeros of $h(\theta_2)$.

$$\theta_2(n+1) = \theta_2(n) - a_n h(\theta_2)|_{\theta_2=\theta_2(n)} \tag{4.5}$$

where $\{a_n, n = 0, 1, 2, ...\}$ is a sequence of positive step sizes.

(2) Repeat the same procedure to find all θ_2 satisfying (4.4).

(3) Plot all the pairs of (θ_1, θ_2).

(4) Check all parts of the plot to find the segments representing the bifurcation boundaries. Then, parts of the curves found in (1) are boundaries of saddle-node bifurcations while parts of the curves found in (2) are boundaries of Hopf bifurcations (if the required transversality conditions are satisfied.)

5 Case Studies

In this section, the numerical **Procedure (P1)** is used to find explicit bifurcation boundaries for several sets of parameters. In order to be able to view the boundaries, we only consider two or three parameters. The procedure is applicable to any number of parameters.

Case I: β_2 and β_5

We first find the bifurcation boundaries for β_2 and β_5 for the system (3.18). Assume that other parameters operate at θ^*. The result is illustrated in Figure 1 in which the dashed line is given by $det(A(\theta)) = 0$, the solid line is the set of parameter pairs satisfying (4.4). The shaded area shows the stable region. All other regions give unstable system (3.18). It can also be seen from Figure 1 that the segment of the dashed line defining the stable region is the boundary of saddle-node bifurcations while the other segment of the same line is not a bifurcation boundary at all. Similarly, the segment of the solid line defining the stable region is the boundary of Hopf bifurcations, that is, Hopf bifurcations occur when parameter values cross this line. The other part of the solid line is not a bifurcation boundary. The stability behavior of (3.18) along the bifurcation boundaries is unclear and is a subject of ongoing research.

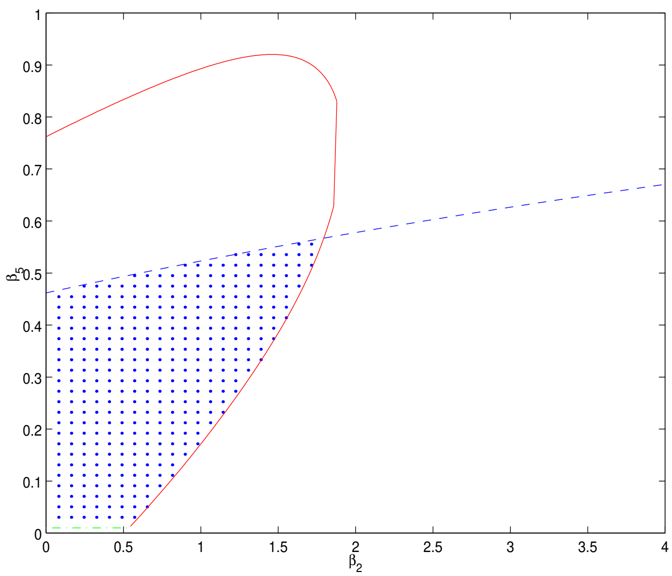
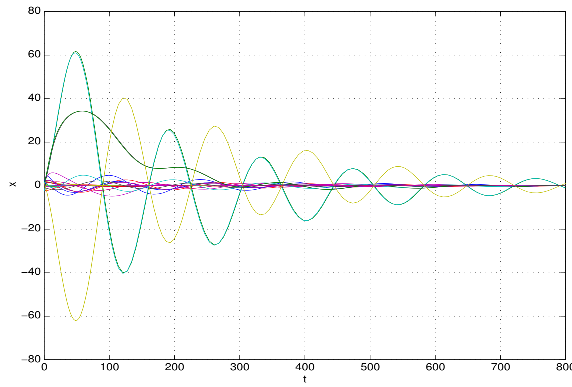


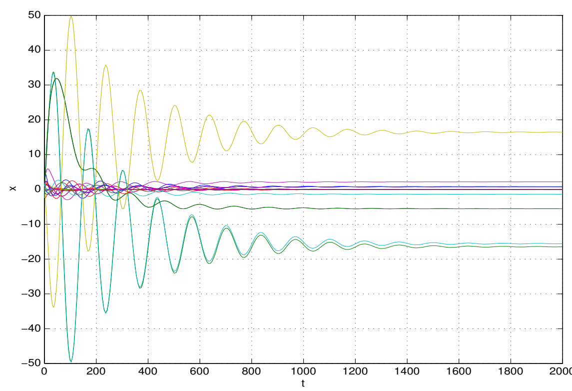
Figure 1. Bifurcation boundaries for β_2 and β_5.

