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Stability analysis of finite difference schemes for inertial oscillations in ocean general circulation models

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Abstract

Numerical time finite difference schemes in widely used ocean general circulation models are systematically examined to ensure the correct and accurate discretization of the Coriolis terms. Two groups of numerical schemes are categorized. One group is suitable for simulating an inertial wave in the ocean with the necessary condition for stability being $F=f\Delta t<1$, where f is the Coriolis parameter and Δt is the integration time step in the model, such as the predictor-corrector Euler scheme, centred difference (leapfrog) scheme, semi-implicit Euler schemes, and leapfrog scheme with a semi-implicit approach. The other group is able to serve as a long-term climate study using a large integration time step which violates $F=f\Delta t<1$ by damping out inertial waves, such as the Cox-Bryan and Oberhuber implicit approaches. Caution should be made in using the Euler forward and other schemes that produce unstable inertial waves; this problem could be serious for a calculation longer than a week. The predictor-corrector scheme is suggested to replace the simple Euler forward scheme. The explicit schemes tend to overestimate the phase frequency, whereas the implicit schemes underestimate it. To better simulate the correct phase frequency (i.e. speed), $F<0.1$ is suggested.

1 Introduction

The inertial oscillation is a ubiquitous feature in the ocean. It generally produces anti-cyclonic oscillatory ocean currents in both hemispheres. In studies of Ekman dynamics due to wind forcing [1], inter-tidal currents, interactions among wind-induced storms, tides and inertial oscillations, and geostrophic adjustment in the ocean and coastal seas, inertial motion plays an important role.

In primitive-equation ocean general circulation models, the inertial mode naturally exists because the Coriolis terms are explicitly included. The Coriolis force not only physically produces a rotating wave and current system, but also introduces a numerical stability constraint ($F<1$, the so-called the inertial constraint) in the finite difference equations. This is the difference between geophysical fluid dynamics and common fluid mechanics with no Coriolis force. Thus, we ask how well a finite difference scheme captures the real physics of this phenomenon (or reproduces the solution of the corresponding differential equation) in the ocean.

When the non-linear terms must be correctly discretized to conserve energy and mass in finite difference schemes in terms of numerical stability, we should pay more attention to the dominant terms in the dynamic equations of interest. In large-scale ocean dynamics, for instance, the geostrophic balance is the first-order approximation in which the pressure gradient and Coriolis terms are comparable and dominant. In coastal, small-scale, tidal dynamics, the inertial-gravity waves, the rotating Poincaré waves and inertial oscillations are the major physical processes in which the pressure gradient, Coriolis, the time derivative and dissipation terms are dominant. Thus, in any numerical model, these four terms (pressure gradient, Coriolis, time derivative, and diffusion terms) must be correctly and precisely differenced to maintain the physics in the original continuum medium. The purpose of this study is to address which numerical time difference scheme is correctly and accurately formulated to simulate the inertial oscillations in the open ocean or to simulate the storm-tide-wind interactions in coastal seas. If researchers are going to use a model as a tool to study and simulate coastal processes, which scheme should they choose?

In this study, we emphasize stability analysis of different numerical schemes in time (rather than in space) and phase error for the inertial oscillation which have been employed in the Cox-Bryan model (a semi-implicit approach for the Coriolis terms with a leapfrog time differencing), the Blumberg-Mellor model (an explicit centred differencing for the Coriolis terms with a leapfrog scheme in time), the Dietrich et al. model (a semi-implicit approach for the Coriolis terms with an Euler forward time differencing), and the Oberhuber model (an implicit approach for the Coriolis terms with a three-time step predictor-corrector scheme in time). As well, we study different approaches such as the explicit Euler forward scheme and a predictor-corrector scheme developed in this study to replace the Euler forward scheme.

In section 2, we start by obtaining the analytical solution for inertial oscillations in the ocean, set up different numerical schemes and perform the stability and phase error analyses. In section 3, we compare the numerical solutions from the different schemes with the analytical solutions to quantify each scheme. Finally, we summarize our findings in section 4.

