

**STABILITY ANALYSIS OF LINEAR CONTROL
SYSTEMS WITH UNCERTAIN PARAMETERS**

by

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ABSTRACT

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In this dissertation, we study stochastic stability of linear systems whose parameters are randomly varying in a certain sense. In particular, we present a new approach to stochastic stability analysis of systems whose system structure is randomly changing among a finite set of possibilities which capture the abrupt changes in systems parameters or sudden failures of system components. These types of systems are referred to as jump linear systems with a finite state Markov chain form process.

We first investigate the properties of various types of moment stability for stochastic jump linear systems, and use large deviation theory to study the relationship between “lower moment” stability and almost sure stability. In particular, we have proved that the region for δ -moment stability is monotonically increasing as δ is decreasing to zero and asymptotically converges to the region for almost sure stability. Roughly speaking, this is equivalent to saying that almost sure stability is equivalent to δ -moment stability for sufficiently small $\delta > 0$. Furthermore, we prove that although the top δ -moment Lyapunov exponent is, in general, not differentiable at zero, it is differentiable at zero from the right and its right derivative at zero is equal to the top Lyapunov exponent. This answers a long standing question in this area. Based on

this analysis, a new Lyapunov function is constructed to obtain a very general sufficient condition for almost sure stability, and this condition is also conjectured to be a necessary condition for almost sure stability. Moreover, a few new approaches for the study of almost sure stability are proposed and some easily-testable conditions for both moment stability and almost sure stability are obtained. Based on the results on almost sure stability and moment stability, the stochastic stabilization problem is also considered and a few future research topics are identified.

This dissertation is the first research work in the current literature to use large deviation theory to study stochastic stability and further represents a systematic study of almost sure stability of jump linear systems with a finite state Markov chain form process. It is our high hope that this work will pave the way for further studies on the almost sure (sample path) stability for stochastic systems.

DEDICATED
TO MY BELOVED PARENTS
AND
MY DEAR WIFE

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TABLE OF CONTENTS

Abstract	ii
Dedication	iv
Acknowledgments	v
Table of contents	vi
Table of figures	viii
CHAPTER I. INTRODUCTION	1
1.1 Modeling and Motivation	1
1.2 Literature Survey	4
1.3 Research Topics	11
CHAPTER II. STABILITY OF DISCRETE-TIME	
JUMP LINEAR SYSTEMS	13
2.1 Introduction	14
2.2 Second Moment Stability	22
2.3 δ -moment Stability Properties	55
2.4 Testable Criteria for δ -moment Stability	100
2.5 Almost Sure Stability	110
2.6 Estimation of Top δ -moment Lyapunov Exponent	135
2.7 Almost Sure Stabilization	138
2.8 Robust stability consideration	145
2.9 Illustrative Examples	154
CHAPTER III. STABILITY OF CONTINUOUS-TIME	

JUMP LINEAR SYSTEMS	163
3.1 Introduction	163
3.2 δ -moment Stability	166
3.3 Testable Criteria for Almost Sure Stability	192
3.4 Estimates of Lyapunov Exponents	215
3.5 Robust Stability Analysis	237
3.6 Almost Sure and Moment Stabilization	243
CHAPTER IV. CONCLUSIONS AND FUTURE	
RESEARCH DIRECTIONS	280
APPENDIX A. SOME PROPERTIES OF MATRIX NORMS	285
APPENDIX B. PROPERTIES OF MATRIX MEASURE	290
REFERENCES	304

Table of Figures

1	Stability region for discrete-time case	155
2	Stability region for continuous-time case	191

古今之成大事业大学问者，必经过三种之境界。

“昨夜西风凋碧树，独上高楼，望尽天涯路。”此第一境也。

“衣带渐宽终不悔，为伊消得人憔悴。”此第二境也。

“众里寻他千百度，回头蓦见，那人正在灯火阑珊处。”

此第三境也。

From history we observe that all big shots all experience the following three states: “Last night, west wind withered green trees. Go up alone on top of high building, watch the road extending far away.” This is the first state. “I became thinner and thinner just for you, I never regret as long as I can get your love before I die.” This is the second state. “I am looking for her everywhere many times but not successful. Suddenly when I look back, she is just where I passed by.” This is the third state. —Guowei Wang

To conceive is a pleasure. —Gauss

But to give birth is painful. —Poincaré

When I (Weil) stopped in Rome to see Vito Volterra and explained my formula to him, he jumped out of his chair and ran to the back of the apartment, crying to his wife: “Mr. Weil has proved a very beautiful theorem!” —André Weil

CHAPTER ONE

1. INTRODUCTION

Stochastic systems have been extensively used as realistic models of physical phenomena in physics, economics, the biological sciences and various engineering disciplines. For large-scale systems, advances in computer technology have presented new challenges in the modeling and analysis of complex distributed communication and control systems. For such systems, the reliability of system components is a major issue. Certain components may fail or delays and interruptions to communications between subsystems may occur. These events can be modeled as random changes in the system structure that can lead to instability of the closed-loop system, resulting in a potentially disastrous situation. In some instances of practical interest, the analysis of complex systems can be reduced to the study of stochastic differential or difference equations where the changes in system structure are modeled by a random process. These systems are inherently nonlinear and random. The study of the stochastic stability properties of these randomly varying systems is an important research topic in support of the design of distributed digital control systems.

1.1. Modeling Issues and Motivation

In recent years, with the trend being toward distributed architectures for real-time control, there is a need to decrease cost while to increase the reliability of a system. An important design approach is the development of fault tolerant control systems. One approach is to design a robust control system which can accommodate component failures, random changes in the system structure

and other external disturbances. For example, in an electrical power system, some generating units may be temporarily out of service thereby reducing the reserve margin, or some regions may be temporarily disconnected from the network due to outages in the transmission network. After some time (which is usually random), the units are available for service or the faults are cleared. An approximate model for this complex (fault-prone) system is to use a jump linear system representation with parameters changing randomly among a finite set of possibilities or modes. In this approach, the reliability principle (e.g., redundancy, standby subsystems, etc) is used to help design better controllers ([1]-[5]). Beard ([6]) examined “self-reorganizing” linear systems which reconfigure themselves to compensate for actuator and sensor failures by identifying the structural changes from a finite set of known possibilities and adapting the system feedback law to achieve the desired closed-loop stability properties. This modeling technique is also used in manufacturing systems to capture the effect of machine failures on the overall performance of a system ([7]).

To convert light energy into electrical energy, large solar electrical power generating systems have been constructed in the Californian desert at Daggett and at Font-Romeu in the French Pyrenees ([7]-[9]). In California, a 10-MW solar-powered generating plant uses a central receiver which consists of several boiling panels in which feedwater is transformed into superheated steam using solar energy. The solar energy is reflected toward the panels by a large field of heliostats. The steam temperature regulator controls the flow of water through the boiler in such a way as to maintain the desired steam conditions at the output. A difficult problem is posed by the motion of the clouds over the heliostats. On a partly cloudy day, the clouds tend to cover the heliostats for

time intervals that are quite short when compared with the dominant system time constants, and they move away just as fast. These sudden changes may be frequent and essentially unpredictable, and thus can be modeled using a discrete random process. It is shown that this complex system can be approximately modeled as a jump linear system with a finite state Markov form process ([7]-[9]).

Distributed systems incorporating integrated communication and control systems have been used in complex dynamical systems like aircraft, spacecraft, automobiles, electrical power systems and manufacturing processes. Usually time-division-multiplexed computer communication networks are used for the exchange of information between spatially distributed plant components as well as for coordination of the diverse control and decision-making functions ([10]-[24]). Due to the time-sharing characteristic of the network, randomly varying, distributed delays can occur in the decision and control feedback loops in addition to the digital sampling and data processing delays that are also introduced. The network induced delays can degrade the dynamic performance of the system, and are a source of potential instability. Ray and his coworkers ([10]-[20]) proposed a modeling framework and provided some preliminary analysis and results. Krtolica et al ([21]) used Kalman and Bertram's ([25]) unified modeling approach to reduce the distributed system to a jump linear system with a finite state Markov chain form process, although the model which results is not directly verifiable. The Ford Motor Company has developed a confidential communication protocol, the Standard Corporate Protocol (SCP), to coordinate the communications between the electrical/electronic modules in a vehicle ([26],[27]). Based on this protocol, in some of our previous work, see Fang et al ([22]-[24]), we developed a queueing model for the SCP network and

studied the stability of the closed-loop control system. A jump linear system model was developed and used to analyze the stochastic stability of the system with random communication delays induced by traffic on the network.

Jump linear systems have also been used to analyze the transient behavior of electrical power networks ([28],[29]), economic policy planning ([30], [31]) and stochastic hybrid system analysis in $C2$ systems ([92],[93]).

1.2. Literature Survey

Many practical problems such as those we discussed in the previous section can be reduced to the study of the following system:

$$\dot{x}(t) = f(x(t), \sigma_t, u(t), t), \quad (1.2.1)$$

or

$$x(k+1) = f(x(k), \sigma_k, u(k), k), \quad (1.2.2)$$

where $\{\sigma_t\}$ or $\{\sigma_k\}$ is a random process characterizing the changes in the system structure, which can be modeled as a finite state Markov or semi-Markov chain, for example. This random process will be referred to as the *form process*. The functions $\{u(t)\}$ or $\{u(k)\}$ represent the control inputs to the system. The control systems (1.2.1) or (1.2.2) will be referred to as *jump control systems*. If the function $f(\cdot, \cdot, \cdot)$ is linear both in the state x and the control u , (1.2.1) or (1.2.2) have the familiar representation

$$\dot{x}(t) = A_t(\sigma_t)x(t) + B_t(\sigma_t)u(t), \quad (1.2.3)$$

or

$$x(k+1) = A_k(\sigma_k)x(k) + B_k(\sigma_k)u(k), \quad (1.2.4)$$

and these systems are referred to as *jump linear control systems*. For the stability analysis of closed-loop systems, for example, where feedback controls $u(t) = -p(x(t), \sigma_t, t)$ or $u(k) = p(x(k), \sigma_k, k)$ are used, the following *jump systems* will be studied:

$$\dot{x}(t) = f(x(t), \sigma_t, t), \quad (1.2.5)$$

or

$$x(k+1) = f(x(k), \sigma_k, k), \quad (1.2.6)$$

The *jump linear systems* that are of interest are of the form:

$$\dot{x}(t) = A_t(\sigma_t)x(t), \quad (1.2.7)$$

or

$$x(k+1) = A_k(\sigma_k)x(k), \quad (1.2.8)$$

The study of these systems can be traced back to the work of Rosenbloom ([133]). Bellman ([134]) studied the second moment stability of jump linear systems using the Kronecker matrix product where the form process is an independent identically distributed random process. Using a stochastic version of Lyapunov's second method, Bertram and Sarachik ([37]) and Kats and Krasovskii ([36]) obtained some sufficient conditions for second moment stability for jump systems with a finite state Markov chain form process. In a series of papers, Krasovskii and Lidskii ([34]) considered jump control systems which are undergoing random variations during the control process and are perturbed by random interference. Bergen ([135]) generalized Bellman's ([134]) idea and studied the second moment stability of continuous-time jump linear systems of type (1.2.7). Then Bharucha ([136]) further generalized this Kronecker product technique to the study p -th moment stability (p is a positive integer) and obtained a necessary and sufficient condition. Bharucha also

showed that asymptotic moment stability is equivalent to exponential moment stability. Morozan ([51]) studied the case where $\{\sigma_t\}$ has piecewise constant sample paths but may not be Markovian. Darhovskii and Leibovich ([66]) investigated systems where the time intervals between jumps are independent identically distributed random variables and the systems' modes are governed by a finite state Markov chain, a necessary and sufficient condition for the system to be second moment stable was derived. Other research works related to the stability of jump linear systems are summarized in Kozin ([38]).

Jump control systems have also been intensively studied from the perspective of controller design. For example, the linear quadratic optimal control designs of (1.2.3) or (1.2.4) attracted considerable attention in the 60's and 70's. Krasovskii and Lidskii ([34]) were the first to consider this problem. Then Florentin ([63]) studied the optimal control design for continuous-time Markov stochastic systems. Sworder ([39]-[50]) generalized the optimal design methods for linear systems to jump linear control systems and obtained the so-called *stochastic maximum principle*, which was also investigated deeply by Wonham ([35]). Sworder and Rogers ([9]) applied this controller design method to a solar thermal central receiver, Blair and Sworder ([31]) used this approach for the regulation of a class of econometric models, Blair and Sworder ([32]), Birdwell et al ([55]) and Chizeck ([64]) used the quadratic regulator design methodology to analyze the reliability of fault-tolerant control systems. Morozan ([52],[53]) considered the infinite-time linear quadratic control problem and introduced the notion of second moment stochastic stability and stabilizability, some very important results were obtained. Chizeck et al ([56]) used the linear quadratic optimization approach to study the stabilization problem and presented some interesting examples. Hopkins ([54]) used a similar idea to

study the simultaneous stabilization problem for continuous-time jump linear systems by introducing a sequence of finite horizon parameter optimization problem. In order to rigorously solve the infinite-time linear quadratic optimal control problem for jump linear systems and understand the asymptotic behavior of the optimal closed-loop systems, Ji and Chizeck ([67]-[72]) studied the stochastic controllability and stochastic observability properties of the system. This solved the linear quadratic optimal design problem when the form process $\{\sigma_t\}$ or $\{\sigma_k\}$ is observable. In some recent works second moment stabilization problems have been investigated, refer to Ji et al ([73]), Feng et al ([74]) and Feng ([75]), where they have proved the equivalence between basic second moment stability concepts. When $\{\sigma_t\}$ or $\{\sigma_k\}$ is not observable, the linear quadratic optimal control problem becomes a dual control problem, which was investigated by Griffiths and Loparo ([80]) and Casiello and Loparo ([81]). Campo and Bar-Shalom ([179]) considered a sub-optimal linear quadratic design for this case. Mariton ([82]-[89]) also studied stochastic controllability, observability, stabilizability and linear quadratic optimal control problems for continuous-time jump linear control systems, this work is discussed in the book ([7]). Hijab ([65]) considered the stabilization and dual control problem for a continuous-time jump linear system with a finite state form process and a sufficient condition for the dual problem to be solvable is obtained: the well-known analytic solvability condition for a partial differential equation. Caines and Chen ([62]) used Hijab's ([65]) PDE technique to obtain an optimal adaptive LQG control policy for jump linear systems with a finite state form process. Šiljak ([90]) and Ladde and Šiljak ([91]) combined jump linear system theory and decentralized control methodology to study the reliability of complex control systems. This work suggests the importance of jump linear systems in the modeling and analysis of large-scale stochastic systems.

Jump system modeling is also relevant to the study of structural stability in civil engineering. For structural engineers, beams and plates are subject to random forces at the boundaries and this can be modeled as a dynamic system with parametric random excitation. Kozin ([38], [178]) reviewed some of stability results for jump systems (1.2.7) whose form process $\{\sigma_t\}$ is a white noise process.

We observe that in order to obtain a reasonable control design, jump control systems should be stochastically stabilizable, i.e., the closed-loop jump systems should be stochastically stable, therefore, the study of stochastic stability of (1.2.5)-(1.2.8) becomes vitally important. A natural approach to the study of stochastic stability is to generalize Lyapunov's second method ([99]) to stochastic systems. A stochastic version of Lyapunov's second method was developed almost simultaneously by Bertran and Sarachik ([37]) in the U.S.A. and Kats and Krasovskii ([36]) in the former U.S.S.R. Kushner ([94]) and Has'minskii ([95]) systematically investigated the stability properties of general stochastic systems. In his comprehensive survey, Kozin ([38]) clarified many confusing stability concepts (which still mislead many researchers today) and gave a nice explanation of the relationship among various stochastic stability concepts. Belle Isle and Kozin ([175]) and Belle Isle ([176]) generalized Lyapunov's second method to derive some sufficient conditions for almost sure sample stability of continuous-time linear differential equations with random time-varying communication delays. Krtolica et al ([21]), Morozan ([52],[53]), Ji et al ([73]), Feng et al ([74]) and Mariton ([7]) used this approach to study the stochastic stability of jump linear systems (1.2.7) or (1.2.8). Hijab ([65]) used this approach to study the stabilizability of jump linear systems.

It is noted by Kozin ([38], [129]) that almost sure stability does not imply moment stability for stochastic systems, and usually second moment stability criteria for almost sure stability are too conservative to be useful in practical applications. This has been illustrated by Mitchell and Kozin ([130]) and Kozin and Sugimoto ([131]), where it is shown that the regions for second moment stability are considerably smaller than the ones for almost sure stability. Practically we are interested in the stability of individual sample paths of the system rather than the stability of the “average system” or the “ensemble of all possible systems” ([176]). Therefore, almost sure stability is a more useful concept than moment stability. The Lyapunov exponent method is, in some sense, the only approach to obtain the tightest criteria for almost sure stability ([96], [144]). Generally, the system is almost surely stable if and only if the top Lyapunov exponent is negative. After the introduction of the concept of Lyapunov exponents by Lyapunov ([99]) in 1892, this concept has been extensively and intensively studied by mathematicians and scientists to investigate the stability properties of dynamic systems. This concept has been generalized to the study of stochastic dynamical systems. The fundamental papers of Lyapunov exponents are due to Furstenberg ([101], [102]), Oseledec ([100]) and Has'minskii ([111]). The intensive research in this area is evidenced by the works reported in the two research monographs by Arnold and his school ([96], [144]), the monographs by Bougerol and Lacroix ([95]) and by Kifer ([98]) and the comprehensive review paper by Gol'dsheid and Margulis ([106]). This approach can be effectively used to study almost sure stability of jump linear systems. Li and Blankenship ([126]) investigated the Lyapunov exponents for a jump linear system whose form process is a Poisson point process, Arnold et al ([123]), Arnold and Kloeden ([124]), Pinsky ([107],[122]), Feng and Loparo

([78]), Loparo and Feng ([79]) studied the Lyapunov exponents and instability properties of the random harmonic oscillator, i.e., the jump linear system (1.2.7) with a telegraphic form process. This can be considered as an approximate model for a single electrical machine connected to an infinite bus, see Loparo and Blankenship ([121]) and Loparo ([182] or [183]). When the form process $\{\sigma_t\}$ or $\{\sigma_k\}$ of the jump linear system (1.2.7) or (1.2.8) is a finite state Markov chain, Feng and Loparo ([76],[77]) obtained a nonrandom spectrum theorem which is an improvement to Oseledec's multiplicative theorem ([100]) for this class of jump linear systems. Leizarowitz ([125], [153]) also considered the Lyapunov exponents of jump linear systems driven by a finite state Markov chain. It seems from this approach that we can completely solve the almost sure stability problem for jump linear systems by determining the sign of the top exponent. However, the computation of the top Lyapunov exponent, even the determination of the sign of the top Lyapunov exponent, is a very complicated and computationally difficult task as demonstrated by Feng ([75]) and Loparo and Feng ([78]). Thus, the estimation of the top Lyapunov exponent becomes a very important research area. Leizarowitz ([153]) obtained some preliminary results on this issue for continuous-time jump linear system (1.2.7) driven by a finite state Markov chain.

Arnold ([115]) discovered a formula connecting the sample path and moment stability properties of jump linear systems (1.2.7) where the form process $\{\sigma_t\}$ is a Poisson random point process. Arnold et al ([116], [117]) generalized this formula. Du and Nhung ([161]) attempted to generalize Arnold's results to a jump linear system (1.2.8) where the form process is independent identically distributed, however the main results are incorrect. Leizarowitz ([153]) heuristically showed that Arnold's ([115]) approach can not be used for

the jump linear system (1.2.7) driven by a finite state Markov chain. However, he did not give a definite answer to whether Arnold's is valid or not for jump linear systems. We can easily see from Arnold's formula that the δ -moment top Lyapunov exponent is differentiable and its derivative at zero is just the top Lyapunov exponent. For one dimensional jump linear system (1.2.7) with a finite state Markov chain, Feng et al ([74]) and Feng ([75]) proved that the region for δ -moment stability monotonically increases and asymptotically converges to the region for almost sure stability as δ approaches to zero from above. This result is very important because it characterizes the close relationship between almost sure stability and moment stability. We believe that this relationship continues to hold for higher dimensional jump linear systems of the types (1.2.7) or (1.2.8) with a finite Markov chain form process, this will be investigated in this dissertation.

1.3. Research Topics

Stochastic stability is a qualitative property basic to all systems natural as well as man-made, and it is usually the first characteristic to be considered in practical applications. This dissertation will be devoted to the study of the stochastic stability of jump linear systems (1.2.7) or (1.2.8) with a finite state Markov chain form process.

In Chapter two, we study the stochastic stability of discrete-time jump linear systems (1.2.8) with a finite state Markov chain. The second moment stability has been studied by many researchers, especially for the system (1.2.8) where $\{\sigma_k\}$ is a finite state independent identically distributed chain. We study the δ -moment stability properties where $\delta > 0$ (not necessarily interger)

among various moment stochastic stability concepts and the relationship between δ -moment stability and almost sure stability. In this way, we obtain a new approach to the study of almost sure (sample path) stability of jump linear systems. Moreover, a new class of Lyapunov functions are proposed to obtain a new sufficient condition for almost sure stability. Some simpler sufficient conditions for both δ -moment and almost sure stability will be given, and stabilization approaches are suggested. Finally, the analytic properties of δ -moment Lyapunov exponents will be studied in this chapter.

Chapter three will be devoted to the study of the stochastic stability of continuous-time jump linear system (1.2.7) with a finite state Markov chain. Since the sampled discrete version of the continuous-time jump linear system is not a type of (1.2.8), the stochastic stability analysis method for discrete-time jump linear system (1.2.8) developed in chapter two can not be applied to the continuous case, some modification is made to obtain some new results. Some incorrect results in the recent literature are corrected and modified and some unresolved problems are solved.

It is very important to note that the moment stability results is closely related to the large deviations in random sequence. By realizing this point, we can use the large deviation theory developed in probability theory and statistics to study the moment stability of jump linear systems. To the author's knowledge, this is the first work that has been done in the current literature to use the large deviation theorem to study the stability of stochastic systems. It seems that this approach is potential for the study of stability of general stochastic systems and should be generalized in the future.

CHAPTER TWO

STABILITY OF DISCRETE-TIME JUMP LINEAR SYSTEMS

This chapter is devoted to the study of the stability of discrete-time jump linear systems with a finite state Markov chain form process. A brief review of stochastic stability results in this direction is given in the first section, then in the second section we discuss the second moment stability, which is the simplest case in stochastic stability analysis. Two approaches has been used here, one is the Lyapunov coupling equations method, and the other is the Kronecker matrix product approach. It can be seen that necessary and sufficient conditions can be obtained for second moment stability. In order to study almost sure stability, δ -moment stability concept is introduced in the third section and equivalence relationship among various concepts of stochastic stability is obtained and the relationship between the region of moment stability and the one of almost sure stability is illustrated. Also the differentiability property of top δ -moment Lyapunov exponent are discussed. Following this study, some testable criteria for δ -moment stability are presented in fourth section. In fifth section, we present new results for almost sure stability and some interesting algebraic testable conditions are obtained. Although the Lyapunov exponents can provide the best possible stability criteria, it is witnessed that the computation of the Lyapunov exponents is extremely difficult and sometimes is impossible, therefore we attempt to give some estimations for the Lyapunov exponents in the sixth section. In seventh section, we briefly discuss the almost sure stabilization problem followed by the eighth section which briefly deals with robust stability of jump linear systems. Finally, we will present

some illustrative examples to show how our results can be applied to solve almost sure stability and moment stability.

2.1. INTRODUCTION

Consider the discrete-time jump linear system in the form

$$x_{k+1} = H(\sigma_k) x_k \tag{2.1.1}$$

where $\{\sigma_k\}$ is a finite state independent identically distributed (iid) random process or a discrete time Markov chain. Stability analysis of systems of this type can be traced back to the work of Rosenbloom ([133]), who was interested in the moment stability properties. Bellman ([134]) was the first to study the moment stability of (2.1.1) with an iid form process using the Kronecker matrix product. Later, Bhuracha ([136]) used Bellman's idea developed in [134] to generalize some old results and studied both asymptotic stability of the mean and exponential stability of the mean.

There is an alternative approach to the study of stochastic stability. Kats and Krasovskii ([36]) and Bertram and Sarachik ([37]) used a stochastic version of Lyapunov's second method to study almost sure stability and moment stability. Unfortunately, constructing an appropriate Lyapunov function is difficult in general, this is a common disadvantage of Lyapunov's second method. Also, in many cases, the criteria obtained from this method are similar to moment stability criteria, which are often too conservative. For certain classes of systems, such as (2.1.1), it is possible to obtain testable stability conditions. For second moment stability, Kats and Krasovskii ([36]) and Morozan ([52]) discovered the necessary and sufficient condition and further proved some equivalence relationship among several concepts of stochastic stability. Ji and

Chizeck ([72]) and Ji et al ([73]) rediscovered such necessary and sufficient conditions and studied the stabilization problem using such conditions. When the form process $\{\sigma_k\}$ is an inhomogenous Markov chain, Krtolica et al ([21]) obtained a necessary and sufficient condition in terms of infinite number of coupling Lyapunov equations, although it is theoretically interesting, it can not be checked in practice.

As Kozin ([38], [120], [129]) pointed out, moment stability implies almost sure stability under fairly general conditions, but the converse is not true. In practical applications, almost sure stability is more than often the desirable property because we can only observe the sample path behavior of the system and the moment stability criteria are sometimes too conservative to be practically useful.

Although Lyapunov exponent techniques may provide necessary and sufficient conditions for almost sure stability ([74],[75],[95],[96],[114],[115],[154]), it is very difficult to compute the top Lyapunov exponent or to obtain good estimates of the top Lyapunov exponent for almost sure stability. As a result, testable conditions are difficult to obtain from this theory.

Arnold et al ([115],[116]) studied the relationship between the top Lyapunov exponent and the δ -moment top Lyapunov exponent for a diffusion process. Using a similar idea, Leizarowitz ([153]) obtained similar results for continuous-time jump linear systems. A general conclusion was that δ -moment stability implies almost sure stability. Thus sufficient conditions for almost sure stability can be obtained through δ -moment stability, which is one of the motivations for study of δ -moment stability. There are many definitions for moment stability: δ -moment stability, exponential δ -moment stability and stochastic

δ -moment stability. Ji et al ([73]) proved that all second moment ($\delta = 2$) stability concepts are equivalent for the system (2.1.1). Feng et al ([74]) showed that all second moment stability concepts are equivalent for the continuous-time jump linear systems, and also proved that for a one dimensional system, the region for δ -moment stability is monotonically converging to the region for almost sure stability as $\delta \downarrow 0^+$. This is tantamount to concluding that almost sure stability is equivalent to δ -moment stability for sufficiently small δ . This is a significant result because the study of almost sure stability can be reduced to the study of δ -moment stability.

Suppose that $\{\sigma_k\}$ is a finite state Markov chain with state space \underline{N} , transition probability matrix $P = (p_{ij})_{N \times N}$ and initial distribution $p = (p_1, \dots, p_N)$ (Certainly, the iid process is a particular Markov chain where each row of P is the same as the initial distribution p). For simplicity, assume that the initial state $x_0 \in \mathcal{R}^n$ is a (nonrandom) constant vector. Let (Ω, \mathcal{F}, P) denote the underlying probability space and let Ξ be the collection of all probability distribution on \underline{N} . Let $e_i \in \Xi$ be the initial distribution concentrated at the i^{th} state, i.e., given by $P\{\sigma_0 = i\} = 1$. Sometimes, we need to signify that some properties are dependent on the choice of the initial distribution of the Markovian form process $\{\sigma_k\}$. If so, for each $\xi \in \Xi$, let P_ξ denote the probability measure for the Markov chain $\{\sigma_k\}$ induced by the initial distribution ξ and E_ξ the expectation with respect to P_ξ . Let $\pi = (\pi_1, \dots, \pi_N)$ be the unique invariant probability distribution for the Markov chain $\{\sigma_k\}$, if the chain possesses a single ergodic (indecomposable) class. Details of Markov chain can be referred to Doob ([137]), Taylor and Karlin ([138]), Loeve ([141]) and Shiriyayev ([150]). For a matrix $C = (c_{ij})$, let $|C| = (|c_{ij}|)$. Some definitions of different stability concepts for jump linear systems are presented next.

DEFINITION 2.1.1:

Let Φ be a subset of Ξ . The jump linear system (2.1.1) with a Markovian form process $\{\sigma_k\}$ as specified above is said to be

- (I). *(asymptotically) δ -moment stable with respect to (w.r.t.) Φ* , if for any $x_0 \in R^n$ and any initial probability distribution $\psi \in \Phi$ of σ_k ,

$$\lim_{k \rightarrow \infty} E \{ \|x_k(x_0, \omega)\|^\delta \} = 0,$$

where $x_k(x_0, \omega)$ is a sample solution of (2.1.1) initial from $x_0 \in R^n$. If $\delta = 2$, we say that the system (2.1.1) is *asymptotically mean square stable w.r.t. Φ* . If $\delta = 1$, we say that the system (2.1.1) is *asymptotically mean stable w.r.t. Φ* . If $\Phi = \Xi$, we simply say the system (2.1.1) is *asymptotically δ -moment stable*. Similar statements apply to the following definitions.

- (II). *exponentially δ -moment stable w.r.t. Φ* , if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Phi$ of σ_k , there exist constants $\alpha, \beta > 0$ independent of x_0 and ψ such that

$$E \{ \|x_k(x_0, \omega)\|^\delta \} \leq \alpha \|x_0\|^\delta e^{-\beta k}, \quad \forall k \geq 0.$$

- (III). *stochastically δ -moment stable w.r.t. Φ* , if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Phi$ of σ_k ,

$$\sum_{k=0}^{\infty} E \{ \|x_k(x_0, \omega)\|^\delta \} < +\infty.$$

- (IV). *almost surely (asymptotically) stable w.r.t. Φ* , if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Xi$ of σ_k ,

$$P \left\{ \lim_{k \rightarrow \infty} \|x_k(x_0, \omega)\| = 0 \right\} = 1.$$

(V). *mean value stable* w.r.t. Φ , if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Phi$ of σ_k ,

$$\lim_{k \rightarrow \infty} E \{x_k(x_0, \omega)\} = 0.$$

In the case when $\{\sigma_k\}$ is actually an iid process with distribution $p = (p_1, \dots, p_N)$, all the above definitions hold with Φ being the singleton set $\Xi = \Phi = \{p\}$. \square

The above definitions are consistent with those given in [73] and [74], and we want to remind the reader of the dependence on the initial probability distribution of the form process $\{\sigma_k\}$ for the Markovian case. The “state” for the jump linear system is the joint process (x_k, σ_k) , even though the initial distribution of the form process may not be known. Thus, it is reasonable that the stability properties as given are independent of the initial distributions. Of course, for a Markov chain with a single ergodic class, the almost sure (sample) stability only depends on the probability measure P_π with the initial distribution π . Then, if the system is P_π -almost surely stable, then it is also almost surely stable (or P_ξ -almost surely stable for any $\xi \in \Xi$). However, this may not be the case for the δ -moment stability. The following simple example illustrates this point and justifies the practical importance of having the stability definitions independent of the initial distribution.

Example 2.1.2:

Consider the scalar system (2.1.1) with $H(1) = h_1 > 0$ and $H(2) = h_2$ with $0 < h_2 < 1$. The form process $\{\sigma_k\}$ has a transition matrix

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 1 \end{pmatrix}$$

and initial distribution $\xi = (\xi_1, \xi_2)$. Clearly, the unique invariant distribution is given by $\pi = e_2 = (0, 1)$ and the system is P_ξ -almost surely stable, regardless of ξ . However, for any $\delta > 0$ and $x_0 \neq 0$, we have

$$\begin{aligned} E_\xi |x_n(\omega, x_0)|^\delta &= E_\xi |H(\sigma_{n-1}) \dots H(\sigma_0)x_0|^\delta \\ &= \sum_{i=1}^2 \xi_i E_{e_i} |H(\sigma_{n-1}) \dots H(\sigma_0)x_0|^\delta. \end{aligned} \quad (2.1.2)$$

Also, we have

$$\begin{aligned} &E_{e_1} |H(\sigma_{n-1}) \dots H(\sigma_0)x_0|^\delta \\ &= \sum_{i_1, \dots, i_{n-1}} p_{1i_1} p_{i_1 i_2} \dots p_{i_{n-2} i_{n-1}} |H(i_{n-1})|^\delta \dots |H(i_1)|^\delta |H(1)|^\delta |x_0|^\delta \\ &\geq \left(\frac{1}{2}\right)^n h_1^{n\delta} |x_0|^\delta = \left(\frac{1}{2} h_1^\delta\right)^n |x_0|^\delta \end{aligned} \quad (2.1.3)$$

and

$$\begin{aligned} &E_{e_2} |H(\sigma_{n-1}) \dots H(\sigma_0)x_0|^\delta \\ &= \sum_{i_1, \dots, i_n} p_{2i_1} p_{i_1 i_2} \dots p_{i_{n-2} i_{n-1}} |H(i_{n-1})|^\delta \dots |H(i_1)|^\delta |H(2)|^\delta |x_0|^\delta \\ &= h_2^{n\delta} |x_0|^\delta = (h_2^\delta)^n |x_0|^\delta \longrightarrow 0, \quad n \rightarrow +\infty \end{aligned} \quad (2.1.4)$$

From (2.1.2), (2.1.3) and (2.1.4), we see that

$$\lim_{n \rightarrow +\infty} E_\pi |x_n(\omega, x_0)|^\delta = \lim_{n \rightarrow +\infty} E_{e_2} |x_n(\omega, x_0)|^\delta = 0.$$

However, for any $\xi = (\xi_1, \xi_2)$ with $\xi_1 > 0$, as long as $h_1^\delta > 2$, we have

$$\lim_{n \rightarrow +\infty} E_\xi |x_n(\omega, x_0)|^\delta \geq \lim_{n \rightarrow +\infty} \xi_1 E_{e_1} |x_n(\omega, x_0)|^\delta = +\infty$$

In this case, the system is “ δ -moment stable”, if $\xi = \pi$, i.e., if the chain $\{\sigma_k\}$ is stationary, and the system is not “ δ -moment stable” for any other initial distribution ξ . Therefore, δ -moment stability with respect to $\Phi = \{\pi\}$ is not a good criteria to be used in practice because a small perturbation of ξ from

π will make the system unstable. The δ -moment stability definition should therefore be “independent” of the initial distribution as given in Definition 2.1.