Of particular interest is the cross point of the two bifurcation boundaries which is approximately $(\beta_2, \beta_5) = (1.785, 0.566)$. At this point the coefficient matrix has three eigenvalues with zero real parts: $s_1 = 0.0000, s_2 = -0.0000 + 0.0336i, s_3 = -0.0000 - 0.0336i$.

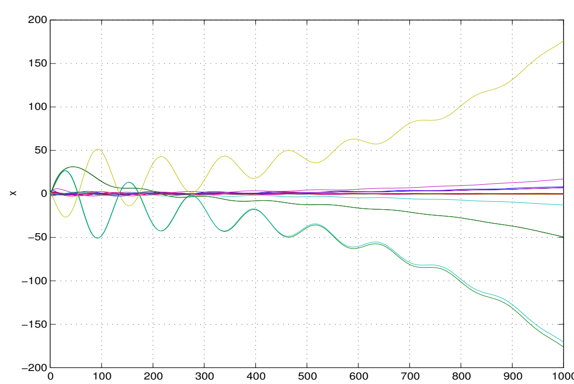
Figures 2-5 illustrate the trajectories of x and phase portraits of (x_1, x_{10}, x_{27}) when the parameters cross the two boundaries.



(a) (β_2, β_5) in stable region

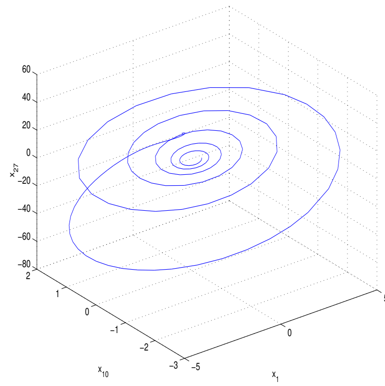


(b) (β_2, β_5) on the saddle-node bifurcation boundary

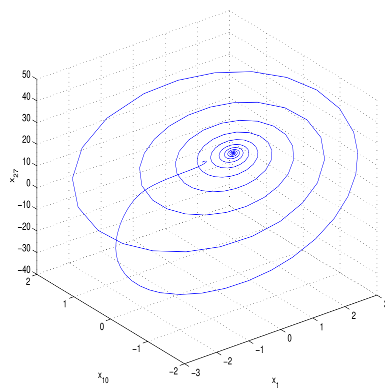


(c) (β_2, β_5) in unstable region after crossing the saddle-node bifurcation boundary

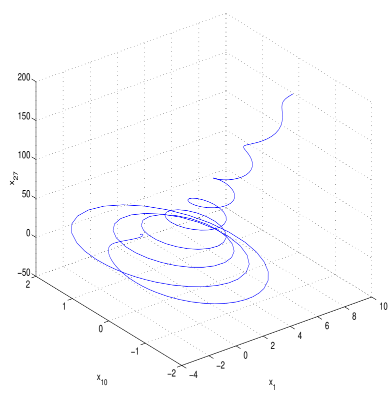
Figure 2. Trajectories when $(\beta_2,\, \beta_5)$ crossing the saddle-node bifurcation boundary



(a) (β_2, β_5) in stable region

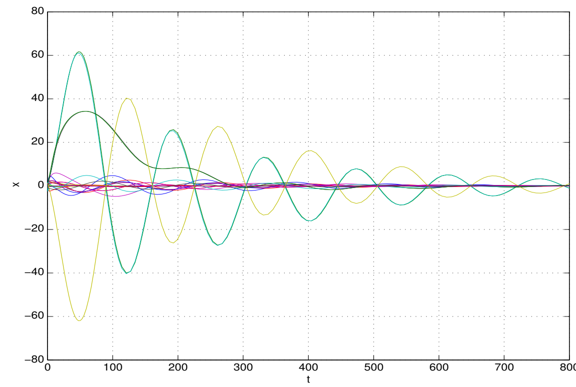


(b) (β_2, β_5) on the saddle-node bifurcation boundary

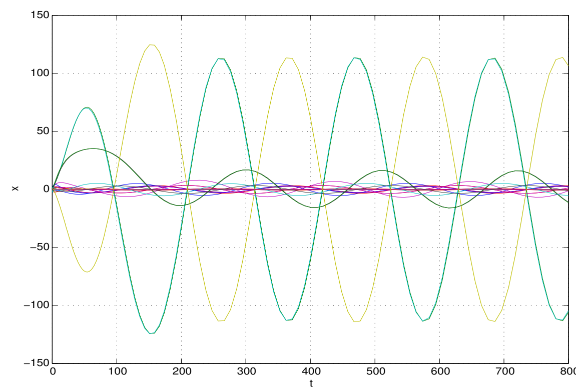


(c) (β_2, β_5) in unstable region after crossing the saddle-node bifurcation boundary