2 Stability and Phase Frequency of Time Finite Difference Schemes

2.1 Governing Equations and Analytical Solution. To focus on finite difference schemes of the Coriolis terms, we choose the linear momentum equations for a one-layer system without any horizontal variability. The equations contain the Rayleigh friction terms as well:

$$\frac{\partial u}{\partial t} - fv = -ru, \qquad \frac{\partial v}{\partial t} + fu = -rv, \tag{1}$$

or using a complex form

$$\frac{\partial w}{\partial t} + (if + r)\, w = 0, \tag{2}$$

where u, v, t, f and r are horizontal velocities in x and y directions, time, the Coriolis parameter, and the linear bottom friction coefficient in units of s^{-1}, respectively, and w=u+iv is the complex velocity with $i=(-1)^{1/2}$. The analytical solutions for u and v subject to the initial conditions, $u_0=1$ and $v_0=0$ ($w_0=1$), are $(u, v)=e^{-rt}[\cos(ft), \sin(ft)]$, or $w=\exp(-rt+i\omega_E t)$, where $\omega_E=-f$ is the exact frequency of the inertial wave. The analytical inertial oscillations (with amplitude of unity if r=0) will be compared with the numerical results using finite difference schemes in both inviscid (r=0) and viscous (r≠0) cases.

2.2 Euler Forward, Semi-Implicit, and Implicit (Euler Backward, Oberhuber Type) Schemes. In some numerical models, eq. (2) is differenced by the values at two time steps using the following scheme:

$$w^{n+1} = (1-R)\, w^{n} - iF[\beta w^{n+1} + (1-\beta)\, w^{n}], \tag{3}$$

where $0 \le \beta \le 1$, $F=f\Delta t$, $R=r\Delta t$, and n and n+1 denote old and new time steps, respectively. The different finite difference equations in component form in this paper were all given in Appendices A-D of Wang and Ikeda [7]. Note that this scheme becomes an Euler forward if β=0, semi-implicit [5] if β=0.5, and Euler backward or implicit scheme [6] if β=1, respectively. The model becomes the inviscid or contains free inertial motion when r=0 (no dissipation).

To obtain the numerical stability criterion, we assume the solution to have a Fourier wave form following Wang [8, see his Appendix A]

$$w = De^{i\omega n\Delta t} = D\lambda^{n}, \tag{4}$$

where $\lambda=\exp(i\omega\Delta t)$, i, ω, Δt, and n are the eigenvalue, the imaginary number, phase (or angular) frequency, integration time step, and time at n, respectively; and $D=U_0+iV_0$ is the constant amplitude of the velocities. If $|\lambda|$ =, <, >1, then the numerical solution is neutral, decaying, growing (unstable) in amplitude. Thus, the stability problem turns to the estimate of the magnitude of the eigenvalue, λ. The phase frequency ω will be analytically derived as well.

Substituting (4) into (3) gives the solution as follows

$$\lambda=\frac{[(1-R)-F^2\beta(1-\beta)]-iF(1-\beta R)}{1+F^2\beta^2} . \tag{5}$$

We observe that if the Coriolis force which introduces truncation error to the finite difference equations is neglected (i.e. F=0), then the true solution to λ of the finite difference equations is $|\lambda|$=1-R for r≠0 and unity for r=0.