In the above example, the form process has a single ergodic class $\{2\}$ as well as a transient state, namely 1. If the form process is irreducible, i.e., satisfies the property that each pair of state communicates, or that the unique invariant distribution π is strictly positive, then, the definitions in definition 2.1.1 are equivalent to the usual stability definitions for a system with a stationary form process. This result is formalized next.

Lemma 2.1.3:

For system (2.1.1) with a finite state and time homogeneous form process, if the chain is irreducible (or indecomposable) with a unique invariant distribution π , then the system is stable in any of above senses if and only if the system is stable in the same sense with respect to $\Phi = \{\pi\}$. \square

Proof: The proof of necessity is trivial. For sufficiency, notice that since $\pi > 0$, it is easy to see that $P_\xi \ll P_\pi$ (P_ξ is absolutely continuous with respect to P_π) for any $\xi \in \Xi$. Thus, P_π -almost sure stability implies P_ξ -almost sure stability. For moment properties, say, δ -moment stability, notice that for any $\xi = (\xi_1, \dots, \xi_N)$,

$$E_\xi \|x_k(\omega, x_0)\|^\delta = \sum_{i=1}^N \xi_i E_{e_i} \|x_k(\omega, x_0)\|^\delta$$

Since $\pi = (\pi_1, \dots, \pi_N) > 0$, $\lim_{k \rightarrow +\infty} E_\pi \|x_k(\omega, x_0)\|^\delta = 0$ implies that $\lim_{k \rightarrow +\infty} E_{e_i} \|x_k(\omega, x_0)\|^\delta = 0$ for all $i \in \underline{N}$. This implies that

$$\lim_{k \rightarrow +\infty} E_\xi \|x_k(\omega, x_0)\|^\delta = 0, \quad \forall \xi \in \Xi.$$

We conclude that if we are dealing with an irreducible Markov chain form process, then it is only necessary to study stability with respect to $\Phi = \{\pi\}$.

2.2. SECOND MOMENT STABILITY

In this section, we study the second moment stability (or mean square stability) of discrete-time jump linear system (2.1.1). As we mentioned earlier, we can use a stochastic version of Lyapunov's second method to study stochastic stability. A natural candidate for a Lyapunov function is an appropriately chosen quadratic form. Morozan ([53]) and Ji et al ([73]) obtained the following necessary and sufficient condition.

Theorem 2.2.1.

Suppose that $\{\sigma_k\}$ is a finite state time homogenous Markov chain with probability transition matrix P , then the system (2.1.1) is second moment stochastically stable if and only if for any given positive matrices $Q(1), Q(2), \dots, Q(N)$, there exists positive definite matrices $P(1), P(2), \dots, P(N)$ such that

$$\sum_{j=1}^N p_{ij} H_i^T P(j) H_i - P(i) = -Q(i), \quad i = 1, 2, \dots, N. \quad (2.2.1)$$

Proof. The Lyapunov function can be chosen as $V(x_k, \sigma_k) = x_k^T P(\sigma_k) x_k$, then the rest of the proof follows from traditional approach (see Ji et al ([73]) for detail) . □

Remark: From Ji et al ([73]), Theorem 2.2.1 is also a necessary and sufficient condition for second moment and exponential second moment stability, see Theorem 2.2.5 given later.

An interesting observation is that the above Lyapunov function has the following feature: x_k is measurable with respect to the σ -algebra generated

by $\sigma_{k-1}, \sigma_{k-2}, \dots$, and the matrix $P(\sigma_k)$ depends only on σ_k . If we use the Lyapunov function $V(x_k, \sigma_k) = x_k^T R(\sigma_{k-1}) x_k$, this leads to the following result.

Theorem 2.2.2.

Suppose that $\{\sigma_k\}$ is a finite state time homogenous Markov chain with probability transition matrix P , then the system (2.1.1) is second moment stochastically stable if and only if for any given positive matrices $S(1), S(2), \dots, S(N)$, there exists positive definite matrices $R(1), R(2), \dots, R(N)$ such that

$$\sum_{j=1}^N p_{ij} H^T(j) R(j) H(j) - R(i) = -S(i), \quad i = 1, 2, \dots, N. \quad (2.2.2)$$

Proof. This can be proved using a similar procedure as in the proof of Theorem 2.2.1 and the Lyapunov function $V(x_k, \sigma_k) = x_k^T R(\sigma_{k-1}) x_k$. \square

Surprisingly, the necessary and sufficient conditions given in Theorem 2.2.1 and 2.2.2 are equivalent. We have

Theorem 2.2.3.

Equation (2.2.1) has a positive definite solution $P(1), P(2), \dots, P(N)$ for some positive definite matrices $Q(1), Q(2), \dots, Q(N)$ if and only if (2.2.2) has a positive definite solution $R(1), R(2), \dots, R(N)$ for some positive definite matrices $S(1), S(2), \dots, S(N)$.

Proof. Suppose that for some positive definite matrices $Q(1), Q(2), \dots, Q(N)$, (2.2.1) has a positive definite solution $P(1), P(2), \dots, P(N)$, let

$$R(i) = \sum_{j=1}^N p_{ij} P(j), \quad S(i) = \sum_{j=1}^N p_{ij} Q(j), \quad i = 1, 2, \dots, N$$

then we have

$$\begin{aligned}
& \sum_{j=1}^N p_{ij} H^T(j) R(j) H(j) - R(i) \\
&= \sum_{j=1}^N p_{ij} H^T(j) \left(\sum_{k=1}^N p_{jk} P(k) \right) H(j) - R(i) \\
&= \sum_{j=1}^N p_{ij} \left(\sum_{k=1}^N p_{jk} H^T(j) P(k) H(j) \right) - R(i) \\
&= \sum_{j=1}^N p_{ij} (P(j) - Q(j)) - R(i) \\
&= - \sum_{j=1}^N p_{ij} Q(j) = -S(i),
\end{aligned}$$

thus, $R(1), R(2), \dots, R(N)$ is a solution to (2.2.2) with the above defined positive matrices $S(1), S(2), \dots, S(N)$.

Conversely, suppose that for some positive definite matrices $S(1), S(2), \dots, S(N)$, (2.2.2) has a positive definite solution $R(1), R(2), \dots, R(N)$. Because of the positive definiteness of $S(1), S(2), \dots, S(N)$, there exists a positive number $\alpha > 0$ such that $S(i) - \alpha I$ is positive definite for any $i \in \{1, 2, \dots, N\}$. Define

$$P(i) = \alpha I + H^T(i) R(i) H(i), \quad Q(i) = \alpha I + H^T(i) (S(i) - \alpha I) H(i),$$

then we have

$$\begin{aligned}
& \sum_{j=1}^N p_{ij} H^T(i) P(j) H(i) - P(i) \\
&= H^T(i) \left(\sum_{j=1}^N p_{ij} P(j) \right) H(i) - P(i) \\
&= H^T(i) \left(\alpha I + \sum_{j=1}^N p_{ij} H^T(j) R(j) H(j) \right) H(i) - P(i)
\end{aligned}$$

$$\begin{aligned}
&= \alpha H^T(i)H(i) + H^T(i)(R(i) - S(i))H(i) - P(i) \\
&= \alpha H^T(i)H(i) + H^T(i)R(i)H(i) - P(i) - H^T(i)S(i)H(i) \\
&= \alpha H^T(i)H(i) - \alpha I - H^T(i)S(i)H(i) = -Q(i),
\end{aligned}$$

and $P(i)$ and $Q(i)$ ($i = 1, 2, \dots, N$) are positive definite, $P(i)$ is a solution to (2.2.1). This completes the proof. \square

Remark: From (2.2.2) and the theory of Lyapunov equations, we can easily obtain that the Schur stability of $\sqrt{p_{ii}}H(i)$ ($i \in \underline{N}$) is a necessary condition for second moment stability.

We call (2.2.1) or (2.2.2) the *Coupled Lyapunov equations*. It is not obvious as to see which of the above two necessary and sufficient conditions is better for practical applications. For the general finite state Markovian case, solving (2.2.1) and (2.2.2) requires solving N coupled matrix equations. However, for some special cases, Theorem 2.2.2 does provide an easier test for stochastic stability. We have

Corollary 2.2.4.

Suppose that $\{\sigma_k\}$ is a finite state independent identically distributed (iid) random sequence with probability distribution $\{p_1, p_2, \dots, p_N\}$, then the system (2.1.1) is second moment stochastically stable if and only if for some positive definite matrix S there exists a positive definite solution R to the following matrix equation

$$\sum_{i=1}^N p_i H^T(i) R H(i) - R = -S.$$

Proof. This is direct consequence of Theorem 2.2.2. \square

Remark: For the iid case, if we apply Ji et al's result Theorem 2.2.1, then we need to solve N coupled Lyapunov equations, which is more complicated than Corollary 2.2.4.

In the above, we only considered second moment stochastic stability. This is not a problem because the following result shows that the above conditions are also necessary and sufficient for the second moment stability and second moment exponential stability.

Theorem 2.2.5 (Morozan [53], Ji et al [73])

Second moment stability, second moment stochastic stability and exponential second moment stability of (2.1.1) with a time-homogenous finite state Markov chain $\{\sigma_k\}$ are equivalent, and all imply almost sure (sample path) stability. \square

As an illustration, we apply Theorem 2.2.2 to the one dimensional case.

Example 2.2.1.

Suppose that $H(i) = a_i$ ($i \in \underline{N}$) are scalars and define

$$A = \begin{pmatrix} p_{11}a_1^2 & p_{12}a_2^2 & \cdots & p_{1N}a_N^2 \\ p_{21}a_1^2 & p_{22}a_2^2 & \cdots & p_{2N}a_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1}a_1^2 & p_{N2}a_2^2 & \cdots & p_{NN}a_N^2 \end{pmatrix}.$$

We want to find a necessary and sufficient condition for (2.1.1) to be second moment stable. Before we proceed, we quote the following result which will be needed in this example: (In this example only, we use the notation $A \geq B$ to denote elementwise inequalities and $\rho(A)$ denotes the spectral radius of a matrix A .)

Lemma A ([149], p. 493) Given a matrix $A \geq 0$, and a vector $x > 0$ satisfying $\alpha x \leq Ax \leq \beta x$ for positive number α and β , then $\alpha \leq \rho(A) \leq \beta$. If $\alpha x < Ax$, then $\alpha < \rho(A)$. If $Ax < \beta x$, then $\rho(A) < \beta$. \square

We start with a necessary condition. Suppose that (2.1.1) is second moment stable, then from Theorem 2.2.2, for $S(1) = S(2) = \dots = S(N) = 1$, there exists positive numbers $R(1), R(2), \dots, R(N)$ such that

$$\sum_{j=1}^N p_{ij} a_j^2 R(j) - R(i) = -1 \quad (i = 1, 2, \dots, N)$$

i.e.,

$$Ay - y = -c \quad (*)$$

where $y = (R(1), R(2), \dots, R(N))^T$ and $c = (1, 1, \dots, 1)^T$. Thus, from (*), we obtain $Ay = y - c < y$. Using Lemma A, we have $\rho(A) < 1$, i.e., A is Schur stable.

Next, we want to prove that this is also sufficient. In fact, suppose that A is Schur stable, i.e., $\rho(A) < 1$. Let $U = (u_{ij})_{N \times N}$, where $u_{ij} = 1$. It is easy to prove that for sufficiently small positive number ϵ , we have $\rho(A + \epsilon U) < 1$, and $A + \epsilon U$ is a positive matrix. By Frobenius-Peeron Theorem ([40]), there exists a positive vector $y > 0$ such that $(A + \epsilon U)y = \rho(A + \epsilon U)y$, i.e.,

$$Ay - y = \rho(A + \epsilon U)y - \epsilon U y - y < y - \epsilon U y - y = -\epsilon U y.$$

Let $R(i) = y_i$ ($i = 1, 2, \dots, N$), which are positive numbers that satisfy

$$\sum_{j=1}^N p_{ij} a_j^2 R(j) - R(i) < 0, \quad (i = 1, 2, \dots, N).$$

Then (2.2.2) is satisfied for this choice of $R(1), \dots, R(N)$ where the positive numbers $S(1), \dots, S(N)$ we suitably choose. From Theorem 2.2.2, we conclude

that (2.1.1) is second moment stable. In this way, we have proved that (2.1.1) is second moment stable if and only if A is Schur stable. \square

Krtolica et al ([21]) studied the second moment exponential stability of (2.1.1) with a time-inhomogenous finite state Markov chain form process $\{\sigma_k\}$, and obtained the following necessary and sufficient condition. In what follows, we use $A \leq B$ (or $A < B$) to denote that $B - A$ is positive semi-definite (or positive definite matrix for any symmetric matrices A and B , and $\underline{m} = \{1, 2, \dots, m\}$ for any integer m).

Theorem 2.2.6 (Krtolica et al [21])

Suppose that $\{\sigma_k\}$ is a time-inhomogenous finite state Markov chain with probability transition matrix $P = (p_{ij}(k))_{N \times N}$, then the system (2.1.1) is exponentially second moment stable if and only if for some positive definite (pd) matrix sequence $Q_k(1), Q_k(2), \dots, Q_k(N)$ ($k = 0, 1, 2, \dots$) satisfying

$$0 < c_1 I \leq Q_k(j) \leq c_2 I \quad (j = 1, 2, \dots, N), \quad \forall k \geq 0$$

for some positive constants c_1 and c_2 , there exist positive definite matrices $P_k(1), P_k(2), \dots, P_k(N)$ such that

$$\sum_{j=1}^N p_{ij}(k+1) H^T(i) P_{k+1}(j) H(i) - P_k(i) = -Q_k(i), \quad i \in \underline{N}, \quad \forall k \geq 0$$

where

$$0 < c_3 I \leq P_k(i) \leq c_4 I, \quad i \in \underline{N}, \quad \forall k \geq 0$$

for some positive constants c_3 and c_4 . \square

From theoretical point of view, this is an interesting result. However, to use this theorem, we need to solve an infinite number of coupled matrix equations, which is not testable in practical applications. Hopefully, the positive

definite solutions $P_k(1), \dots, P_k(N)$ may converge as k goes to infinity so that a finite set of algebraic conditions can be obtained. If the probability transition matrix is periodic in k , we can obtain the following testable condition.

Theorem 2.2.7.

Suppose that $\{\sigma_k\}$ is a finite state Markov chain with probability transition matrix $P = \Pi_k = (p_{ij}(k))$ satisfying $\Pi_{k+p} = \Pi_k$, then (2.1.1) is exponentially second moment stable if and only if for some positive definite matrices $Q_1(j), Q_2(j), \dots, Q_p(j)$ ($j \in \underline{N}$), there exists positive definite matrices $P_1(j), P_2(j), \dots, P_p(j)$ ($j \in \underline{N}$) such that the following hold:

$$\begin{aligned} \sum_{j=1}^N p_{ij}(l) H^T(i) P_{l+1}(j) H(i) - P_l(i) &= -Q_l(i), \quad l \in \underline{p-1} \\ \sum_{j=1}^N p_{ij}(p) H^T(i) P_1(j) H(i) - P_p(i) &= -Q_p(i), \quad i \in \underline{N}. \end{aligned} \tag{2.2.3}$$

In order to prove this, we need the following result.

Lemma 2.2.8.

- (1). If A and B are positive definite matrices, then there exists a nonsingular matrix T such that $T^T A T = I$ and $T^T B T$ is a diagonal matrix (T^T denotes the matrix transpose of T);
- (2). If A , B and $A - B$ are positive definite matrices, then there exists a nonsingular matrix T such that $T^T (A - B) T = I - \Lambda(BA^{-1})$, where $\Lambda(X)$ denotes the diagonal matrix whose diagonal elements are the eigenvalues of the matrix X ;

(3). If A , B and $A - B$ are positive definite matrices, then

$$0 < \lambda_{\min}(BA^{-1}) \leq \lambda(BA^{-1}) \leq \lambda_{\max}(BA^{-1}) < 1,$$

$$\lambda_{\min}(BA^{-1})x^T Ax \leq x^T Bx \leq \lambda_{\max}(BA^{-1})x^T Ax, \quad \forall x \in R^n.$$

Proof of Lemma 2.2.8. (1). Since $A > 0$, there exists a nonsingular T_1 such that $A = T_1^T T_1$. Because $T_1^{-T} B T_1^{-1} > 0$, there exists an orthogonal matrix T_2 satisfying $T_2^T T_2 = I$ and $T_2^T (T_1^{-T} B T_1^{-1}) T_2$ is diagonal. Let $T = T_1^{-1} T_2$, which is nonsingular, then we have $T^T A T = I$ and $T^T B T$ is diagonal.

(2). Let T be the matrix chosen in (1), then we have

$$T^T (A - B) T = I - T_2^T (T_1^{-T} B T_1^{-1}) T_2.$$

Since $T_2^T = T_2^{-1}$, it suffices to show that $\sigma(T_1^T B T_1^{-1}) = \sigma(BA^{-1})$, where $\sigma(X)$ denotes the spectrum of the matrix X . In fact, using the property of a similarity transformation, we have

$$\begin{aligned} \sigma(T_1^{-T} B T_1^{-1}) &= \sigma(T_1^T (T_1^{-T} B T_1^{-1}) T_1^{-T}) \\ &= \sigma(B T_1^{-1} T_1^{-T}) = \sigma(B (T_1^T T_1)^{-1}) = \sigma(BA^{-1}). \end{aligned}$$

(3). From (2), there exists a nonsingular T such that $T^T (A - B) T = I - \Lambda(BA^{-1})$. Since $A - B > 0$, hence $1 - \sigma(BA^{-1}) > 0$, i.e., $\lambda_{\max}(BA^{-1}) < 1$. It is also easy to prove that $\lambda_{\min}(BA^{-1}) > 0$ because of $B > 0$;

For any $x \neq 0$, we have

$$\frac{x^T B x}{x^T A x} = \frac{(T_1 x)^T (T_1^{-T} B T_1^{-1}) (T_1 x)}{(T_1 x)^T (T_1 x)},$$

we have

$$\lambda_{\min}(T_1^{-T} B T_1^T) \leq \frac{x^T B x}{x^T A x} \leq \lambda_{\max}(T_1^{-T} B T_1^T).$$

Because $\sigma(T_1^{-T} B T_1^T) = \sigma(BA^{-1})$, we have

$$\lambda_{\min}(BA^{-1}) \leq \frac{x^T B x}{x^T A x} \leq \lambda_{\max}(BA^{-1}),$$

which implies the desired inequalities. This completes the proof of Lemma 2.2.8. □

Now we can prove Theorem 2.2.7.

Proof of Theorem 2.2.7. Sufficiency: Define

$$\bar{P}_{kp+l}(j) = P_l(j), \quad j \in \underline{N}, \quad k \geq 0$$

$$\bar{Q}_{kp+l}(j) = Q_l(j), \quad l \in \underline{p-1}.$$

and define the following Lyapunov function

$$V(x_k, \sigma_k) = x_k^T \bar{P}_k(\sigma_k) x_k.$$

Since $P_l(j)$ ($j \in \underline{N}$, $l \in \underline{p}$) are positive definite, $V(\cdot, \cdot)$ is always a positive definite function.

Since

$$\begin{aligned} \Delta V(x_k, \sigma_k) &\stackrel{\text{def}}{=} V(x_{k+1}, \sigma_{k+1}) - V(x_k, \sigma_k) \\ &= x_k^T (H^T(\sigma_k) \bar{P}_{k+1}(\sigma_{k+1}) H(\sigma_k) - \bar{P}_k(\sigma_k)) x_k, \end{aligned}$$

we can obtain

$$\begin{aligned} &E(\Delta V(x_k, \sigma_k) | x_k = x, \sigma_k = i) \\ &= x^T E(H^T(\sigma_k) \bar{P}_{k+1}(\sigma_{k+1}) H(\sigma_k) - \bar{P}_k(\sigma_k) | \sigma_k = i) x. \end{aligned}$$

For any $k \geq 0$ and $l \in \underline{p-1}$, we obtain

$$\begin{aligned}
& E(\Delta V(x_{kp+l}, \sigma_{kp+l}) | x_{kp+l} = x, \sigma_{kp+l} = i) \\
&= x^T [E(H^T(\sigma_{kp+l}) \bar{P}_{kp+l+1}(\sigma_{kp+l+1}) H(\sigma_{kp+l}) | \sigma_{kp+l}) - \bar{P}_{kp+l}(i)] \\
&= x^T H^T(i) \left[\sum_{j=1}^N P(\sigma_{kp+l+1} = j | \sigma_{kp+l} = i) \bar{P}_{kp+l+1}(j) \right] H(i)x - x^T \bar{P}_{kp+l}(i)x \\
&= x^T H^T(i) \left[\sum_{j=1}^T P(\sigma_{l+1} = j | \sigma_l = i) P_{l+1} \right] H(i)x - x^T P_l(i)x \\
&= x^T H^T(i) \left[\sum_{j=1}^N p_{ij}(l) P_{l+1}(j) \right] H(i)x - x^T P_l(i)x \\
&= -x^T Q_l(i)x.
\end{aligned} \tag{2.2.4}$$

and for $l = p$, we have

$$\begin{aligned}
& E(\Delta V(x_{kp+p}, \sigma_{kp+p}) | x_{kp+p} = x, \sigma_{kp+p} = i) \\
&= x^T \left[\sum_{j=1}^N H^T(i) P(\sigma_{kp+p+1} = j | \sigma_{kp+p} = i) \bar{P}_{kp+p+1}(j) H(i) - \bar{P}_{kp+p}(i) \right] x \\
&= x^T \left[\sum_{j=1}^N H^T(i) p_{ij}(p) P_1(j) H(i) - P_p(i) \right] x \\
&= -x^T Q_p(i)x.
\end{aligned} \tag{2.2.5}$$

Let

$$\begin{aligned}
\mu_1 &= \min \{ \lambda_{\min}(Q_l(j) P_l(j)^{-1}) : 1 \leq l \leq p, 1 \leq j \leq N \} \\
\mu_2 &= \max \{ \lambda_{\max}(Q_l(j) P_l(j)^{-1}) : 1 \leq l \leq p, 1 \leq j \leq N \}.
\end{aligned}$$

From (2.2.3), we have $P_l(j) > Q_l(j)$ ($l \in \underline{p}$ and $j \in \underline{N}$), hence from Lemma 2.2.8, we obtain $0 < \mu_1 \leq \mu_2 < 1$. From (2.2.4) and (2.2.5), we have

$$\begin{aligned}
& E(\Delta V(x_{kp+l}, \sigma_{kp+l}) | x_{kp+l} = x, \sigma_{kp+l} = i) \\
&\leq -\mu_1 x^T P_l(i)x = -\mu_1 V(x, i), \quad l \in \underline{p-1}, i \in \underline{N}.
\end{aligned}$$

and

$$\begin{aligned} & E(\Delta V(x_{kp+p}, \sigma_{kp+p}) | x_{kp+p} = x, \sigma_{kp+p} = i) \\ & \leq -\mu_1 x^T P_p(i) x = \mu_1 V(x, i), \quad i \in \underline{N}. \end{aligned}$$

Combining the above two inequalities, we claim that for any $i \in \underline{N}$ and $k \geq 0$, we have

$$E(\Delta V(x_k, \sigma_k) | x_k = x, \sigma_k = i) \leq -\mu_1 V(x, i).$$

i.e.,

$$E(V(x_{k+1}, \sigma_{k+1}) | x_k = x, \sigma_k = i) \leq (1 - \mu_1) V(x, i).$$

This implies

$$E(V(x_{k+1}, \sigma_{k+1})) \leq (1 - \mu_1) E(V(x_k, \sigma_k)). \quad (2.2.6)$$

Let

$$\begin{aligned} \lambda_1 &= \min \{ \lambda_{\min}(P_l(j)) : l \in \underline{p}, j \in \underline{N} \}, \\ \lambda_2 &= \max \{ \lambda_{\max}(P_l(j)) : l \in \underline{p}, j \in \underline{N} \}. \end{aligned}$$

It is easy to show that $\lambda_2 \geq \lambda_1 > 0$ and from (2.2.6),

$$E\|x_k\|^2 \leq \frac{E\{V(x_k, \sigma_k)\}}{\lambda_1} \leq \frac{\lambda_2}{\lambda_1} E\|x_0\|^2 (1 - \mu_1)^k.$$

This implies that (2.1.1) is exponentially second moment stable.

Necessity: For any given positive definite matrices $Q_l(j)$ ($j \in \underline{N}$ and $l \in \underline{p}$), define $\bar{Q}_k(j)$ as before. Define

$$\begin{aligned} \Phi(m, k) &= H(\sigma_{m-1}) H(\sigma_{m-2}) \cdots H(\sigma_k), \quad m > k \\ \tilde{P}_k^K &= \sum_{m=k}^K \Phi^T(m, k) \bar{Q}_m(\sigma_m) \Phi(m, k), \quad k \geq 0 \\ \nu_1 &= \min \{ \lambda_{\min}(Q_l(j)) : l \in \underline{p}, j \in \underline{N} \} \\ \nu_2 &= \max \{ \lambda_{\max}(Q_l(j)) : l \in \underline{p}, j \in \underline{N} \}. \end{aligned}$$

We want to prove first that for all k

$$\tilde{P}_k^\infty = \sum_{m=k}^{\infty} \Phi^T(m, k) \bar{Q}_m(\sigma_m) \Phi(m, k)$$

are elements of the Hilbert space $\mathcal{L}^2(\Omega, \mathcal{F}, P)$. Suppose that (2.1.1) is exponentially second moment stable, i.e., there exists a positive $\alpha < 1$ and a $B > 0$ such that

$$E\|x_m\|^2 = E\|\Phi(m, k)z\|^2 \leq B\|x_k\|^2\alpha^{m-k}, \quad m > k,$$

then we have

$$\begin{aligned} E\|\tilde{P}_k^K\|^2 &= \sup_{\|x\|=1} E x^T (\tilde{P}_k^K)^T \tilde{P}_k^K x \left(\sup_{\|x\|=1} E x^T \tilde{P}_k^K x \right)^2 \\ &= \left(\sup_{\|x\|=1} E x^T \sum_{m=k}^K \Phi^T(m, k) \overline{Q}_m(\sigma_m) \Phi(m, k) x \right)^2 \\ &\leq \nu_2^2 \left(\sum_{m=k}^K E(\Phi(m, k)x)^T (\Phi(m, k)x) \right)^2 \\ &\leq \nu_2^2 B^2 \left(\|x\|^2 \sum_{m=k}^K \alpha^{m-k} \right)^2 \leq \frac{\nu_2^2 B^2 \|x\|}{1 - \alpha} < +\infty. \end{aligned}$$

This implies that $\tilde{P}_k^\infty \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$. Define

$$\tilde{P}_k = \tilde{P}_k^\infty = \sum_{m=k}^{\infty} \Phi^T(m, k) \overline{Q}(\sigma_m) \Phi(m, k), \quad k \geq 0$$

then we have

$$\tilde{P}_k = H^T(\sigma_k) \tilde{P}_{k+1} H(\sigma_k) + \overline{Q}_k(\sigma_k) \quad (k \geq 0).$$

Define $\tilde{P}_k(i) = E(\tilde{P}_k | \sigma_k = i)$ for $k \geq 0$, then

$$\tilde{P}_k(i) = H^T(i) E(\tilde{P}_{k+1} | \sigma_k = i) H(i) + \overline{Q}_k(i), \quad i \in \underline{N}, \quad (2.2.7)$$

and

$$\begin{aligned}
E(\tilde{P}_{k+1}|\sigma_k = i) &= \sum_{j=1}^N E(\tilde{P}_{k+1}|\sigma_{k+1} = j, \sigma_k = i)P(\sigma_{k+1} = j|\sigma_k = i) \\
&= \sum_{j=1}^N p_{ij}(k)E(\tilde{P}_{k+1}|\sigma_{k+1} = j) \\
&= \sum_{j=1}^N p_{ij}(k)\tilde{P}_{k+1}(j).
\end{aligned} \tag{2.2.8}$$

Next, we only need to prove that $\tilde{P}_k(i)$ is periodic in k with period p . In fact, we have

$$\begin{aligned}
\tilde{P}_{k+p}(i) &= E(\tilde{P}_{k+p}|\sigma_{k+p} = i) \\
&= E\left(\sum_{m=k+p}^{\infty} \Phi^T(m, k+p)\overline{Q}_m(\sigma_m)\Phi(m, k+p)|\sigma_{k+p} = i\right) \\
&= E\left(\sum_{m=k}^{\infty} \Phi^T(m+p, k+p)\overline{Q}_{m+p}(\sigma_{m+p})\Phi(m+p, k+p)|\sigma_{k+p} = i\right) \\
&= \sum_{m=k}^{\infty} E(H^T(\sigma_{k+p})\cdots H^T(\sigma_{m+p})\overline{Q}_{m+p}(\sigma_{m+p}) \\
&\quad \times H(\sigma_{m+p})\cdots H(\sigma_{k+p})|\sigma_{k+p} = i) \\
&= \sum_{m=k}^{\infty} \sum_{i_1, \dots, i_{m-k}} H^T(i)H^T(i_1)\cdots H^T(i_{m-k})\overline{Q}_{m+p}(i_{m-k})H(i_{m-k})\cdots H(i_1) \\
&\quad \times H(i)P(\sigma_{k+p+1} = i_1, \dots, \sigma_{m+p} = i_{m-k}|\sigma_{k+p} = i) \\
&= \sum_{m=k}^{\infty} \sum_{i_1, \dots, i_{m-k}} H^T(i)H^T(i_1)\cdots H^T(i_{m-k})\overline{Q}_m(\sigma_m)H(i_{m-k})\cdots H(i_1) \\
&\quad \times H(i)P(\sigma_{k+1} = i_1, \dots, \sigma_m = i_{m-k}|\sigma_k = i) \\
&= \sum_{m=k}^{\infty} E(H^T(\sigma_k)\cdots H^T(\sigma_m)\overline{Q}_m(\sigma_m)H(\sigma_m)\cdots H(\sigma_k)|d_k) = \tilde{P}_k(i).
\end{aligned}$$

In the above, we have used the periodicity of $\overline{Q}_k(i)$ in k and the fact that

$$\begin{aligned}
&P(\sigma_{k+p+1} = i_1, \dots, \sigma_{m+p} = i_{m-k}|\sigma_{k+p} = i) \\
&= P(\sigma_{k+1} = i_1, \dots, \sigma_m = i_{m-k}|\sigma_k = i).
\end{aligned}$$

This can be proved in the following:

$$\begin{aligned}
& P(\sigma_{k+p+1} = i_1, \dots, \sigma_{m+p} = i_{m-k} | \sigma_{k+p} = i) \\
&= P(\sigma_{k+p+1} = i_1 | \sigma_{k+p} = i) P(\sigma_{k+p+2} = i_2 | \sigma_{k+p+1} = i_1, \sigma_{k+p} = i) \times \dots \\
&\quad \times P(\sigma_{m+p} = i_{m-k} | \sigma_{m+p-1} = i_{m-k-1}, \dots, \sigma_{k+p+1} = i_1, \sigma_{k+p} = i) \\
&= P(\sigma_{k+p+1} = i_1 | \sigma_{k+p} = i) P(\sigma_{k+p+2} = i_2 | \sigma_{k+p+1} = i_1) \times \dots \\
&\quad \times P(\sigma_{m+p} = i_{m-k} | \sigma_{m+p-1} = \sigma_{m-k-1}) \\
&= P(\sigma_{k+1} = i_1 | \sigma_k = i) P(\sigma_{k+2} = i_2 | \sigma_{k+1} = i_1) \times \dots \\
&\quad \times P(\sigma_m = i_{m-k} | \sigma_{m-1} = \sigma_{m-k-1}) \\
&= P(\sigma_{k+1} = i_1, \dots, \sigma_m = i_{m-k} | \sigma_k = i)
\end{aligned}$$

in which we have used the periodicity of Π_k . Summarizing the above, we conclude that $P_l(j) = \tilde{P}_l(j)$ ($j \in \underline{N}$ and $l \in \underline{p}$) is positive definite solution to the matrix equation (2.2.3). This completes the proof of Theorem 2.2.7. \square

Remark. Theorem 2.2.7 gives a practically testable conditions for exponential second moment stability. When the form process $\{\sigma_k\}$ is time-homogenous, that is, $p = 1$, our condition (2.2.3) is equivalent to Ji et al's ([73]) necessary and sufficient condition (Theorem 2.2.1). Moreover, when p becomes infinitely large, (2.2.3) is equivalent to Krtolica et al's results ([21]) and Theorem 2.2.7 may be used as an approximation to a more general case.

One approach to obtaining a solution to a Lyapunov matrix equation is to introduce the Kronecker product. It is still true for coupled Lyapunov equations. One natural question is whether we can transform the above necessary and sufficient conditions for (exponential) second moment stability to conditions in terms of the Kronecker product, which may be more direct than solving a set of coupled Lyapunov equations. The answer to this question is affirmative. This approach was first explored for second moment stability of (2.1.1)

with an iid form process $\{\sigma_k\}$ by Bellman ([134]) and generalized by many others ([38]). For the basics of Kronecker products, the reader is referred to the excellent book by Horn and Johnson ([149]). Let $A = (a_{ij})_{m \times n}$ be a real or complex matrix, and define the linear operator $\text{vec}(\cdot)$ by

$$\text{vec}(A) = [a_{11}, a_{21}, \dots, a_{m1}, a_{12}, \dots, a_{m2}, \dots, a_{1n}, \dots, a_{mn}]^T.$$

To develop our main results, we need the following lemma.

Lemma 2.2.9. (Horn and Johnson [149])

(a). $\text{vec}(AX) = (I \otimes A)\text{vec}(X)$, $\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X)$;

(b). If $A_1XB_1 + \dots + A_kXB_k = C$, then

$$[B_1^T \otimes A_1 + \dots + B_k^T \otimes A_k]\text{vec}(X) = \text{vec}(C);$$

(c). $\text{vec}(AX + YB) = (I \otimes A)\text{vec}(X) + (B^T \otimes I)\text{vec}(Y)$. □

The following result is a general sufficient condition for exponential second moment stability for the system (2.1.1) with a finite state Markov chain form process.