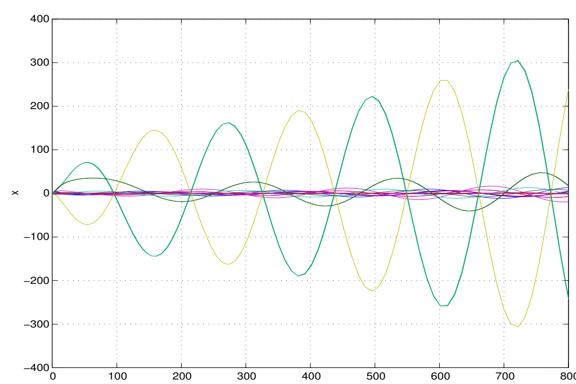
Figure 3. Phase portrait of (x_1, x_{10}, x_{27}) when $(\beta_2,\ \beta_5)$ crossing the saddle-node bifurcation boundary



(a) (β_2, β_5) in stable region

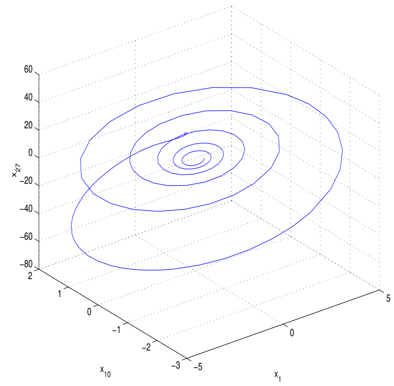


(b) (β_2, β_5) on the Hopf bifurcation boundary

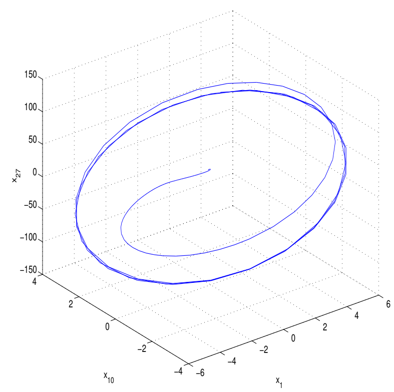


(c) (β_2, β_5) in unstable region after crossing the Hopf bifurcation boundary

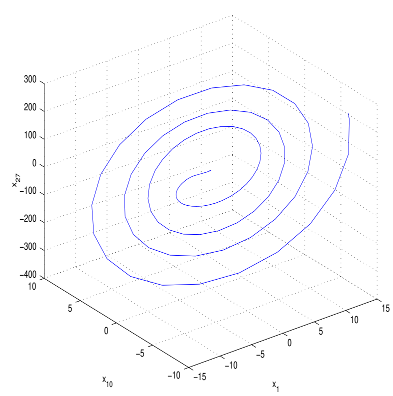
Figure 4. Trajectories when $(\beta_2,\ \beta_5)$ crossing the Hopf bifurcation boundary



(a) (β_2, β_5) in stable region



(b) (β_2, β_5) on the Hopf bifurcation boundary



(c) (β_2, β_5) in unstable region after crossing the Hopf bifurcation boundary

Figure 5. Phase portrait of (x_1, x_{10}, x_{27}) when $(\beta_2,\ \beta_5)$ crossing the Hopf bifurcation boundary

Case II: β_2, β_5, and β_{15}

We now add the parameter β_{15} into Case I. Use again **Procedure (P1)**, we find the surface of the bifurcation boundary for β_2, β_5, and β_{15} as shown in Figure 6.

