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Stability Analysis of Fixed-Point Digital Filters using Computer Generated Lyapunov Functions- Part II: Wave Digital Filters and Lattice Digital Filters

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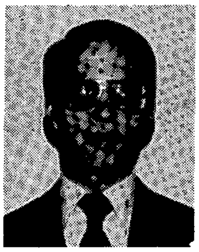

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Stability Analysis of Fixed-Point Digital Filters Using Computer Generated Lyapunov Functions—Part II: Wave Digital Filters and Lattice Digital Filters

KELVIN T. ERICKSON, MEMBER, IEEE, AND ANTHONY N. MICHEL, FELLOW, IEEE

***Abstract*—In a companion paper [4], we utilize the *constructive stability algorithm* of Brayton and Tong in the stability analysis of fixed-point digital filters which are in the direct form and in the coupled form. We continue this work in the present paper by considering wave digital filters and lattice digital filters. We believe that the results of the present paper and its companion paper demonstrate that the *constructive algorithm* constitutes an *effective* and *general* approach in the qualitative analysis of fixed-pointed digital filters.**

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I. Introduction

IN THE companion paper [4], we first showed how the constructive stability algorithm of Brayton and Tong [2], [3] may be applied in the stability analysis of rather broad classes of fixed-point digital filters which may be endowed with various types of quantization and overflow nonlinearities. We then considered, in particular, direct form digital filters and coupled form digital filters. Our objective was to determine a region in the parameter plane of a given digital filter for which the zero-input digital filter is globally asymptotically stable, and consequently, does not possess any zero-input limit cycles. The results in [4], which use only *one* approach of stability analysis, seem rather encouraging when compared to many of the existing

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corresponding qualitative results, which make use of a variety of diverse methods of analysis.

In this paper, we continue the stability analysis of fixed-point digital filters via the constructive stability algorithm by considering a class of wave digital filters and a class of lattice digital filters. As in [4], we attempt to compare the method of analysis advanced herein with other existing methods and results. Furthermore, we establish new stability results for some of the classes of filters which we consider. We believe that the results of the present paper, combined with those given in [4], demonstrate that the constructive algorithm constitutes indeed a very powerful tool in the qualitative analysis of second-order fixed-point digital filters.

Throughout this paper we employ the notation, preliminary results, and some of the main results established in [4]. Before reading this paper, it is essential that reference be made to sections II, III-A, and III-B of [4].

This paper consists of four sections. In Section II we develop the extreme matrices for the classes of filters which we consider, while in Section III we utilize the constructive algorithm in the stability analysis of these filters and we make comparisons with existing stability methods and results. The paper is closed in Section IV with appropriate concluding remarks.

II. Extreme Matrices for Wave Digital Filters and Lattice Digital Filters

In this section, which consists of two parts, we put the classes of filters which we consider into suitable forms to make it possible to apply the constructive stability algorithm. In Section II-A we deal with wave digital filters while in Section II-B we address lattice digital filters.

A. Wave Digital Filters

Wave digital filters are a class of low-sensitivity digital filter structures first advanced by Fettweis [5]. These structures can be synthesized from equally terminated LC analog filters by replacing the analog elements by appropriate digital realizations. Wave digital filters are either full-synchronic or half-synchronic. In a full-synchronic filter, the arithmetic operations are carried out, at least in principle, simultaneously at periodically recurring instants. In a half-synchronic filter, the various arithmetic operations are still carried out at the same rate, but do not take place simultaneously, even in principle. We only consider full-synchronic wave digital filters, since most conventional digital filters are full-synchronic. A general wave digital filter is characterized by an n-port network, as illustrated in Fig. 1. Since wave digital filters constitute a class of filters, we only consider a specific example of a wave filter synthesized from an LC network in the next subsection.

1) Specific Wave Digital Filters Considered: The wave digital filter structure which we will examine is based on a general second-order low-pass LC filter shown in Fig. 2. This section can represent many types of filters, e.g., Butterworth or Chebychev.

Following the synthesis procedure in Antoniou [1], we identify the series and parallel interconnection, as shown in Fig. 3(a). The wave digital filter is then formed with one parallel wire interconnection and one series wire interconnection, as in Fig. 3(b). The resulting structure, in terms of delays, adders, and multipliers is shown in Fig. 4. The state equations for the linear wave digital filter with zero input ($a_1 = a_2 = 0$) are

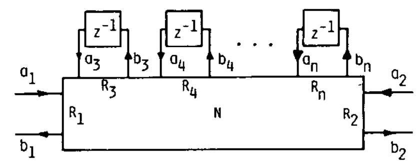

Fig. 1. General full-synchronic wave digital filter.

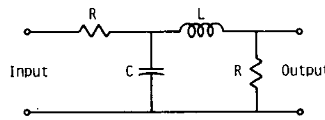

Fig. 2. General second-order LC low-pass analog filter.

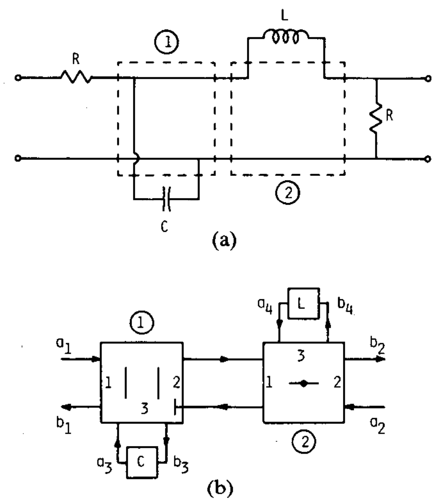

Fig. 3. Synthesis of second-order LC low-pass wave digital filter. (a) Identification of wire interconnections. (b) Wave digital filter.

$$x_1(k+1) = c_{11}x_1(k) + c_{12}x_2(k)$$
$$x_2(k+1) = c_{21}x_1(k) + c_{22}x_2(k)$$

where,

$$\begin{aligned} c_{11} &= -1 - m_1(2+m_2) \\ c_{12} &= m_2 \\ c_{21} &= -m_1(2+m_2+m_3) \\ c_{22} &= 1 + m_2 + m_3 \end{aligned} \tag{1}$$

where,

$$\begin{aligned} m_1 &= \frac{-2a}{1+2a} \\ m_2 &= \frac{-1}{1+a+b+2ab} \\ m_3 &= \frac{-(1+a)}{1+a+b+2ab} = m_2(1+a) \end{aligned} \tag{2}$$

and where $a = C/T$, $b = L/T$, and T denotes the sample period of the filter. (For details concerning the evaluation of m_1, m_2, m_3, refer, e.g., to [1], [3a]).