If β=0, the scheme becomes the Euler forward scheme. Taking the absolute value or modulus of (5) gives $|\lambda|=(1+F^2)^{1/2}>1$, if r=0; and $|\lambda|=[(1-R)^2+F^2]^{1/2}$, if r≠0. Now we can see that the Euler forward scheme is unconditionally unstable for inertial oscillations if there is no friction (r=0). However, with r≠0, the model solution might be stable if dissipation of the numerical model is taken to be large enough to damp out the numerical instability due to improper discretization of the Coriolis terms only if $1-(1-F^2)^{1/2} \leq R \leq 1+(1-F^2)^{1/2}$ [the solution is obtained by solving the quadratic equation on R: $(1-R)^2+F^2\leq 1$]. If F^2 is much less than unity, the stability condition is $F^2/2\leq R\leq 2-F^2/2$. The lower limit ($R\geq F^2/2$) gives small but effective friction. Once $f=10^{-4}$ s^{-1} and Δt=240 s are chosen, associated with this friction is the e-folding time scale $T\sim r^{-1}$ s~9.6 days. The large viscosity necessary in the finite difference equations may ruin a physical process with time scale longer than 10 days, such as mesoscale eddies with time scales from two weeks to months by seriously smoothing gravity waves and Rossby waves, etc. Thus, a proper numerical scheme must be carefully examined to replace this unstable scheme which will be discussed shortly.

If β=0.5 and 1, the scheme becomes the semi-implicit one [5] and the Euler backward or implicit one [6]. The similar analysis was applied, and the results and the frequency error were listed in Table 1.

In summary, the Euler forward scheme (β=0) is unconditionally unstable for the inertial mode in the absence of friction: the larger the time step, the faster the model blows up, as will be discussed in the next section. By contrast, the Euler backward scheme (β=1) always dampens inertial waves [6]: the larger the time step, the faster the inertial oscillations are damped out. The semi-implicit scheme (β=0.5) is the most accurate [5].

2.3 Predictor-Corrector Scheme. In order to extend integration time of the Euler forward scheme without changing the two-time step algorithms, a predictor step and a corrector step are differenced respectively as follows:

$$\begin{aligned} w^{n+1*} &= (1-R)\,w^n-iFw^n, \\ w^{n+1} &= (1-R)\,w^n-iF[\beta w^{n+1*}+(1-\beta)\,w^n]\,, \end{aligned} \tag{6}$$

where the superscript star denotes the predicted value. Substituting the predictor step into the corrector step in (6) leads to

$$w^{n+1}=[(1-R)-F^2\beta-iF(1-R)\beta-iF(1-\beta)]\,w^n, \tag{7}$$

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Table 1: A summary of stability analysis and phase frequency errors of different time finite differencing schemes for inertial oscillations. $F=f\Delta t$ and $R=r\Delta t$.

Numerical Scheme	Parameter and its Scheme	Modulus of Eigenvalue: $\|\lambda\|=$ (r=0)	Amplitude Stability	Normalized Frequency $\omega/(-f)=$
Sec. 2.2: Euler Schemes	$\beta=0$: Euler Forward	$(1+F^2)^{1/2}>1$	Uncond. Unstable	F^{-1} arctan{ $F(1-\beta R)$ $/[1-R-\beta(1-\beta)F^2]$}
	$\beta=0.5$: semi-implicit	1	Neutral	
	$\beta=1$: Implicit or Euler Backward	$(1+F^2)^{-1/2}<1$	Serious Damping	
Sec. 2.3: Predictor-Corrector Scheme	$\beta=0$	$(1+F^2)^{1/2}>1$	Uncond. Unstable	F^{-1} arctan{ $F(1-\beta R)$ $/[1-R-\beta F^2]$ }
	$\beta=0.5$	$(1+F^4/4)^{1/2}>1$	Weakly Unstable	
	$\beta=.5+\varepsilon, \varepsilon=F^2/8$ $+O(F^4/16)$	1	Neutral	
	$\beta=1$	$(1+F^2+F^4)^{1/2}<$ 1, if $F<1$	Weak Damping	
Sec. 2.4: Leapfrog Scheme		1	Neutral	F^{-1} arcsin{ $F/(1-R)$ }
Sec. 2.5: Leapfrog Euler Scheme	$\beta=0$	$(1+4F^2)^{1/2}>1$	Uncond. Unstable	$(2F)^{-1}$ arctan{ $2F[(1-\beta)+\beta(1-2R)]/[1-2R-4\beta(1-\beta)F^2]$ }
	$\beta=0.5$	1	Neutral	
	$\beta=1$	$(1+4F^2)^{-1/2}<1$	Serious Damping	

where $0\leq\beta\leq 1$. To do the stability analysis, substituting (4) into (7) gives thefollowing equation for the eigenvalue: $\lambda=(1-R-\beta F^2)-iF(1-\beta R)$; and taking the absolute value or modulus gives $|\lambda|=[(1-R-\beta F^2)^2+F^2(1-\beta R)^2]^{1/2}$.