Theorem 2.2.10.

Suppose that $\{\sigma_k\}$ is a finite state Markov chain with probability transition matrix $\Pi_k = (p_{ij}(k))$, then the system (2.1.1) is exponentially second moment stable if the deterministic matrix product sequence $\{\prod_{i=1}^k A_i\}$ is exponentially convergent to the zero matrix, where

$$A_k = \begin{pmatrix} H(1) \otimes H(1) & & & \\ & H(2) \otimes H(2) & & \\ & & \ddots & \\ & & & H(N) \otimes H(N) \end{pmatrix} (\Pi_{k-1}^T \otimes I).$$

Proof. Let $G(i) = H(i) \otimes H(i)$, from $x_{k+1}x_{k+1}^T = H(\sigma_k)x_kx_k^T H^T(\sigma_k)$, using Lemma 2.2.9, we obtain

$$\begin{aligned} \text{vec}(x_{k+1}x_{k+1}^T) &= (H(\sigma_k) \otimes H(\sigma_k))\text{vec}(x_kx_k^T) \\ &= G(\sigma_k)\text{vec}(x_kx_k^T) = \cdots = G(\sigma_k)G(\sigma_{k-1}) \cdots G(\sigma_0)\text{vec}(x_0x_0^T). \end{aligned}$$

Let $y_k = \text{vec}(x_kx_k^T)$, and (p_1, p_2, \dots, p_N) is any initial probability distribution.

From the above equality, we have

$$\begin{aligned} Ey_{k+1} &= E(G(\sigma_k)G(\sigma_{k-1}) \cdots G(\sigma_0)y_0) \\ &= \left(\sum_{i_0, i_1, \dots, i_k=1}^N p_{i_0}p_{i_0i_1}(0) \cdots p_{i_{k-1}i_k}(k-1)G(i_k)G(i_{k-1}) \cdots G(i_0) \right) y_0 \\ &= \left(\sum_{i_0, \dots, i_k} [p_{i_{k-1}i_k}(k-1)G(i_k)] \cdots [p_{i_0i_1}(0)G(i_1)][p_{i_0}G(i_0)] \right) y_0 \\ &= (I, I, \dots, I)A_k A_{k-1} \cdots A_1 \begin{pmatrix} p_1 G(1)y_0 \\ p_2 G(2)y_0 \\ \vdots \\ p_N G(N)y_0 \end{pmatrix}, \end{aligned} \tag{2.2.9}$$

where I is the identity matrix with appropriate dimension and

$$\begin{aligned} A_k &= \begin{pmatrix} p_{11}(k-1)G(1) & p_{21}(k-1)G(1) & \cdots & p_{1N}(k-1)G(1) \\ p_{12}(k-1)G(2) & p_{22}(k-1)G(2) & \cdots & p_{N2}(k-1)G(2) \\ \vdots & \vdots & \ddots & \vdots \\ p_{1N}(k-1)G(N) & p_{2N}(k-1)G(N) & \cdots & p_{NN}(k-1)G(N) \end{pmatrix} \\ &= \text{diag}\{G(1), G(2), \dots, G(N)\}(\Pi_{k-1}^T \otimes I). \end{aligned}$$

Thus, if the matrix product $\{\prod_{i=1}^k A_i\}$ is exponentially convergent to the zero matrix, then from (2.2.9), we conclude that Ey_k is exponentially convergent to zero. Hence, from the relation

$$E\|x_k\|^2 = Ex_k^T x_k = \text{tr}(Ex_kx_k^T) = \text{tr}(y_k),$$

we can conclude that $E\|x_k\|^2$ is exponentially convergent to zero for any initial probability distribution. This implies that (2.1.1) is exponentially second moment stable. \square

This theorem provides a test procedure for exponential second moment stability, from which we can obtain some testable condition. From a close look at the stability criterion, we can observe that second moment stability of a stochastic system can be reduced to the stability of a deterministic system in the form: $y_{k+1} = A_k y_k$, where the system matrix A_k is given as above, thus the complexity of the stability problem is greatly reduced. Moreover, we can clearly see how the system matrices $H(1), H(2), \dots, H(N)$ and the probability transition matrix Π_k affect the second moment stability of (2.1.1). If Π_k is periodic in k , then A_k is also periodic with the same period. If $\{\sigma_k\}$ is time-homogenous, A_k is a constant matrix. If Π_k can be approximated by Π , which is also a probability transition matrix, then A_k can be approximated by A . For all these three cases, we can expect simpler second moment stability criteria. These ideas are formalized in the next corollary.

Corollary 2.2.11.

- (a). Suppose that $\{\sigma_k\}$ is a time-homogenous finite state Markov chain with probability transition matrix $P = (p_{ij})$, then the system (2.1.1) is (stochastically, exponentially) second moment stable if the matrix

$$A = \text{diag}\{H(1) \otimes H(1), H(2) \otimes H(2), \dots, H(N) \otimes H(N)\}(P^T \otimes I)$$

is Schur stable, i.e., its eigenvalues are strictly inside the unit circle in complex plane.

(b). Suppose that the probability transition matrix Π_k is periodic with period p , then the system (2.1.1) is exponentially second moment stable if $A_p A_{p-1} \cdots A_1$ is Schur stable.

(c). Suppose that the probability transition matrix Π_k can be approximated by Π , which is also a probability transition matrix, then the system (2.1.1) is exponentially second moment stable if the matrix

$$A(\Pi) = \text{diag}\{H(1) \otimes H(1), H(2) \otimes H(2), \dots, H(N) \otimes H(N)\}(\Pi^T \otimes I)$$

is Schur stable.

Proof. (a). If $\{\sigma_k\}$ is time-homogenous, then $\Pi_k = P$, thus $A_k = A$, so $\prod_{i=1}^k A_i = A^k$. If A is Schur stable, then A^k is exponentially convergent to zero and therefore the condition of Theorem 2.2.10 is satisfied and the result (a) of Corollary 2.2.11 follows.

(b). If Π_k is periodic with period p , then $A_{kp+r} = A_r$ for any $k \geq 0$ and $0 \leq r < p$. Thus, let $k = mp + r$, we have

$$\begin{aligned} \prod_{i=1}^k A_i &= A_{mp+r} \cdots A_{mp+1} \prod_{i=1}^m (A_{ip} A_{ip-1} \cdots A_{(i-1)p}) \\ &= (A_r \cdots A_1) (A_p \cdots A_1)^m. \end{aligned}$$

Hence, if $A_p \cdots A_1$ is Schur stable, then the product $\{\prod_{i=1}^k A_i\}$ is exponentially convergent, from Theorem 2.2.10, we conclude that (2.1.1) is exponentially second moment stable.

(c). We first notice the following fact ([149]): Given a matrix A , for any $\epsilon > 0$, there exists a matrix norm $\|\cdot\|$ such that $\|A\| \leq \rho(A) + \epsilon$. Since $A(\Pi)$ is stable, we have $\rho(A(\Pi)) < 1$. Using the above fact, there exists a matrix norm

$\|\cdot\|$ such that $\|A(\Pi)\| < 1$. Moreover, there is a positive number $\rho < 1$ such that $\|A(\Pi)\| < \rho < 1$. From this hypothesis, we have $\lim_{k \rightarrow \infty} \Pi_k = \Pi$, hence $\lim_{k \rightarrow \infty} A_k = A(\Pi)$. From continuity of matrix norm, we have

$$\lim_{k \rightarrow \infty} \|A_k\| = \left\| \lim_{k \rightarrow \infty} A_k \right\| = \|A(\Pi)\| < \rho,$$

thus, there exists $K > 0$ such that for any $k \geq K$, we have $\|A_k\| < \rho$, consequently, we have

$$\left\| \prod_{i=1}^k A_i \right\| \leq \prod_{i=1}^k \|A_i\| \leq \left(\prod_{i=1}^K \|A_i\| \right) \rho^{k-K}.$$

Therefore, the sequence $\{\prod_{i=1}^k A_i\}$ is exponentially convergent, and from Theorem 2.2.10, (2.1.1) is exponentially second moment stable. This completes the proof of Corollary 2.2.11. \square

Remark:

- (1). (c) of Corollary 2.2.11 is extremely important, because in many practical applications, the probability transition matrix converges to a stationary matrix. In this case, (c) states that we do not need to test the stability of the deterministic time-varying system, which is very difficult, or test the definiteness of solutions of an infinite number of coupled Lyapunov equations as in Krtolica et al's results (Theorem 2.2.6), we only need to test the time-homogenous case where the probability transition matrix is replaced by its stationary limit. In this way, we have transformed an impractical test criterion into one that can be computed.
- (2). Compared with Theorem 2.2.1 or Theorem 2.2.2, one advantage of (a) of Corollary 2.2.11 is a potential reduction in computations. In principle, solving coupled Lyapunov equations using the Kronecker product

approach requires representing matrices as expanded vectors and transforming the coupled Lyapunov equations into a linear equation with coefficient matrix B . From the solution of the linear matrix equation we can obtain the matrix $P(1), P(2), \dots, P(N)$, then we check the definiteness of $P(1), \dots, P(N)$. However, it is easy to show that B is stable if and only if A is stable (details can be found later in this section), so we only need to check the eigenvalues of B to determine if (2.1.1) is second moment exponentially stable. In this way, the computations may be reduced.

- (3). We conjecture that the sufficient condition in (a) is also necessary. However, we have not been able to give a rigorous proof for this. From Example 2.2.1, we know that for one dimensional jump linear system, this conjecture is true. It will be seen that this conjecture is also true for the iid case. \square

For the case that $\{\sigma_k\}$ is an iid form process, we obtain the following much simpler testable criterion.

Corollary 2.2.12.

Suppose that $\{\sigma_k\}$ is a finite state iid form process with probability distribution $\{p_1, p_2, \dots, p_N\}$, then a necessary and sufficient condition for (2.1.1) to be exponentially second moment stable if and only if the matrix

$$A_0 = p_1 H(1) \otimes H(1) + p_2 H(2) \otimes H(2) + \dots + p_N H(N) \otimes H(N)$$

is Schur stable.

Proof. *Sufficiency:* Let

$$T = \begin{pmatrix} I & I & \cdots & I \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{pmatrix}.$$

Then T is nonsingular and it is easy to verify that

$$TAT^{-1} = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ p_2G(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_NG(N) & 0 & \cdots & 0 \end{pmatrix},$$

so A is Schur stable if and only if A_0 is Schur stable. From Corollary 2.2.11, the system (2.1.1) is exponentially second moment stable.

Necessity: As we noticed earlier, we have

$$\text{vec}(x_{k+1}x_{k+1}^T) = (H(\sigma_k) \otimes H(\sigma_k))\text{vec}(x_kx_k).$$

Because σ_k and x_k are independent, from the above equality, we obtain

$$\begin{aligned} y_{k+1} &= E\{H(\sigma_k) \otimes H(\sigma_k)\}y_k \\ &= (p_1H(1) \otimes H(1) + \cdots + p_NH(N) \otimes H(N))y_k = A_0y_k. \end{aligned}$$

From the relation $E\|x_k\|^2 = \text{tr}E\{x_kx_k^T\}$, it is easy to prove that (2.1.1) is exponentially second moment stable if and only if y_k converges to zero exponentially. Thus A_0 is Schur stable. \square

Remark:

(1). The proof of necessity is in fact a direct simple proof of Corollary 2.2.12.

The purpose to give the above proof of the sufficiency is to illustrate that the iid case is really a special case of Markov chain case. Corollary 2.2.12

states that the sufficient condition in (a) of Corollary 2.2.11 is also a necessary condition for the iid case.

- (2). For a finite state ergodic Markov chain, there exists a unique invariant measure $\pi = (\pi_1, \dots, \pi_N)$, and when the initial distribution is this invariant measure, then the stationary chain behaves like the iid chain. One may conjecture that for a finite state Markov chain case, when the initial distribution is $\{\pi_1, \pi_2, \dots, \pi_N\}$, Corollary 2.2.12 is still valid, i.e., (2.1.1) is exponentially second moment stable if and only if $\pi_1 H(1) \otimes H(1) + \dots + \pi_N H(N) \otimes H(N)$ is Schur stable. Unfortunately, this is not true. For example, let $H(1) = \sqrt{1.9}$ and $H(2) = \sqrt{0.5}$, the probability transition matrix is given by $P = \begin{pmatrix} 0.1 & 0.9 \\ 0.8 & 0.2 \end{pmatrix}$. Using this data in (a) of Corollary 2.2.11, we have $A = \begin{pmatrix} 0.19 & 1.52 \\ 0.45 & 0.1 \end{pmatrix}$, whose eigenvalues are 0.973 and -0.683, so A is stable, and (2.1.1) is second moment stable. The unique (ergodic) invariant measure is $(8/17, 9/17)$, and

$$\pi_1 H(1) \otimes H(1) + \pi_2 H(2) \otimes H(2) = \frac{8}{17} \times 1.9 + \frac{9}{17} \times 0.5 = \frac{19.7}{17} > 1.$$

This implies that the conjecture do not provide a necessary condition. Whether the conjecture provides a sufficient condition is still an open question. \square

There is close relationship between the criteria derived from the coupled Lyapunov equations and the Kronecker product formulation. In fact, the Kronecker product approach gives a method for solving the coupled Lyapunov equations. To illustrate this, we only consider the time-homogenous Markovian case. Applying Lemma 2.2.9 to (2.2.1), we obtain

$$\sum_{j=1}^N p_{ij} (H^T(i) \otimes H^T(i)) \text{vec}(P(j)) - \text{vec}(P(i)) = -\text{vec}(Q(i)), \quad i \in \underline{N}$$

which is equivalent to the following matrix equation:

$$\begin{aligned} & \left[\begin{pmatrix} p_{11}G^T(1) & p_{12}G^T(1) & \cdots & p_{1N}G^T(1) \\ p_{21}G^T(2) & p_{22}G^T(2) & \cdots & p_{2N}G^T(2) \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1}G^T(N) & p_{N2}G^T(N) & \cdots & p_{NN}G^T(N) \end{pmatrix} - I \right] \begin{pmatrix} \text{vec}(P(1)) \\ \text{vec}(P(2)) \\ \vdots \\ \text{vec}(P(N)) \end{pmatrix} \\ & = - \begin{pmatrix} \text{vec}(Q(1)) \\ \text{vec}(Q(2)) \\ \vdots \\ \text{vec}(Q(N)) \end{pmatrix}. \end{aligned}$$

Let \bar{A} denote the the first matrix in the coefficient matrix of the above matrix equation, $X = (\text{vec}(P(1)), \dots, \text{vec}(P(N)))^T$ and $Y = (\text{vec}(Q(1)), \dots, \text{vec}(Q(N)))^T$, then the above matrix equation becomes

$$(\bar{A} - I)X = -Y. \quad (2.2.10)$$

It can be verified that

$$\bar{A} = ((P^T \otimes I) \text{diag}\{G(1), G(2), \dots, G(N)\})^T.$$

Using the fact that UV and VU have the same nonzero eigenvalues for any matrices U and V , and that U and U^T have the same eigenvalues, we can show that \bar{A} and A have the same eigenvalues. Thus, if A is Schur stable, then $\bar{A} - I$ is nonsingular, and the equation (2.2.10), i.e., (2.2.1) has a unique solution.

Suppose that (2.2.1) has a solution, we can prove that

$$X^T \bar{A} X < X^T X. \quad (2.2.11)$$

In fact, $X^T(\bar{A} - I)X = -X^T Y$ and

$$X^T Y = \sum_{j=1}^N (\text{vec}(P(j)))^T \text{vec}(Q(j)) = \sum_{j=1}^N \text{tr}(P(j)Q(j)) > 0,$$

where we have used the fact that $\text{tr}(UV) > 0$ for any positive definite matrices U and V . Thus, (2.2.11) follows. From (2.2.11), it may be possible to prove that the eigenvalues of \overline{A} are strictly inside the unit circle. This issue will be investigated in the future.

Although Corollary 2.2.12 provides a necessary and sufficient condition for exponential second moment stability for the iid case, the matrix A_0 is an $n^2 \times n^2$ matrix, when n becomes large, determining the Schur stability of A_0 becomes difficult. The following result gives a much easier sufficient condition for second moment stability.

Theorem 2.2.13.

The system (2.1.1) with an iid form process $\{\sigma_k\}$ with common probability distribution is $\{p_1, p_2, \dots, p_N\}$ is exponentially second moment stable, if one of the following conditions holds:

- (1). $H \stackrel{\text{def}}{=} E \{H^T(\sigma_0)H(\sigma_0)\} = \sum_{j=1}^N p_j H^T(j)H(j)$ is a stable matrix, i.e., $\rho(H) < 1$.
- (2). $p_1 \lambda_1 + p_2 \lambda_2 + \dots + p_N \lambda_N < 1$ with $\lambda_i = \lambda_{\max}(H(i)^T H(i)) = \rho(H(i)^T H(i))$ as defined before.
- (3). $H(i)^T H(i)$ is stable matrix, i.e., $\rho(H^T(i)H(i)) < 1$.
- (4). If there exists a matrix norm $\|\cdot\|$ such that

$$p_1 \|H(1)\|^2 + p_2 \|H(2)\|^2 + \dots + p_N \|H(N)\|^2 < 1.$$

□

Proof. Let $\sigma = \rho(H)$. For any $x \in \mathcal{R}^n$, we have $x^T H x \leq \sigma x^T x$. It follows that

$$\begin{aligned} E x_k^T x_k &= E \{ x_{k-1}^T H^T(\sigma_{k-1}) H(\sigma_{k-1}) x_{k-1} \} \\ &= E \left\{ E \{ x_{k-1}^T H^T(\sigma_{k-1}) H(\sigma_{k-1}) x_{k-1} \mid x_{k-1} \} \right\} \\ &= E \{ x_{k-1}^T H x_{k-1} \} \leq \sigma E \{ x_{k-1}^T x_{k-1} \} \leq \dots \leq \sigma^k E \{ x_0^T x_0 \}. \end{aligned}$$

Thus, $\rho(H) < 1$ implies that (2.1.1) is (exponentially) second moment stable. This proves that (1) is sufficient for second moment stability. Since $H(i)^T H(i)$ and H are positive semi-definite, let $\|\cdot\|_2$ denote the spectral norm or 2-norm, then $\rho(H) = \|H\|_2$ and $\lambda_i = \|H(i)^T H(i)\|_2$. Then

$$\rho(H) = \|H\|_2 = \left\| \sum_{i=1}^N p_i H(i)^T H(i) \right\|_2 \leq \sum_{i=1}^N p_i \|H(i)^T H(i)\|_2 = \sum_{i=1}^N p_i \lambda_i.$$

It is clear that the condition (2) implies (1). This shows the sufficiency of (2). Similarly, the condition (3) clearly implies (2) and thus, (3) is also sufficient. From the inequality: $\|x_{k+1}\| \leq \|H(\sigma_k)\| \|x_k\|$ and using the one-dimensional result of Corollary 2.2.12, the proof is completed. \square

For the finite state Markov chain case, the similar results can be obtained. The idea is to use a matrix norm to reduce a high-dimensional jump linear system to a one-dimensional jump linear system, then the result in Example 2.2.1 can be used to obtain a simple sufficient condition. This is formalized in the next result.

Theorem 2.2.14. If there exists a matrix norm $\|\cdot\|$ such that the matrix

$$\begin{pmatrix} p_{11} \|H(1)\|^2 & p_{21} \|H(1)\|^2 & \cdots & p_{N1} \|H(1)\|^2 \\ p_{12} \|H(2)\|^2 & p_{22} \|H(2)\|^2 & \cdots & p_{N2} \|H(2)\|^2 \\ \vdots & \vdots & \ddots & \vdots \\ p_{1N} \|H(N)\|^2 & p_{2N} \|H(N)\|^2 & \cdots & p_{NN} \|H(N)\|^2 \end{pmatrix}$$

is Schur stable, then the system (2.1.1) is exponentially second moment stable.

Proof. From the matrix norm property, we have $\|x_{k+1}\| \leq \|H(\sigma_k)\| \|x_k\|$. Let $a(\sigma_k) = \|H(\sigma_k)\|$ and $y_k = \|x_k\|$, then if the system $y_{k+1} = a(\sigma_k)y_k$ is exponentially second moment stable, then the system (2.1.1) is also exponentially second moment stable. Applying (a) of Corollary 2.2.11 to the system $y_{k+1} = a(\sigma_k)y_k$, we can complete the proof. \square

Remark: Theorem 2.2.14 depend on the choice of the matrix norm. From [149], we know that different norms give different approximations to the spectral radius, and it can be seen that the matrix A is Schur stable if and only if there exists a matrix norm $\|\cdot\|$ such that $\|A\| < 1$. From this point of view, Theorem 2.2.14 is not too conservative.

To illustrate the applications of criteria developed in this section, we present the following examples.

Example 2.2.2 ([56])

Consider the one-dimensional jump linear system

$$x_{k+1} = a(\sigma_k)x_k, \quad x_0 \text{ is given.}$$

Here the form process is a 7 state Markov chain with the following probability transition matrix ([56])

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ p_{21} & 0 & p_{23} & 0 & 0 & p_{26} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{53} & 0 & p_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & p_{72} & 0 & 0 & 0 & 0 & p_{77} \end{pmatrix}.$$

It is easy to see that $\{6\}$ is an absorbing state, $\{3, 4\}$ is a communicating class and $\{1, 2, 5, 7\}$ are transient states. The problem is to find conditions for second moment stability.

Here, we give a simpler procedure to solve this problem. We want to use Corollary 2.2.11 and Example 2.2.1 to give a necessary and sufficient condition for second moment stability. From Corollary 2.2.11, the test matrix A is given by

$$A = \begin{pmatrix} 0 & p_{21}a^2(1) & 0 & 0 & 0 & 0 & 0 \\ a^2(2) & 0 & 0 & 0 & 0 & 0 & p_{72}a^2(2) \\ 0 & p_{23}a^2(3) & 0 & a^2(3) & p_{53}a^2(3) & 0 & 0 \\ 0 & 0 & a^2(4) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{55}a^2(5) & 0 & 0 \\ 0 & p_{26}a^2(6) & 0 & 0 & 0 & a^2(6) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^2(7) \end{pmatrix}.$$

It is easy to compute

$$\begin{aligned} \det(\lambda I - A) &= (\lambda - p_{21}a^2(1)a^2(2))(\lambda^2 - a^2(3)a^2(4))(\lambda - p_{55}a^2(5)) \\ &\quad \times (\lambda - a^2(6))(\lambda - p_{77}a^2(7)). \end{aligned}$$

A is Schur stable if and only if $p_{21}a^2(1)a^2(2) < 1$, $a^2(3)a^2(4) < 1$, $p_{55}a^2(5) < 1$, $a^2(6) < 1$ and $p_{77}a^2(7) < 1$, which is also a necessary and sufficient condition for (2.1.1) to be second moment stable. This is the exact result obtained in [56] using a different approach.

Example 2.2.3: Stability in each mode does not guarantee second moment stability

Let

$$H(1) = \begin{pmatrix} 0.5 & 10 \\ 0 & 0.5 \end{pmatrix}, \quad H(2) = \begin{pmatrix} 0.5 & 0 \\ 10 & 0.5 \end{pmatrix},$$

which are Schur stable matrices.

Assume that the probability transition matrix is $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In this case, choose $Q(1) = Q(2) = I$. Using this data in Theorem 2.2.1, we obtain the solution

$$P(1) = \begin{pmatrix} 0.9981 & -0.0503 \\ -0.0503 & -0.0075 \end{pmatrix}, \quad P(2) = \begin{pmatrix} -0.0075 & -0.0503 \\ -0.0503 & 0.9981 \end{pmatrix}$$

which are not positive definite. From Theorem 2.2.1, we obtain that (2.1.1) is not second moment stable although its mode matrices $H(1), H(2)$ are stable. Notice also that it is easy to compute that the eigenvalues of the test matrix A in (a) of Corollary 2.2.11 are 0.25, 0.25, -0.25 , -0.25 , 0.0006, -0.0006 , 100.4994 and -100.4994 , hence A is not Schur stable.

Assume that the form process is a two state iid chain with the probability transition matrix $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$. If we want to use Theorem 2.2.1 with $Q(1) = Q(2) = I$, we need to compute the solution of the coupled Lyapunov equations. This yields,

$$P(1) = \begin{pmatrix} 0.9970 & -0.0807 \\ -0.0807 & -1.0212 \end{pmatrix}, \quad P(2) = \begin{pmatrix} -1.0212 & -0.0807 \\ -0.0807 & 0.9970 \end{pmatrix}$$

which are not positive definite, hence (2.1.1) is not second moment stable. Moreover, the test matrix A in Corollary 2.2.11 has 50.7451 and -49.75 as its eigenvalues, and it is not Schur stable. Since the form process is iid, we can use the simpler test criterion given in Corollary 2.2.4. Choosing $S = I$, we obtain the solution of the coupled Lyapunov equation in Corollary 2.2.4 as

$$R = \begin{pmatrix} -0.0121 & -0.0807 \\ -0.0807 & -0.0121 \end{pmatrix}$$

which is not positive definite. From Corollary 2.2.4, we conclude that (2.1.1) is not second moment stable. We can also use Corollary 2.2.12 to solve this

problem. It is easy to show by direct computation that A_0 in Corollary 2.2.12 has the following eigenvalues: 0.25, -0.2451 , -49.75 and 50.7451 , thus A_0 is not Schur stable. From Corollary 2.2.12, we know that (2.1.1) is not second moment stable.

Assume that the form process has the probability transition matrix $P = \begin{pmatrix} 0.2 & 0.8 \\ 0.1 & 0.9 \end{pmatrix}$. Solving the coupled Lyapunov equations in Theorem 2.2.1, we obtain

$$P(1) = \begin{pmatrix} 0.9787 & -0.4575 \\ -0.4573 & -9.0346 \end{pmatrix}, \quad P(2) = \begin{pmatrix} -0.3512 & -0.0436 \\ -0.0436 & 0.9989 \end{pmatrix}$$

which are not positive definite, hence (2.1.1) is not second moment stable. Moreover, the test matrix A has eigenvalue 28.9686 , hence A is not stable. This case is very interesting: From the probability transition matrix P , we notice that with greater probability the system (2.1.1) stays in the mode 2, which is a stable mode. Intuitively, the system should be second moment stable. However, this is not the case as indicated by the computations. An explanation of this phenomenon is that second moment stability is an average property, and very rare events (switching to mode 1) can accumulate and lead to instability). In fact, this can happen when $\{\sigma_k\}$ is iid. Choose $p_1 = 0.1$ and $p_2 = 0.9$, then the test matrix A_0 in Corollary 2.2.12 has an eigenvalue equal to 30.6422 , thus A_0 is not stable, hence (2.1.1) is not second moment stable. In fact, we have used computational tests to obtain the following: For the system (2.1.1) with a two state iid chain having probability distribution (p_1, p_2) , we have that (2.1.1) is second moment stable if $0 \leq p_1 \leq 0.00003$, and (2.1.1) is not second moment stable if $0.00004 \leq p_1 \leq 0.99996$.

Example 2.2.4: Instability of individual modes does not imply second moment instability

Let

$$H(1) = \begin{pmatrix} 1 & -1 \\ 0 & 0.5 \end{pmatrix}, \quad H(2) = \begin{pmatrix} 0.5 & 1 \\ 0 & 1 \end{pmatrix}.$$

Assume that $\{\sigma_k\}$ is a two state Markov chain with probability transition matrix $P = \begin{pmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{pmatrix}$. After a simple computation, we obtain that the eigenvalue of the test matrix A in Corollary 2.2.11 are 0.5695, -0.2195 , 0.5168, -0.2418 , -0.25 , 0.5, 0.5 and -0.25 , thus A is Schur stable, from Corollary 2.2.11, we conclude that (2.1.1) is second moment stable. Of course, we can also solve the coupled Lyapunov equations in Theorem 2.2.1 with $Q(1) = Q(2) = I$ and obtain:

$$P(1) = \begin{pmatrix} 3.1429 & -2.2857 \\ -2.2857 & 4.6964 \end{pmatrix}, \quad P(2) = \begin{pmatrix} 1.7143 & 0.5714 \\ 0.5714 & 5.2321 \end{pmatrix},$$

which are positive definite. From Theorem 2.2.1, (2.1.1) is second moment stable.

Assume that the form process $\{\sigma_k\}$ is time-inhomogenous two state Markov chain with the probability transition matrix

$$\Pi_k = \begin{pmatrix} 0.3 + e^{-(k+1)} & 0.7 - e^{-(k+1)} \\ 0.8 - \frac{\sin^2 k}{(k+2)^2} & 0.2 + \frac{\sin^2 k}{(k+2)^2} \end{pmatrix}.$$

To use Krtolica et al's ([21]) result (Theorem 2.2.6), we need to solve infinite number of matrix equations, which is practically impossible. However, using Corollary 2.2.11 we do not need to do this, we only need to use the steady-state probability transition matrix, which is

$$P = \lim_{k \rightarrow \infty} \Pi_k = \begin{pmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{pmatrix}$$

From the previous discussion, we know that (2.1.1) with the probability transition matrix P is second moment stable. From Corollary 2.2.11, we conclude

that the system (2.1.1) with the time-inhomogenous Markov chain having the probability transition matrix Π_k is exponentially second moment stable.

Example 2.2.5: Reliable Control System Design ([55])

This example is from Birdwell et al ([55]), which deals with reliable control system design. The control system is described by

$$x_{k+1} = Ax_k + B(\sigma_k)u_k, \quad (2.2.12)$$

where

$$A = \begin{pmatrix} 2.71828 & 0 \\ 0 & 0.36788 \end{pmatrix}, \quad B(1) = \begin{pmatrix} 1.71828 & 1.72828 \\ -0.63212 & 0.63212 \end{pmatrix},$$

$$B(2) = \begin{pmatrix} 0 & 1.71828 \\ 0 & 0.63212 \end{pmatrix}, \quad B(3) = \begin{pmatrix} 1.71828 & 0 \\ -0.63212 & 0 \end{pmatrix}, \quad B(4) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This model captures the failure/repair events for a reliable system with two actuators, in which actuators may fail and need to be repaired. State 1 of $\{\sigma_k\}$ represents the case that both actuators work well, states 2 and 3 represent the case where one of actuators fails and has to be repaired, and state 4 represents the case where both actuators fail. Let p_f and p_r denotes the failure rate and repair rate, where the actuator repair and failure events are independent, then the probability transition matrix P is given by

$$\begin{pmatrix} (1-p_f)^2 & (1-p_f)p_r & (1-p_f)p_r & p_r^2 \\ p_f(1-p_f) & (1-p_f)(1-p_r) & p_r p_f & p_r(1-p_r) \\ p_f(1-p_f) & p_r p_f & (1-p_f)(1-p_r) & p_r(1-p_r) \\ p_f^2 & (1-p_r)p_f & (1-p_r)p_f & (1-p_r)^2 \end{pmatrix}.$$

The problem is to find a feedback control law $u_k = G(\sigma_k)x_k$ so that the closed-loop control system is reliable, i.e., the expected quadratic cost is finite, which is equivalent generically to the exponentially second moment stability of the closed-loop system.

Assume that $p_r = 0.1$ and $p_f = 0.9$, and choose a feedback control $u_k = G(\sigma_k)x_k$, where

$$G(i) = \begin{pmatrix} -0.8890 & 0.04222 \\ -0.7752 & -0.9914 \end{pmatrix} \quad (i = 1, 2, 3, 4).$$

Then the closed-loop individual mode system matrices are given by $H(\sigma_k) = A + B(\sigma_k)G(\sigma_k)$, which are

$$H(1) = \begin{pmatrix} -0.1490 & -1.6409 \\ 0.0719 & -0.2855 \end{pmatrix}, \quad H(2) = \begin{pmatrix} 1.3785 & -1.7134 \\ -0.49 & -0.2588 \end{pmatrix},$$

$$H(3) = \begin{pmatrix} 1.1907 & 0.0725 \\ 0.5620 & 0.3412 \end{pmatrix}, \quad H(4) = \begin{pmatrix} 2.71828 & 0 \\ 0 & 0.36788 \end{pmatrix}.$$

Solving the coupled equations in Theorem 2.2.1 with $Q(i) = I$ ($i = 1, 2, 3, 4$), we obtain

$$P(1) = \begin{pmatrix} 1.2809 & 1.3087 \\ 1.3087 & 23.0683 \end{pmatrix}, \quad P(2) = \begin{pmatrix} 3.1538 & -1.5980 \\ -1.5980 & 3.6128 \end{pmatrix},$$

$$P(3) = \begin{pmatrix} 3.0268 & 0.5840 \\ 0.5840 & 1.3098 \end{pmatrix}, \quad P(4) = \begin{pmatrix} 1.6751 & 0.0030 \\ 0.0030 & 1.0393 \end{pmatrix},$$

which are all positive definite, thus from Theorem 2.2.1 the closed-loop system (2.2.12) with the given control law is exponentially second moment stable, hence the system is reliable with this control law. This is also very easily tested by Corollary 2.2.11, in fact the eigenvalues of the test matrix A in Corollary 2.2.11 are 0.6359, -0.1943 , 0.0654, -0.0653 and the rest are zero, hence A is stable. From Corollary 2.2.11, we also conclude that the closed-loop control system of (2.2.12) is exponentially second moment stable. Birdwell et al used a different initial distribution for $\{\sigma_k\}$ to test the reliability of the system. Since the results developed in this paper are valid for any initial distribution, we do not need to test the reliability for any initial distribution, we can conclude that the system with the above control law is reliable for any initial distribution. It is easy to show numerically that if we change $p_r = 0.7$ and $p_f = 0.3$, then the above control law is no longer a reliable one.