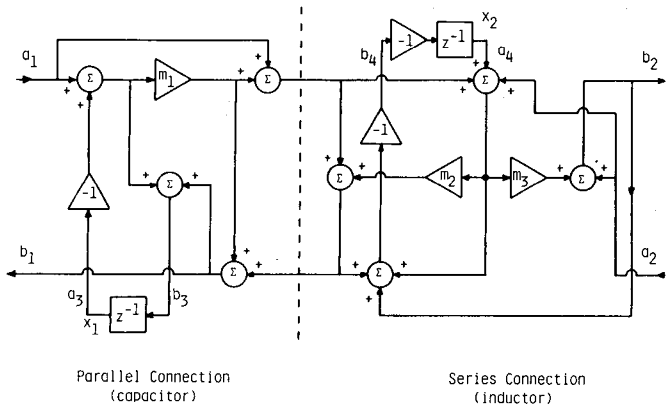

Fig. 4. Linear wave digital filter structure for specific example.

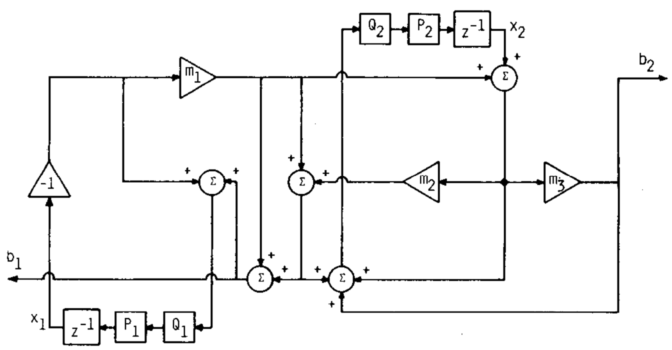
Fig. 5. Wave digital filter with two quantizers.

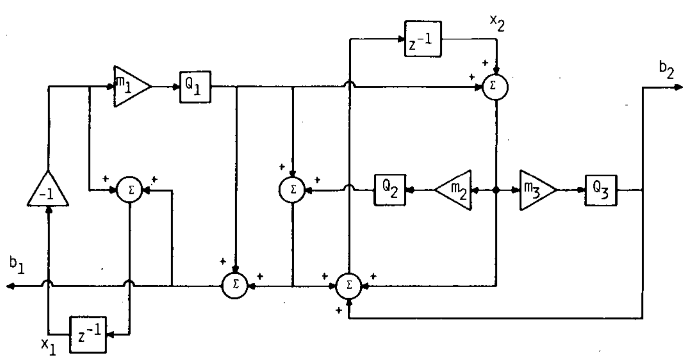
Fig. 6. Wave digital filter with three quantizers.

For the passive LC network of Fig. 2, $L>0$ and $C>0$ and thus the wave digital filter parameters are also, $a>0$ and $b>0$. Fettweis [6] shows that all wave digital filters derived from classical LC networks are also pseudopassive, and, therefore, globally asymptotically stable, when infinite wordlength is used. Therefore, for the example considered here, the linear wave digital filter is globally asymptotically stable when $a>0$ and $b>0$.

We consider two possible structures for a nonlinear wave digital filter. Quantization and overflow nonlinearities can be applied at the states of the filter. This two quantizer structure is shown in Fig. 5. This structure has received previous attention by other authors. We also consider quantization after each multiplication, as shown in Fig. 6. In this case, there are three quantizers. This structure is a more realistic implementation of the actual filter using a microprocessor. We do not consider overflow nonlinearities due to the large number of adders. We next present the procedure used to generate the extreme matrices which are used by the constructive algorithm.

2) Two Quantizers: The wave digital filter structure which we consider is shown in Fig. 5. As in the direct form and coupled form filters [4], the quantization and overflow nonlinearities are considered together. Under this assumption, the state equations are

$$x_1(k+1)=f_1[c_{11}x_1(k)+c_{12}x_2(k)]$$
$$x_2(k+1)=f_2[c_{21}x_1(k)+c_{22}x_2(k)]$$

where $f_1(\cdot)$ and $f_2(\cdot)$ are the combined overflow and quantization nonlinearities. The coefficients c_{ij}, $i,j=1,2$ are given in (1) and (2).

Following the technique outlined in [4, sect. III], the state equations are written as

$$x(k+1)=M(x(k))x(k)$$

where,

$$M(x(k))=\begin{bmatrix}\Phi_1(x)c_{11} & \Phi_1(x)c_{12}\\ \Phi_1(x)c_{21} & \Phi_2(x)c_{22}\end{bmatrix}$$

and

$$\Phi_1(x)=\frac{f_1[c_{11}x_1+c_{12}x_2]}{c_{11}x_1+c_{12}x_2}$$
$$\Phi_2(x)=\frac{f_2[c_{21}x_1+c_{22}x_2]}{c_{21}x_1+c_{22}x_2}.$$

The functions Φ_1 and Φ_2 are bounded by constants,

$$\alpha_1\leqslant\Phi_1(x)\leqslant\alpha_2$$
$$\beta_1\leqslant\Phi_2(x)\leqslant\beta_2.$$

In our case,

$$\alpha_1=\beta_1=k_0$$
$$\alpha_2=\beta_2=k_q$$

where k_0 and k_q are defined in [4] by (14) and (13), respectively. The extreme matrices of the set $\boldsymbol{M}$ are

$$E(\boldsymbol{M})=\left\{\begin{bmatrix}\alpha_i c_{11} & \alpha_i c_{12}\\ \beta_j c_{21} & \beta_j c_{22}\end{bmatrix}, i,j=1,2\right\}. \tag{3}$$

(Refer to [4] for the definition of $\boldsymbol{M}$.) Therefore, the constructive algorithm uses four extreme matrices for each point in the $a-b$ parameter plane. If the overflow nonlinearities are absent, then $\alpha_1=\beta_1=0$ and the set of extreme matrices for this case is the same as for the filter with saturation or zeroing overflow nonlinearities.