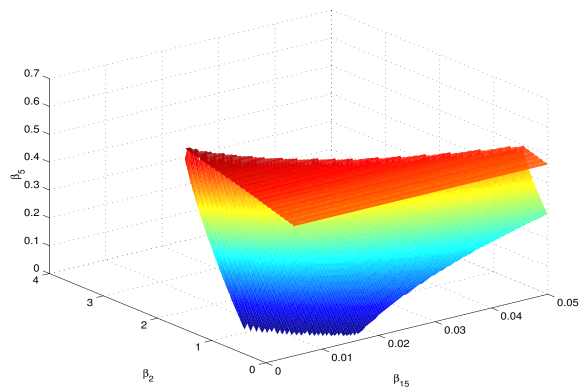


Figure 6. Bifurcation boundary for β_2, β_5 and β_{15}.

Case III: γ_8 and β_{15}

In this case, we find bifurcation boundaries for parameters γ_8 and β_{15}. Assume that other parameters operate at θ^*. The result is illustrated in Figure 7 in which only Hopf bifurcations occur. The shaded area shows the stable region.

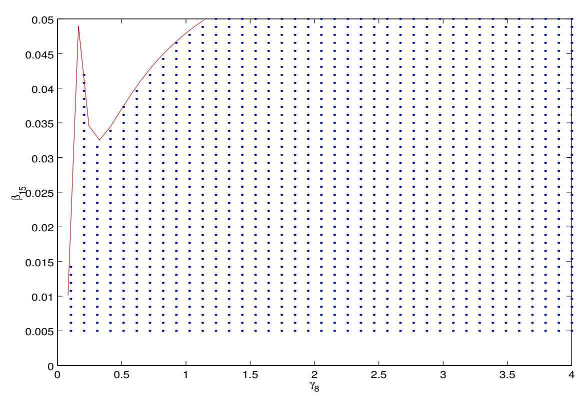
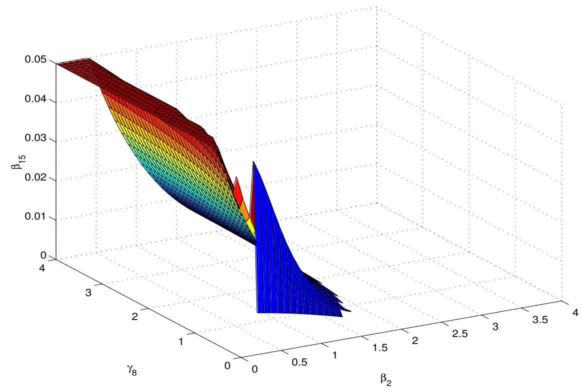


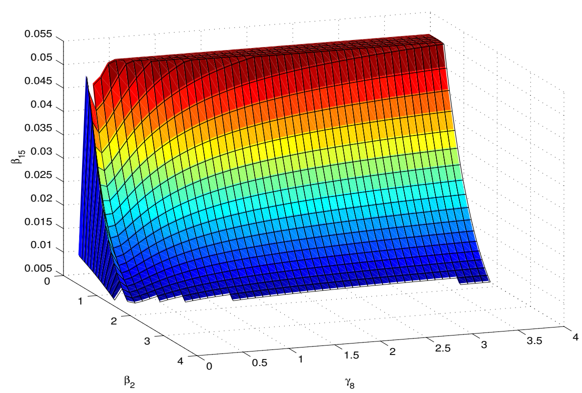
Figure 7. Bifurcation boundary for γ_8 and β_{15}.

Case IV: γ_8, β_{15}, and β_2

In this case, we consider the three dimensional bifurcation boundary for γ_8, β_{15}, and β_2. Similar to Case III, only Hopf bifurcation occurs for the three parameters. The following figure illustrates the boundary viewed from two different directions.



(a)



(b)

Figure 8. Bifurcation boundary for γ_8, β_{15} and β_2.

6 Conclusion

We have found that bifurcations exist within the plausible range of parameter values for the Bergstrom, Nowman and Wymer continuous-time macroeconometric model, and we have successfully located and drawn the bifurcation boundaries. A trajectory simulation of the linearized model for different settings of the parameter values shows that the behavior of the system is consistent with the prediction of bifurcation theory. We confirm this finding at parameter settings within the stable region, the unstable region, and on the bifurcation boundary.

This paper reports on the first results from an ongoing research project. Based upon our current results, we now plan to explore further cases of system behavior when the parameters are set exactly on the bifurcation boundaries. We also plan to investigate whether any of the parameter settings within the unstable region can support chaos. In short, the current results are only a first step, but are critical as motivation for the future research we now contemplate.

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