2.3. δ -MOMENT STABILITY PROPERTIES

As it is noted by Mitchell and Kozin ([130]) that the second moment stability regions for second moment stability for a class of stochastic differential systems are comparatively smaller than the region for almost sure stability. This is also the same case for the jump linear systems. In this section, we will study various types of δ -moment stability for the jump linear system (2.1.1). For $\delta = 2$, Ji et al ([73]) proved that for the system (2.1.1) with a finite state Markov form process, second moment stability, second moment exponential stability and second moment stochastic stability are equivalent, and all of these stability definitions imply almost sure stability. Feng et al ([74]) obtained the same result for continuous-time jump linear systems and further proved that for one dimensional systems, almost sure stability is equivalent to δ -moment stability for sufficiently small δ . Next, we will prove that for any positive δ , δ -moment stability, exponential δ -moment stability and stochastic δ -moment stability are also equivalent, and all of these stability definitions imply almost sure stability. Consequently, a much simpler proof of the results for $\delta = 2$ in Ji et al's work ([73]) is provided. We will also prove that almost sure stability of (2.1.1) is equivalent to δ -moment stability for sufficiently small δ , thus the conjecture given in Feng et al ([74], [75]) is proved for discrete-time systems. For diffusion processes, Arnold et al ([116]) obtained a relationship between the Lyapunov exponents and the δ -moment Lyapunov exponents. We obtain a similar result for systems of the type (2.1.1).

We shall work on the system (2.1.1) with a Markov form process exclusively. The iid case is a special case, except for the following consideration: Recall that in the Markovian case, we require the stability properties are independent of the initial distributions $\xi \in \Xi$. Thus, we have to justify that

when $\{\sigma_k\}$ is indeed an iid sequence with a common probability distribution $p = (p_1, \dots, p_N)$ and is interpreted as a Markov chain with transition matrix $P = (p', p' \dots, p')'$, the seemingly strong stability concepts (with respect to arbitrary initial distributions) for the Markovian case are coincident with that for iid case. This, however, directly follows from the observation that stability is an asymptotic property, and if $\{\eta_k\}_{k=0}^{+\infty}$ is a Markov chain with the transition matrix $P = (p', p', \dots, p')'$ and any initial distribution, then $\{\eta_{k+1}\}_{k=0}^{+\infty}$ is an iid sequence with the common distribution p . We begin with the equivalence of various δ -moment stability properties:

Theorem 2.3.1:

For system (2.1.1) with a Markov form process (which is finite state and time-homogeneous), δ -moment stability, exponential δ -moment stability and stochastic δ -moment stability are all equivalent.

Proof. It is easy to show that exponential δ -moment stability implies stochastic δ -moment stability which then implies δ -moment stability. Thus, to prove the equivalence, it is sufficient to show that δ -moment stability implies exponential δ -moment stability.

According to the definition in section 2.1, if (2.1.1) is δ -moment stable, then for any initial distribution $\xi \in \Xi$,

$$\lim_{n \rightarrow \infty} E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta = 0$$

where E_ξ denotes the expectation with respect to P_ξ , the probability measure induced by ξ for $\{\sigma_k\}$.

Let $\xi_i = (p_{i1}, \dots, p_{iN})$ be the i -th row of the matrix P . This defines a probability distribution for the process $\{\sigma_k\}$. Let ξ_0 be any initial distribution in Ξ with $\xi_0 = \{p_1, p_2, \dots, p_N\}$. From the δ -moment stability, we have

$$\lim_{n \rightarrow \infty} \max_{0 \leq i \leq N} E_{\xi_i} \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta = 0$$

Then for any $0 < r < 1$ given, there exists an integer m so that

$$\max_{0 \leq i \leq N} E_{\xi_i} \|H(\sigma_{m-1}) \cdots H(\sigma_0)\|^\delta \leq r \quad (2.3.1)$$

Also, there exists an $M > 0$ such that for any $0 \leq q < m$ and any k , we have

$$\max_{0 \leq i \leq N} E_{\xi_i} \|H(\sigma_{k+q}) \cdots H(\sigma_k)\|^\delta \leq M \quad (2.3.2)$$

In arriving at (2.3.2), we have used the time homogeneous property of $\{\sigma_k\}$.

Let $k = pm + q$, where $0 \leq q < m$, then we obtain, using the time homogeneous property again,

$$\begin{aligned} & E_{\xi_0} \|H(\sigma_k) \cdots H(\sigma_0)\|^\delta \\ & \leq E_{\xi_0} \|H(\sigma_{pm+q}) \cdots H(\sigma_{pm})\|^\delta \|H(\sigma_{pm-1}) \cdots H(\sigma_{(p-1)m})\|^\delta \times \cdots \\ & \quad \times \|H(\sigma_{m-1}) \cdots H(\sigma_0)\|^\delta \\ & = \sum_{i_0, \dots, i_{pm+q}} p_{i_0} p_{i_0 i_1} \cdots p_{i_{pm+q-1} i_{pm+q}} \|H(i_{pm+q}) \cdots H(i_{pm})\|^\delta \\ & \quad \times \|H(i_{pm-1}) \cdots H(i_{(p-1)m})\|^\delta \cdots \|H(i_{m-1}) \cdots H(i_0)\|^\delta \\ & = \sum_{i_0, \dots, i_{pm-1}} p_{i_0} p_{i_0 i_1} \cdots p_{i_{pm-2} i_{pm-1}} \|H(i_{pm-1}) \cdots H(i_{(p-1)m})\|^\delta \times \cdots \\ & \quad \times \|H(i_{m-1}) \cdots H(i_0)\|^\delta \times \\ & \quad \times \left(\sum_{i_{pm}, \dots, i_{pm+q}} p_{i_{pm-1} i_{pm}} \cdots p_{i_{pm+q-1} i_{pm+q}} \|H(i_{pm+q}) \cdots H(i_{pm})\|^\delta \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i_0, \dots, i_{pm-1}} p_{i_0} p_{i_0 i_1} \cdots p_{i_{pm-2} i_{pm-1}} \|H(i_{pm-1}) \cdots H(i_{(p-1)m})\|^\delta \times \cdots \\
&\quad \times \|H(i_{m-1}) \cdots H(i_0)\|^\delta \times E_{\xi_{i_{pm-1}}} \|H(\sigma_q) \cdots H(\sigma_0)\| \\
&\leq \max_{1 \leq i \leq N} E_{\xi_i} \|H(\sigma_q) \cdots H(\sigma_0)\|^\delta \\
&\quad \times \sum_{i_0, \dots, i_{pm-1}} p_{i_0} p_{i_0 i_1} \cdots p_{i_{pm-2} i_{pm-1}} \|H(i_{pm-1}) \cdots H(i_{(p-1)m})\|^\delta \times \cdots \\
&\quad \quad \times \|H(i_{m-1}) \cdots H(i_0)\|^\delta \\
&\leq \cdots \leq \max_{1 \leq i \leq N} E_{\xi_i} \|H(\sigma_q) \cdots H(\sigma_0)\|^\delta \\
&\quad \times \max_{1 \leq i \leq N} E_{\xi_i} \|H(\sigma_{m-1}) \cdots H(\sigma_0)\|^\delta \times \cdots \times \max_{1 \leq i \leq N} E_{\xi_i} \|H(\sigma_{m-1}) \cdots H(\sigma_0)\|^\delta \\
&= \max_{1 \leq i \leq N} E_{\xi_i} \|H(\sigma_q) \cdots H(\sigma_0)\|^\delta \times \cdots \times \left[\max_{1 \leq i \leq N} E_{\xi_i} \|H(\sigma_{m-1}) \cdots H(\sigma_0)\|^\delta \right]^p \\
&\leq M r^p = M (r^{1/m})^{pm} \leq M_1 r_1^k
\end{aligned}$$

where $M_1 = M r^{-q/m}$ and $r_1 = r^{1/m}$. We conclude from this that (2.1.1) is exponentially δ -moment stable. This completes the proof. \square

The above theorem establishes the equivalence of various δ -moment stability properties. This is a generalization of the result of Ji et al [73] for second moment stability ($\delta = 2$). Next, we study the relationship between moment and almost sure stability. First of all, we prove some basic results for the general nonlinear stochastic system

$$x_{k+1} = f(\omega, x_k), \quad x_0 \in \mathcal{R}^n. \quad (2.3.3)$$

All stability concepts for (2.3.3) are similarly defined as for the jump linear system, in an obvious way. Furthermore, we make the following definition:

Definition 2.3.2:

The stochastic system (2.3.3) is said to be weakly exponentially stable in probability, if for any $\epsilon > 0$, there exist $M(\epsilon) > 0$ and $0 < \gamma < 1$ (independent of ϵ) such that for all $k \geq 0$,

$$P\{\|x_k(\omega, x_0)\| \geq \epsilon\} \leq M(\epsilon)\gamma^k, \quad \forall x_0 \in \mathcal{R}^n.$$

□

Proposition 2.3.3:

The following statements hold for the stochastic system (2.3.3).

- (i) If (2.3.3) is exponentially δ -moment stable, then it is stochastically δ -moment stable, which in turn implies that (2.3.3) is almost surely stable.
- (ii) If (2.3.3) is exponentially δ -moment stable, then, it is weakly exponentially stable in probability. Furthermore, suppose that there exists $G > 0$ such that $\|x_k(\omega, x_0)\| \leq \|x_0\|G^k$ almost surely for all x_0 and k . Then, (2.3.3) is weakly exponentially stable in probability implies that (2.3.3) is δ' -moment stability for some $\delta' > 0$. □

Proof. If the system is exponentially δ -moment stable, then there exists $M > 0$ and $0 < \gamma < 1$, such that $E\|x_k\|^\delta \leq M\gamma^k$. Thus,

$$\sum_{k=0}^{+\infty} E\|x_k\|^\delta \leq \sum_{k=0}^{+\infty} M\gamma^k = \frac{M}{1-\gamma} < +\infty,$$

i.e., the system is stochastically δ -moment stable. Now, assume (2.3.3) is stochastically δ -moment stable. Let $\xi = \overline{\lim}_{k \rightarrow \infty} \|x_k\|$, then from Markov's

inequality, we have that for any $c > 0$, the following holds:

$$\begin{aligned} P(\xi \geq c) &= P(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} (\|x_m\| \geq c)) \leq P(\cup_{m=n}^{\infty} (\|x_m\| \geq c)) \\ &\leq \sum_{m=n}^{\infty} P(\|x_m\| \geq c) \leq \frac{1}{c^\delta} \sum_{m=n}^{\infty} E\|x_m\|^\delta \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Thus with any $c > 0$, $P(\xi \geq c) = 0$, from which we obtain

$$P(\xi > 0) = P\left(\bigcup_{n=1}^{\infty} (\xi > \frac{1}{n})\right) \leq \sum_{n=1}^{\infty} P(\xi > \frac{1}{n}) = 0. \quad (2.3.4)$$

It follows that $P(\xi = 0) = 1$. This proves that $\lim_{k \rightarrow \infty} x_k = 0$ almost surely and the proof of (i) is complete.

For (ii), suppose (2.3.3) is exponentially δ -moment stable, i.e., there exists $M > 0$ and $0 < \gamma < 1$, such that $E\|x_k\|^\delta \leq M\gamma^k$. From Markov's inequality, we obtain for any $\varepsilon > 0$,

$$P(\|x_k\| \geq \varepsilon) \leq \frac{1}{\varepsilon^\delta} E\|x_k\|^\delta \leq \frac{M}{\varepsilon^\delta} \gamma^k$$

which implies that (2.3.3) is exponentially stable in probability.

Assume $\|x_k\| \leq \|x_0\|G^k$. If (2.3.3) is exponentially stable in probability, then for all $\varepsilon > 0$, there exists $0 < \gamma_1 < 1$ and $M(\varepsilon) > 0$ such that

$$P(\|x_k\| \geq \varepsilon) \leq M(\varepsilon)\gamma_1^k.$$

It follows that

$$\begin{aligned} E\|x_k\|^\delta &= \int_{(\|x_k\| \geq \varepsilon)} \|x_k\|^\delta P(d\omega) + \int_{(\|x_k\| < \varepsilon)} \|x_k\|^\delta P(d\omega) \\ &\leq \|x_0\|^\delta G^{k\delta} P(\|x_k\| \geq \varepsilon) + \varepsilon^\delta \leq M(\varepsilon)\|x_0\|^\delta (G^\delta \gamma_1)^k + \varepsilon^\delta \end{aligned}$$

Since $\lim_{\delta \rightarrow 0} G^\delta \gamma_1 = \gamma_1 < 1$, there is $\delta' > 0$ and $\gamma < 1$ such that $G^{\delta'} \gamma_1 < \gamma$.

Hence,

$$E\|x_k\|^{\delta'} \leq M(\varepsilon)\|x_0\|^{\delta'} \gamma^k + \varepsilon^{\delta'}$$

Therefore, $\lim_{k \rightarrow \infty} E \|x_k\|^{\delta'} \leq \epsilon^{\delta'}$. However, as $\epsilon > 0$ is arbitrary but fixed, this implies that $\lim_{k \rightarrow +\infty} E \|x_k\|^{\delta'} = 0$, and the system is δ' -moment stable.

□

Theorem 2.3.1 established that for the jump linear system (2.1.1), exponential δ -moment, stochastic δ -moment and δ -moment stability are all equivalent. A direct consequence of the above proposition is the following corollary:

Corollary 2.3.4.

For the jump linear system (2.1.1) with a Markov form process, weak exponential stability in probability is equivalent to δ -moment stability for some $\delta > 0$ and they both imply almost sure stability.

□

To continue the analysis, we briefly discuss the top Lyapunov exponent for jump linear systems: without any specification, for the remainder of this section, the Markovian form process $\{\sigma_k\}$ is assumed to possess a single ergodic (indecomposable) class. We let $\log(\cdot)$ denote the extended real-valued function from $[0, +\infty]$ to $[-\infty, +\infty]$, defined by

$$\log(x) = \begin{cases} \log(x), & \text{if } x \in (0, +\infty), \\ -\infty, & \text{if } x = 0, \\ +\infty, & \text{if } x = +\infty. \end{cases}$$

For system (2.1.1), we define the *top Lyapunov exponent* for each $\xi \in \Xi$ as the extended real value

$$\alpha_\xi = \overline{\lim}_{n \rightarrow +\infty} \frac{1}{n} E_\xi \log \|H(\sigma_n) \dots H(\sigma_0)\| \quad (2.3.5)$$

and

$$\alpha = \alpha_\pi = \lim_{n \rightarrow +\infty} \frac{1}{n} E_\pi \log \|H(\sigma_n) \cdots H(\sigma_0)\| \quad (2.3.5')$$

if the indicated limit exists.

Lemma 2.3.5.

For the system (2.1.1) with a Markov form process with a single ergodic class,

(i) For any $\xi \in \Xi$, $\alpha_\xi \in [-\infty, +\infty)$ and $\alpha_\xi = -\infty$ if $P_\xi(A) > 0$ with

$$A \stackrel{\text{def}}{=} \{\omega \in \Omega : \exists \text{ finite } n \text{ such that } \|H(\sigma_n) \cdots H(\sigma_0)\| = 0\}.$$

The limit $\alpha = \alpha_\pi$ in (2.3.5') exists (possibly infinite).

(ii)

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| = \alpha \quad P_\pi - \text{a.s.} \quad (2.3.6)$$

(iii) For any $\xi \in \Xi$ satisfying $\xi \ll \pi$, we have $\alpha_\xi \leq \alpha_\pi$.

Furthermore, if $H(j)$ is nonsingular for all $j \in \underline{N}$ and $\xi \ll \pi$, then

(iii) $\alpha_\xi = \lim_{n \rightarrow +\infty} \frac{1}{n} E_\xi \log \|H(\sigma_n) \cdots H(\sigma_0)\|$ is finite and $\alpha_\xi = \alpha_\pi = \alpha$.

(iv) $\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| = \alpha$, P_π -a.s.. There is a proper subspace L of \mathcal{R}^n such that for any $\xi \ll \pi$ and $x \in \mathcal{R}^n \setminus L$, we have

$$\overline{\lim}_{k \rightarrow +\infty} \frac{1}{k} \log \|H(\sigma_k) \cdots H(\sigma_0)x\| = \alpha, \quad P_\xi - \text{a.s.}$$

□

Proof. (i). Let $G = \max_{1 \leq k \leq N} \|H(k)\|$ with $H(k) = (h_{ij}(k))$ for all $k \in \underline{N}$.

Let $\chi_n = \log \|H(\sigma_n) \cdots H(\sigma_0)\|$. Then,

$$\chi_n \leq \log \|H(\sigma_n)\| \cdots \|H(\sigma_0)\| \leq \log G^{n+1} = (n+1) \log G \quad (2.3.7)$$

It follows that $\alpha_\xi \leq \log G < +\infty$. If $P_\xi(A) > 0$, clearly, $\alpha_\xi = -\infty$. Actually, $P_\xi(A) > 0$ implies that there is a $j < +\infty$ such that $P_\xi(A_j) > 0$ with $A_j \stackrel{\text{def}}{=} \{\omega \in \Omega : H(\sigma_k) \cdots H(\sigma_0) = 0, \forall k \geq j\}$. It follows that

$$\begin{aligned} \frac{1}{k} E_\xi \chi_k &= E_\xi \left(\frac{1}{k} \chi_k \right) = \int_{A_j} \frac{1}{k} \chi_k P_\xi(d\omega) + \int_{A_j^c} \frac{1}{k} \chi_k P_\xi(d\omega) \\ &\leq \frac{k+1}{k} (\log G) P_\xi(A_j^c) + \int_{A_j} \frac{1}{k} \chi_k P_\xi(d\omega) = -\infty, \quad \forall k > j. \end{aligned}$$

Thus, $\alpha_\xi = \lim_{k \rightarrow +\infty} k^{-1} E_\xi \chi_k = -\infty$.

To show the limit α exists, let $a_n = E_\pi \log \|H(\sigma_{n-1}) \cdots H(\sigma_0)\|$. We show that a_n is a subadditive sequence, i.e., $a_{m+n} \leq a_n + a_m$ for all m and n . Actually, by stationarity,

$$\begin{aligned} a_{n+m} &= E_\pi \log \|H(\sigma_{n+m-1}) \cdots H(\sigma_0)\| \\ &\leq E_\pi \log \{ \|H(\sigma_{n+m-1}) \cdots H(\sigma_n)\| \|H(\sigma_{n-1}) \cdots H(\sigma_0)\| \} \\ &= E_\pi \log \|H(\sigma_{n+m-1}) \cdots H(\sigma_n)\| + E_\pi \log \|H(\sigma_{n-1}) \cdots H(\sigma_0)\| \\ &= a_n + a_m. \end{aligned}$$

If there exists m_0 such that $a_{m_0} = -\infty$, then from (2.3.7), we have $a_n = -\infty$ for $n \geq m_0$, thus $\alpha = -\infty$. Otherwise, for any n , a_n is finite. For any $m > 0$, and $n = pm + q$, $0 \leq q < m$, we have

$$\frac{a_n}{n} = \frac{a_{pm+q}}{pm+q} \leq \frac{p}{pm+q} a_m + \frac{a_q}{pm+q}.$$

From this we obtain that $\overline{\lim}_{n \rightarrow \infty} \frac{a_n}{n} \leq \frac{a_m}{m}$, hence $\overline{\lim}_{n \rightarrow \infty} \frac{a_n}{n} \leq \underline{\lim}_{m \rightarrow \infty} \frac{a_m}{m}$, therefore, $\lim_{n \rightarrow \infty} \frac{a_n}{n}$ exists. In fact, we have

$$\begin{aligned} -\infty \leq \alpha &= \lim_{n \rightarrow \infty} \frac{1}{n} E_\pi \log \|H(\sigma_n) \cdots H(\sigma_0)\| \\ &= \inf_{n \geq 1} \frac{1}{n} E_\pi \log \|H(\sigma_n) \cdots H(\sigma_1)\| \leq \log G < +\infty. \end{aligned}$$

This proves (i).

(ii). we need to show (2.3.6) holds. Suppose that $\{\sigma_k\}$ is stationary, i.e., with initial distribution π . For any $m > 0$ arbitrarily given, let $n = pm + q$, where $0 \leq q < m$, we obtain

$$\begin{aligned} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| &\leq \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_{pm})\| \\ &\quad + \frac{p}{n} \frac{1}{p} \sum_{i=0}^{p-1} \log \|H(\sigma_{(i+1)m-1}) \cdots H(\sigma_{im})\|. \end{aligned} \tag{2.3.8}$$

With the initial distribution π , the process $\{\sigma_k\}$ is a stationary and irreducible Markov chain. Therefore, for any $m > 0$ given, the process $\{\tilde{\sigma}_i \stackrel{\text{def}}{=} (\sigma_{(i+1)m-1}, \dots, \sigma_{im})\}_{i=0}^{+\infty}$ is also a finite state, time homogeneous, irreducible and stationary Markov chain with a unique invariant distribution $\tilde{\pi}$. If there exists m_0 such that $E_\pi \log \|H(\sigma_{m_0-1}) \cdots H(\sigma_0)\| = -\infty$, then $H(\sigma_{m_0-1}) \cdots H(\sigma_0) = 0$ with positive (P_π or $P_{\tilde{\pi}}$) probability and $\alpha = -\infty$. Also, there exists $j_0, j_1, \dots, j_{m_0-1}$ such that $H(j_{m_0-1}) \cdots H(j_0) = 0$ and the state (j_{m_0-1}, \dots, j_0) is positive recurrent for the chain $\{\tilde{\sigma}_i\}$. It follows that for P_π -almost all $\omega \in \Omega$, $\|H(\sigma_n) \cdots H(\sigma_0)\| = 0$ for sufficiently large n , hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| = -\infty = \alpha, \quad P_\pi\text{-a.s.}$$

and henceforth (2.3.6) is valid. Suppose the above does not happen, i.e., for any $m > 0$, $E_\pi \log \|H(\sigma_m) \cdots H(\sigma_0)\|$ is finite. From the Law of Large Numbers, we have

$$\begin{aligned} \lim_{p \rightarrow \infty} \frac{1}{p} \sum_{i=0}^{p-1} \log \|H(\sigma_{(i+1)m-1}) \cdots H(\sigma_{im})\| \\ = E_{\bar{\pi}} \log \|H(\sigma_{m-1}) \cdots H(\sigma_0)\| = E_\pi \log \|H(\sigma_{m-1}) \cdots H(\sigma_0)\|. \end{aligned} \quad (2.3.9)$$

Moreover, $\log \|H(\sigma_n) \cdots H(\sigma_{pm})\| \leq (q+1) \log G$ is bounded from above. From (2.3.8) and (2.3.9), we obtain

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \leq \frac{1}{m} E_\pi \log \|H(\sigma_{m-1}) \cdots H(\sigma_0)\| \quad P_\pi - \text{a.s.}$$

Hence,

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \\ \leq \inf_{m \geq 1} \frac{1}{m} E_\pi \log \|H(\sigma_{m-1}) \cdots H(\sigma_0)\| = \alpha. \quad P_\pi - \text{a.s.} \end{aligned} \quad (2.3.10)$$

Next, let

$$\gamma_n = \frac{1}{n} \sum_{i=0}^n \log \|H(\sigma_i)\| - \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\|$$

Clearly, $\gamma_n \geq 0$. From the Law of Large Numbers, we have

$$\underline{\lim}_{n \rightarrow \infty} \gamma_n = E_\pi \log \|H(\sigma_0)\| - \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\|$$

and

$$\begin{aligned} E_\pi \gamma_n &= \frac{1}{n} \sum_{i=0}^n E_\pi \log \|H(\sigma_i)\| - \frac{1}{n} E_\pi \log \|H(\sigma_n) \cdots H(\sigma_0)\| \\ &\xrightarrow{n \rightarrow \infty} E_\pi \log \|H(\sigma_0)\| - \alpha. \end{aligned}$$

From Fatou's lemma, we obtain

$$\begin{aligned} 0 \leq E_\pi \underline{\lim}_{n \rightarrow \infty} \gamma_n &= E_\pi \log \|H(\sigma_0)\| - E_\pi \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \\ &\leq \underline{\lim}_{n \rightarrow \infty} E_\pi \gamma_n = E_\pi \log \|H(\sigma_0)\| - \alpha. \end{aligned}$$

Therefore,

$$E_\pi \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \geq \alpha. \quad (2.3.11)$$

From (2.3.10) and (2.3.11), we obtain the desired equality

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| = \alpha. \quad P_\pi - \text{a.s.}$$

This completes the proof of (ii).

(iii). Since $H(j)$ is nonsingular, the condition of Oseledec's Theorem ([100]) is satisfied and the limit in (2.3.6) exists, i.e.,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| = \alpha = \alpha_\pi, \quad P_\pi - \text{a.s.}$$

which is finite. We need to show that the limit in (2.3.5) exists and $\alpha_\xi = \alpha$.

However, by the nonsingularity of $H(j)$ and (2.3.7), we obtain

$$-\infty < -2|\log g| \leq \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \leq 2|\log G| < +\infty.$$

where $g = \min_{1 \leq k \leq N} \|H^{-1}(k)\|^{-1} > 0$. From this, the fact that $P_\xi \ll P_\pi$ (since $\xi \ll \pi$), and the Dominated Convergence Theorem, we have

$$\begin{aligned} \alpha_\xi &= \lim_{n \rightarrow +\infty} \frac{1}{n} E_\xi \log \|H(\sigma_n) \cdots H(\sigma_0)\| \\ &= E_\xi \left\{ \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \right\} = \alpha. \end{aligned}$$

(v). The first statement follows from Oseledec's Theorem as in (iv). The second statement follows from the nonrandom spectrum theorem proved in ([76]). \square

Remarks:

(1). In (i), we conjecture that the condition $P_\xi(A) > 0$ is also necessary for $\alpha_\xi = -\infty$. If $\{\sigma_k\}$ is not irreducible, α_ξ may not be a good quantity for

almost sure stability. For example, let $H(1) = 0$ and $H(2) = 10$, and $P = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 1 \end{pmatrix}$. It is easy to check that $\alpha_\xi = -\infty < 0$. However, the system (2.1.1) with the above structure is not almost surely stable. We also conjecture that for any $\xi \ll \pi$, $\alpha_\xi = \alpha_\pi$ without the invertibility assumption of $H(i)$. Unfortunately, we have not been able to find a rigorous proof for this conjecture.

- (2). The top Lyapunov exponent α is closely related to the almost sure stability property of (2.1.1). Clearly, when $\alpha < 0$, (2.1.1) is P_π -almost surely stable, and by the fact that there exists a single ergodic class, this implies that the system is almost surely stable (for any initial distribution ξ). When $\alpha > 0$, it is easy to show that (2.1.1) is not almost surely stable. A question arises at the “bifurcation” point when $\alpha = 0$. We conjecture that when $\alpha = 0$, the system (2.1.1) is not almost surely stable. A rigorous proof of this is currently under research. However, the following simple example illustrates our intuition that $\alpha = 0$ implies that (2.1.1) is not almost surely stable.

Example 2.3.6:

Consider a scalar ($n = 1$) jump linear system with an iid form process. Assume $H(i) \neq 0$ for all $i \in \underline{N}$. Let (p_1, \dots, p_N) be the common distribution of σ_k . In this case, by the Law of Large Numbers, we have

$$\begin{aligned} \alpha &= \lim_{n \rightarrow +\infty} \frac{1}{n} \log |H(\sigma_n) \dots H(\sigma_0)| \\ &= \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^n \log |H(\sigma_i)| = E \log |H(\sigma_i)| = \sum_{i=1}^N p_i \log |H(i)| \quad \text{a.s.} \end{aligned}$$

Suppose that $\alpha = 0$. Define $\eta_k = \log |H(\sigma_k)|$. Then, $\{\eta_k\}$ is an iid process with $E\{\eta_k\} = \alpha = 0$ and $E\{\eta_k^2\} = \sigma^2 < +\infty$. Thus, by the Law of Iterative Logarithm, we obtain

$$P\left\{\overline{\lim}_{n \rightarrow +\infty} \frac{\sum_{k=0}^{n-1} \eta_k}{\sqrt{2\sigma^2 n \log \log n}} = 1\right\} = 1$$

It follows that

$$\overline{\lim}_{n \rightarrow +\infty} \sum_{k=0}^{n-1} \eta_k = \overline{\lim}_{n \rightarrow +\infty} \log |H(\sigma_{n-1}) \dots H(\sigma_0)| = +\infty \quad \text{a.s.}$$

Therefore, we end up with

$$\overline{\lim}_{n \rightarrow +\infty} |x_n(\omega, x_0)| = \overline{\lim}_{n \rightarrow +\infty} |H(\sigma_{n-1}) \dots H(\sigma_0)x_0| = +\infty \quad \text{a.s.}$$

for all $x_0 \neq 0$. The system is not almost surely stable. \square

Next, we begin to study the relationship between δ -moment and almost sure stability. The δ -moment stability region Σ^δ and the almost sure stability region Σ^a in the parameter space of jump linear system are defined by

$$\Sigma^\delta = \{(H(1), \dots, H(N)) : (2.1.1) \text{ is } \delta - \text{moment stable.}\}$$

and

$$\Sigma^a = \{(H(1), \dots, H(N)) : (2.1.1) \text{ is almost surely stable.}\}$$

respectively. From the above, we see that we can decompose Σ^a into the disjoint union of the form

$$\Sigma^a = \Sigma_-^a \cup \Sigma_0^a$$

with

$$\Sigma_-^a \stackrel{\text{def}}{=} \Sigma^a \cap \{(H(1), \dots, H(N)) : \alpha < 0\}$$

and

$$\Sigma_0^a \stackrel{\text{def}}{=} \Sigma^a \cap \{(H(1), \dots, H(N)) : \alpha = 0\}$$

If the conjecture given before Definition 2.3.6 is true, then we expect $\Sigma_0^a = \phi$, i.e., $\Sigma^a = \Sigma_-^a$. The following theorem illustrates the relationship between the region of almost sure stability and the region of δ -moment stability:

Theorem 2.3.7.

For the system (2.1.1) with a finite state Markov form process, we have

- (i) For any $0 < \delta_1 \leq \delta_2$, $\Sigma^{\delta_2} \subseteq \Sigma^{\delta_1}$. $\Sigma^\delta \subseteq \Sigma^a$ for all $\delta > 0$ and Σ^δ is an open set.
- (ii) If $\{\sigma_k\}$ is irreducible, i.e., each pair of states communicates, then

$$\Sigma_-^a = \lim_{\delta \downarrow 0^+} \Sigma^\delta \stackrel{\text{def}}{=} \bigcup_{\delta > 0} \Sigma^\delta \subseteq \Sigma^a$$

□

Before we prove this theorem, we want to make some comments. The above result simply states that the δ -moment stability region Σ^δ is monotonically increasing to, roughly speaking, the almost sure stability region Σ^a from the interior as δ goes to 0 from above. This is a significant result which generalizes the results reported in Feng et al ([74]).

To prove the theorem, we first establish some fundamental results .

Lemma 2.3.8.

- (a). For any random variable ξ , the function $F(x)$ defined by

$$F(x) = (E\|\xi\|^x)^{\frac{1}{x}}$$

is nondecreasing on $(0, +\infty)$ whenever it is well defined.

- (b). Suppose $\{\sigma_k\}$ is a finite state and irreducible Markov chain with probability transition matrix P , let P_s denote the submatrix of P obtained by deleting the l -th row and the l -th column for some $l \in \underline{N}$, then the spectral radius $\rho(P_s)$ of P_s is strictly less than one, i.e., there exists a $B > 0$ and $0 \leq r_1 < 1$ such that

$$\|P_s^k\| \leq Br_1^k \quad (\forall k \geq 0).$$

□

Proof. (a). For any x and y satisfying $0 < x \leq y$, let $\alpha = y/x$, then $\alpha \geq 1$, and $\phi(x) = x^\alpha$ is a convex function of x . Using Jensen's inequality, we have

$$\phi(E\|\xi\|^x) \leq E\phi(\|\xi\|^x)$$

From which, we obtain

$$(E\|\xi\|^x)^{\frac{1}{x}} \leq (E\|\xi\|^y)^{\frac{1}{y}},$$

thus $F(x) \leq F(y)$, which means that $F(x)$ is nondecreasing.

(b). Let $P = (p_{ij})_{N \times N}$. Without loss of generality, we assume that P_s is the submatrix of P obtained by deleting the first row and first column of P , define

$$\bar{P} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & p_{11} & p_{12} & \dots & p_{1N} \\ p_{21} & 0 & p_{22} & \dots & p_{2N} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ p_{N1} & 0 & p_{N2} & \dots & p_{NN} \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & P_s \end{pmatrix}$$

where X, Y and Z are block matrices with appropriate dimensions. It is obvious that \bar{P} is also a stochastic matrix, thus we can form a new $N + 1$ state Markov

chain $\{\bar{\sigma}_k\}$ with probability transition matrix \bar{P} . Since $\{\sigma_k\}$ is an irreducible chain, there exists $i \in \underline{N} \setminus \{1\}$ such that $p_{i1} > 0$, thus $\{\bar{\sigma}_k\}$ is a Markov chain with the absorbing state 1 and the transient states $2, \dots, N+1$, therefore there is no cyclically transferring subclass of states in the ergodic class $\{1\}$ for the Markov chain $\{\bar{\sigma}_k\}$. From [137], we have

$$Q = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{l=1}^k \bar{P}^l = \lim_{k \rightarrow \infty} \bar{P}^k.$$

Since $2, \dots, N+1$ are transient states of $\{\bar{\sigma}_k\}$, it follows that

$$Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

Moreover, it is easy to prove that

$$\bar{P}^k = \begin{pmatrix} * & * \\ * & * + P_s^k \end{pmatrix}$$

where $*$ denotes a nonnegative matrix because of the nonnegativity of \bar{P} . From the structure of Q and the above discussion, we obtain that $\lim_{k \rightarrow \infty} P_s^k = 0$, hence $\rho(P_s) < 1$. The rest of (b) can be easily obtained (see [148]). This completes the proof. \square

Lemma 2.3.9. (Large Deviation Theorem)

Let $\{\sigma_n\}$ be a finite state time homogeneous and irreducible Markov chain with a unique invariant distribution π . For any fixed integer $m > 1$, let

$$\Phi((j+1)m, jm) = H(\sigma_{(j+1)m-1}) \cdots H(\sigma_{jm}),$$

$$\Delta_m = E_\pi \{\log \|\Phi(m-1, 0)\|\}.$$

Suppose that for any k , $H(\sigma_k)H(\sigma_{k-1})\cdots H(\sigma_1) \neq 0$, then, for any $\varepsilon > 0$, there exist $M, \delta > 0$ such that

$$P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi((j+1)m, jm)\| \geq \Delta_m + \varepsilon \right) \leq M \exp(-\delta p).$$

Proof. The large deviation result of this Lemma is a consequence of Theorem IV.1 of [155] and we will verify the hypothesis of [155] with the aid of a similar procedure as in [152]. Let $\Phi((j+1)m, jm)$ be defined as in the Lemma. Define

$$Y_p = \sum_{j=0}^{p-1} \log \|\Phi((j+1)m, jm)\|,$$

$$c_p(\delta) = \frac{1}{p} \log E_\pi \{ \exp(\delta Y_p) \}.$$

Then we have

$$c_p(\delta) = \log E_\pi \prod_{j=0}^{p-1} \|\Phi((j+1)m, jm)\|^\delta.$$

We first show that $c(\delta) = \lim_{p \rightarrow +\infty} c_p(\delta)$ exists for all $\delta \in \mathcal{R}$ (where the limit is possibly be $+\infty$).