3) Three Quantizers: The wave digital filter structure which we consider is shown in Fig. 6. Note that only quantization nonlinearities are present in this filter. The state equations for this structure are

$$x_1(k+1)=-x_1(k)+2Q_1[-m_1x_1(k)]+Q_2\{m_2Q_1[-m_1x_1(k)]\}+Q_2[m_2x_2(k)]$$
$$x_2(k+1)=2Q_1[-m_1x_1(k)]+Q_2\{m_2Q_1[-m_1x_1(k)]\}+Q_3\{m_3Q_1[-m_1x_1(k)]\}+Q_2[m_2x_2(k)]+Q_3[m_3x_2(k)]+x_2(k).$$

To apply the constructive stability algorithm, we write the state equations as

$$x(k+1) = M(x(k))x(k).$$

By defining,

$$\Phi_1(x) = \frac{Q_1[-m_1x_1]}{-m_1x_1}$$

$$\Phi_2(x) = \frac{Q_2[m_2x_2]}{m_2x_2}$$

$$\Phi_3(x) = \frac{Q_3[m_3x_2]}{m_3x_2}$$

$$\Phi_4(x) = \frac{Q_2\{m_2Q_1[-m_1x_1]\}}{m_2Q_1[-m_1x_1]}$$

$$\Phi_5(x) = \frac{Q_3\{m_3Q_1[-m_1x_1]\}}{m_3Q_1[-m_1x_1]}$$

$$\Phi_6(x) = -1 - 2m_1\Phi_1(x) - m_1m_2\Phi_1(x)\Phi_4(x)$$

$$\Phi_7(x) = -2m_1\Phi_1(x) - m_1m_2\Phi_1(x)\Phi_4(x) - m_1m_3\Phi_1(x)\Phi_5(x)$$

$$\Phi_8(x) = 1 + m_2\Phi_2(x) + m_3\Phi_3(x)$$

we can write $M(x(k))$ as

$$M(x(k)) = \begin{bmatrix} \Phi_6(x) & m_2\Phi_2(x) \\ \Phi_7(x) & \Phi_8(x) \end{bmatrix}.$$

The functions $\Phi_i(x)$, $i = 1, \cdots, 5$, are bounded by constants,

$$\alpha_1 \leqslant \Phi_1(x) \leqslant \alpha_2$$

$$\beta_1 \leqslant \Phi_2(x) \leqslant \beta_2$$

$$\gamma_1 \leqslant \Phi_3(x) \leqslant \gamma_2$$

$$\beta_1 \leqslant \Phi_4(x) \leqslant \beta_2$$

$$\gamma_1 \leqslant \Phi_5(x) \leqslant \gamma_2$$

where,

$$\alpha_1 = b_1 = \gamma_1 = 0$$

$$\alpha_2 = \beta_2 = \gamma_2 = k_q.$$

The functions $\Phi_1(x)\Phi_4(x)$ and $\Phi_1(x)\Phi_5(x)$ are also bounded by constants

$$\delta\Phi_1 \leqslant \Phi_1(x)\Phi_4(x) \leqslant \delta_2$$

$$\epsilon_1 \leqslant \Phi_1(x)\Phi_5(x) \leqslant \epsilon_2$$

where,

$$\delta_1 = \epsilon_1 = 0$$

$$\delta_2 = \epsilon_2 = k_q^2.$$

The functions $\Phi_6(x)$, $\Phi_7(x)$, and $\Phi_8(x)$ are also bounded by constants,

$$\zeta_1 \leqslant \Phi_6(x) \leqslant \zeta_2$$

$$\eta_1 \leqslant \Phi_7(x) \leqslant \eta_2$$

$$\theta_1 \leqslant \Phi_8(x) \leqslant \theta_2$$

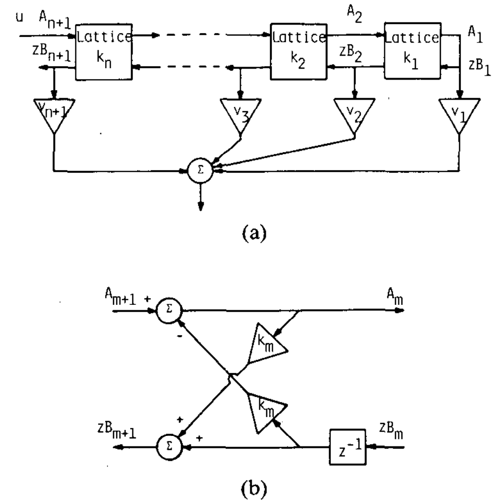

Fig. 7. General lattice digital filter structure. (a) General lattice structure. (b) Two multiplier lattice section.

where,

$$\zeta_1 = \min\{[-1 - 2m_1\alpha_i - m_1m_2\delta_j], \ i, j = 1,2\}$$

$$\zeta_2 = \max\{[-1 - 2m_1\alpha_i - m_1m_2\delta_j], \ i, j = 1,2\}$$

$$\eta_1 = \min\{[-2m_1\alpha_i - m_1m_2\delta_j - m_1m_3\epsilon_k], \ i, j, k = 1,2\}$$

$$\eta_2 = \max\{[-2m_1\alpha_1 - m_1m_2\delta_j - m_1m_3\epsilon_k], \ i, j, k = 1,2\}$$

$$\theta_1 = \min\{[1 + m_2\beta_i + m_3\gamma_j], \ i, j = 1,2\}$$

$$\theta_2 = \max\{[1 + m_2\beta_i + m_3\gamma_j], \ i, j = 1,2\}.$$

Therefore, we write the extreme matrices of the set $\boldsymbol{M}$ as

$$E(\boldsymbol{M}) = \left\{ \begin{bmatrix} \zeta_i & m_2\beta_j \\ \eta_k & \theta_l \end{bmatrix}, \ i, j, k, l = 1,2 \right\}. \tag{4}$$

Thus for each individual point in the $a - b$ parameter plane, the constructive algorithm uses sixteen matrices.

B. Lattice Digital Filters

Since their introduction by Itakura and Saito [12], lattice digital filters have been used extensively in the area of speech and signal processing [9]. A general lattice filter is shown in Fig. 7(a) as a cascade of lattice sections. The particular lattice structure we consider is the two multiplier lattice of Gray and Markel [11]. One section of this type of lattice filter is shown in Fig. 7(b). Gray and Markel [11] have shown that the linear digital lattice filter will have all of its poles within the unit circle (and thus will be globally asymptotically stable), if and only if all of the k_m parameters satisfy

$$|k_m| < 1, \qquad m = 1, 2, \cdots, n.$$

We investigate two possible structures for the second-order lattice digital filter. In the first structure, the quantization and overflow nonlinearities are applied at the states of the filter. We consider this first structure since it has been studied previously. This second-order filter structure is shown in Fig. 8. In the second structure, quantization is assumed to take place after each multiplication and over-

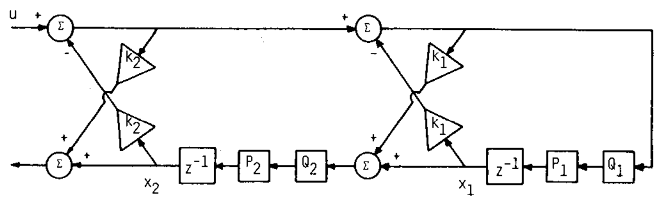

Fig. 8. Lattice digital filter with two quantizers.