If r=0 we have the results shown in Table 1, where ε is a small positive number. The scheme with $\beta=0$ is unconditionally unstable, with the modulus being proportional to $1+F^2/2$, similar to the Euler forward scheme. The scheme with $\beta=1$ slightly damps out the inertial mode if $F<1$, and the amplitude retains unity (i.e. neutral) if $F=1$. Note that when $F>1$, this predictor-corrector scheme becomes an unstable scheme.

We can see that if $\beta=0.5$, the scheme is still weakly unstable, even though accuracy has been raised two orders higher than the Euler forward scheme. To ensure that the eigenvalues fall inside the unit circle [10], we obtain $\varepsilon=F^2/8+O(F^4/16)$ by solving the quadratic equation $1-(2\beta-1)F^2-\beta^2F^4=1$ on ε [7], where $\beta=1/2+\varepsilon$. For instance, if $f=10^{-4}$ s^{-1}, $\Delta t=10^3$ s, then $\varepsilon=0.0012$.

Thus $\beta=0.5+\varepsilon$ (=0.50012 in this example) ensures this scheme to be conditionally ($F\leq1$) neutral for inertial oscillations. Using this idea, we are able to modify the Euler forward scheme in a model to be a neutral scheme with the necessary conditions for stability being $F\leq1$ and $\beta=0.5+F^2/8+O(F^4/16)$.

2.4 Leapfrog or Centred (Blumberg-Mellor Type) Scheme. To complete this investigation, we include the centred (3-timestep) difference schemes used by the Blumberg-Mellor model [4] and the Cox-Bryan model [2,3] as discussed in the next section. Eq. (2) can be discretized using the leapfrog time scheme as follows: $w^{n+1}=(1-2R)w^{n-1}-i2Fw^n$. Using the same method by substituting (4) into this finite difference equation, we obtain the result listed in Table 1.

2.5 Cox-Bryan Scheme. Last, we discuss the numerical scheme used by the Cox-Bryan model [2,3]. Eq. (2) can be discretized using the averaged velocity values, between times n+1 and n-1, in the Coriolis terms as follows: $w^{n+1}=(1-2R)w^{n-1}-i2F[\beta w^{n+1}+(1-\beta)w^{n-1}]$. We note that the scheme with $\beta=0$, 0.5, 1 (the parameter α in the Cox-Bryan model [3]) corresponds to the Euler forward scheme, semi-implicit scheme and the fully implicit scheme [3], respectively. Substituting (4) into this finite difference equation and using the same method as before, we gives the results in Table 1.

2.6 Phase Frequency Errors. We have examined all the frequency errors introduced by the various finite difference schemes associated with the Coriolis terms [7]. Here we only present the results from the predictor-corrector scheme (Fig. 1). We see that the semi-implicit scheme ($\beta=0.5$) preserves the best phase information of inertial oscillation if $F<1$. The scheme with $\beta=0$ accelerates the phase frequency, whereas the scheme with $\beta=1$ slows down it with a jump at $F=1$ where the scheme generates a wave propagating in the opposite direction to the true wave ($\omega/\text{-}f<0$). Overall, the phase frequency can be well preserved when $F<0.1$. When $F\rightarrow\infty$, $\omega/\text{-}f\rightarrow0$, the simulated phase frequency is much slower than the true solution.