Let S denote the state space of the Markov chain $\{\sigma_k\}$. For any sequence $l = (i_0, i_1, \dots, i_{m-1}) \in S^m$ and $\delta \in \mathcal{R}$, write

$$\Lambda(l; \delta) = \Lambda(i_0, i_1, \dots, i_{m-1}; \delta) = \|H(i_{m-1})H(i_{m-2})\cdots H(i_0)\|^\delta$$

Then, we have $\Lambda(l; \delta) > 0$ due to the fact that $H(\sigma_k)\cdots H(\sigma_0) \neq 0$ for any $k \geq 0$. Since $\{\sigma_k\}$ is irreducible, it follows that the chain $\tilde{r}_j = (\sigma_{(j+1)m-1}, \dots, \sigma_{jm})$ for $j = 0, 1, \dots$ is also an irreducible Markov chain with state space

$$\tilde{S} = \{(i_0, \dots, i_{m-1}) \in S^m : P_\pi(\sigma_{k+m-1} = i_{m-1}, \dots, \sigma_k = i_0) > 0, \exists k \geq 0\}.$$

Suppose that $|\tilde{S}| = \tilde{N}$ and that we have ordered the states in a certain way so that for $k \in \{1, 2, \dots, \tilde{N}\}$, with $\Lambda(k; \delta)$ defined accordingly. Now, consider that

$$\begin{aligned}
& E_\pi \left\{ \prod_{j=0}^{p-1} \|\Phi((j+1)m, jm)\|^\delta \right\} \\
&= \sum_{i_0, \dots, i_{p-1}} \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{p-2} i_{p-1}} \times \prod_{j=0}^{p-1} \|\Phi((j+1)m, jm)\|^\delta \\
&= \sum_{i_0, \dots, i_{p-1}} \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{p-2} i_{p-1}} \Lambda(i_0, i_1, \dots, i_{p-1}; \delta) \\
&\times \Lambda(i_m, i_{m+1}, \dots, i_{2m-1}; \delta) \cdots \Lambda(i_{(p-1)m}, i_{(p-1)m+1}, \dots, i_{pm-1}; \delta) \\
&= E_{\hat{\pi}} \left\{ \prod_{j=0}^{p-1} \Lambda(\tilde{r}_j; \delta) \right\} = \sum_{l_0, \dots, l_{p-1}} \hat{\pi}_{l_0} \hat{p}_{l_0 l_1} \cdots \hat{p}_{l_{p-2} l_{p-1}} \prod_{l=0}^{p-1} \Lambda(l; \delta) \\
&= \sum_{j_0, \dots, j_{p-1}} (\hat{\pi}_{j_0} \Lambda(j_0; \delta)) (\hat{p}_{j_0 j_1} \Lambda(j_1, \delta)) \cdots (\hat{p}_{j_{p-2} j_{p-1}} \Lambda(j_{p-1}; \delta)) \\
&= x^T (B(\delta))^{p-2} y.
\end{aligned} \tag{A.1}$$

Where in (A.1) $\hat{P} = (\hat{p}_{jl})_{\tilde{N} \times \tilde{N}}$ is the transition matrix of \tilde{r}_k , $\hat{\pi} = (\hat{\pi}_1, \dots, \hat{\pi}_{\tilde{N}})$ is its initial distribution induced by π , and

$$\begin{aligned}
0 \neq x^T &\stackrel{\text{def}}{=} (\hat{\pi}_1 \Lambda(1; \delta), \dots, \hat{\pi}_{\tilde{N}} \Lambda(\tilde{N}; \delta)) \geq 0 \\
y^T &= (1, 1, \dots, 1) > 0 \\
B(\delta) &= (\hat{p}_{jl} \Lambda(l; \delta))_{\tilde{N} \times \tilde{N}} \geq 0.
\end{aligned}$$

Since the transition matrix \hat{P} is irreducible and $\Lambda(l; \delta) > 0$, we see that $B(\delta)$ is an irreducible nonnegative matrix. Next, we show that with $c_p(\delta) = \log x^T B(\delta)^{p-2} y$, $c(\delta) = \lim_{p \rightarrow \infty} \frac{1}{p} c_p(\delta)$ exists and is differentiable at any point where it is defined. Since $B(\delta)$ is an irreducible nonnegative matrix, from matrix theory [148], there exists a positive vector v such that $B(\delta)v = \rho(B(\delta))v$. Since v and y are positive vectors, there exists positive numbers $L > 0$ and

$U > 0$ such that $Lv \leq y \leq Uv$ and $x^T v > 0$, where the last inequality follows from the nonnegativity of $x \neq 0$. Thus, we have

$$\begin{aligned} c_p(\delta) &\leq \log x^T B^{p-2}(Uv) = \log\{Ux^T \rho(B(\delta))^{p-2}v\} \\ &= \log U + (p-2) \log \rho(B(\delta)) + \log x^T v, \end{aligned}$$

from which we obtain that

$$\overline{\lim}_{p \rightarrow \infty} \frac{1}{p} c_p(\delta) \leq \log \rho(B(\delta)). \quad (\text{A.2})$$

In a similar way, we have

$$c_p(\delta) \geq \log L + (p-2) \log \rho(B(\delta)) + \log x^T v,$$

from which we have

$$\underline{\lim}_{p \rightarrow \infty} \frac{1}{p} c_p(\delta) \geq \log \rho(B(\delta)). \quad (\text{A.3})$$

(A.2) and (A.3) yield

$$c(\delta) \stackrel{\text{def}}{=} \lim_{p \rightarrow \infty} \frac{1}{p} c_p(\delta) = \log \rho(B(\delta)).$$

Due to the continuity of $B(\delta)$, $\mathcal{D}(c) \stackrel{\text{def}}{=} \{\delta \in \mathcal{R} : c(\delta) < +\infty\}$ is a nonempty open interval containing 0, and c is a closed convex function. Furthermore, because $B(\delta)$ is irreducible, $\rho(B(\delta))$ is simple [148]. Due to the differentiability of $B(\delta)$, it follows that $c(\delta)$ is differentiable [160]. Therefore, From Theorem IV.1 of [155], we obtain that there exists τ such that for any $\epsilon > 0$ there is $\eta = \eta(\epsilon) > 0$ such that

$$P_\pi \left(\left| \frac{1}{p} Y_p - \tau \right| \geq \epsilon \right) \leq \exp(-\eta p) \quad (\text{A.4})$$

for p large. However, by the Law of Large Numbers, we should have $\tau = \Delta_m$. Thus, Lemma 4.9 follows from (A.4) directly. \square

Lemma 2.3.10.

If there exists an $\epsilon < 0$ such that

$$\lim_{n \rightarrow \infty} P_\xi \left(\frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \geq \epsilon \right) = 0,$$

where the convergence is exponential, then there exists a δ such that

$$\lim_{k \rightarrow +\infty} E_\xi \|x_k(\omega, x_0)\|^\delta = 0,$$

for any x_0 . □

Proof. Let $\mathcal{A}, \mathcal{A}^c \in \mathcal{F}$ be defined as

$$\begin{aligned} \mathcal{A} &= \left(\omega \in \Omega \mid \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \geq \epsilon \right) \\ \mathcal{A}^c &= \left(\omega \in \Omega \mid \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| < \epsilon \right) \end{aligned}$$

With $G = \max_{1 \leq j \leq N} \|H(j)\| + 1$, we have

$$\|H(\sigma_n) \cdots H(\sigma_0)\| \leq \|H(\sigma_n)\| \cdots \|H(\sigma_0)\| \leq G^{n+1}.$$

Notice that for any $\omega \in \mathcal{A}^c$,

$$\frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| < \epsilon \Rightarrow \|H(\sigma_n) \cdots H(\sigma_0)\| \leq e^{\epsilon n}$$

and by the hypothesis, there are $M_2 > 0$ and $0 < \gamma < 1$ such that

$$P_\xi(\mathcal{A}) \leq M_2 \gamma^n$$

Therefore,

$$\begin{aligned} & E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta \\ &= \int_{\mathcal{A}} \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta P_\xi(d\omega) + \int_{\mathcal{A}^c} \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta P_\xi(d\omega) \\ &\leq G^{\delta(n+1)} P_\xi(\mathcal{A}) + e^{\delta \epsilon n} \leq M_2 G (G^\delta \gamma)^n + e^{\delta \epsilon n} \end{aligned}$$

Since $\lim_{\delta \rightarrow 0} G^\delta \gamma = \gamma < 1$, there exists a $\delta > 0$ such that $0 < G^\delta \gamma < 1$.

Moreover, as $\varepsilon < 0$, it follows that

$$\lim_{n \rightarrow \infty} E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta = 0.$$

□

Now, we are ready to prove Theorem 2.3.7.

Proof of Theorem 2.3.7: (i). For any $\delta_2 > \delta_1 > 0$, if $(H(1), H(2), \dots, H(N)) \in \Sigma^{\delta_2}$, then the system (2.1.1) is δ_2 -moment stable, i.e., $\lim_{k \rightarrow \infty} E_\xi \|x_k\|^{\delta_2} = 0$ for all $\xi \in \Xi$. From Lemma 2.3.8, we have

$$(E_\xi \|x_k\|^{\delta_1})^{\frac{1}{\delta_1}} \leq (E_\xi \|x_k\|^{\delta_2})^{\frac{1}{\delta_2}}.$$

Then, $\lim_{k \rightarrow \infty} E_\xi \|x_k\|^{\delta_1} = 0$ for all $\xi \in \Xi$ and $(H(1), H(2), \dots, H(N)) \in \Sigma^{\delta_1}$. Hence, $\Sigma^{\delta_2} \subseteq \Sigma^{\delta_1}$.

For any $\delta > 0$, if $(H(1), H(2), \dots, H(N)) \in \Sigma^\delta$, then (2.1.1) is δ -moment stable. From Corollary 2.3.4, the system (2.1.1) is almost surely stable, i.e., $(H(1), H(2), \dots, H(N)) \in \Sigma^a$. Thus $\Sigma^\delta \subseteq \Sigma^a$.

Now, we want to prove that Σ^δ is open. For any $(H(1), \dots, H(N)) \in \Sigma^\delta$, then (2.1.1) is δ -moment stable, i.e., $\lim_{n \rightarrow \infty} E \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta = 0$ for any initial distribution. Using the same notations as in the proof of Theorem 2.3.1, for $0 < \rho < 1$, there exists an $m > 0$ such that

$$\max_{0 \leq i \leq N} E_{\xi_i} \|H(\sigma_{m-1}) \cdots H(\sigma_0)\|^\delta \leq \rho.$$

Here, ξ_i is the i -th row of P . For any $r > 0$, $\rho < r < 1$, and since the left side of the above inequality is a continuous function in $(H(1), \dots, H(N))$,

there exists an open neighborhood U of $(H(1), \dots, H(N))$ such that for any $(\bar{H}(1), \dots, \bar{H}(N)) \in U$,

$$\max_{0 \leq i \leq N} E_{\xi_i} \|\bar{H}(\sigma_{m-1}) \cdots \bar{H}(\sigma_0)\|^\delta \leq r < 1.$$

Following the same procedure as in the proof of Theorem 2.3.1, we can prove that the system (2.1.1) with the system mode matrices $\bar{H}(1), \dots, \bar{H}(N)$ is exponentially δ -moment stable, hence $(\bar{H}(1), \dots, \bar{H}(N)) \in \Sigma^\delta$, hence $U \subset \Sigma^\delta$. This implies that Σ^δ is an open set.

(ii). Recall that the top Lyapunov exponent α satisfies

$$\begin{aligned} & \overline{\lim}_{k \rightarrow \infty} \frac{1}{k} \log \|H(\sigma_k) \cdots H(\sigma_0)\| \\ & = \alpha = \lim_{k \rightarrow \infty} \frac{1}{k} E_\pi (\log \|H(\sigma_k) \cdots H(\sigma_0)\|) \quad P_\pi - \text{a.s.} \end{aligned}$$

Thus, if $(H(1), \dots, H(N)) \in \Sigma_-^a = \Sigma^a \cap \{(H(1), \dots, H(N)) : \alpha < 0\}$, then $\alpha < 0$. In this case, it is sufficient to show that there exists a $\delta > 0$ such that (2.1.1) is δ -moment stable.

Suppose that $\alpha < 0$ (α may be $-\infty$), and with a slight abuse of notation, there exists a finite α and $\epsilon_0 > 0$ satisfying $\alpha + \epsilon_0 < 0$ such that

$$\Delta_m \stackrel{\text{def}}{=} E_\pi (\log \|H(\sigma_{m-1}) \cdots H(\sigma_0)\|) < m(\alpha + \epsilon_0) \quad (2.3.12)$$

Case (a). If there exists an $m > 0$ such that $\Delta_m = -\infty$, with

$$\Delta_m = \sum_{i_1, i_2, \dots, i_m} \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{m-1} i_m} \log \|H(i_m) \cdots H(i_1)\|,$$

there exists j_1, j_2, \dots, j_m such that $P_\pi \{\sigma_{m-1} = j_m, \dots, \sigma_0 = j_1\} > 0$, $\|H(j_m) \cdots H(j_1)\| = 0$, i.e., $H(j_m) \cdots H(j_1) = 0$. In this case, we want to

show that there exists a $\delta > 0$ such that (2.1.1) is δ -moment stable. Let $G = \max_{1 \leq k \leq N} \|H(k)\|$, we have

$$\begin{aligned}
& E_\pi \|H(\sigma_{pm-1}) \cdots H(\sigma_0)\|^\delta \\
&= \sum_{i_1, i_2, \dots, i_{pm}} \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{p-1} m+1 i_{pm}} \|H(i_{pm}) \cdots H(i_1)\|^\delta \\
&\leq \sum_{i_1, i_2, \dots, i_{pm}} \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{p-1} m+1 i_{pm}} \|H(i_{pm}) \cdots H(i_{(p-1)m+1})\|^\delta \times \cdots \\
&\quad \times \|H(i_m) \cdots H(i_1)\|^\delta \tag{*} \\
&\leq G^{\delta pm} \sum_{\substack{(i_{rm}, \dots, i_{(r-1)m+1}) \neq (j_m, \dots, j_1) \\ 1 \leq r \leq p}} \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{p-1} m+1 i_{pm}}.
\end{aligned}$$

We first prove a special case: $m = 1$. Without loss of generality, we can assume that $j_1 = 1$, in this case, the summation on the right hand side of (*) is

$$\begin{aligned}
& \sum_{\substack{(i_{rm}, \dots, i_{(r-1)m+1}) \neq (j_m, \dots, j_1) \\ 1 \leq r \leq p}} \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{p-1} m+1 i_{pm}} \\
&= \sum_{i_0, i_1, \dots, i_k=2}^N \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{p-1} m+1 i_{pm}} = (\pi_2, \dots, \pi_N) P_s^{pm} c
\end{aligned}$$

where $c = (1, \dots, 1)'$. From (b) of Lemma 2.3.8 and (*), there exists $B_1 > 0$ and $0 < r_1 < 1$ such that

$$E_\pi \|H(\sigma_{pm-1}) \cdots H(\sigma_0)\|^\delta \leq B_1 G^{\delta pm} r_1^{pm} = B_1 (G^\delta r_1)^{pm}.$$

Note that $\lim_{\delta \rightarrow 0} G^{\delta m} r_1 = r_1 < 1$, there exists a $\delta > 0$ such that $G^\delta r_1 < 1$. Thus, for such $\delta > 0$, we have

$$\lim_{p \rightarrow \infty} E_\pi \|H(\sigma_{pm}) \cdots H(\sigma_0)\|^\delta = 0$$

for $m = 1$. Following a similar procedure as given in the proof of Theorem 2.3.1, we can conclude that (2.1) is δ -moment stable, thus $(H(1), \dots, H(N)) \in \Sigma^\delta$.

For $m > 1$, define $x_p = (\sigma_{pm}, \dots, \sigma_{(p-1)m+1})$. Because $\{\sigma_k\}$ is an irreducible finite state Markov chain, x_p is also a finite state irreducible Markov chain and (j_m, \dots, j_1) is one state of $\{x_p\}$. Following a similar procedure as given for the $m = 1$ case, there exists $\delta > 0$ such that (2.1.1) is δ -moment stable, i.e., $(H(1), \dots, H(N)) \in \Sigma^\delta$.

Case (b). If for any m , $\Delta_m > -\infty$, we want to show next that this implies that for some $\epsilon < 0$,

$$\lim_{n \rightarrow \infty} P_\pi \left(\frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \geq \epsilon \right) = 0 \quad (2.3.13),$$

where the convergence is exponential.

For the $m > 0$ satisfying (2.3.12), let $n = pm + q$, where $p \geq 0$ and $0 \leq q < m$, we then have

$$\begin{aligned} & \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \\ & \leq \frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_{pm})\| + \frac{1}{n} \log \|H(\sigma_{pm-1}) \cdots H(\sigma_0)\| \\ & \leq \frac{1}{n} \sum_{i=pm}^n \log \|H(\sigma_i)\| + \frac{1}{n} \sum_{j=0}^{p-1} \log \|H(\sigma_{(j+1)m-1}) \cdots H(\sigma_{jm})\| \end{aligned}$$

We again use the notation $G = \max_{1 \leq j \leq N} \|H(j)\|$, then from the above, we obtain

$$\frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \leq \frac{q}{n} \log G + \frac{p}{n} \frac{1}{p} \sum_{j=0}^{p-1} \log \|H(\sigma_{(j+1)m-1}) \cdots H(\sigma_{jm})\| \quad (2.3.14)$$

For $\epsilon < 0$ to be determined later, we have

$$\begin{aligned} & P_\pi \left(\frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \geq \epsilon \right) \\ & \leq P_\pi \left(\frac{q}{n} \log G + \frac{p}{n} \frac{1}{p} \sum_{j=0}^{p-1} \log \|H(\sigma_{(j+1)m-1}) \cdots H(\sigma_{jm})\| \geq \epsilon \right) \\ & = P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|H(\sigma_{(j+1)m-1}) \cdots H(\sigma_{jm})\| \geq \frac{n}{p} \left(\epsilon - \frac{q}{n} \log G \right) \right) \end{aligned} \quad (2.3.15)$$

Notice that for fixed m , $\lim_{p \rightarrow \infty} \frac{n}{p} = m$, and $\lim_{p \rightarrow \infty} \frac{n}{p}(\varepsilon - \frac{q}{n} \log G) = m\varepsilon$.

Define $\tilde{\sigma}_j = (\sigma_{(j+1)m-1}, \dots, \sigma_{jm})$ for $j = 0, 1, 2, \dots$. Since $\{\sigma_k\}$ is a finite state, time homogeneous and irreducible Markov chain, $\{\tilde{\sigma}_j\}_{j=0}^{+\infty}$ is also a finite state, time homogeneous and irreducible Markov chain. Let $\tilde{\pi}$ denote the unique invariant distribution of $\{\tilde{\sigma}_j\}$. Let $\underline{N}^m \stackrel{\text{def}}{=} \underline{N} \times \dots \times \underline{N}$ and $f : \underline{N}^m \rightarrow \mathcal{R}$ be defined as

$$f(\hat{\sigma}) = \log \|H(\hat{\sigma}_{m-1}) \cdots H(\hat{\sigma}_0)\|, \quad \hat{\sigma} = (\hat{\sigma}_{m-1}, \dots, \hat{\sigma}_0) \in \underline{N}.$$

Then, we have

$$E_{\tilde{\pi}} f(\tilde{\sigma}_0) = E_{\tilde{\pi}} \log \|H(\sigma_{m-1}) \cdots H(\sigma_0)\| = E_{\pi} \log \|H(\sigma_{m-1}) \cdots H(\sigma_0)\| = \Delta_m. \quad (2.3.16)$$

Since $\alpha + \varepsilon_0 < 0$, there exists an $\varepsilon_1 > 0$ such that $\alpha + \varepsilon_0 + 2\varepsilon_1 < 0$. Choose $\varepsilon = \alpha + \varepsilon_0 + 2\varepsilon_1$. Then, there exists a $K > 0$ such that for any $n \geq K$, we have $\frac{n}{p}(\varepsilon - \frac{q}{n} \log G) \geq m(\varepsilon - \varepsilon_1)$.

Then, from (2.3.12), (2.3.15) and (2.3.16), we obtain for $n \geq K$,

$$\begin{aligned} & P_{\pi} \left(\frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \geq \varepsilon \right) \\ & \stackrel{(2.3.15)}{\leq} P_{\pi} \left(\frac{1}{p} \sum_{j=0}^{p-1} f(\tilde{\sigma}_j) \geq m(\varepsilon - \varepsilon_1) \right) \\ & = P_{\tilde{\pi}} \left(\frac{1}{p} \sum_{j=0}^{p-1} f(\tilde{\sigma}_j) \geq m(\alpha + \varepsilon_0) + m\varepsilon_1 \right) \\ & \stackrel{(2.3.13), (2.3.16)}{\leq} P_{\tilde{\pi}} \left(\frac{1}{p} \sum_{j=0}^{p-1} f(\tilde{\sigma}_j) \geq \Delta_m + m\varepsilon_1 \right) \end{aligned} \quad (2.3.17)$$

Due to the fact that $\Delta_m > -\infty$ for any m , it is easy to verify that $H(\sigma_k)H(\sigma_{k-1}) \cdots H(\sigma_0) \neq 0$ for any k . From the large deviation theorem (Lemma 2.3.9), there exists $M_1 > 0$ and $0 < \gamma_1 < 1$, such that for p large,

$$P_{\tilde{\pi}} \left(\frac{1}{p} \sum_{j=0}^{p-1} f(\tilde{\sigma}_j) \geq \Delta_m + m\varepsilon_1 \right) \leq M_1 \gamma_1^p. \quad (2.3.18)$$

Thus, for n large, from (2.3.17) and (2.3.18), we obtain

$$\begin{aligned} P_{\pi} \left(\frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \geq \varepsilon \right) \\ \leq M_1 \gamma_1^p = (M_1 \gamma_1^{-q/m}) (\gamma_1^{1/m})^{pm+q} \leq M_2 \gamma^n \end{aligned}$$

where $\gamma = \gamma_1^{1/m}$ and $M_2 = M_1 \gamma_1^{-1}$. This establishes (2.3.13) for the case when $\alpha \neq -\infty$

Now, from Lemma 2.3.10 and (2.3.13), there exists a $\delta > 0$ such that

$$\lim_{k \rightarrow +\infty} E_{\pi} \|x_k(\omega, x_0)\|^\delta = 0, \quad \forall x_0 \in \mathcal{R}^n. \quad (2.3.19)$$

Since $\{\sigma_k\}$ is irreducible and $\pi > 0$, from Lemma 2.3.3, the system (2.1.1) is δ -moment stable for any initial distribution $\xi \in \Xi$.

Summarizing the above, we have proved that if $(H(1), \dots, H(N)) \in \Sigma_-^a$, then there exists a $\delta_0 > 0$ such that $(H(1), \dots, H(N)) \in \Sigma^{\delta_0}$. Notice that Σ^δ (as a set) is monotonically increasing as δ decreases. Therefore we conclude that

$$\Sigma_-^a \subseteq \lim_{\delta \rightarrow 0} \Sigma^\delta = \cup_{\delta > 0} \Sigma^\delta \subseteq \Sigma^a.$$

Next, we will show that the first set containment \subseteq can be replaced by a set equality $=$. In fact, if $(H(1), \dots, H(N)) \in \Sigma^\delta$ for $\delta > 0$, then (2.1.1) is δ -moment stable. From Theorem 2.3.1, we know that (2.1.1) is

also exponentially δ -moment stable, i.e., there exists an $M > 0$ and $0 < \rho < 1$ such that $E\|H(\sigma_n) \cdots H(\sigma_1)\|^\delta \leq M\rho^n$, thus we have

$$g(\delta) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \log E\|H(\sigma_n) \cdots H(\sigma_0)\|^\delta \leq \log \rho < 0.$$

From Jensen's inequality, we obtain that

$$\begin{aligned} \delta\alpha &= \lim_{n \rightarrow \infty} \frac{1}{n} E \log \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \log E\|H(\sigma_n) \cdots H(\sigma_0)\|^\delta \leq \log \rho < 0. \end{aligned}$$

Thus, $\alpha < 0$, i.e., $(H(1), \dots, H(N)) \in \Sigma_-^a$. This completes the proof of (ii). \square

Up to now, we have seen that there exists a close relationship between the stability regions of almost sure stability and δ -moment stability. A critical look at the proof of Theorem 2.3.7 reveals that we have used the properties of top Lyapunov exponent and top δ -moment Lyapunov exponent, which are in fact the best characterization for almost sure stability and δ -moment stability. Therefore, in what follows, we will investigate the relationship between the top Lyapunov exponent and the δ -moment Lyapunov exponent. Similar results to those obtained by Arnold et al ([115],[116]) for stochastic differential equations of Ito type are obtained.

Recall that the top Lyapunov exponent as defined in (2.3.5) is given by

$$\alpha_\xi = \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} E_\xi \log \|H(\sigma_n) \cdots H(\sigma_0)\|,$$

which has the properties given in Lemma 2.3.5. The δ -moment Lyapunov exponent can be defined as the extended real-valued function $\beta(\cdot, \cdot) : \mathcal{R} \times \Xi \rightarrow \overline{\mathcal{R}} = [-\infty, +\infty]$ given by

$$\beta(\delta, \xi) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta \quad (2.3.20)$$

if the indicated limit exists. We may also define

$$\begin{aligned}\bar{\beta}(\delta, \xi) &= \overline{\lim}_{n \rightarrow +\infty} \frac{1}{n} \log E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta \\ \underline{\beta}(\delta, \xi) &= \underline{\lim}_{n \rightarrow +\infty} \frac{1}{n} \log E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta.\end{aligned}\tag{2.3.20'}$$

Define $\beta(0, \xi) = \underline{\beta}(0, \xi) = \bar{\beta}(0, \xi) = 0$. For simplicity, we assume that the matrices $H(j)$ are nonsingular for all $j \in \underline{N}$. We have the following properties of the δ -moment Lyapunov exponent(s):

Proposition 2.3.11.

Assume that the $H(j)$ are invertible matrices for all $j \in \underline{N}$. For any $\xi \in \Xi$, we have

(i) $-\infty < \underline{\beta}(\delta, \xi) \leq \bar{\beta}(\delta, \xi) < +\infty$.

(ii) $\delta\alpha_\xi \leq \bar{\beta}(\delta, \xi)$. If $\xi \ll \pi$, then, $\delta\alpha_\xi = \delta\alpha_\pi \leq \underline{\beta}(\delta, \xi)$. □

Proof: (i). Since $H(1), \dots, H(N)$ are nonsingular, let

$$G = \max_{1 \leq k \leq N} \|H(k)\|, \quad g = \min_{1 \leq k \leq N} \|H^{-1}(k)\|^{-1}, \quad \text{with } \|H^{-1}(k)\|^{-1} \leq \|H(k)\|$$

for any k , we have for $\delta > 0$,

$$-\infty < -2\delta |\log g| \leq \frac{1}{n} \log E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta \leq 2\delta |\log G| < +\infty,$$

and for $\delta < 0$,

$$-\infty < 2\delta |\log G| \leq \frac{1}{n} \log E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta \leq -2\delta |\log g| < +\infty.$$

These inequalities imply that $-\infty < \underline{\beta}(\delta, \xi) \leq \bar{\beta}(\delta, \xi) < +\infty$.

(ii). $\log x$ is a concave function on $(0, +\infty)$, and using Jensen's inequality, we obtain

$$\frac{1}{n} E_\xi \log \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta \leq \frac{1}{n} \log E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta$$

Taking the limit superum on both sides, we have $\delta\alpha_\xi \leq \overline{\beta}(\delta, \xi)$. If $\xi \ll \pi$, taking the limit infimum and applying Lemma 2.3.5, we have $\delta\alpha_\xi = \delta\alpha_\pi \leq \underline{\beta}(\delta, \xi)$. \square

Lemma 2.3.12:

(i) Suppose that $f(x)$ is continuous, and for any x_1, x_2 ,

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$$

holds, then $f(x)$ is convex.

(ii) If $f(x)$ is convex, then it is also continuous. \square

Proof: This is a well-known result from calculus. \square

Theorem 2.3.13:

If $H(1), \dots, H(N)$ are invertible, then

(i) For any $\xi \in \Xi$ fixed, $\overline{\beta}(\cdot, \xi)$ is a convex function defined on \mathcal{R} .

(ii). For any $\xi \in \Xi$ fixed, $\overline{\beta}(\delta, \xi)/\delta$ is nondecreasing on $\mathcal{R} \setminus \{0\}$. Let $\dot{\overline{\beta}}(\delta, \xi) = \frac{d}{d\delta} \overline{\beta}(\delta, \xi)$, then $\dot{\overline{\beta}}(0^-, \xi) \leq \alpha_\xi \leq \dot{\overline{\beta}}(0^+, \xi)$ and $\dot{\overline{\beta}}(0, \xi) = \alpha_\xi$ whenever it exists.

(iii) For any $\delta \in \mathcal{R}$ fixed, $\underline{\beta}(\delta, \cdot)$ is a concave function defined on Ξ . \square

Proof: (i). Define $f(x) = \log E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^x$. Then for any $x_1, x_2 \in \mathcal{R}$, from the Cauchy-Schwartz inequality, we have

$$\begin{aligned} f\left(\frac{x_1 + x_2}{2}\right) &= \log \left\{ E_\xi \left[(\|H(\sigma_n) \cdots H(\sigma_0)\|^{x_1})^{1/2} (\|H(\sigma_n) \cdots H(\sigma_0)\|^{x_2})^{1/2} \right]^2 \right\} \\ &\leq \log \left\{ (E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^{x_1})^{1/2} (E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^{x_2})^{1/2} \right\} \\ &= \frac{f(x_1) + f(x_2)}{2} \end{aligned} \tag{2.3.21}$$

It is easy to show that $f(x)$ is continuous on \mathcal{R} from the invertibility of $H(1), \dots, H(N)$. From (2.3.21) and Lemma 2.3.12, we know that $f(x)$ is convex. Thus for any $0 \leq \lambda \leq 1$, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

Multiplying both sides of the inequality by $1/n$ and taking the limit supremum, we obtain

$$\bar{\beta}(\lambda x_1 + (1 - \lambda)x_2, \xi) \leq \lambda \bar{\beta}(x_1, \xi) + (1 - \lambda) \bar{\beta}(x_2, \xi)$$

which implies that $\bar{\beta}(x, \xi)$ is convex.

(ii). Since $\log x$ is monotonically increasing on $(0, +\infty)$, from Lemma 2.3.8, we have that $\bar{\beta}(\delta, \xi)/\delta$ is nondecreasing on $(0, +\infty)$. When δ takes values in $(-\infty, 0)$, replacing δ by $-\delta$ and using Lemma 2.3.8, we obtain that $\bar{\beta}(\delta, \xi)/\delta$ is also nondecreasing on $(-\infty, 0)$. By (ii) of Proposition 2.3.11, we have for any δ , $\delta \alpha_\xi \leq \bar{\beta}(\delta, \xi)$. Thus, for $\delta > 0$, we have

$$\frac{\bar{\beta}(-\delta, \xi)}{-\delta} \leq \alpha_\xi \leq \frac{\bar{\beta}(\delta, \xi)}{\delta}$$

Taking the limit $\delta \downarrow 0^+$, we have (ii).

(iii). For $\lambda \in [0, 1]$ and $\xi, \zeta \in \Xi$, it is easy to verify that $P_{\lambda\xi+(1-\lambda)\zeta} = \lambda P_\xi + (1-\lambda)P_\zeta$. Then, since $\log(\cdot)$ is concave, we obtain

$$\begin{aligned} & \frac{1}{n} \log E_{\lambda\xi+(1-\lambda)\zeta} \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta \\ &= \frac{1}{n} \log(\lambda E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta + (1-\lambda) E_\zeta \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta) \\ &\geq \frac{\lambda}{n} \log E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta + \frac{1-\lambda}{n} \log E_\zeta \|H(\sigma_n) \cdots H(\sigma_0)\|^\delta. \end{aligned}$$

Taking the limit infimum on both sides, we have

$$\underline{\beta}(\delta, \lambda\xi + (1-\lambda)\zeta) \geq \lambda \underline{\beta}(\delta, \xi) + (1-\lambda) \underline{\beta}(\delta, \zeta),$$

i.e., $\underline{\beta}(\delta, \cdot)$ is a concave function. \square

It is evident that Theorem 2.3.13 is also valid for a more general class of systems as long as the indicated expectations exist. In the theorem we require that the δ -moment top Lyapunov exponent is differentiable at zero. For diffusion process, Arnold et al ([115],[116]) proved that this assumption is not only true, they have proved that the δ -moment top Lyapunov exponent is analytic at zero and its derivative at zero is the top Lyapunov exponent! From this observation, one may conjecture that it is true for the jump linear systems. Leizarowitz ([153]) heuristically argued that this conjecture can not be proved by Arnold et al's ([115], [116]) approach, and he did not give a definite answer to this conjecture. Du and Nhung ([161]) attempted to generalize this to the random matrix product case, they consider the following discrete-time system

$$x(n+1) = A_n x_n, \quad x_0 = x(0) \tag{2.3.22}$$

where $(A_n)_{n \geq 1}$ is a sequence of $Gl(d, R)$ -valued i.i.d. random variables with common distribution μ . The δ -moment Lyapunov exponent is defined to be

$$g(\delta, x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E \|x(n, x_0)\|^\delta.$$

It has been proved in ([161]) that for the system (2.3.22) the limit in the definition of moment Lyapunov exponent exists and also corresponding to a finite number of real numbers, the moment Lyapunov exponents, there exists a sequence of subspaces that defines a filtration. The main result in [161] claimed that the function $g(\delta, x_0)$ is differentiable with respect to δ at $\delta = 0$ for any fixed $x_0, 0 \neq x_0 \in \mathbf{R}^d$ and if $x_0 \in L_i \setminus L_{i-1}$, then $g'(0, x_0) = \lambda_i$, where λ_i and L_i are the Lyapunov exponents and the corresponding filtration, respectively. Unfortunately, this is not true for the system (2.3.22) as we will show in the following counterexample, this shows that Arnold's formula is not valid for jump linear system, and the analytic properties of δ -moment Lyapunov exponents are clarified. Using standard notations, which is different from the notation used by Du and Nhung ([161]), let $\lambda_r < \lambda_{r-1} < \dots < \lambda_1$ denote the sample Lyapunov exponents, where λ_1 is the top sample top Lyapunov exponent, and $\{0\} = L_r \subset L_{i-1} \subset \dots \subset L_1$ denotes the filtration.