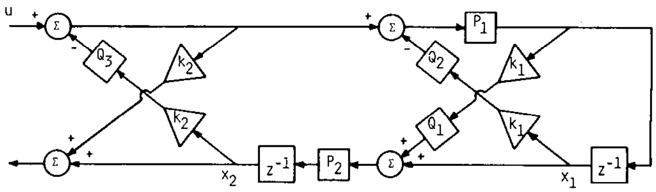

Fig. 9. Lattice digital filter with three quantizers.

flow is placed after each addition. This structure with three quantizers is a more realistic implementation of the actual filter using a fixed-point microprocessor and is shown in Fig. 9.

1) Two Quantizers: The structure of the second order lattice digital filter with two quantizers is shown in Fig. 8. Again, we consider the quantization and overflow nonlinearities together. The state equations for the structure are

$$x_1(k+1)=f_1[-k_1x_1(k)-k_2x_2(k)]$$
$$x_2(k+1)=f_2[(1-k_1^2)x_1(k)-k_1k_2x_2(k)]$$

where $f_1(\cdot)$ and $f_2(\cdot)$ are the combined quantization and overflow nonlinearities.

Following the technique outlined in [4, sect. III], the state equations are written as

$$x(k+1)=M(x(k))x(k).$$

By defining

$$\Phi_1(x)=\frac{f_1[-k_1x_1-k_2x_2]}{-k_1x_1-k_2x_2}$$

$$\Phi_2(x)=\frac{f_2[(1-k_1^2)x_1-k_1k_2x_2]}{(1-k_1^2)x_1-k_1k_2x_2}$$

the matrix $M(x(k))$ can be written as

$$M(x(k))=\begin{bmatrix}-k_1\Phi_1(x) & -k_2\Phi_1(x)\\ (1-k_1^2)\Phi_2(x) & -k_1k_2\Phi_2(x)\end{bmatrix}.$$

The function $\Phi_1(x)$ and $\Phi_2(x)$ are bounded by constants

$$\alpha_1\leqslant\Phi_1(k)\leqslant\alpha_2$$
$$\beta_1\leqslant\Phi_2(x)\leqslant\beta_2$$

where,

$$\alpha_1=\beta_1=k_0$$
$$\alpha_2=\beta_2=k_q.$$

The extreme matrices of the set $\boldsymbol{M}$ are

$$E(\boldsymbol{M})=\left\{\begin{bmatrix}-k_1\alpha_i & -k_2\alpha_i\\ (1-k_1^2)\beta_j & -k_1k_2\beta_j\end{bmatrix}, i,j=1,2\right\}. \quad (5)$$

Thus for each individual point in the k_1-k_2 parameter plane, the constructive algorithm uses four extreme matrices. If the overflow nonlinearities are absent, then $\alpha_1=\beta_1=0$ and the set of extreme matrices for this case is the same as for the filter with two saturation or zeroing overflow nonlinearities.

2) Three Quantizers: The lattice digital filter which we consider is shown in Fig. 9. The state equations for this digital filter are

$$x_1(k+1)=P_1\{-Q_2[k_1x_1(k)]-Q_3[k_2x_2(k)]\}$$
$$x_2(k+1)=P_2\{x_1(k)+Q_1[k_1P_1(-Q_2[k_1x_1(k)] -Q_3[k_2x_1(k)])]\}.$$

To apply the constructive stability algorith, we write the state equations as

$$x(k+1)=M(x(k))x(k).$$

The matrix $M(x(k))$ is given by

$$M(x(k))=\begin{bmatrix}-k_1\Phi_1(x)\Phi_4(x) & -k_2\Phi_2(x)\Phi_4(x)\\ \Phi_5(x)[1-k_1^2\Phi_1(x)\Phi_3(x)\Phi_4(x)] & -k_1k_2\Phi_2(x)\Phi_3(x)\Phi_4(x)\Phi_5(x)\end{bmatrix}$$

where,

$$\Phi_1(x)=\frac{Q_2[k_1x_1]}{k_1x_1}$$

$$\Phi_2(x)=\frac{Q_3[k_2x_2]}{k_2x_2}$$

$$\Phi_3(x)=\frac{Q_1[k_1P_1(-Q_2[k_1x_1]-Q_3[k_2x_2])]}{k_1P_1(-Q_2[k_1x_1]-Q_3[k_2x_2])}$$

$$\Phi_4(x)=\frac{P_1\{-Q_2[k_1x_1]-Q_3[k_2x_2]\}}{-Q_2[k_1x_1]-Q_3[k_2x_2]}$$

$$\Phi_5(x)=\frac{P_2\{x_1+Q_1[k_1P_1(-Q_2[k_1x_1]-Q_3[k_2x_2])]\}}{x_1+Q_1[k_1P_1(-Q_2[k_1x_1]-Q_3[k_2x_2])]}.$$

The functions $\Phi_i(x)$, $i=1,\cdots,5$ are bounded by constants

$$\alpha_1\leqslant\Phi_1(x)\leqslant\alpha_2$$
$$\beta_1\leqslant\Phi_2(x)\leqslant\beta_2$$
$$\gamma_1\leqslant\Phi_3(x)\leqslant\gamma_2$$
$$\delta_1\leqslant\Phi_4(x)\leqslant\delta_2$$
$$\epsilon_1\leqslant\Phi_5(x)\leqslant\epsilon_2$$

where,

$$\alpha_1=\beta_1=\gamma_1=0$$
$$\alpha_2=\beta_2=\gamma_2=k_q$$
$$\delta_1=\epsilon_1=k_0$$
$$\delta_2=\epsilon_2=1.$$

Combinations of these functions are also bounded by constants, i.e.,

$$\begin{aligned}\zeta_1 &\leqslant \Phi_1(x)\Phi_4(x) \leqslant \zeta_2\\ \eta_1 &\leqslant \Phi_2(x)\Phi_4(x) \leqslant \eta_2\\ \theta_1 &\leqslant \Phi_2(x)\Phi_3(x)\Phi_4(x)\Phi_5(x) \leqslant \theta_2\\ \lambda_1 &\leqslant \Phi_5(x)\left[1-k_1^2\Phi_1(x)\Phi_3(x)\Phi_4(x)\right] \leqslant \lambda_2\end{aligned} \tag{6}$$

where, for the nonlinearities which we consider,

$$\begin{aligned}\zeta_1 &= \eta_1 = k_0 k_q\\ \zeta_2 &= \eta_2 = k_q\\ \theta_1 &= k_0 k_q^2\\ \theta_2 &= k_q^2\\ \lambda_1 &= \min\left[\epsilon_i\left(1-k_1^2\alpha_j\gamma_k\delta_l\right),\ i,j,k,l=1,2\right]\\ \lambda_2 &= \max\left[\epsilon_i\left(1-k_1^2\alpha_j\gamma_k\delta_l\right),\ i,j,k,l=1,2\right].\end{aligned}$$

Thus the extreme matrices of the set $\boldsymbol{M}$ are

$$E(\boldsymbol{M}) = \left\{\begin{bmatrix} -k_1\zeta_i & -k_2\eta_j\\ \lambda_k & -k_1k_2\theta_l\end{bmatrix},\ i,j,k,l=1,2\right\}. \tag{7}$$

In this case, the constructive algorithm uses sixteen extreme matrices for every point in the k_1-k_2 parameter plane.