3 Comparisons between the Numerical and Exact Solutions

To compare the solutions from finite difference equations with the differential equations, we set, in both differential and finite difference equations, $f=10^{-4}$ s^{-1} and $r=0$ for the inviscid case and $r=2.5\times10^{-6}$ (10^{-7}) s^{-1} for the viscous case, which corresponds to the e-folding (decaying) time scale of 4.5 (45) days.

3.1 Inviscid Cases: r=0. We carried out the following runs using $\Delta t=240$ s, then $F=f\Delta t=0.024<1$. Fig. 2 shows the comparisons between the exact or differential solutions (solid lines) and finite difference or numerical solutions (dashed lines) for $\beta=0$ (upper panel) as discussed in section 2.2, which corresponds to the Euler forward scheme. Fig. 2 similarly shows the comparisons from the semi-implicit (middle panel) and implicit (lower panel) schemes, respectively. We clearly see that the Euler forward scheme ($\beta=0$) is unconditionally unstable, while the semi-implicit scheme ($\beta=0.5$) does the

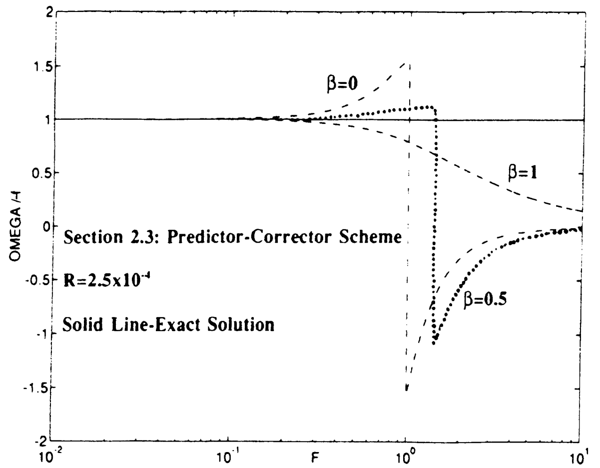

Fig. 1. The normalized phase speeds of the predictor-corrector scheme (the ratio of phase speed from the differencing scheme to that from the exact solution) against F=fΔt and β=0, 0.5 and 1. for R=rΔt=2.5x10^{-4}. The values lower/higher than the solid line (ω/-f) indicate the under/over-estimation by the numerical scheme. Note that the negative values indicate the waves propagating in the opposite direction to the true waves (true solutions).

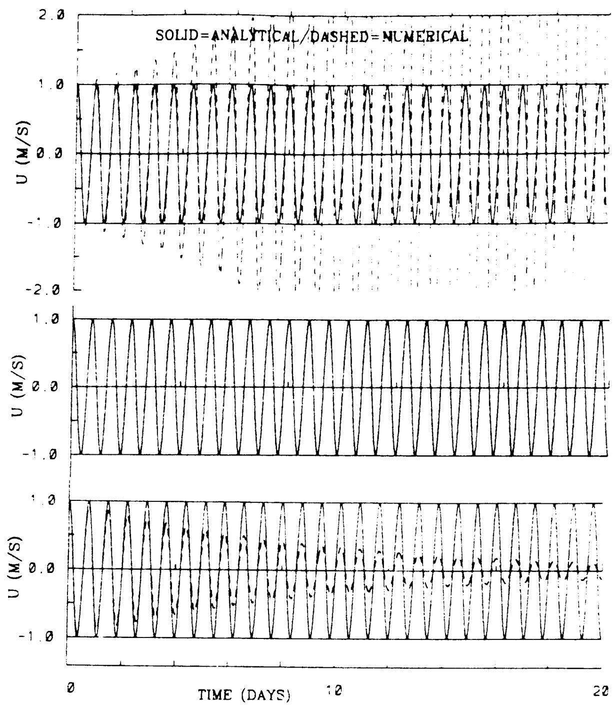

Fig. 2. The time series of inertial current velocity, u only, from the analytical (exact) solutions (solid curves) and numerical solutions (dashed curves) of the Euler forward scheme with β=0 (upper panel), semi-implicit scheme with β=0.5 (middle panel) and Euler forward scheme with β=1, as discussed in section 2.2. The integration time step is 240 s.

best job, and the implicit scheme [6] damps out inertial oscillations.