Counterexample. Suppose that $\{\sigma_n\}$ is a two-state i.i.d. random process with the common distribution $\pi_1 = P(\sigma_n = 1) = 0.5, \pi_2 = P(\sigma_n = 2) = 0.5$, let

$$A(1) = \begin{pmatrix} a(1) & 0 \\ 0 & b(1) \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad A(2) = \begin{pmatrix} a(2) & 0 \\ 0 & b(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the system (2.3.22) with $A_n = A(\sigma_n)$. It is easy to verify that $\{A(\sigma_n)\}$ is an i.i.d. random matrix sequence satisfying the conditions of Theorem 2.2 in [161]. We want to show that the top moment Lyapunov exponent $g(\delta) = \max_{x_0 \neq 0} g(\delta, x_0)$ is not differentiable at $\delta = 0$. A straightforward computation of the top moment Lyapunov exponent yields

$$g(\delta) = \lim_{k \rightarrow \infty} \frac{1}{k} \log E \|A(\sigma_k) \cdots A(\sigma_1)\|_2^\delta,$$

where $\|\cdot\|_2$ denote the spectral norm induced by Euclidean vector norm and \log is the logarithm with natural base. Let

$$g_1(\delta) = \lim_{k \rightarrow \infty} \frac{1}{k} \log |a(\sigma_k) \cdots a(\sigma_1)|^\delta,$$

$$g_2(\delta) = \lim_{k \rightarrow \infty} \frac{1}{k} \log |b(\sigma_k) \cdots b(\sigma_1)|^\delta.$$

We first want to show that $g(\delta) = \max\{g_1(\delta), g_2(\delta)\}$. In fact, from

$$\begin{aligned} \|A(\sigma_k) \cdots A(\sigma_1)\|_2 &= \sqrt{\max\left\{\left|\prod_{i=1}^k a(\sigma_i)\right|^2, \left|\prod_{i=1}^k b(\sigma_i)\right|^2\right\}} \\ &= \max\left\{\prod_{i=1}^k |a(\sigma_i)|, \prod_{i=1}^k |b(\sigma_i)|\right\}, \end{aligned}$$

we obtain

$$E\|A(\sigma_k) \cdots A(\sigma_1)\|_2^\delta = E \max\left\{\prod_{i=1}^k |a(\sigma_i)|^\delta, \prod_{i=1}^k |b(\sigma_i)|^\delta\right\}.$$

Thus, it is easy to prove that $g(\delta) \geq \max\{g_1(\delta), g_2(\delta)\}$. Using the trivial inequality $\max\{x, y\} \leq x + y$ ($\forall x, y \geq 0$), we have

$$\begin{aligned} &E \max\left\{\prod_{i=1}^k |a(\sigma_i)|^\delta, \prod_{i=1}^k |b(\sigma_i)|^\delta\right\} \\ &\leq E \left\{\prod_{i=1}^k |a(\sigma_i)|^\delta + \prod_{i=1}^k |b(\sigma_i)|^\delta\right\} \\ &= \prod_{i=1}^k E|a(\sigma_i)|^\delta + \prod_{i=1}^k E|b(\sigma_i)|^\delta \\ &= (\pi_1|a(1)|^\delta + \pi_2|a(2)|^\delta)^k + (\pi_1|b(1)|^\delta + \pi_2|b(2)|^\delta)^k. \end{aligned}$$

Thus, we have

$$\begin{aligned} g(\delta) &\leq \lim_{k \rightarrow \infty} \frac{1}{k} \log [(\pi_1|a(1)|^\delta + \pi_2|a(2)|^\delta)^k + (\pi_1|b(1)|^\delta + \pi_2|b(2)|^\delta)^k] \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{k} \log \left(2 \times [\max\{(\pi_1|a(1)|^\delta + \pi_2|a(2)|^\delta), (\pi_1|b(1)|^\delta + \pi_2|b(2)|^\delta)\}]^k\right) \\ &= \max\{\log(\pi_1|a(1)|^\delta + \pi_2|a(2)|^\delta), \log(\pi_1|b(1)|^\delta + \pi_2|b(2)|^\delta)\} \\ &= \max\{g_1(\delta), g_2(\delta)\}. \end{aligned}$$

Therefore, we have proved that $g(\delta) = \max\{g_1(\delta), g_2(\delta)\}$. For the current example, taking $a(1) = a$, $a(2) = b(1) = b(2) = 1$ and $\pi_1 = \pi_2 = 0.5$ into the above formula, we finally arrive at

$$\begin{aligned} g(\delta) &= \max \left\{ \log(\pi_1 |a(1)|^\delta + \pi_2 |a(2)|^\delta), \log(\pi_1 |b(1)|^\delta + \pi_2 |b(2)|^\delta) \right\} \\ &= \max \left\{ \log \left(\frac{a^\delta + 1}{2} \right), 0 \right\} \end{aligned}$$

If $a > 1$, we have

$$g(\delta) = \begin{cases} \log \left(\frac{a^\delta + 1}{2} \right), & \delta \geq 0 \\ 0, & \delta < 0 \end{cases}$$

from which we obtain $g'(0+) = \lim_{\delta \downarrow 0+} \frac{d}{d\delta} (\log(a^\delta + 1)/2) = \frac{1}{2} \log a$ and $g'(0-) = 0$. Therefore, $g'(0+) \neq g'(0-)$, and $g(\delta)$ is not differentiable even though $A(1)$ and $A(2)$ are invertible. This shows that Du and Nhung's result is incorrect. Notice, however, that the top Lyapunov exponent $\lambda_1 = \frac{1}{2} \log a$ for $a \geq 1$ and $g'(0+) = \lambda_1$. It is also easy to check that for $0 < a < 1$, the top Lyapunov exponent is $\lambda_1 = 0$, $g'(0+) = 0$ and $g'(0-) = \frac{1}{2} \log a$, hence $g'(0+) = \lambda_1$ is still valid. From this observation, we may conjecture that Du and Nhung's result may be modified to: $g'(0+) = \lambda_1$. In Theorem 2.3.13, we proved that $g(\delta)$ has left and right derivative at $\delta = 0$, thus $g'(0+)$ is meaningful. We will show that under certain conditions, the above modification of Du and Nhung's result is true. Although their result is incorrect in general, for a certain class of random matrix products their result is valid. What is necessary is that a "regularity condition" is satisfied. In what follows, we assume that $\{\sigma_k\}$ is irreducible with ergodic probability measure π and the individual mode matrices $A(1), A(2), \dots, A(N)$ are invertible. This eliminates the so-called "deadbeat" case.

Definition. The top Lyapunov exponent and the top δ -moment Lyapunov exponent of system (2.1.1) are defined, respectively, as

$$\alpha = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|A(\sigma_n) \cdots A(\sigma_1)\|,$$

$$g(\delta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E_\pi \|A(\sigma_n) \cdots A(\sigma_1)\|^\delta.$$

The exponents α and $g(\delta)$ are the top sample Lyapunov exponent and top δ -moment Lyapunov exponent, respectively, i.e.,

$$\alpha = \max_{x_0 \neq 0} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|x(k, x_0)\|,$$

$$g(\delta) = \max_{x_0 \neq 0} \lim_{k \rightarrow \infty} \frac{1}{k} \log E_\pi \|x(k, x_0)\|^\delta,$$

where $x(k, x_0)$ is a sample solution of (2.1.1) with initial condition $x(0, x_0) = x_0$. It can be easily proved that for any initial probability distribution ξ , we have

$$\overline{\beta}(\delta, \xi) = \underline{\beta}(\delta, \xi) = \beta(\delta, \xi) = g(\delta), \quad \alpha_\xi = \alpha_\pi = \alpha.$$

Theorem 2.3.1.

For $\delta \geq 0$, the δ -moment Lyapunov exponent $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0+) = \alpha$.

Proof. As we have already proven from Lemma 2.3.5:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|A(\sigma_n) \cdots A(\sigma_1)\| = \lim_{n \rightarrow \infty} \frac{1}{n} E_\pi \log \|A(\sigma_n) \cdots A(\sigma_1)\| = \alpha$$

almost surely. Thus for any $\varepsilon > 0$ and $\varepsilon_1 > 0$, there exists an $m > 0$ such that

$$E_\pi \log \|A(\sigma_m) \cdots A(\sigma_1)\| < m\alpha + m\varepsilon \tag{2.3.23}$$

For sufficiently large p , we have

$$\frac{1}{pm} \log \|A(\sigma_{pm}) \cdots A(\sigma_1)\| \leq \frac{1}{pm} \sum_{j=0}^{p-1} \log \|A(\sigma_{(j+1)m}) \cdots A(\sigma_{jm+1})\|.$$

Thus, for any $\varepsilon > 0$, we have

$$\begin{aligned} & P \left(\frac{1}{pm} \log \|A(\sigma_{pm}) \cdots A(\sigma_1)\| \geq \alpha + \varepsilon + \varepsilon_1 \right) \\ & \leq P \left(\frac{1}{pm} \sum_{j=0}^{p-1} \log \|A(\sigma_{(j+1)m}) \cdots A(\sigma_{jm+1})\| \geq \alpha + \varepsilon + \varepsilon_1 \right) \\ & = P \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|A(\sigma_{(j+1)m}) \cdots A(\sigma_{jm+1})\| \geq m(\alpha + \varepsilon) + m\varepsilon_1 \right) \\ & \leq P \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|A(\sigma_{(j+1)m}) \cdots A(\sigma_{jm+1})\| \geq E \log \|A(\sigma_m) \cdots A(\sigma_1)\| + m\varepsilon_1 \right) \end{aligned} \tag{2.3.24}$$

where in the last inequality, we have used (2.3.23). Using the Large Deviation Theorem (Lemma 2.3.9), there exists r_1 satisfying $0 \leq r_1 < 1$ and an $M_1 > 0$ such that

$$\begin{aligned} & P \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|A(\sigma_{(j+1)m}) \cdots A(\sigma_{jm+1})\| \geq E_\pi \log \|A(\sigma_m) \cdots A(\sigma_1)\| + m\varepsilon_1 \right) \\ & \leq M_1 r_1^p = M_1 (r_1^{1/m})^{pm}. \end{aligned}$$

From this and (2.3.24), we have

$$P \left(\frac{1}{pm} \log \|A(\sigma_{pm}) \cdots A(\sigma_1)\| \geq \alpha + \varepsilon + \varepsilon_1 \right) \leq M_1 (r_1^{1/m})^{pm} \tag{2.3.25}$$

Since $\{\sigma_n\}$ is a finite state Markov chain, there exists an $M > 0$ such that $\|A(\sigma_n)\| \leq M$. Define the sets

$$\begin{aligned} \mathcal{A} &= \left(\omega : \frac{1}{pm} \log \|A(\sigma_{pm}) \cdots A(\sigma_1)\| \geq \alpha + \varepsilon + \varepsilon_1 \right) \\ \mathcal{A}^c &= \left(\omega : \frac{1}{pm} \log \|A(\sigma_{pm}) \cdots A(\sigma_1)\| < \alpha + \varepsilon + \varepsilon_1 \right) \end{aligned}$$

Then on \mathcal{A}^c ,

$$\|A(\sigma_{pm}) \cdots A(\sigma_1)\| < e^{pm(\alpha + \varepsilon + \varepsilon_1)},$$

and on \mathcal{A} ,

$$\|A(\sigma_{pm}) \cdots A(\sigma_1)\| \leq M^{pm}.$$

Thus, we have

$$\begin{aligned} & E_\pi \|A(\sigma_{pm}) \cdots A(\sigma_1)\|^\delta \\ &= \int_{\mathcal{A}} \|A(\sigma_{pm}) \cdots A(\sigma_1)\|^\delta P(d\omega) + \int_{\mathcal{A}^c} \|A(\sigma_{pm}) \cdots A(\sigma_1)\|^\delta P(d\omega) \quad (2.3.26) \\ &\leq M^{pm} P(\mathcal{A}) + e^{\delta pm(\alpha + \varepsilon + \varepsilon_1)} \leq M_1 (M^\delta r_1^{1/m})^{pm} + e^{\delta pm(\alpha + \varepsilon + \varepsilon_1)}. \end{aligned}$$

Since $\lim_{\delta \rightarrow 0} M^\delta r_1^{1/m} = r_1^{1/m} < 1$, we choose $\delta > 0$ sufficiently small so that there exists a ρ satisfying $0 \leq \rho < 1$, and $M^\delta r_1^{1/m} \leq \rho < 1$.

If $\alpha \geq 0$, then from (2.3.26) we have for sufficiently large p ,

$$\begin{aligned} \frac{1}{pm} \log E_\pi \|A(\sigma_{pm}) \cdots A(\sigma_1)\|^\delta &\leq \frac{1}{pm} \log [M_1 \rho^{pm} + e^{\delta pm(\alpha + \varepsilon + \varepsilon_1)}] \\ &\leq \frac{1}{pm} \log [1 + e^{\delta pm(\alpha + \varepsilon + \varepsilon_1)}] \leq \frac{1}{pm} \log [2e^{\delta pm(\alpha + \varepsilon + \varepsilon_1)}] \\ &= \frac{\log 2}{pm} + \delta(\alpha + \varepsilon + \varepsilon_1). \end{aligned}$$

Letting $p \rightarrow \infty$, we have $\frac{g(\delta)}{\delta} \leq \alpha + \varepsilon + \varepsilon_1$. Because of the arbitrary choice of ε and ε_1 and the fact that $\lim_{\delta \downarrow 0+} g(\delta)/\delta = g'(0+)$, we have that $g'(0+) \leq \alpha$. It is already known that $g'(0+) \geq \alpha$, therefore, we have $g'(0+) = \alpha$ for $\alpha \geq 0$.

Suppose that $\alpha < 0$, then we choose $\beta > -\alpha$, define $B(j) = A(j)e^\beta$ ($j = 1, 2, \dots, N$). Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|B(\sigma_n) \cdots B(\sigma_1)\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|A(\sigma_n) \cdots A(\sigma_1)\| + \beta = \alpha + \beta > 0$$

and

$$G(\delta) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \log E_\pi \|B(\sigma_n) \cdots B(\sigma_1)\|^\delta = g(\delta) + \beta\delta.$$

Using the result for $\alpha \geq 0$, we have that $G(\delta)$ is differentiable from the right at $\delta = 0$ and $G'(0+) = \alpha + \beta$. Since $G'(\delta) = g'(\delta) + \beta$, we have that $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0+) = \alpha$. This completes the proof. \square

In [161], only i.i.d. random matrix products were discussed, the above result is more general in certain sense. The following result is concerned with the finite state i.i.d. case.

Corollary 2.3.15.

If $A(1), A(2), \dots, A(N)$ are invertible matrices and $\{\sigma_k\}$ is a finite state i.i.d. random sequence with state space $S = \{1, 2, \dots, N\}$ and with common probability distribution $\pi = \{\pi_1, \pi_2, \dots, \pi_N\}$, then the moment Lyapunov exponent $g(\delta)$ of system (2.1.1) is differentiable from the right at $\delta = 0$ and $g'(0+) = \alpha$. \square

Applying Corollary 2.3.15 to the *Counterexample* illustrates the validity of this result.

Remark: The proof of Du and Nhung ([161]) can not be used or modified to prove Corollary 2.3.15. There exists a misprint in equation (2.9) of [161]: $2^{l\delta}$ should be replaced by $2^{-l\delta}$. Also, the proof of Du and Nhung used Hasminskii's idea (see the proof of Lemma 4.1 on page 205 of [95]). However, Hasminskii used the condition that the corresponding stochastic system is “stable in probability” to obtain a similar inequality as (2.9) in [161]. According to Hasminskii ([95]), (2.1.1) is stable in probability if, for every $\varepsilon > 0$ and $\rho > 0$, there exists an $r \geq 0$ such that if $k > 0$ and $\|x_0\| < r$, we have $P\{\|x(k, x_0)\| > \varepsilon\} < \rho$. However, in [161], it does not appear that (2.8) is “stable in probability” in

the sense of Hasminskii. Of course, we have that (2.8) is almost surely stable, but it is not so easy to prove that (2.8) is “stable in probability” in the sense of Hasminskii. Therefore, the crucial inequality (2.9) in [161] is questionable! In our proof, however, we use the fact that the moment Lyapunov exponent is closely related to a large deviation property, and we use the large deviation theorem developed in our earlier work on stochastic stability of jump linear systems.

If $A(1), A(2), \dots, A(N)$ are invertible, then the definition of the δ -moment Lyapunov exponent $g(\delta)$ can be extended to the whole real line, i.e., $g(\delta)$ is well-defined in \mathbf{R} . Theoretically, it is interesting to study the differentiability of $g(\delta)$ at $\delta = 0$. As we have already noticed for system (2.1.1), $g(\delta)$ is not necessarily differentiable at $\delta = 0$. However, under certain conditions, differentiability is guaranteed. We introduce the following:

Regularity Condition (RC):

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \|A^{-1}(\sigma_1)A^{-1}(\sigma_2) \cdots A^{-1}(\sigma_k)\| = -\alpha,$$

where α is the top Lyapunov exponent of system (2.1.1).

This regularity condition is not surprising. If the system (2.1.1) is regular in the sense of [96] or [97], then the above regularity condition is satisfied. Given (RC), we can prove the following theorem.

Theorem 2.3.16

Suppose that $A(1), \dots, A(N)$ are invertible, and the above Regularity Condition (RC) holds, then $g(\delta)$ is differentiable at $\delta = 0$ and $g'(0) = \alpha$.

Proof. From Theorem 2.3.14, we have proved that $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0+) = \alpha$, so it is sufficient to prove that $g(\delta)$ is differentiable from the left and $g'(0-) = \alpha$. Using a similar argument as in Lemma 2.3.5, together with RC, we have that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \log \|A^{-1}(\sigma_1) \cdots A^{-1}(\sigma_n)\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} E_\pi \log \|A^{-1}(\sigma_1) \cdots A^{-1}(\sigma_n)\| = -\alpha. \end{aligned}$$

For any $\varepsilon_1 > 0$, there exists an $m > 0$ such that

$$m(\alpha - \varepsilon_1) \leq E_\pi \{-\log \|A^{-1}(\sigma_1) \cdots A^{-1}(\sigma_m)\|\} \leq m(\alpha + \varepsilon_1).$$

For any $\varepsilon_2 > 0$, let $d = \alpha - \varepsilon_1 - \varepsilon_2$. Then using the inequality

$$\begin{aligned} 1 = \|I\| &= \|A(\sigma_n) \cdots A(\sigma_1) A^{-1}(\sigma_1) \cdots A^{-1}(\sigma_n)\| \\ &\leq \|A(\sigma_n) \cdots A(\sigma_1)\| \|A^{-1}(\sigma_1) \cdots A^{-1}(\sigma_n)\|, \end{aligned}$$

we have

$$\frac{1}{pm} \log \|A(\sigma_{pm}) \cdots A(\sigma_1)\| \geq -\frac{1}{m} \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|A^{-1}(\sigma_{jm+1}) \cdots A^{-1}(\sigma_{(j+1)m})\| \right).$$

From this, we can obtain the following

$$\begin{aligned} & P \left(\frac{1}{pm} \log \|A(\sigma_{pm}) \cdots A(\sigma_1)\| < d \right) \\ & \leq P \left(-\frac{1}{m} \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|A^{-1}(\sigma_{jm+1}) \cdots A^{-1}(\sigma_{(j+1)m})\| \right) < d \right) \\ & = P \left(\frac{1}{p} \sum_{j=0}^{p-1} \{-\log \|A^{-1}(\sigma_{jm+1}) \cdots A^{-1}(\sigma_{(j+1)m})\|\} < m(\alpha - \varepsilon_1) - m\varepsilon_2 \right) \\ & \leq P \left(\frac{1}{p} \sum_{j=0}^{p-1} \{-\log \|A^{-1}(\sigma_{jm+1}) \cdots A^{-1}(\sigma_{(j+1)m})\|\} \right. \\ & \quad \left. < E_\pi \{-\log \|A^{-1}(\sigma_1) \cdots A^{-1}(\sigma_m)\|\} - m\varepsilon_2 \right) \end{aligned}$$

From this and the Large Deviation Theorem (Lemma 2.3.9), there exists an $M_2 > 0$ and $\rho < 1$ such that

$$P\left(\frac{1}{pm} \log \|A(\sigma_{pm}) \cdots A(\sigma_1)\| < d\right) \leq M_2 \rho^p. \quad (2.3.27)$$

Since $A(1), \dots, A(N)$ are invertible, there exists $M_1 > 0$ such that $\|A^{-1}(\sigma_n)\| \leq M_1$, and for $\delta < 0$

$$\begin{aligned} \|A(\sigma_n) \cdots A(\sigma_1)\|^\delta &\leq (\|A^{-1}(\sigma_1) \cdots A^{-1}(\sigma_n)\|^{-1})^\delta \\ &\leq (\|A^{-1}(\sigma_1)\| \cdots \|A^{-1}(\sigma_n)\|)^{-\delta} \leq M_1^{-n\delta}. \end{aligned}$$

Following the same procedure as in the proof of Theorem 2.3.14, for $\delta < 0$, we obtain

$$\begin{aligned} E\|A(\sigma_{pm}) \cdots A(\sigma_1)\|^\delta &\leq M_1^{-pm\delta} P\left(\frac{1}{pm} \log \|A(\sigma_{pm}) \cdots A(\sigma_1)\| < d\right) + e^{dpm\delta} \\ &\leq M_2(M_1^{-m\delta} \rho)^p + e^{dpm\delta}. \end{aligned} \quad (2.3.28)$$

If $\alpha < 0$, then $d < 0$ and $\delta d > 0$ and we choose $|\delta|$ sufficiently small so that $M_2(M_1^{-m\delta} \rho)^p < 1$ for sufficiently large p (this is possible because $\lim_{\delta \rightarrow 0} M_1^{-m\delta} \rho = \rho < 1$). From (2.3.28), we have

$$E\|A(\sigma_{pm}) \cdots A(\sigma_1)\|^\delta \leq 1 + e^{dpm\delta} \leq 2e^{dpm\delta}.$$

From this we obtain,

$$g(\delta) = \lim_{p \rightarrow \infty} \frac{1}{pm} \log E\|A(\sigma_{pm}) \cdots A(\sigma_1)\|^\delta \leq \lim_{p \rightarrow \infty} \frac{1}{pm} \log (2e^{dpm\delta}) = d\delta.$$

Since $\delta < 0$, we have

$$\frac{g(\delta)}{\delta} \geq d = \alpha - \varepsilon_1 - \varepsilon_2.$$

Because of the arbitrary choice of ε_1 and ε_2 and the fact that $g(\delta)/\delta$ is a nondecreasing function, we conclude that $g'(0-) \geq \alpha$ for $\alpha < 0$. From the fact that $g(\delta)/\delta \leq \alpha$ for $\delta < 0$, we have $g'(0-) = \alpha$ for $\alpha < 0$.

If $\alpha \geq 0$, we choose $\beta > \alpha$ and consider the sequence $\{B(\sigma_n)\}$ where $B(\sigma_n) = A(\sigma_n)e^{-\beta}$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|B(\sigma_n) \cdots B(\sigma_1)\| = \alpha - \beta < 0,$$

and

$$G(\delta) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \log E \|B(\sigma_n) \cdots B(\sigma_1)\|^\delta = g(\delta) - \beta\delta,$$

and $B(1), B(2), \dots, B(N)$ are invertible and the system (2.1.1) with system matrix $B(\sigma_k)$ satisfies RC. Using the previous result, $G'(0-) = \alpha - \beta$, from which we obtain that $g'(0-) = \alpha$. This completes the proof. \square

Remark: From this theorem, we can see that for the finite state i.i.d. random matrix product, if the corresponding system (2.1.1) satisfies the regularity condition, Du and Nhung's result is true. Consider the *Counterexample* again, it is easy to compute that for $a > 1$, almost surely, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|A(\sigma_k) \cdots A(\sigma_1)\| &= \frac{1}{2} \log a, \\ \lim_{k \rightarrow \infty} \frac{1}{k} \log \|A^{-1}(\sigma_1) \cdots A^{-1}(\sigma_k)\| &= 0. \end{aligned}$$

this means that the regularity condition RC does not hold for this example.

For a bounded independent identically distributed (iid) random matrix sequence $\{A_n\}$, using the Large Deviation Theorem discussed in [155], we may obtain the following result.

Proposition 2.3.17.

Suppose that $\{A_n\}$ is an i.i.d. matrix sequence and there exists an $M > 0$ such that $\|A_n\| \leq M$, if the integrability condition: $E \log^+ \|A_1\| < +\infty$ holds, then the δ -moment Lyapunov exponent

$$g(\delta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E \|A_n \cdots A_1\|^\delta$$

is differentiable from the right at $\delta = 0$ and $g'(0+)$ is equal to the top Lyapunov exponent

$$\alpha = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|A_n \cdots A_1\|.$$

□

Remark: Suppose that $\{A_n\}$ is an i.i.d. random invertible matrix sequence which satisfies the integrability condition:

$$\int_{\Omega} \log^+ \|A_1\| + \log^+ \|A_1^{-1}\| P(d\omega) < +\infty,$$

and the top Lyapunov exponent and the δ -moment top Lyapunov exponent are defined as before, then the above procedure can be extended to prove that $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0+) = \alpha$. Under the following hypothesis: there exists an $M > 0$ and $\delta > 0$ such that

$$\lim_{n \rightarrow \infty} \int_{\left(\frac{1}{n} \sum_{i=1}^n \log \|A_i\| \geq M\right)} e^{\delta \sum_{i=1}^n \log \|A_i\|} P(d\omega) = 0,$$

the proof follows from the proof of Theorem 2.3.14 just presented and the Large Deviation Theorem developed by Ellis ([155]). This can be seen from the following observation: For any $m > 0$ chosen in a similar manner as in the proof of Theorem 2.3.14, define the sets

$$\begin{aligned} \mathcal{A} &= \left(\omega : \frac{1}{pm} \log \|A_{pm} \cdots A_1\| \geq d \right) \\ \mathcal{A}^c &= \left(\omega : \frac{1}{pm} \log \|A_{pm} \cdots A_1\| < d \right) \\ \mathcal{B} &= \left(\omega : \frac{1}{pm} \sum_{i=1}^{pm} \log \|A_i\| \geq M \right) \\ \mathcal{B}^c &= \left(\omega : \frac{1}{pm} \sum_{i=1}^{pm} \log \|A_i\| < M \right). \end{aligned}$$

Then we have

$$\begin{aligned}
E\|A_{pm} \cdots A_1\|^\delta &= \int_{\mathcal{A}} \|A_{pm} \cdots A_1\|^\delta P(d\omega) + \int_{\mathcal{A}^c} \|A_{pm} \cdots A_1\|^\delta \\
&= \int_{\mathcal{A} \cap \mathcal{B}} \|A_{pm} \cdots A_1\|^\delta P(d\omega) + \int_{\mathcal{A} \cap \mathcal{B}^c} \|A_{pm} \cdots A_1\|^\delta P(d\omega) \\
&\quad + \int_{\mathcal{A}^c} \|A_{pm} \cdots A_1\|^\delta P(d\omega) \\
&\leq \int_{\mathcal{B}} \|A_{pm} \cdots A_1\|^\delta P(d\omega) + \int_{\mathcal{A} \cap \mathcal{B}^c} \|A_{pm} \cdots A_1\|^\delta P(d\omega) \tag{2.3.29} \\
&\quad + \int_{\mathcal{A}^c} \|A_{pm} \cdots A_1\|^\delta P(d\omega) \\
&\leq \int_{\mathcal{B}} e^{\delta \sum_{i=1}^{pm} \log \|A_i\|} P(d\omega) + \int_{\mathcal{A} \cap \mathcal{B}^c} \|A_{pm} \cdots A_1\|^\delta P(d\omega) \\
&\quad + \int_{\mathcal{A}^c} \|A_{pm} \cdots A_1\|^\delta P(d\omega)
\end{aligned}$$

If $\omega \in \mathcal{B}^c$, then $\|A_{pm} \cdots A_1\| \leq e^{pmM}$, thus we can use the technique in the proof of Theorem 2.3.14 to deal with the last two terms of (2.3.29). The first term of (2.3.29) is dealt with by the hypothesis. Unfortunately, we have not been able to find a rigorous proof for the hypothesis. The Large Deviation Theorem has positive confirmation of this hypothesis. We will study this issue in the future.

Although Theorem 2.3.14, Theorem 2.3.16 only consider the top Lyapunov sample and δ -moment exponents, using the random spectrum theory ([75] and [77]), we can prove similar relationships between the sample Lyapunov exponents and the δ -moment Lyapunov exponents. The subspaces L_i , which defines a filtration of the state space, are invariant subspaces of \mathbf{R}^n with respect to the corresponding system (2.1.1). The details will be left to the readers.

2.4. Testable Criteria for δ -moment Stability

In previous section, we have studied relationship between almost sure stability and δ -moment stability and that we observed that almost sure stability can be recovered by δ -moment stability for sufficiently small positive δ . However, except $\delta = 2$, i.e., except second moment stability, it is difficult to obtain a necessary and sufficient condition for δ -moment stability. Therefore, in this section, we present some testable δ -moment stability criteria for the jump linear systems (2.1.1). In the rest of this section, we use the following notations: $\lambda_i = \lambda_{\max}(H^T(i)H(i))$, and $\rho(A)$ denotes the spectral radius of matrix A .

Theorem 2.4.1.

Suppose that $\{\sigma_k\}$ is a finite iid form process with probability distribution, then the system (2.1.1) is δ -moment stable, if one of the following conditions holds:

- (a) $p_1\lambda_1^{\delta/2} + p_2\lambda_2^{\delta/2} + \cdots + p_N\lambda_N^{\delta/2} < 1$. Where $\lambda_i = \lambda_{\max}(H(i)'H(i))$ as defined previously.
- (b) If there exists an induced matrix norm $\|\cdot\|$ such that

$$p_1\|H(1)\|^\delta + p_2\|H(2)\|^\delta + \cdots + p_N\|H(N)\|^\delta < 1$$

Particularly, for a scalar system ($n = 1$), (a) and/or (b) is necessary for δ -moment stability. □

Proof: For any positive semi-definite matrix A , we have $(x'Ax)^\delta \leq (\lambda_{max}(A))^\delta (x'x)^\delta$ for all $x \in \mathcal{R}^n$. Then, it follows that

$$\begin{aligned}
E\|x_k\|^\delta &= E \left\{ x'_{k-1} H'(\sigma_{k-1}) H(\sigma_{k-1}) x_{k-1} \right\}^{\delta/2} \\
&\leq E \left\{ \lambda_{max}^{\delta/2} (H'(\sigma_{k-1}) H(\sigma_{k-1})) (x'_{k-1} x_{k-1})^{\delta/2} \right\} \\
&= E \left\{ \lambda_{max}^{\delta/2} (H'(\sigma_{k-1}) H(\sigma_{k-1})) \right\} E\|x_{k-1}\|^\delta \\
&= E \left\{ \lambda_{max}^{\delta/2} (H'(\sigma_0) H(\sigma_0)) \right\} E\|x_{k-1}\|^\delta \\
&\leq \dots \leq \left(E \lambda_{max}^{\delta/2} (H'(\sigma_0) H(\sigma_0)) \right)^k E\|x_0\|^\delta
\end{aligned}$$

where we have used the property that σ_{k-1} and x_{k-1} are independent. By definition

$$E \lambda_{max}^{\delta/2} (H'(\sigma_0) H(\sigma_0)) = p_1 \lambda_1^{\delta/2} + p_2 \lambda_2^{\delta/2} + \dots + p_N \lambda_N^{\delta/2}.$$

Which yields

$$E\|x_k\|^\delta \leq (p_1 \lambda_1^{\delta/2} + \dots + p_N \lambda_N^{\delta/2})^k E\|x_0\|^\delta.$$

Therefore, if (a) holds, then the system (2.1) is δ -moment stable. Similarly, for the sufficiency of (b), we have

$$\begin{aligned}
E\|x_k\|^\delta &= E\{\|H(\sigma_{k-1}) \dots H(\sigma_0) x_0\|^\delta\} \\
&\leq E\{\|H(\sigma_{k-1})\|^\delta\} \dots E\{\|H(\sigma_0)\|^\delta\} E\{\|x_0\|^\delta\} \\
&= (E\{\|H(\sigma_0)\|^\delta\})^k E\{\|x_0\|^\delta\} \\
&= (p_1 \|H(1)\|^\delta + \dots + p_N \|H(N)\|^\delta)^k E\{\|x_0\|^\delta\}
\end{aligned}$$

The result follows directly. Finally, we observe that for a scalar system, the conditions (a) and (b) are the same. Hence, we need only to show that (a) is

also a necessary condition for δ -moment stability in the scalar case. However, in this case, $H(i)$ for $i \in \underline{N}$ are scalars and we obtain

$$\begin{aligned} E\|x_k\|^\delta &= E\{\|H(\sigma_{k-1})\|^\delta \cdots \|H(\sigma_0)\|^\delta \|x_0\|\} \\ &= (E\|H(\sigma_0)\|^\delta)^k E\|x_0\|^\delta \\ &= (p_1\lambda_1^{\delta/2} + \cdots + \lambda_N^{\delta/2})^k E\|x_0\|^\delta \end{aligned}$$

Hence $E\|x_k\|^\delta \rightarrow 0$ if and only if $p_1\lambda_1^{\delta/2} + p_2\lambda_2^{\delta/2} + \cdots + \lambda_N^{\delta/2} < 1$. This completes the proof. \square

From the proof, we observe that the assumption that $\{\sigma_k\}$ is independent identically distributed is crucial. It is not so easily to generalize the above result to Markovian case. In stead, we have the following result for the Markovian case.