If the overflow nonlinearities are absent, the set of extreme matrices is not the same as for the filter with saturation or zeroing overflow nonlinearities. For this case, the constants that bound the combinations of the functions in (6) are

$$\begin{aligned}\zeta_1 &= \eta_1 = \theta_1 = 0\\ \zeta_2 &= \eta_2 = k_q\\ \theta_2 &= k_q^2\\ \lambda_1 &= 1-k_1^2k_q^2\\ \lambda_2 &= 1.\end{aligned}$$

III. Stability Results by the Constructive Algorithm

In this section, we present the stability results obtained by applying the constructive algorithm to the different nonlinear filter structures enumerated in Section II. We also compare these results with existing results and with results obtained by other methods of stability analysis (such as the absolute stability theorem of Jury and Lee [13]). As mentioned previously, our analysis by the constructive algorithm yields sufficient conditions for global asymptotic stability in terms of the parameters of a given filter under zero-input conditions. These results constitute of course also sufficient conditions for the absence of zero-input limit cycles.

The present section consists of two parts. First, we consider wave digital filters and then we treat lattice digital filters.

A. Wave Digital Filters

For the specific class of wave digital filters presented in Section II, we consider the filters implemented with two or three quantizers. Apparently, the only known stability results apply to this wave digital filter implemented with two truncation quantizers. However, we can apply the Jury–Lee absolute stability criterion (see [4, th. 4]) to some of the other cases to obtain stability results. These stability results are compared with the stability results obtained by the constructive algorithm.

1) Two Quantizers

a) Truncation quantizers: Evidently, the only known stability analysis for wave digital filters is due to Fettweis and Meerkötter [7], [8]. These workers use the concept of a stored pseudoenergy to establish the complete stability of wave digital filters. (The precise definition of complete stability is given below.) The stored pseudoenergy function plays the role of a Lyapunov function in their proof. The nonlinear arithmetic operations, i.e., overflow and quantization, are assumed to be applied to the signals b_i ($i=3,4,\cdots,n$) in Fig. 1. These signals correspond to the states of the specific example which we consider in Fig. 4.

Before we recall the stability results of Fettweis and Meerkötter, we require their definition of complete stability. Consider a general wave digital filter such as the one of Fig. 1. Arbitrary initial conditions are present in the filter at a certain initial time, t_0. The inputs to the filter are zero for all time greater than t_0, i.e., $a_1(t_0+k)=a_2(t_0+k)=0$ for all $k \geqslant 0$. The wave digital filter is said to be *completely stable* if the signals $b_i(t_0+k)$, $i=1,\cdots,n$ become permanently zero for all $k \geqslant k_1+t_0$ for some $k_1 \geqslant 0$. Clearly, complete stability implies that the filter is free of any limit cycles.

The stability result of Fettweis and Meerkötter is stated without proof. The interested reader is referred to [7] and [8] for details.

Theorem 1: The wave digital filter of Fig. 1 is completely stable if:

1) the linear n-port network is pseudopassive [6],

2) the linear n-port network is free of any limit cycles under zero-input conditions, and

3) the nonlinearities $f_i(\cdot)$ at b_i ($i=3,4,\cdots,n$) satisfy the conditions

$$|f_i(b_i)| \leqslant b_i$$

$$|f_i(b_i)| = |b_i| \qquad \text{implies} \quad f_i(b_i) = b_i.$$

This theorem applies to the specific wave digital filter we consider, since all linear wave digital filters derived from LC networks are pseudopassive and globally asymptotically stable [6]. The nonlinearities which will satisfy condition 3) of Theorem 1 are truncation quantizers with any of the overflow characteristics that we consider. Thus Theorem 1 shows that the specific wave digital filter which we consider here is free of limit cycles for any parameter values when truncation quantization with any overflow is applied at the states of the filter.

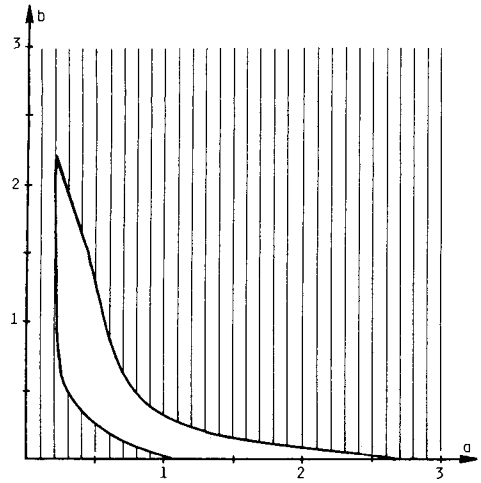

Fig. 10. Region where the specific wave filter with two roundoff quantizers and no overflow is g.a.s. by theorem 4 in [4].

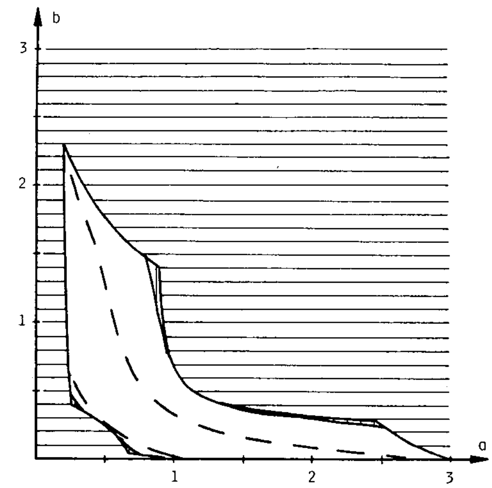

Fig. 11. Region where the specific wave filter with two roundoff quantizers and no overflow is g.a.s. by the constructive algorithm.