To emphasize the timestep-dependent growth of the Euler forward scheme, we conducted a series of 10-day runs using different time steps, Δt, equal to 10, 50, 100, 500 and 1000 seconds (see Table 2). We see that the larger the time step used in the Euler forward scheme, the faster the inertial oscillations grow. However, a time step of 10 seconds can be used for a simulation less than 10 days (Table 2).

Table 2. The timestep-dependent growth of the eigenvalue $\lambda^n=[(1-R)^2+F^2]^{n/2}$ at the end of a 10-day simulation using the Euler forward scheme, where $F=f\Delta t$ with $f=10^{-4}$ s^{-1}. The values in parentheses are the normalized eigenvalues by their true solutions $(1-R)^n$ or the normalized amplifications.

Total time steps (n)	86400	17280	8640	1728	864	
Time Step, Δt (sec)	10	50	100	500	1000	True solution
r=0	1.04	1.24	1.54	8.65	73.59	1.0
$r=2.5\times 10^{-7}$ s^{-1}	0.958 (1.04)	1.14 (1.24)	1.413 (1.54)	7.934 (8.65)	67.56 (73.66)	0.917
$r=2.5\times 10^{-6}$ s^{-1}	0.44 (1.04)	0.523 (1.24)	0.649 (1.54)	3.652 (8.67)	31.27 (74.23)	0.421

3.2 Viscous Cases: $r\neq 0$. To confirm that the above findings are still valid for the viscous fluid, we compare the results in the viscous case, which is closer to reality. In the following runs, we set $\Delta t=240$ s. Similar to the inviscid case, we also conducted the amplitude growth of inertial oscillations calculated from the Euler forward scheme using different time steps for both small and large viscosity cases: $r=2.5\times10^{-7}$ s^{-1} and $r=2.5\times10^{-6}$ s^{-1} (Table 2). As we see, the normalized amplitudes (amplification) are the same as those with r=0, independent of viscosity values.

4 Conclusions

Based on the careful investigations in the preceding sections, we can apply Table 1 as our summary:

1. To correctly simulate inertial waves and the geostrophic adjustment process in the ocean and coastal seas, only four types of numerical schemes are suggested to apply to this purpose. These are 1) the semi-implicit scheme for the Coriolis terms with forward differencing in time with $\beta=0.5$ [5], 2) the predictor-corrector scheme with $\beta=0.5+\varepsilon$ ($\varepsilon=F^2/8$), 3) the Blumberg-Mellor scheme [4], and 4) the Cox-Bryan scheme with $\beta=0.5$ only [2,3].
2. The Euler forward scheme with $\beta=0$, the predictor-corrector scheme with $\beta=0$ and the Cox-Byran scheme [3] with $\beta=0$, are unconditionally unstable and must be used with caution by modellers. We have developed

the predictor-corrector scheme with β>0.5 to replace the unstable Euler forward scheme.

3. For the climate study, the long integration time step violating F<1 in the model is usually used. Thus, the fully implicit numerical scheme for the Coriolis terms must be used to remove this inertial constraint (i.e. F<1) by damping inertial waves. Only two schemes, Cox-Bryan with β>0.5 [3] and Oberhuber [6] schemes, can serve this purpose.

4. To correctly model the phase frequency of the inertial wave, F<0.1 is strongly suggested. Generally speaking, the explicit schemes [4] overestimate the phase frequency while the implicit schemes [3,6] underestimate it when 0.1<F<1. The semi-implicit scheme [5] does the best job when F<1. When F becomes large, the simulated phase frequency is much slower than the true solution, i.e. $\omega/\text{-}f \rightarrow 0$.

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