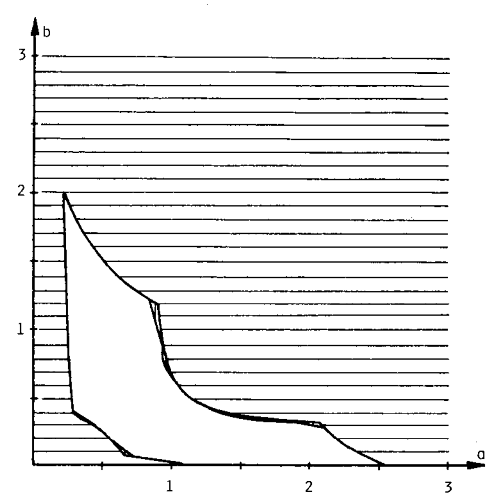

Fig. 12. Region where the specific wave filter with two roundoff quantizers and triangular overflow is g.a.s. by the constructive algorithm.

To apply the constructive algorithm to the wave digital filter structure with two quantizers (Fig. 5), we use the extreme matrices determined by (3). The constructive algorithm shows that the filter is globally asymptotically stable for truncation quantization with any overflow for parameters a and b satisfying, say

$$0 < a \leqslant 100$$
$$0 < b \leqslant 100. \tag{8}$$

We did not try larger values of the parameters a and b because the run time of the computer program increased significantly as a and b are increased in value. However, this region does cover any reasonable values of these parameters, since the larger values of a and b imply that the sampling frequency of the filter is fairly high compared to the cutoff frequency of the filter. Our result shows that the filter is free of limit cycles for any of the parameters in the region defined by (8) and thus essentially agrees with existing results.

b) Roundoff quantizers: There do not seem to exist results for the stability of wave digital filters when roundoff quantization is used at the states. However, for comparison purposes we apply the absolute stability criterion of Jury and Lee to this case. Applying theorem 4 in [4] to the wave digital filter with two quantizers, as shown in Fig. 5, the matrix $G(z)$ may be written as

$$G(z)=\begin{bmatrix} -c_{11}z^{-1} & -c_{12}z^{-1} \\ -c_{21}z^{-1} & -c_{22}z^{-1} \end{bmatrix}$$

where c_{11}, c_{12}, c_{21}, and c_{22} are determined by (1) and (2). The matrix $H(z)$, given by

$$H(z)=\begin{bmatrix} \frac{2}{k_{11}}-c_{11}\left(z^{-1}+\overline{z^{-1}}\right) & -c_{12}z^{-1}-c_{21}\overline{z^{-1}} \\ -c_{21}z^{-1}-c_{12}\overline{z^{-1}} & \frac{2}{k_{22}}-c_{22}\left(z^{-1}+\overline{z^{-1}}\right) \end{bmatrix}$$

must be positive definite for $|z|=1$. For two roundoff quantizers, $k_{11}=k_{22}=2$. The region in the parameter plane where the filter is globally asymptotically stable is presented as the unhatched region in Fig. 10.

To apply the constructive algorithm to the wave digital filter structure with two quantizers (Fig. 5), we use the extreme matrices determined by (3). The regions in the parameter plane where the digital filter is globally asymptotically stable by the constructive algorithm for all cases are shown in Figs. 11 and 13. Horizontal hatching indicates the region where at least one extreme matrix has an eigenvalue with a magnitude greater than one. Although only a portion of the first quadrant is shown, this horizontally hatched region extends to at least $a=b=100$, which is the most extensive region we examined. Vertical hatching indicates the rest of the region where we can make no conclusion about the stability of the filter.

As can be seen from Fig. 11, the constructive algorithm yields a less conservative result than the application of theorem 4 in [4]. All of the results obtained for the roundoff quantization in conjunction with overflow seem to be new results (see Figs. 12, 13).

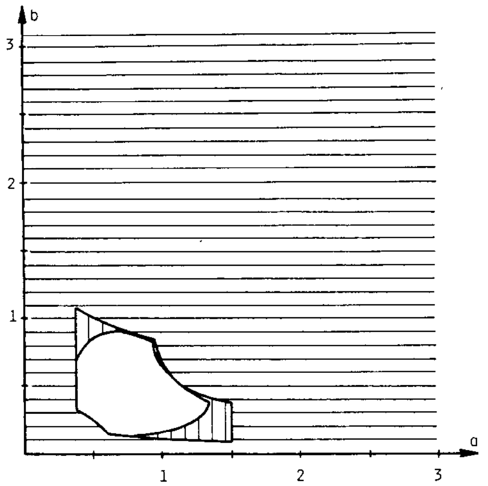

Fig. 13. Region where the specific wave filter with two roundoff quantizers and two's complement overflow is g.a.s. by the constructive algorithm.

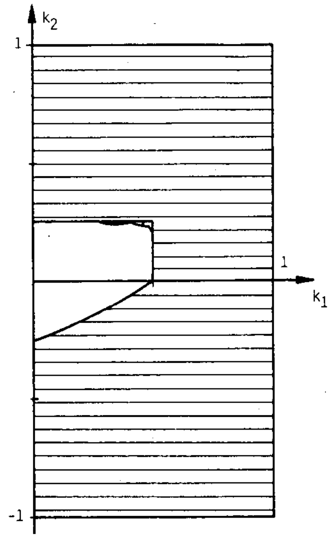

Fig. 14. Region where the lattice filter with two roundoff quantizers and saturation, zeroing or no overflow is g.a.s. by the constructive algorithm.

2) Three Quantizers: To apply the constructive stability algorithm to the wave digital filter structure with three quantizers (Fig. 6), we used the extreme matrices given in (4). For truncation or roundoff quantization, the constructive algorithm failed to determine any region in the parameter plane where the filter is globally asymptotically stable. Although no details are given here, application of the Jury–Lee criterion [4, th. 4] also failed to determine any region in the parameter plane for $a > 0$ and $b > 0$ where the filter is globally asymptotically stable.

B. Lattice Digital Filters

For the class of second-order lattice digital filters, we consider the filter implemented with two or three quantizers. Apparently, the only known stability results apply to the lattice filter with two truncation quantizers. However, application of the Jury–Lee criterion (theorem 4 in [4]) yields some stability results for filters with three quantizers. These results are then compared with the stability results obtained by the constructive algorithm.

1) Two Quantizers:

a) Truncation quantizers: Gray [10] uses energy analogies to determine the absence of limit cycles in nonlinear lattice digital filters. The approach is similar to that of Fettweis and Meerkötter [7]. Gray shows that the nonlinear lattice digital filter will be free of limit cycles whenever the linear filter is globally asymptotically stable if truncation quantization and overflow nonlinearities are applied at each section output (i.e., at A_m and B_{m+1} of Fig. 7(b)). This result applies to any of the overflow characteristics that we are considering [4, fig. 2]. When applied to a second-order lattice filter, this result shows that no limit cycles exist, even if truncation quantization and overflow are applied only at the states of Fig. 8.

To apply the constructive algorithm to the stability analysis of the lattice structure with two quantizers, we use the extreme matrices determined by (5). With truncation quantization and any overflow nonlinearity, the constructive algorithm shows that this nonlinear lattice filter is globally asymptotically stable in the region of the $k_1 - k_2$ parameter plane determined by

$$|k_1| < 1$$
$$|k_2| < 1.$$

This result agrees with the result of Gray [10].

b) Roundoff quantizers: There do not appear to be any stability results for the second-order lattice digital filter implemented with two roundoff quantizers (Fig. 8). Although no details are given here, application of the Jury–Lee criterion yields no region in the parameter plane where this filter is globally asymptotically stable. To apply the constructive algorithm to the lattice structure with two roundoff quantizers, we use the extreme matrices determined by (5). The regions in the $k_1 - k_2$ parameter plane where this filter is globally asymptotically stable by the constructive algorithm are shown as the unhatched regions in Figs. 14–16. A horizontally hatched region indicates the region where at least one extreme matrix has an eigenvalue with a magnitude which is greater than one. Vertical hatching indicates the rest of the region where the constructive algorithm does not yield any conclusive results concerning global asymptotic stability. These regions are symmetric about the k_2-axis. These results appear to be new.

2) Three Quantizers: There do not seem to be any existing stability results for lattice digital filters with three quantizers (Fig. 9). However, for purposes of comparison, we apply the absolute stability criterion of Jury and Lee [4, th. 4] to this structure without the overflow nonlinearities. Applying theorem 4 in [4] to the lattice digital filter with three quantizers, as shown in Fig. 9, we obtain the matrix $G(z)$,

$$G(z) = \begin{bmatrix} 0 & k_1 & k_1 \\ 0 & k_1 z^{-1} & k_1 z^{-1} \\ -k_2 z^{-1} & k_2 z^{-2} & k_2 z^{-2} \end{bmatrix}.$$

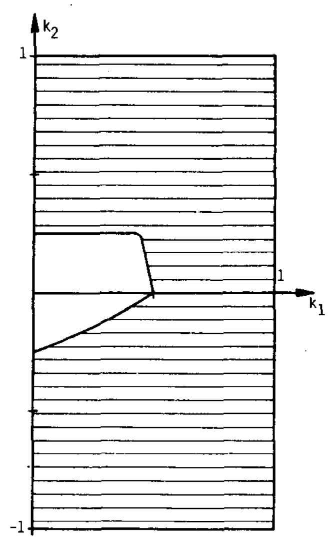

Fig. 15. Region where the lattice filter with two roundoff quantizers and triangular overflow is g.a.s. by the constructive algorithm.

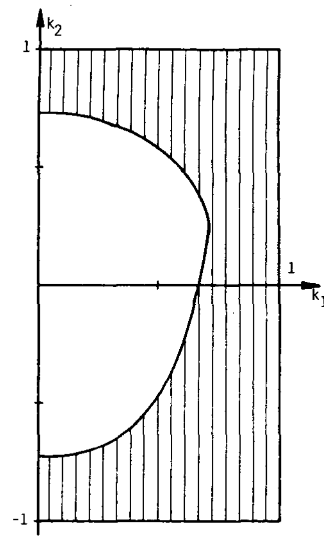

Fig. 17. Region where the lattice filter with three truncation quantizers and no overflow is g.a.s. by theorem 4 in [4].

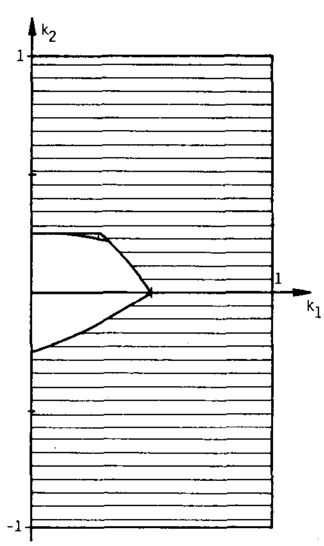

Fig. 16. Region where the lattice filter with two roundoff quantizers and two's complement overflow is g.a.s. by the constructive algorithm.

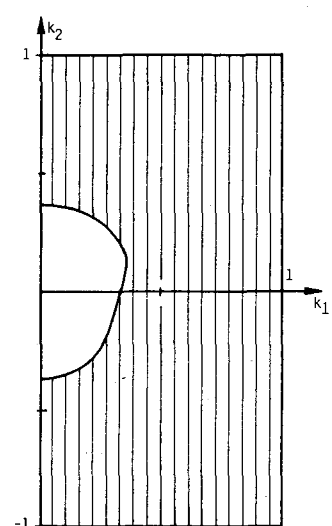

Fig. 18. Region where the lattice filter with three roundoff quantizers and no overflow is g.a.s. by theorem 4 in [4].

The matrix $H(z)$, given by

$$H(z)=\begin{bmatrix} \frac{2}{k_{11}} & k_1 & k_1-k_2\overline{z^{-1}} \\ k_1 & \frac{2}{k_{22}}+k_1\left(z^{-1}+\overline{z^{-1}}\right) & k_1z^{-1}+k_2\overline{z^{-2}} \\ k_1-k_2z^{-1} & k_1\overline{z^{-1}}+k_2z^{-2} & \frac{2}{k_{33}}+k_2\left(z^{-2}+\overline{z^{-2}}\right) \end{bmatrix}$$

must be positive definite for $|z|=1$. For three truncation quantizers, $k_{11}=k_{22}=k_{33}=1$. The region in the parameter plane where the filter is globally asymptotically stable is shown as the unhatched region in Fig. 17. For three roundoff quantizers, $k_{11}=k_{22}=k_{33}=2$. The region where the filter is globally asymptotically stable for this case is presented as the unhatched region in Fig. 18. These regions are symmetric about the k_2-axis.

To apply the constructive algorithm to the lattice structure with three quantizers (Fig. 9), we use the extreme matrices determined in (7). The regions in the parameter plane where the filter is globally asymptotically stable for all cases are presented in Figs. 19–26. The horizontally hatched regions are those regions where at least one extreme matrix has an eigenvalue whose magnitude is greater than one. Vertical hatching indicates the rest of the region where no conclusion can be drawn concerning the global stability of the filter by the constructive algorithm.

As can be seen in Figs. 19 and 23, less conservative results are obtained by the constructive algorithm than by

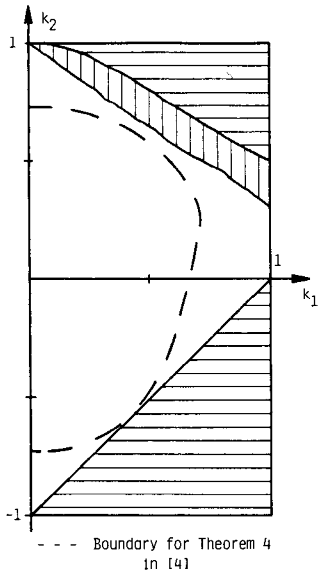

Fig. 19. Region where the lattice filter with three truncation quantizers and no overflow is g.a.s. by the constructive algorithm.

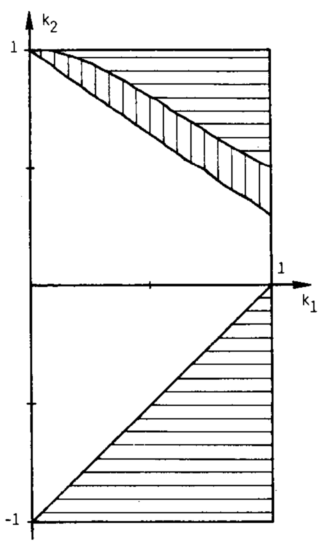

Fig. 20. Region where the lattice filter with three truncation quantizers and saturation or zeroing overflow is g.a.s. by the constructive algorithm.

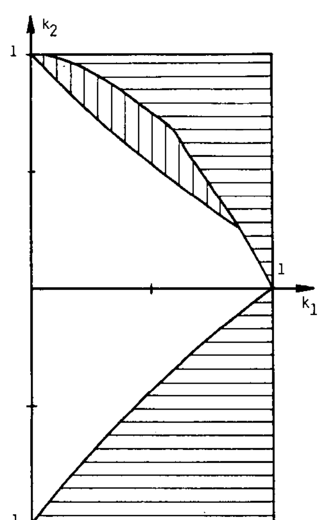

Fig. 21. Region where the lattice filter with three truncaton quantizers and triangular overflow is g.a.s. by the constructive algorithm.

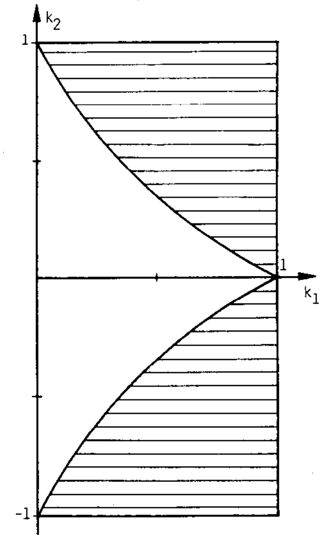

Fig. 22. Region where the lattice filter with three truncation quantizers and two's complement overflow is g.a.s. by the constructive algorithm.

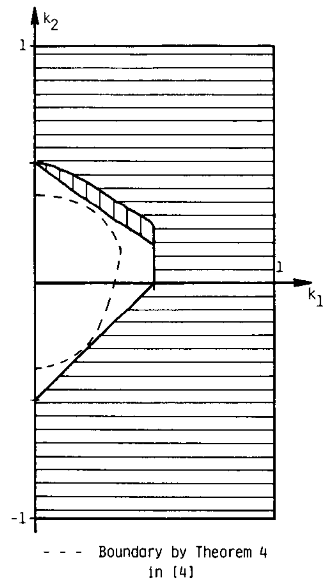

Fig. 23. Region where the lattice filter with three roundoff quantizers and no overflow is g.a.s. by the constructive algorithm.

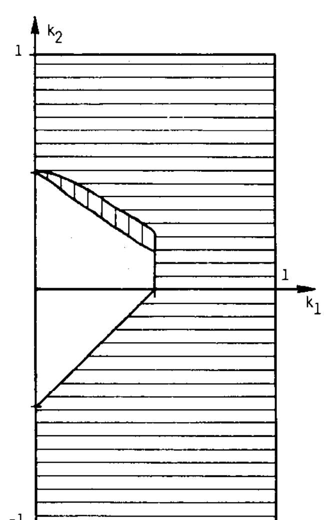

Fig. 24. Region where the lattice filter with three roundoff quantizers and saturation or zeroing overflow is g.a.s. by the constructive algorithm.

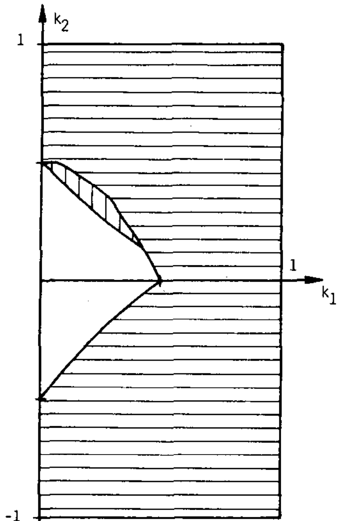

Fig. 25. Region where the lattice filter with three roundoff quantizers and triangular overflow is g.a.s. by the constructive algorithm.

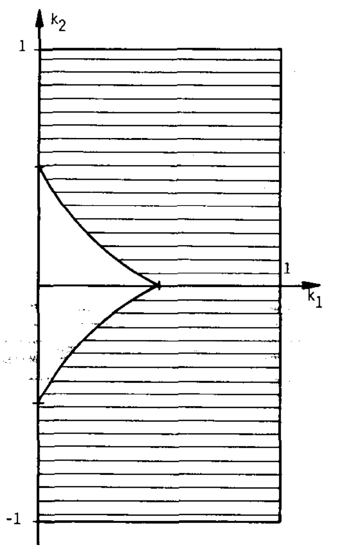

Fig. 26. Region were the lattice filter with three roundoff quantizers and two's complement overflow is g.a.s. by the constructive algorithm.

application of the Jury–Lee criterion. All of the stability results for the filters with overflow nonlinearities seem to be new.

IV. Concluding Remarks

In this paper, we applied the *constructive stability algorithm* of Brayton and Tong to the stability analysis of several classes of second-order fixed-point wave digital filters and lattice digital filters. Our objective was to determine regions in the parameter planes of these filters for which these filters are globally asymptotically stable. In particular, our results yield sufficient conditions under which a filter, with an appropriate set of parameters, does not possess any zero-input limit cycles.

We demonstrated that the present results yield tight stability bounds by comparing them with existing results and also, with results obtained by other methods (such as absolute stability results). Many of the results which we obtained for the various filter structures considered appear to be new.

We believe that the results of this paper, combined with the results obtained in [4] demonstrate that the *constructive algorithm* offers and *effective* and *general* approach for the qualitative analysis of fixed-point digital filters.

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