Theorem 2.4.2:

Suppose that the form process $\{\sigma_k\}$ in (2.1.1) is a finite state Markov chain with the probability transition matrix P . Let $D = \text{diag}\{\|H(1)\|^\delta, \|H(2)\|^\delta, \dots, \|H(N)\|^\delta\}$, where $\|\cdot\|$ is any given matrix norm. Then, a sufficient condition for (2.1.1) to be δ -moment stable is that DP is a stable matrix, i.e., $\rho(DP) < 1$. Furthermore, for a scalar system ($n = 1$), if the strong mixing condition $P = (p_{ij})_{N \times N} > 0$ is satisfied, i.e. $p_{ij} > 0$ for all $i, j \in \underline{N}$, then, $\rho(DP) < 1$ is also a necessary condition for δ -moment stability. \square

Proof: Let $a_i = \|H(i)\|^\delta$. Then,

$$\begin{aligned} E\|H(\sigma_k) \cdots H(\sigma_0)\|^\delta &\leq E\|H(\sigma_k)\|^\delta \cdots \|H(\sigma_0)\|^\delta \\ &= \sum_{i_0, i_1, \dots, i_k} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \|H(i_k)\|^\delta \cdots \|H(i_0)\|^\delta \end{aligned}$$

$$\begin{aligned}
&= \sum_{i_0, i_1, \dots, i_k} p_{i_0} p_{i_0 i_1} \cdots p_{i_{k-1} i_k} a_{i_k} \cdots a_{i_1} a_{i_0} \\
&= \sum_{i_0, i_1, \dots, i_k} (p_{i_0} a_{i_0}) (p_{i_0 i_1} a_{i_1}) \cdots (p_{i_{k-1} i_k} a_{i_k}) \\
&= (\pi_0 D)(PD) \cdots (PD)(Pb) = \pi_0 (DP)^k b
\end{aligned}$$

where $b = (a_1, a_2, \dots, a_N)'$ and $\pi_0 = (p_1, p_2, \dots, p_N)$ is the initial probability distribution. If DP is stable, we have $\lim_{k \rightarrow \infty} (DP)^k = 0$. It follows that $\lim_{k \rightarrow \infty} E\|H(\sigma_k) \cdots H(\sigma_0)\|^\delta = 0$. This implies that (2.1.1) is δ -moment stable. This proves the first statement. For the scalar case, without loss of generality, we assume that $a_i = \|H(i)\|^\delta \neq 0$. According to the strong mixing assumption, P is a positive matrix, so is DP . Since PD and DP have the same eigenvalues, DP is stable if and only if PD is stable. Let λ_{\max} be the largest eigenvalue of PD , which is in fact the spectral radius of PD , then from Perron's theorem ([148]), there exists a positive (row) eigenvector x for λ_{\max} , i.e., $x'PD = \lambda_{\max} x'$. As (2.1.1) is δ -moment stable for any initial distribution, choose initial distributions which are the rows of P , we then obtain

$$\lim_{k \rightarrow \infty} P(DP)^k b = \lim_{k \rightarrow +\infty} (PD)^{k+1} D^{-1} b = \lim_{k \rightarrow \infty} (PD)^{k+1} e = 0$$

where $e = (1, 1, \dots, 1)'$. Therefore, for $x = (x_1, x_2, \dots, x_N)'$,

$\lim_{k \rightarrow \infty} x'(PD)^{k+1} e = 0$. It follows that

$$\lim_{k \rightarrow +\infty} x'(PD)^{k+1} e = \lim_{k \rightarrow +\infty} \lambda_{\max}^{k+1} x' e = \lim_{k \rightarrow +\infty} \lambda_{\max}^{k+1} (x_1 + \cdots + x_N) = 0.$$

Because $x_1 + \cdots + x_N > 0$, this implies that $\lim_{k \rightarrow \infty} \lambda_{\max}^{k+1} = 0$ and $\lambda_{\max} < 1$.

That is, DP is a stable matrix. This completes the proof. \square

In fact, we can use the results for second moment stability to give a simpler proof for Theorem 2.4.2. For any matrix norm $\|\cdot\|$, any positive number δ and any initial distribution, we have

$$\begin{aligned}
E\|x_{k+1}\|^\delta &\leq E\|H(\sigma_k)\|^\delta \|x_k\|^\delta \\
&\leq \cdots \leq E\{\|H(\sigma_k)\|^\delta \cdots \|H(\sigma_0)\|^\delta \|x_0\|^\delta\}.
\end{aligned} \tag{2.4.1}$$

If we define

$$B(\sigma_k) = \|H(\sigma_k)\|^{\delta/2}, \quad (2.4.2)$$

and let y_k denotes the solution of $y_{k+1} = B(\sigma_k)y_k$. From (2.4.1), we can obtain that

$$E\|x_k\|^\delta \leq E|y_k|^2,$$

from which we obtain that if (2.4.2) is second moment stable, then (2.1.1) is δ -moment stable. From Corollary 2.2.11, we obtain that if

$$A = \text{diag}\{B(1) \otimes B(1), \dots, B(N) \otimes B(N)\}(P^T \otimes I) = DP^T$$

is schur stable, then (2.4.2) is second moment stable. However, as we know that

$$\rho(DP^T) = \rho((DP^T)^T) = \rho(PD^T) = \rho(PD) = \rho(DP).$$

Hence, if DP is schur stable, i.e., $\rho(DP) < 1$, then (2.1.1) is δ -moment stable.

We know that an iid sequence is a special type of Markov chain. To be more specific, if $\{\sigma_k\}_{k=0}^{+\infty}$ is an iid sequence with common distribution $p = (p_1, \dots, p_N)$ and $\{\xi_j\}_{j=1}^{+\infty}$ is a Markov chain with a transition matrix $P = (p_{ij})_{N \times N}$ and initial distribution π_0 , then $\{\sigma_k\}_{k=0}^{+\infty}$ and $\{\xi_j\}_{j=1}^{+\infty}$ have the same set of finite dimensional distributions as long as $p_{ij} = p_j$ for all $i, j \in \underline{N}$, regardless of π_0 . Therefore, one can identify the two sequences $\{\sigma_k\}_{k=1}^{+\infty}$ and $\{\xi_k\}_{k=1}^{+\infty}$ with each other. From this, it is expected that we can recover the δ -moment stability criteria for the iid case from the above result. We justify this claim next.

Lemma 2.4.3.

Let A be a nonnegative matrix, i.e., a matrix with nonnegative entries. Then for any positive vector x where $x = (x_1, \dots, x_n)'$, we have

$$\min_{1 \leq i \leq n} \frac{1}{x_i} \sum_{j=1}^n a_{ij} x_j \leq \rho(A) \leq \max_{1 \leq i \leq n} \frac{1}{x_i} \sum_{j=1}^n a_{ij} x_j \quad (2.4.3)$$

Furthermore, if there exists a positive vector x so that $Ax = \lambda x$ for a positive λ , then $\rho(A) = \lambda$.

Proof: (2.4.3) can be found in ([148]). Assume that $Ax = \lambda x$ for some x and λ positive. Then, $\sum_{j=1}^n a_{ij} x_j = \lambda x_i$. By applying (2.4.3), we obtain $\rho(A) = \lambda$. \square

Proposition 2.4.4:

Let $P = (p_{ij})_{N \times N}$ be a transition matrix satisfying $p_{ij} = p_j > 0$ for all $i, j \in \underline{N}$. Then, DP is stable if and only if

$$p_1 \|H(1)\|^\delta + \dots + p_N \|H(N)\|^\delta < 1, \quad (2.4.4)$$

i.e., for an i.i.d. form process with common distribution (p_1, \dots, p_N) , a sufficient condition for δ -moment stability is (2.4.4). \square

Proof: Let $A = (DP)'$. Then, A is nonnegative and the following equation holds:

$$\begin{aligned} A \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} &= \begin{pmatrix} a_1 p_1 & a_2 p_1 & \dots & a_N p_1 \\ a_1 p_2 & a_2 p_2 & \dots & a_N p_2 \\ \vdots & \vdots & & \vdots \\ a_1 p_N & a_2 p_N & \dots & a_N p_N \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} \\ &= \begin{pmatrix} (p_1 a_1 + \dots + p_N a_N) p_1 \\ (p_1 a_1 + \dots + p_N a_N) p_2 \\ \vdots \\ (p_1 a_1 + \dots + p_N a_N) p_N \end{pmatrix} = (p_1 a_1 + \dots + p_N a_N) \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} \end{aligned}$$

From Lemma 3.3, $p_1 a_1 + \dots + p_N a_N$ is the spectral radius of $A = DP$. Thus DP is stable if and only if

$$p_1 a_1 + \dots + p_N a_N = p_1 \|H(1)\|^\delta + p_2 \|H(2)\|^\delta + \dots + p_N \|H(N)\|^\delta < 1.$$

□

We know that for an ergodic Markovian form process, the almost sure stability property is independent of the initial distribution. From the above results, it is very tempting to conjecture that this may also be true for δ -moment stability. In particular, for a scalar system with an ergodic Markovian form process, one may conjecture that the δ -moment stability condition is $\pi_1 a_1 + \dots + \pi_N a_N < 1$ with $\pi = (\pi_1, \dots, \pi_N)$ being the unique invariant distribution (a_i is defined as above). Equivalently, the stability of DP may be equivalent to $\pi_1 a_1 + \dots + \pi_N a_N < 1$. Unfortunately, this is not true as illustrated by the following example.

Example 2.4.5:

Let $H(1) = 1.9$ and $H(2) = 0.5$, and $P = \begin{pmatrix} 0.1 & 0.9 \\ 0.8 & 0.2 \end{pmatrix}$. For the first moment ($\delta = 1$), it is easy to see that the eigenvalues of DP are 0.973 and -0.683 . Hence DP is stable and (2.1.1) is first moment stable from Theorem 2.4.3. However, the unique invariant measure is $(8/17, 9/17)$ and for $\delta = 1$, $\pi_1 a_1 + \pi_2 a_2 = (8/17) \times 1.9 + (9/17) \times 0.5 = 19.7/17 > 1$. □

Remarks:

- (1). For iid case, Corollary 2.2.12 states that the system is second moment stable if and only if the matrix $\sum_{i=1}^N p_i H(i) \otimes H(i)$ is schur stable. For Markov case, it is tempting to conjecture that (2.1.1) is second moment

stable if and only if the matrix $\sum_{i=1}^N \pi_i H(i) \otimes H(i)$ is schur stable, where $\{\pi_1, \dots, \pi_N\}$ is the unique invariant measure of the form process $\{\sigma_k\}$. Unfortunately, this is not true. In example 2.4.5, we take $H(1) = \sqrt{1.9}$ and $H(2) = \sqrt{0.5}$, following the same procedure, we can prove that DP is schur stable, hence (2.1.1) is second moment stable. However, we have

$$\sum_{i=1}^2 \pi_i H(i) \otimes H(i) = \pi_1 H(1)^2 + \pi_2 H(2)^2 = \frac{8}{17} \times 1.9 + \frac{9}{17} \times 0.5 = \frac{19.5}{17} > 1.$$

This shows that the schur stability of the matrix $\sum_{i=1}^N \pi_i H(i) \otimes H(i)$ is not necessary;

- (2). Although the schur stability of DP does not imply $\pi_1 a_1 + \dots + \pi_N a_N < 1$, we conjecture the converse is true, i.e., if

$$\pi_1 \|H(\sigma_k)\|^\delta + \pi_2 \|H(2)\|^\delta + \dots + \pi_N \|H(N)\|^\delta < 1,$$

then DP is schur stable, hence the system (2.1.1) is δ -moment stable for a ergodic form process $\{\sigma_k\}$. Up to now, we have not been able to find a rigorous proof yet.

The above criteria for δ -moment stability are easily testable. In the remainder of this section, as an illustrative example, we study the relationship between δ -moment stability and almost sure stability for a scalar jump linear system with an iid finite state chain which has common probability measure $\{p_1, p_2, \dots, p_N\}$. As we proved in section 2.3, the stability regions Σ^δ in the space of system parameters associated with δ -moment stability are monotonically increasing as δ decreases to 0 and Σ^δ tends to the almost sure stability region Σ^a monotonically from the interior of Σ^a . To verify our results in section 2.3, we use a direct elementary approach.

Lemma 2.4.6:

For $\lambda_i \geq 0$, $i \in \underline{N}$ fixed, define a function $f : \mathcal{R} \rightarrow \mathcal{R}$ by

$$f(x) = \begin{cases} (p_1 \lambda_1^x + p_2 \lambda_2^x + \cdots + p_N \lambda_N^x)^{1/x}, & \text{if } x \neq 0 \\ \lambda_1^{p_1} \lambda_2^{p_2} \cdots \lambda_N^{p_N}, & \text{if } x = 0 \end{cases}.$$

Then, $f(x)$ is continuous and nondecreasing on $[0, \infty)$. \square

Proof: It is easy to see that $f(x)$ is continuous on $\mathcal{R} \setminus \{0\}$. Hence, it suffices to show that it is continuous at $x = 0$. In fact, using L'Hospital's rule, we have

$$\begin{aligned} \lim_{x \rightarrow 0} (p_1 \lambda_1^x + \cdots + p_N \lambda_N^x)^{1/x} &= \exp \left(\lim_{x \rightarrow 0} \frac{\log(p_1 \lambda_1^x + \cdots + p_N \lambda_N^x)}{x} \right) \\ &= \exp \left(\lim_{x \rightarrow 0} \frac{p_1 \lambda_1^x \log \lambda_1 + \cdots + p_N \lambda_N^x \log \lambda_N}{p_1 \lambda_1^x + \cdots + p_N \lambda_N^x} \right) \\ &= \lambda_1^{p_1} \lambda_2^{p_2} \cdots \lambda_N^{p_N} \end{aligned}$$

Thus $f(x)$ is continuous at 0. Next, we show that f is nondecreasing on $[0, +\infty)$. For any $x_1, x_2 \in [0, +\infty)$ with $x_1 < x_2$, let $\beta = x_1/x_2 < 1$ and $g(x) \stackrel{\text{def}}{=} x^\beta$. As $\beta < 1$, $g(x)$ is concave. From Jensen's inequality, we obtain

$$\sum_{i=1}^N p_i g(\lambda_i^{x_2}) \leq g \left(\sum_{i=1}^N p_i \lambda_i^{x_2} \right)$$

From this, we have $f(x_1) \leq f(x_2)$. Thus $f(x)$ is nondecreasing on $[0, +\infty)$. This completes the proof. \square

Now, let $\lambda_i = \|H(i)\|$ where $H(i)$ is a scalar matrix for $i \in \underline{N}$, define

$$\Sigma^a = \{(\lambda_1, \lambda_2, \dots, \lambda_N) \mid \lambda_1^{p_1} \lambda_2^{p_2} \cdots \lambda_N^{p_N} < 1, \lambda_j \geq 0\}$$

$$\Sigma^\delta = \{(\lambda_1, \lambda_2, \dots, \lambda_N) \mid p_1 \lambda_1^\delta + p_2 \lambda_2^\delta + \cdots + p_N \lambda_N^\delta < 1, \lambda_j \geq 0\}$$

By the result Theorem 2.4.1 and a result developed later (Theorem 2.5.1), we see that Σ^a is the almost sure stability region and Σ^δ is the δ -moment stability

region in the space of parameters of $(H(1), \dots, H(N))' \in \mathcal{R}^N$. We have the following important theorem for scalar jump linear system (2.1.1):

Theorem 2.4.7:

For the scalar jump linear system (2.1.1) with form process,

(a) For any $\delta_1, \delta_2 > 0$ with $\delta_1 < \delta_2$, we have $\Sigma^{\delta_2} \subset \Sigma^{\delta_1} \subset \Sigma^a$.

(b) $\lim_{\delta \rightarrow 0} \Sigma^\delta = \bigcup_{\delta > 0} \Sigma^\delta = \Sigma^a$. □

Proof: The proof is a direct application of Lemma 2.4.6. □

Remark: This theorem verify our analysis in section 2.3 from a special case using a more direct approach. It is also easy to obtain that the top δ -moment Lyapunov exponent is given by

$$g(\delta) = \log(p_1 |H(1)|^\delta + p_2 |H(2)|^\delta + \dots + p_N |H(N)|^\delta).$$

Thus, if the individual modes (scalar) $H(1), H(2), \dots, H(N)$ are not all zero, then $g(\delta)$ is differentiable (in fact analytic). This verify the right differentiability results in section 2.3.

2.5. Almost Sure Stability

It is common knowledge that in practical applications, what we observe is sample path behavior rather than moment behavior, therefore almost sure (sample path) stability is much more desirable than any moment stability. However, the analysis of almost sure stability is much more difficult than moment stability, this is why in current literature there are much more results for moment stability (especially second moment stability). Although moment stability implies almost sure stability, the stability criteria for high moments (say, second moment) as almost sure stability criteria are too conservative to be useful in practical applications. From our discussions in sections 2.3 and 2.4, we observed that “lower” moment stability also implies almost sure stability and almost sure stability can be recovered from sufficient lower moments, therefore it is reasonable to use lower moment stability to study almost sure stability. In this section, we will devote our effort to the study of almost sure stability. The following notation will be used throughout the remainder of this section. Let $\underline{N} \stackrel{\text{def}}{=} \{1, 2, \dots, N\}$ and $\Delta_i = \det(H(i))$ for $i \in \underline{N}$. Let $\lambda(A)$ denote the collection of all eigenvalues of the matrix A , i.e., the spectrum of A , and $\rho(A)$ denote the spectral radius of A , i.e., $\rho(A) = \max\{|\lambda| : \lambda \in \lambda(A)\}$. Let $\mu_i = \rho(H(i))$ be the spectral radius of $H(i)$ for $i \in \underline{N}$. If A is a positive semi-definite matrix, let $\lambda_{\max}(A) = \max\{\lambda : \lambda \in \lambda(A)\}$ denote the largest eigenvalue of A , and let $\lambda_i = \lambda_{\max}(H'(i)H(i))$ for $i \in \underline{N}$, where $'$ denotes matrix transpose. Let $I_i(x)$ be the indicator function which is defined as

$$I_i(x) = \begin{cases} 1, & \text{if } x = i; \\ 0, & \text{otherwise.} \end{cases}$$

Finally, for (η_1, \dots, η_N) given with $\eta_i \geq 0$, define the set of matrices

$$\mathcal{K}(\eta_1, \dots, \eta_N) = \{(A_1, A_2, \dots, A_N) : A_i \in \mathcal{R}^{n \times n}, \lambda_{\max}(A_i' A_i) \leq \eta_i, \forall i \in \underline{N}\}. \quad (2.5.1)$$

If (2.1.1) is almost surely stable for any $(H(1), \dots, H(N)) \in \mathcal{K}(\eta_1, \dots, \eta_N)$, we say (2.1.1) is almost surely $\mathcal{K}(\eta_1, \dots, \eta_N)$ stable.

We first give some results for iid cases.

Theorem 2.5.1 (almost sure stability properties):

For the jump linear system (2.1.1) with a finite state iid form process with probability distribution $\{p_1, p_2, \dots, p_N\}$, we have

(1) A sufficient condition for almost sure stability is

$$\lambda_1^{p_1} \lambda_2^{p_2} \dots \lambda_N^{p_N} < 1. \quad (2.5.2)$$

In particular, for a scalar system (the dimension of the state space $n = 1$), the above condition is also necessary.

(2) If $\Delta_1^{p_1} \Delta_2^{p_2} \dots \Delta_N^{p_N} \geq 1$, then for almost all (with respect to Lebesgue measure) $x_0 \in \mathcal{R}^n$, $\lim_{t \rightarrow +\infty} \|x_t\| \neq 0$ and thus, the system is not almost surely stable.

(3) A necessary and sufficient condition for the system to be almost surely for any $(H(1), H(2), \dots, H(N)) \in \mathcal{K}(\eta_1, \dots, \eta_N)$ stable is

$$\eta_1^{p_1} \eta_2^{p_2} \dots \eta_N^{p_N} < 1. \quad (2.5.3)$$

□

Before we prove Theorem 2.5.1, we make some comments about the results. Note that $\sqrt{\lambda_i}$ is the largest singular value of $H(i)$ and (1) (equation (2.5.2)) is only a sufficient condition for a.s. (almost sure) stability, which becomes a necessary condition only for scalar systems. Condition (2) implies that

$\Delta_1^{p_1} \Delta_2^{p_2} \dots \Delta_N^{p_N} < 1$ is a necessary condition for a.s. stability, Condition (3) is a necessary and sufficient condition for robust a.s. stability. Another interpretation of (3) is that the condition (2.5.2) becomes necessary and sufficient for a.s. $\mathcal{K}(\lambda_1, \dots, \lambda_N)$ stability. All these conditions are easily testable and examples are presented after the proof of the theorem to illustrate the range of applicability of these results. To prove the theorem, we need the following lemma which is a fundamental result of stochastic processes and its proof can be found in, say, (Shiryayev [150]).

Lemma 2.5.2 (Law of the Iterative Logarithm):

Let $\{\xi_k\}$ be a sequence of independent identically distributed random variables with $E\xi_k = 0$ and $E\xi_k^2 = \sigma^2 < +\infty$, then

$$P\left(\overline{\lim}_{n \rightarrow \infty} \frac{\sum_{i=1}^n \xi_i}{\sqrt{2\sigma^2 n \log \log n}} = 1\right) = 1$$

$$P\left(\underline{\lim}_{n \rightarrow \infty} \frac{\sum_{i=1}^n \xi_i}{\sqrt{2\sigma^2 n \log \log n}} = -1\right) = 1.$$

□

Proof of Theorem 2.5.1: For any k and x , it is easy to show that

$$x' H'(\sigma_k) H(\sigma_k) x \leq \lambda_1^{I_1(\sigma_k)} \lambda_2^{I_2(\sigma_k)} \dots \lambda_N^{I_N(\sigma_k)} x' x. \quad (2.5.4)$$

Thus, we have

$$\begin{aligned} x'_k x_k &= x'_0 H'(\sigma_0) H(\sigma_0) \dots H'(\sigma_{k-1}) H(\sigma_{k-1}) \dots H(\sigma_0) x_0 \\ &\leq \lambda_1^{I_1(\sigma_{k-1})} \dots \lambda_N^{I_N(\sigma_{k-1})} x'_0 H'(\sigma_0) \dots H'(\sigma_{k-2}) H(\sigma_{k-2}) \dots H(\sigma_0) x_0 \\ &\leq \dots \leq \lambda_1^{\sum_{i=0}^{k-1} I_1(\sigma_i)} \lambda_2^{\sum_{i=0}^{k-1} I_2(\sigma_i)} \dots \lambda_N^{\sum_{i=0}^{k-1} I_N(\sigma_i)} x'_0 x_0 \\ &= \left(\lambda_1^{\frac{1}{k} \sum_{i=0}^{k-1} I_1(\sigma_i)} \dots \lambda_N^{\frac{1}{k} \sum_{i=0}^{k-1} I_N(\sigma_i)} \right)^k x'_0 x_0. \end{aligned} \quad (2.5.5)$$

Since $\{\sigma_k\}$ is an i.i.d. sequence, for each $j \in \underline{N}$, $\{I_j(\sigma_k)\}$ is also an i.i.d. sequence. Therefore from the Law of Large Numbers ([150], p.366), we have

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} I_j(\sigma_i) = E\{I_j(\sigma_i)\} = p_j \quad \text{a.s.} \quad \forall j \in \underline{N}.$$

It follows that

$$\lim_{k \rightarrow \infty} \lambda_1^{\frac{1}{k} \sum_{i=0}^{k-1} I_1(\sigma_i)} \lambda_2^{\frac{1}{k} \sum_{i=0}^{k-1} I_2(\sigma_i)} \dots \lambda_N^{\frac{1}{k} \sum_{i=0}^{k-1} I_N(\sigma_i)} = \lambda_1^{p_1} \lambda_2^{p_2} \dots \lambda_N^{p_N} \quad \text{a.s.}$$

From (2.5.2), the right hand side of the above inequality is less than 1, almost surely, and there exist a $k_0 = k_0(\omega)$ and $\lambda < 1$, so that

$$x'_k x_k \leq \lambda^k x'_0 x_0, \quad \forall k \geq k_0 \quad \text{a.s.} \quad (2.5.6)$$

From this we conclude that the system (2.1) is almost surely (exponentially) stable. This establishes the first statement of (1).

Next, we apply Lemma 2.5.2 to show that (2.5.2) is also necessary when $n = 1$. In this case, all the inequalities in (2.5.4)-(2.5.6) are actually equalities. It is easy to see that if $\lambda_1^{p_1} \lambda_2^{p_2} \dots \lambda_N^{p_N} > 1$, we have $\lim_{n \rightarrow \infty} x_k^2 = +\infty$ a.s. for any $x_0 \neq 0$, which contradicts the almost sure stability. Suppose that $\lambda_1^{p_1} \lambda_2^{p_2} \dots \lambda_N^{p_N} = 1$. Then, we have $\sum_{i=1}^N p_i \log \lambda_i = 0$ and also

$$\begin{aligned} x_k^2 &= \lambda_1^{\sum_{i=0}^{k-1} I_1(\sigma_i)} \dots \lambda_N^{\sum_{i=0}^{k-1} I_N(\sigma_i)} x_0^2 \\ &= \exp \left(\sum_{i=0}^{k-1} \sum_{j=1}^N I_j(\sigma_i) \log \lambda_j \right) x_0^2 \end{aligned}$$

Let $\xi_i = \sum_{j=1}^N I_j(\sigma_i) \log \lambda_j$, with $\{\sigma_i\}$ an i.i.d. sequence, $\{\xi_i\}$ is also an i.i.d. sequence. Thus, we obtain

$$E\xi_i = \sum_{j=1}^N (\log \lambda_j) E I_j(\sigma_i) = \sum_{j=1}^N (\log \lambda_j) p_j = 0$$

and

$$E\xi_i^2 = E \left(\sum_{j=1}^N (\log \lambda_j) I_j(\sigma_i) \right)^2 = \sum_{j=1}^N p_j (\log \lambda_j)^2 < +\infty$$

From Lemma 2.5.2, we obtain that

$$\overline{\lim}_{k \rightarrow \infty} \frac{\sum_{i=0}^{k-1} \xi_i}{\sqrt{2\sigma^2 \log \log n}} = 1 \quad \text{a.s.}$$

Thus, $\sum_{i=0}^{k-1} \xi_i$ is unbounded from above almost surely and so is x_k^2 . This contradicts the almost sure stability assumption and completes the proof of (1).

For (3), the sufficiency directly follows from (1). The necessity is proved next. Take $H(j) = \sqrt{\eta_j} I$ for $j \in \underline{N}$, where I denotes the identity matrix. Then $(H(1), \dots, H(N)) \in \mathcal{K}(\eta_1, \dots, \eta_N)$ and

$$x'_k x_k = \eta_1^{\sum_{i=0}^{k-1} I_1(\sigma_i)} \dots \eta_N^{\sum_{i=0}^{k-1} I_N(\sigma_i)} x'_0 x_0$$

Using the same arguments as in the proof of the necessity of (2.5.2) for the scalar case, we see that a.s. stability of (2.1.1) with $H(j) = \sqrt{\eta_j} I$ implies

$$\eta_1^{p_1} \eta_2^{p_2} \dots \eta_N^{p_N} < 1.$$

This completes the proof of (3).

Finally, we show (2) by contradiction. Suppose that $\lim_{k \rightarrow +\infty} x_k = 0$ a.s. for all $x_0 \in S \subset \mathcal{R}^n$ where S is a subset of R^n which has positive Lebesgue measure. Then, there is a basis $\{v_1, \dots, v_n\}$ of \mathcal{R}^n so that for each j ,

$$\lim_{k \rightarrow +\infty} H(\sigma_k) H(\sigma_{k-1}) \dots H(\sigma_0) v_j = 0, \quad \text{a.s.}$$

It follows that

$$\lim_{k \rightarrow +\infty} H(\sigma_k) H(\sigma_{k-1}) \dots H(\sigma_0) [v_1, \dots, v_n] = 0 \in \mathcal{R}^{n \times n} \quad \text{a.s.}$$

and therefore,

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \det(H(\sigma_k)H(\sigma_{k-1}) \dots H(\sigma_0)) \\ &= \lim_{k \rightarrow +\infty} \left(\Delta_1^{\frac{1}{k} \sum_{i=0}^k I_1(\sigma_i)} \dots \Delta_N^{\frac{1}{k} \sum_{i=0}^k I_N(\sigma_i)} \right)^k = 0 \text{ a.s..} \end{aligned}$$

Using a similar argument as in the proof of the necessity of (2.5.2) for the scalar case (using Lemma 2.5.2 and the law of large number), we conclude that

$$\Delta_1^{p_1} \dots \Delta_N^{p_N} < 1.$$

This is the desired contradiction. \square

From Lemma 2.5.2, we can conclude that the sequence $\sum_{i=0}^{k-1} \xi_i$ is not bounded from below, hence there exists a subsequence $\{n_k\}$ so that $x_{n_k}^2 \rightarrow 0$ ($k \rightarrow \infty$) almost surely. So a scalar system (2.1.1) is neither almost surely stable nor almost surely unstable when $\lambda_1^{p_1} \lambda_2^{p_2} \dots \lambda_N^{p_N} = 1$. Second, note that the $\lambda_1, \dots, \lambda_N$ are closely related to the matrix norm induced by the vector 2-norm in \mathcal{R}^n . This suggest the following more general sufficient condition for a.s. stability:

Theorem 2.5.3:

Suppose that $\{\sigma_k\}$ is a finite state ergodic Markov chain with the unique ergodic measure $\pi = \{\pi_1, \pi_2, \dots, \pi_N\}$. If there exists an matrix norm $\|\cdot\|$, such that

$$\|H(1)\|^{\pi_1} \|H(2)\|^{\pi_2} \dots \|H(N)\|^{\pi_N} < 1,$$

then the system (2.1.1) is almost surely stable. In particular, for one dimensional system (2.1.1), a necessary and sufficient condition for (2.1.1) to be almost sure stable is

$$a_1^{\pi_1} a_2^{\pi_2} \dots a_N^{\pi_N} < 1, \quad \text{where } a_i = \|H(i)\|.$$

Proof: Since

$$\|H(\sigma_k) \cdots H(\sigma_1)\| \leq \|H(\sigma_k)\| \cdots \|H(\sigma_1)\|,$$

following arguments similar to those used in the proof of (1) of Theorem 2.5.1, we can obtain the result. \square

Next, we comment on the importance of this proposition. From [145] or Appendix A, we know that the spectral radius $\rho(A)$ of a matrix A is given by

$$\rho(A) = \inf_{|\cdot| \in \mathcal{N}} \left[\sup_{x \in C^n} (|Ax|/|x|) \right]$$

where \mathcal{N} is the set of all vector norms on C^n . This means that for any $\varepsilon > 0$, there exists an induced matrix norm $\|\cdot\|$ so that $\|A\| \leq \rho(A) + \varepsilon$. It follows that A is a stable matrix if and only if there exists an induced matrix norm so that $\|A\| < 1$. Hence the suitable selection of a vector norm may improve the efficiency of the result of Theorem 2.5.3. One possible consideration is as follows: For any nonsingular matrix T , the vector norm $|x| = |Tx|_2$ induces a matrix norm given by $\|A\| = \|TAT^{-1}\|_2$, where 2 denotes the 2–norm, i.e., the Euclidean norm. Thus we can try to solve the following optimization problem:

$$\min_T \|TH(1)T^{-1}\|_2^{\pi_1} \|TH(2)T^{-1}\|_2^{\pi_2} \cdots \|TH(N)T^{-1}\|_2^{\pi_N}.$$

If there is an optimal solution T^* and the optimal value of the objective function is less than unity, then (2.1.1) is almost surely stable. On the other hand, for a given T ,

$$D_T \stackrel{\text{def}}{=} \{(H(1), \dots, H(N)) : \|TH(1)T^{-1}\|_2^{\pi_1} \cdots \|TH(N)T^{-1}\|_2^{\pi_N} < 1\}$$

defines a stability region in the parameter space $\overbrace{\mathcal{R}^{n \times n} \times \mathcal{R}^{n \times n} \times \cdots \times \mathcal{R}^{n \times n}}^{N \text{ times}}$. One may try to maximize the volume of D_T by proper choice of T . This

is a common procedure in applying Lyapunov's second method, i.e., try to select a suitable Lyapunov function from a given family so as to maximize the associated stability domain. If the form process $\{\sigma_k\}$ is iid and we use 2–norm to induce the matrix norm, we can obtain (1) of Theorem 2.5.1. The following example demonstrates the application of the above results:

Example 2.5.4: Suppose $\{\sigma_k\}$ is a two-state iid chain. Let

$$H(1) = \begin{pmatrix} 0.9 & 1 \\ 0 & 0.9 \end{pmatrix}, \quad H(2) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad p_1 = p_2 = 0.5.$$

As $\lambda_1 = \lambda_{\max}(H(1)'H(1)) = 2.3546$, $\lambda_2 = \lambda_{\max}(H(2)'H(2)) = 1$ and $\lambda_1^{p_1} \lambda_2^{p_2} = 1.5345 > 1$, we can not use Theorem 2.5.1 directly.

Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}$, then we obtain

$$\begin{aligned} \|PH(1)P^{-1}\|^{p_1} \|PH(2)P^{-1}\|^{p_2} &= \left\| \begin{pmatrix} 0.9 & 0.01 \\ 0 & 0.9 \end{pmatrix} \right\|^{0.5} \left\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\|^{0.5} \\ &= 0.951 < 1. \end{aligned}$$

Using Theorem 2.5.3, we conclude that the system (2.1.1) is a.s. stable. \square

From Theorem 2.5.3 and Appendix A, we can see that the above sufficient conditions really depends on the structure of $H(1), \dots, H(N)$. In order to get a feeling of how the structures of $H(1), H(2), \dots, H(N)$ to affect the stochastic stability, we study some special classes of jump linear systems.

Theorem 2.5.5:

For the system (2.1.1) with $\{\sigma_k\}$ a finite state independent identically distributed random process with common probability distribution $\{p_1, p_2, \dots, p_N\}$, assume that each of the $n \times n$ matrices $H(1), H(2), \dots, H(N)$ can be

simultaneously transformed by a similarity transformation to upper triangular form with the diagonal elements $\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{n,j}$ for $j \in \underline{N}$. Then, a necessary and sufficient condition for almost sure stability is

$$|\lambda_{i,1}^{p_1} \lambda_{i,2}^{p_2} \cdots \lambda_{i,N}^{p_N}| < 1 \quad \forall i = 1, 2, \dots, n. \quad (2.5.7)$$

In particular, if $H(1), H(2), \dots, H(N)$ pairwise commute, then there exists a unitary matrix T such that

$$T^{-1}H(j)T = \begin{pmatrix} \lambda_{1,j} & * & \cdots & * \\ & \lambda_{2,j} & \cdots & * \\ & & \ddots & * \\ & & & \lambda_{n,j} \end{pmatrix} \quad \forall j = 1, 2, \dots, N,$$

and (2.5.7) is a necessary and sufficient condition for almost sure stability. If $\{\sigma_k\}$ is a finite state ergodic Markov chain with the unique ergodic measure $\pi = \{\pi_1, \pi_2, \dots, \pi_N\}$, then the above claims are still valid with the substitution of p_i by π_i ($i \in \underline{N}$). \square

To prove this theorem, we need the following lemma, which infers stability of (2.1.1) from the stability of a dominating system.

Lemma 2.5.6:

For (2.1.1) with an iid form process, let (p_1, \dots, p_N) be the common probability distribution of $\{\sigma_k\}$.

- (i) Let $H(r) = (h_{ij}(r))$ and $\bar{H}(r) = (\bar{h}_{ij}(r))$ with $|h_{ij}(r)| \leq \bar{h}_{ij}(r)$ for all $r \in \underline{N}$. If the dominant system defined by

$$x_{k+1} = \bar{H}(\sigma_k) x_k, \quad x(0) = x_0 \quad (2.5.8)$$

is almost surely stable, then the system (2.1.1) is almost surely stable.

(ii) Let $|H(r)| = (|h_{ij}(r)|)$ for $r = 1, 2, \dots, N$. If

$$E|H(\sigma_0)| = p_1|H(1)| + p_2|H(2)| + \dots + p_N|H(N)|$$

is a stable matrix, then the system (2.1.1) is almost surely stable. In particular, if $H(1), H(2), \dots, H(N)$ are nonnegative matrices, then the system (2.1.1) is almost surely stable if

$$EH(\sigma_0) = p_1H(1) + p_2H(2) + \dots + p_NH(N)$$

is a stable matrix. □

Proof: To show (i), it is sufficient to prove that if $\bar{H}(\sigma_k)\bar{H}(\sigma_{k-1})\dots\bar{H}(\sigma_0)$ converges to zero almost surely, then $H(\sigma_k)H(\sigma_{k-1})\dots H(\sigma_1)$ also converges to zero almost surely. However, $|H(\sigma_k)H(\sigma_{k-1})\dots H(\sigma_0)| \leq \bar{H}(\sigma_k)\bar{H}(\sigma_{k-1})\dots\bar{H}(\sigma_0)$ and the result follows directly.

For (ii), let $G_k = |H(\sigma_k)| \dots |H(\sigma_0)|$. If $E|H(\sigma_0)|$ is stable, then by the iid property, $EG_k = (E|H(\sigma_0)|)^{k+1}$ exponentially converges to zero. Thus, with $G_k = (g_{ij}(k))$, there exists an $M > 0$ and $0 < r < 1$ so that $Eg_{ij}(k) \leq Mr^k$. Let $g_{ij} = \overline{\lim}_{k \rightarrow \infty} g_{ij}(k)$, since G_k is nonnegative matrix, $g_{ij}(k) \geq 0$. Hence, $g_{ij} \geq 0$. For any positive $c > 0$, by Chebyshev's inequality,

$$\begin{aligned} P\{g_{ij} > c\} &= P\{\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} \{g_{ij}(n) > c\}\} \leq P\{\cup_{n=m}^{\infty} \{g_{ij}(n) > c\}\} \\ &\leq \sum_{n=m}^{\infty} P\{g_{ij}(n) > c\} \leq \frac{1}{c} \sum_{n=m}^{\infty} Eg_{ij}(n) = \frac{1}{c} \sum_{n=m}^{\infty} Mr^n \end{aligned}$$

and letting m go to infinity, we obtain that $P\{g_{ij} > c\} = 0$ for any $c > 0$. From this, we can obtain that $P\{g_{ij} = 0\} = 1$. Hence, $\lim_{k \rightarrow \infty} g_{ij}(k) = 0$ almost surely. This means that $x_{k+1} = |H(\sigma_k)|x_k$ is almost surely stable. From (i), we conclude that (2.1.1) is almost surely stable. In the case when $H(j)$ is nonnegative for all $j \in \underline{N}$, we have $\bar{H}(j) = H(j)$ and the result follows. □

Remark: This Lemma gives very simple criteria for almost sure stability of the system (2.1.1) with an iid form process. Notice that $EH(\sigma_0)$ is stable if and only if (2.1.1) is mean value stable. Thus, from (ii) we see that for (2.1.1) with nonnegative form matrices, mean value stability implies (but is not necessarily equivalent to) almost sure stability, which is not true in general (see the examples which are given later).

□

Proof of Theorem 2.5.5: Without loss of generality, we can assume that $H(1), H(2), \dots, H(N)$ are all upper triangular matrices. Let b be the upper bound of the absolute values of the off-diagonal elements of $H(1), \dots, H(N)$, and with a slight abuse of notation, let $\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{n,j}$ be the absolute values of the diagonal elements of $H(j)$ for $j \in \underline{N}$. To prove the sufficiency, from Lemma 2.5.6, it suffices to show that if

$$\lambda_{i,1}^{p_1} \lambda_{i,2}^{p_2} \cdots \lambda_{i,N}^{p_N} < 1 \quad \forall i = 1, 2, \dots, n$$

then the dominating system defined by

$$x_{k+1} = \bar{H}(\sigma_k) x_k \tag{2.5.9}$$

with

$$\bar{H}(j) = \begin{pmatrix} \lambda_{1,j} & b & \cdots & b \\ & \lambda_{2,j} & \cdots & b \\ & & \ddots & \vdots \\ & & & \lambda_{n,j} \end{pmatrix}, \quad j = 1, 2, \dots, N,$$

is almost surely stable. Let

$$G_k = \bar{H}(\sigma_k) \cdots \bar{H}(\sigma_0) = \begin{pmatrix} g_{11}^{(k)} & g_{12}^{(k)} & \cdots & g_{1n}^{(k)} \\ & g_{22}^{(k)} & \cdots & g_{2n}^{(k)} \\ & & \ddots & \vdots \\ & & & g_{nn}^{(k)} \end{pmatrix}$$

The almost sure stability of (2.5.9) is equivalent to the almost sure convergence of G_k to the zero matrix. Let $\lambda_1^{(\sigma_k)}, \lambda_2^{(\sigma_k)}, \dots, \lambda_n^{(\sigma_k)}$ be the diagonal elements of $\bar{H}(\sigma_k)$. From $G_k = H(\sigma_k) G_{k-1}$ and the triangular form of matrices $H(\sigma_k)$, we obtain the recursive equations for the last column of G_k :

$$\begin{aligned}
g_{1n}^{(k)} &= \lambda_1^{(\sigma_k)} g_{1n}^{(k-1)} + b(g_{2n}^{(k-1)} + \dots + g_{nn}^{(k-1)}) \\
g_{2n}^{(k)} &= \lambda_2^{(\sigma_k)} g_{2n}^{(k-1)} + b(g_{3n}^{(k-1)} + \dots + g_{nn}^{(k-1)}) \\
&\vdots \\
g_{(n-1)n}^{(k)} &= \lambda_{n-1}^{(\sigma_k)} g_{(n-1)n}^{(k-1)} + b g_{nn}^{(k-1)} \\
g_{nn}^{(k)} &= \lambda_n^{(\sigma_k)} g_{nn}^{(k-1)}.
\end{aligned} \tag{2.5.10}$$

To proceed, we use induction on the dimension n . If $n = 1$, it is easy to show that $G_k \rightarrow 0$ almost surely and the result is valid. Suppose that it is true for $n - 1$. Because of the triangular structure of G_j and the induction hypothesis, it is sufficient to show that when $\dim G_k = n$, the elements of the last column of the matrix G_k are converging to zero almost surely. From the last equation of (2.5.10), we have

$$\begin{aligned}
g_{nn}^{(k)} &= \lambda_n^{(\sigma_k)} \lambda_n^{(\sigma_{k-1})} \dots \lambda_n^{(\sigma_1)} g_{nn}^{(0)} \\
&= \left(\lambda_{n,1}^{\frac{1}{k}} \sum_{i=1}^k I_1(\sigma_i) \dots \lambda_{n,N}^{\frac{1}{k}} \sum_{i=1}^k I_N(\sigma_i) \right)^k g_{nn}^{(0)}
\end{aligned} \tag{2.5.11}$$

where $I_r(j) = \delta_{rj}$ is the Kronecker delta (indicator) function. By hypothesis $\lambda_{n,1}^{p_1} \lambda_{n,2}^{p_2} \dots \lambda_{n,N}^{p_N} < 1$ and the Law of Large Numbers, there exists a $0 < \rho_n < 1$ and $M_n(\omega) > 0$, such that for all k ,

$$|g_{nn}^{(k)}| \leq M_n(\omega) \rho_n^k$$

Now, we use an induction argument on the index j of the elements $g_{jn}^{(k)}$ to show that for each $1 \leq i \leq n$, there exists $M_i(\omega)$ and $0 < \rho_i < 1$ with $M_i(\omega)$ a polynomial of finite degree in the variable k so that

$$|g_{in}^{(k)}| \leq M_i(\omega) \rho_i^k. \tag{2.5.12}$$

From this, we can conclude that $g_{in}^{(k)}$ goes to zero almost surely for each $1 \leq i \leq n$. We proceed as follows: Suppose that for some $1 < j < n$, there exists $M_j(\omega), \dots, M_n(\omega) > 0$ and $\rho_j, \dots, \rho_n < 1$, such that

$$|g_{in}^{(k)}| \leq M_i(\omega)\rho_i^k \quad \forall i = j, j+1, \dots, n,$$

We show that (2.5.12) holds for $i = j-1$. From the $(j-1)^{\text{th}}$ equation of (2.5.10), we have

$$\begin{aligned} g_{(j-1)n}^{(k)} &= \lambda_{j-1}^{(\sigma_k)} \lambda_{j-1}^{(\sigma_{k-1})} \cdots \lambda_{j-1}^{(\sigma_1)} g_{(j-1)n}^{(0)} \\ &\quad + \sum_{l=1}^{k-1} b \lambda_{j-1}^{(\sigma_k)} \cdots \lambda_{j-1}^{(\sigma_{l+1})} (g_{jn}^{(l-1)} + \cdots + g_{nn}^{(l-1)}) \\ &\quad + b(g_{jn}^{(k-1)} + \cdots + g_{nn}^{(k-1)}) \end{aligned} \quad (2.5.13)$$

We see that the last term in (2.5.13) almost surely converges to zero at an exponential rate because of the induction hypothesis. Actually, let $\overline{M}_{j-1}(\omega) = \max_{j \leq i \leq n} M_i(\omega)$, and $\overline{\rho}_{j-1} = \max_{j \leq i \leq n} \rho_i$. We have

$$|b(g_{jn}^{(k-1)} + \cdots + g_{nn}^{(k-1)})| \leq nb \overline{M}_{j-1}(\omega) \overline{\rho}_{j-1}^{k-1}. \quad (2.5.14)$$

By the Law of Large Numbers, similar to (2.5.11), we also have for the first term in (2.5.13) that there exist $\widetilde{M}_{j-1}(\omega)$ and $0 < \widetilde{\rho}_{j-1} < 1$ such that

$$|\lambda_{j-1}^{(\sigma_k)} \lambda_{j-1}^{(\sigma_{k-1})} \cdots \lambda_{j-1}^{(\sigma_1)} g_{(j-1)n}^{(0)}| \leq \widetilde{M}_{j-1}(\omega) \widetilde{\rho}_{j-1}^k. \quad (2.5.15)$$

For the second term in (2.5.13), consider that

$$\lambda_{j-1}^{(\sigma_k)} \cdots \lambda_{j-1}^{(\sigma_{l+1})} = \left(\lambda_{j-1,1}^{\frac{1}{k-l}} \sum_{i=l+1}^{k-1} I_1(\sigma_i) \cdots \lambda_{j-1,N}^{\frac{1}{k-l}} \sum_{i=l+1}^{k-1} I_N(\sigma_i) \right)^{(k-l)}$$

Again, by the Law of Large Numbers and the hypothesis of the theorem, there exist $\widehat{M}_{j-1}(\omega) > 0$ and $0 < \widehat{\rho}_{j-1} < 1$ such that

$$\lambda_{j-1}^{(\sigma_k)} \cdots \lambda_{j-1}^{(\sigma_{l+1})} \leq \widehat{M}_{j-1}(\omega) \widehat{\rho}_{j-1}^{k-l} \quad \text{a.s.}$$

Let $\rho_{j-1} = \max\{\bar{\rho}_{j-1}, \tilde{\rho}_{j-1}, \hat{\rho}_{j-1}\}$ and $M'_{j-1}(\omega) = \max\{\bar{M}_{j-1}(\omega), \hat{M}_{j-1}(\omega)\}$.

Then, it follows that the second term in (2.5.13) satisfies the inequality

$$\begin{aligned}
& \left| \sum_{l=1}^{k-1} b \lambda_{j-1}^{(\sigma_k)} \cdots \lambda_{j-1}^{(\sigma_{l+1})} (g_{jn}^{(l-1)} + \cdots + g_{nn}^{(l-1)}) \right| \\
& \leq \sum_{l=1}^{k-1} b (\hat{M}_{j-1}(\omega) \hat{\rho}_{j-1}^{k-l}) (M_j(\omega) \rho_2^l + \cdots + M_n(\omega) \rho_n^l) \quad (2.5.16) \\
& \leq nb M'_{j-1}{}^2(\omega) \sum_{l=1}^{k-1} \rho_{j-1}^k \leq nb M'_j{}^2(\omega) k \rho_{j-1}^k \text{ a.s.}
\end{aligned}$$

Combining (2.5.14), (2.5.15) and (2.5.16), we conclude that there exists $M_{j-1}(\omega)$ (which is a finite degree polynomial in k) such that

$$|g_{(j-1)n}^{(k)}| \leq M_{j-1}(\omega) \rho_{j-1}^k \xrightarrow{k \rightarrow \infty} 0 \text{ a.s.}$$

Induction on the dimension n guarantees that the system (2.5.7) and hence the system (2.1.1) is almost surely stable for arbitrary dimension, if the conditions of the theorem are satisfied. This completes the proof of sufficiency.

Necessity can be proved by observing that if the diagonal elements of the product of matrices G_k converges to zero as k goes to infinity, then using the Law of Large Numbers again, the conditions of the theorem are satisfied.

If $H(1), H(2), \dots, H(N)$ pairwise commute, then from Horn and Johnson ([148]) that there exists a unitary matrix T such that they can be transformed by this similarity transformation to the upper triangular forms, then applying (2.5.7), we obtain the proof.

A close look at the proof enables us to prove the Markovian case in a similar fashion. This is left to the reader. \square

Remarks:

- (1). We may prove Theorem 2.5.5 directly from the system (2.1.1) with an iid or Markovian form process using Theorem 2.5.1 for the one dimensional case, the procedure will be the same as the above. Notice that from the above proof, under the condition of Theorem 2.5.5, the system is actually almost surely exponentially stable.
- (2). Suppose that $\{\sigma_k\}$ is a finite state ergodic Markov or iid form process with ergodic measure π and $H(1), \dots, H(N)$ pairwise commute, let $\rho_i = \rho(H(i))$ ($i \in \underline{N}$), then a simple sufficient condition for (2.1.1) to be almost surely stable is

$$\rho_1^{\pi_1} \rho_2^{\pi_2} \cdots \rho_N^{\pi_N} < 1.$$

Furthermore, we can easily see that if the individual modes are stable, then the jump linear system is almost surely stable. This is true only for the commuting structure, in general this does not hold.

Lyapunov's second method is a very important technique for the study of stability of dynamical systems. Hasminskii [95] used a stochastic version of this theory to study the stability of stochastic systems. In what follows, we use $\|\cdot\|$ to denote any vector norm on \mathbf{R}^n , $\|x\|$ for $x \in \mathbf{R}^n$ and also any induced matrix norm on $\mathbf{R}^{n \times n}$, $\|A\|$ for $A \in \mathbf{R}^{n \times n}$. We use $\|\cdot\|_2$ to denote the 2-norm, i.e., the Euclidean norm.

Lemma 2.5.7: (Hasminskii [95], pp.214-215)

Let $\{A_n\}$ be an iid matrix sequence, a sufficient condition for

$$E\|A_n A_{n-1} \cdots A_1\|^\delta \xrightarrow{n \rightarrow \infty} 0$$

is that there exists a positive definite function $f(x)$, homogenous of degree δ , such that the function $Ef(A_1x) - f(x)$ is negative definite of degree δ , i.e., there exists $K > 0$ such that $Ef(A_1x) - f(x) \leq -K\|x\|^\delta$. \square

From this we obtain:

Theorem 2.5.8:

Let $\{\sigma_k\}$ be a finite state iid sequence with a common probability distribution $\{p_1, p_2, \dots, p_N\}$. Define

$$\begin{aligned}\mu_{max}^\delta &= \max_{\|x\|=1} \{p_1\|H(1)x\|^\delta + p_2\|H(2)x\|^\delta + \dots + p_N\|H(N)x\|^\delta\} \\ \mu_{min}^\delta &= \min_{\|x\|=1} \{p_1\|H(1)x\|^\delta + p_2\|H(2)x\|^\delta + \dots + p_N\|H(N)x\|^\delta\}.\end{aligned}$$

Then, the system (2.1.1) is δ -moment stable if $\mu_{max}^\delta < 1$, and the system (2.1.1) is δ -moment unstable if $\mu_{min}^\delta \geq 1$. \square

Proof: Let $f(x) = \|x\|^\delta$, then

$$\begin{aligned}Ef(H(\sigma_1)x) - f(x) &= p_1\|H(1)x\|^\delta + p_2\|H(2)x\|^\delta + \dots + p_N\|H(N)x\|^\delta - \|x\|^\delta \\ &= \|x\|^\delta \left(p_1\|H(1)\frac{x}{\|x\|}\|^\delta + \dots + p_N\|H(N)\frac{x}{\|x\|}\|^\delta - 1 \right) \\ &\leq \|x\|^\delta (\mu_{max}^\delta - 1)\end{aligned}$$

Then from Lemma 2.5.7, if $\mu_{max}^\delta < 1$, the system (2.1.1) is δ -moment stable. In a similar fashion, we can prove that if $\mu_{min}^\delta \geq 1$, the system is δ -moment unstable. \square

From the relationship between almost sure and δ -moment stability it is shown that δ -moment stability implies almost sure stability. From this fact, together with the above theorem, we obtain the following criterion for the almost sure stability of (2.1.1).

Theorem 2.5.9:

Let $\{\sigma_k\}$ be a finite state iid sequence with common distribution $\{p_1, p_2, \dots, p_N\}$, then the system (2.1.1) is almost surely stable if $\sigma_{max} < 1$, where $\sigma_{max} = \max_{\|x\|=1} \{\|H(1)x\|^{p_1} \cdots \|H(N)x\|^{p_N}\}$.

□

Proof: If $\sigma_{max} < 1$, we claim that there exists a $\delta > 0$ such that $\mu_{max}^\delta < 1$. Otherwise, for any k , $\delta = \frac{1}{k}$, there exists x_k satisfying $\|x_k\| = 1$, such that

$$p_1 \|H(1)x_k\|^{1/k} + \cdots + p_N \|H(N)x_k\|^{1/k} \geq 1. \quad (2.5.17)$$

Without loss of generality (because the unit sphere is compact and $\{x_k\}$ is sequence in the sphere), we can assume that there exists an x_0 satisfying $\|x_0\| = 1$, such that $\lim_{k \rightarrow \infty} x_k = x_0$. Hence for any j , $\lim_{k \rightarrow \infty} \|H(j)x_k\| = \|H(j)x_0\|$. It follows that for any $\varepsilon > 0$, and sufficiently large k , we have

$$\|H(j)x_k\| \leq \|H(j)x_0\| + \varepsilon \quad (2.5.18)$$

(2.5.17) and (2.5.18) gives

$$\begin{aligned} 1 &\leq \left(p_1 \|H(1)x_k\|^{1/k} + \cdots + \|H(N)x_k\|^{1/k} \right)^k \\ &\leq \left(p_1 (\|H(1)x_0\| + \varepsilon)^{1/k} + \cdots + p_N (\|H(N)x_0\| + \varepsilon)^{1/k} \right)^k \end{aligned}$$

Letting $k \rightarrow \infty$, we obtain

$$1 \leq (\|H(1)x_0\| + \varepsilon)^{p_1} \cdots (\|H(N)x_0\| + \varepsilon)^{p_N},$$

and letting $\varepsilon \rightarrow 0$, we obtain

$$\|H(1)x_0\|^{p_1} \|H(2)x_0\|^{p_2} \cdots \|H(N)x_0\|^{p_N} \geq 1.$$

This means that $\sigma_{max} \geq 1$, which contradicts the assumption. Therefore there exists $\delta > 0$ such that $\mu_{max}^\delta < 1$ and from Theorem 2.5.8, the system is δ -moment stable. Since for any $\delta > 0$ δ -moment stability implies almost sure stability, (2.1.1) is almost surely stable. This completes the proof. \square

The following example illustrates that this criterion is better than Theorem 2.5.1.

Example 2.5.10:

$$H(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H(2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, p_1 = p_2 = 0.5.$$

Since $\|H(1)\|_2 = \|H(2)\|_2 = 1$, we can not guarantee the almost sure stability of the system from Theorem 2.5.1. However,

$$\sigma_{max}^2 = \max_{\|x\|_2=1} \|H(1)x\|_2 \|H(2)x\|_2 = \max_{0 \leq \theta \leq 2\pi} |\sin \theta \cos \theta| \leq \frac{1}{2}$$

and from Theorem 2.5.9, we obtain that the system is almost surely stable. \square

In the derivation of Theorem 2.5.9, we have used the Lyapunov function $V(x) = \|x\|^\delta$. This leads us to consider some more general Lyapunov function. As we know that quadratic form is very useful in the study of linear systems, one may wonder if we can use small moment of Lyapunov function so that we deal with the δ -moment stability problem, then like obtaining Theorem 2.5.9, one derives some more general almost sure stability criteria. Fortunately, the answer is positive and we will explore this next.

Theorem 2.5.11.

Suppose that $\{\sigma_k\}$ is a finite state Markov chain with transition matrix $P = (p_{ij})$. If there exist positive definite (pd) matrices $P(1), P(2), \dots, P(N)$ such that

$$\sup_{\|x\|_2=1} \prod_{j=1}^N \left(\frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} \right)^{p_{ij}} < 1, \quad \forall i \in \underline{N} \quad (2.5.19)$$

then the system (2.1.1) is almost sure stable.

Proof. Define the Lyapunov function

$$V(x_k, \sigma_k) = (x_k^T P(\sigma_k) x_k)^{\delta/2}.$$

Then, we have

$$\begin{aligned} \Delta V(x, i) &= E \left\{ (x_{k+1}^T P(\sigma_{k+1}) x_{k+1})^{\delta/2} \mid x_k = x, \sigma_k = i \right\} - (x^T P(i) x)^{\delta/2} \\ &= E \left\{ (x^T H^T(\sigma_k) P(\sigma_{k+1}) H(\sigma_k) x)^{\delta/2} \mid \sigma_k = i \right\} - (x^T P(i) x)^{\delta/2} \\ &= \sum_{j=1}^N p_{ij} (x^T H^T(i) P(j) H(i) x)^{\delta/2} - (x^T P(i) x)^{\delta/2} \\ &= (x^T P(i) x)^{\delta/2} \left[\sum_{j=1}^N p_{ij} \left(\frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} \right)^{\delta/2} - 1 \right] \\ &= (x^T P(i) x)^{\delta/2} \left\{ \left[\left(\sum_{j=1}^N p_{ij} \left(\frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} \right)^{\delta/2} \right)^{2/\delta} \right]^{\delta/2} - 1 \right\} \end{aligned} \quad (2.5.20)$$

As $\sum_{j=1}^N p_{ij} = 1$, we have

$$\lim_{\delta \rightarrow 0} \left[\sum_{j=1}^N p_{ij} \left(\frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} \right)^{\delta/2} \right]^{2/\delta} = \prod_{j=1}^N \left(\frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} \right)^{p_{ij}} \quad (2.5.21)$$

Suppose that (2.5.19) holds, there exists ρ , $0 < \rho < 1$, such that

$$\sup_{\|x\|=1} \prod_{j=1}^N \left(\frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} \right)^{p_{ij}} < \rho < 1 \quad (2.5.22)$$

Then, we can obtain that there exists a $\delta > 0$ such that for any $x \in \mathbb{R}^n$ satisfying $\|x\| = 1$ and for any $i \in \underline{N}$, the following holds

$$\left[\sum_{j=1}^N p_{ij} \left(\frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} \right)^{\delta/2} \right]^{2/\delta} < \rho. \quad (2.5.23)$$

In fact, suppose that this is not true, then for any $\delta > 0$, there exists $i \in \underline{N}$ and x satisfying $\|x\| = 1$ such that

$$\left[\sum_{j=1}^N p_{ij} \left(\frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} \right)^{\delta/2} \right]^{2/\delta} \geq \rho.$$

As \underline{N} is a finite set and $S^n \triangleq \{x \mid \|x\| = 1\}$ is compact, without loss of generality, we can choose $i \in \underline{N}$ and a convergent sequence δ_k satisfying $\delta_k \downarrow 0+$ and a convergent sequence x_k satisfying $\lim_{k \rightarrow \infty} x_k = x_0$ and $\|x_0\| = 1$ such that

$$\left[\sum_{j=1}^N p_{ij} \left(\frac{x_0^T H^T(i) P(j) H(i) x_0}{x_0^T P(i) x_0} \right)^{\delta_k/2} \right]^{2/\delta_k} \geq \rho. \quad (2.5.24)$$

For notational simplicity, let

$$M_{ij}(x) = \frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x}.$$

Since $M_{ij}(x)$ is continuous on S^n , for any $\varepsilon > 0$, there exists $K > 0$ such that whenever $k > K$, we have $M_{ij}(x_k) \leq M_{ij}(x_0) + \varepsilon$. From (2.5.24), we obtain

$$\left(\sum_{j=1}^N p_{ij} (M_{ij}(x_0) + \varepsilon)^{\delta_k/2} \right)^{2/\delta_k} \geq \rho.$$

From this and (2.5.21), we have

$$\prod_{j=1}^N (M_{ij} + \varepsilon)^{p_{ij}} \geq \rho.$$

Letting ε go to zero, we obtain

$$\prod_{j=1}^N M_{ij}^{p_{ij}}(x_0) \geq \rho.$$

This contradicts to (2.5.22), thus the claim in (2.5.23) is proved.

Taking (2.5.23) into (2.5.20), we obtain

$$\Delta V(x, i) \leq (x^T P(i)x)^{\delta/2} (\rho^{\delta/2} - 1) = -(1 - \rho^{\delta/2})(x^T P(i)x)^{\delta/2}$$

for any $x \in R^n$. From stochastic version of Lyapunov's second method, we conclude that (2.1.1) is almost surely stable. This completes the proof. \square

From this theorem, we can obtain the following easy-to-test criterion.

Corollary 2.5.12:

Suppose that $\{\sigma_k\}$ is a finite state Markov chain with probability transition matrix $P = (p_{ij})$. If there exists pd matrices $P(1), P(2), \dots, P(N)$ such that

$$\prod_{j=1}^N \lambda_{\max} (H^T(i)P(j)H(i)P^{-1}(i))^{p_{ij}} < 1, \quad (i = 1, 2, \dots, N),$$

then (2.1.1) is almost surely stable.

Proof. Using the following fact: $\max_{x \neq 0} \frac{x^T Q x}{x^T P x} = \lambda_{\max}(QP^{-1})$ and maximizing each term in the product of (2.1.1), we can obtain the proof. \square

Next, we want to show that Theorem 2.5.11 provides a very general sufficient condition for almost sure stability of the system (2.1.1). For one

dimensional system, the sufficient condition in Theorem 2.5.11 is also necessary. And if (2.1.1) is second moment stable, then (2.5.19) is also necessary.

Corollary 2.5.13:

Suppose that (2.1.1) is one dimensional system with $|H(i)| \triangleq a_i \neq 0$ ($i = 1, 2, \dots, N$) and that $\{\sigma_k\}$ is a finite state irreducible Markov chain with ergodic measure π , then a necessary and sufficient condition for (2.1.1) to almost surely stable is that there exists N positive numbers $P(1), P(2), \dots, P(N)$ such that (2.5.19) holds.

Proof: We only need to prove the necessity. We first prove that $\text{Im}(P - I) = \{z \in \mathbf{R}^N : \pi z = 0\}$, where $\text{Im}(A)$ denotes the image of linear mapping A . In fact, $\forall u \in \text{Im}(P - I)$, there exists a $v \in \mathbf{R}^N$ such that $u = (P - I)v$, which implies, together with the identity $\pi P = \pi$, that $\pi u = \pi(P - I)v = (\pi P - \pi)v = 0$, hence $u \in \{z : \pi z = 0\}$, this implies that $\text{Im}(P - I) \subseteq \{z : \pi z = 0\}$. However, π is the unique solution to the equation

$$\begin{cases} \pi(P - I) = 0 \\ \pi(1, \dots, 1) = 1 \end{cases}$$

we obtain that $\text{rank}(P - I) = N - 1$, so $\dim(\text{Im}(P - I)) = N - 1$. Moreover, $\dim(\{z : \pi z = 0\}) = N - 1$, we conclude that $\text{Im}(P - I) = \{z : \pi z = 0\}$. Now let us choose

$$z = -a + \frac{\pi a}{\pi \pi^T} \pi^T \in \mathbf{R}^N,$$

where $a = (\log a_1, \log a_2, \dots, \log a_N)^T$. Since $\pi z = -\pi a + \pi a = 0$, i.e., $z \in \{z : \pi z = 0\} = \text{Im}(P - I)$, thus there exists a $y \in \mathbf{R}^N$ such that $z = (P - I)y$, i.e.,

$$(P - I)y + a = \frac{\pi a}{\pi \pi^T} \pi^T. \quad (2.5.25)$$

Suppose that (2.1.1) is almost surely stable, then from Theorem 2.5.3, we have

$$a_1^{\pi_1} a_2^{\pi_2} \cdots a_N^{\pi_N} < 1, \text{ i.e.,}$$

$$\pi_1 \log a_1 + \pi_2 \log a_2 + \cdots + \pi_N \log a_N < 0, \text{ or } \pi a < 0. \quad (2.5.26)$$

Now choose $P(i) = e^{2y_i} > 0$ ($i \in \underline{N}$), from above inequality, from (2.5.25) and (2.5.26) we have

$$(P - I) \begin{pmatrix} \log \sqrt{P(1)} \\ \log \sqrt{P(2)} \\ \vdots \\ \log \sqrt{P(N)} \end{pmatrix} + \begin{pmatrix} \log a_1 \\ \log a_2 \\ \vdots \\ \log a_N \end{pmatrix} < 0,$$

which is equivalent to

$$\sum_{j=1}^N p_{ij} \log \sqrt{\frac{P(j)}{P(i)}} + \sum_{j=1}^N p_{ij} \log a_i < 0, \quad \forall i \in \underline{N}.$$

Thus, we finally arrive at

$$\prod_{j=1}^N \left(\frac{P(j)}{P(i)} a_i^2 \right)^{p_{ij}} < 1, \quad \forall i \in \underline{N}.$$

This completes the proof of the necessity. \square

Corollary 2.5.14.

If (2.1.1) is second moment stable, then there exists pd matrices $P(1)$, $P(2), \dots, P(N)$ such that (2.5.19) holds.

Proof. Suppose that (2.1.1) is second moment stable, then from Theorem 2.2.1, there exists pd matrices $P(1), P(2), \dots, P(N)$ such that

$$\sum_{j=1}^N p_{ij} H^T(i) P(j) H(i) - P(i) = -I, \quad i = 1, 2, \dots, N.$$

For any $x \neq 0$, we have

$$\sum_{j=1}^N p_{ij} x^T H^T(i) P(j) H(i) x = x^T P(i) x - x^T x,$$

or

$$\sum_{j=1}^N p_{ij} \frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} = 1 - \frac{x^T x}{x^T P(i) x}.$$

Using the inequality

$$\beta_1^{\alpha_1} \beta_2^{\alpha_2} \cdots \beta_N^{\alpha_N} \leq \alpha_1 \beta_1 + \alpha_2 \beta_2 + \cdots + \alpha_N \beta_N, \quad (\alpha_i \geq 0, \sum_{i=1}^N \alpha_i = 1),$$

we obtain

$$\prod_{j=1}^N \left(\frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} \right)^{p_{ij}} \leq 1 - \frac{x^T x}{x^T P(i) x} \leq 1 - \lambda_{\min}(P^{-1}(i)) < 1.$$

From this, we can conclude that (2.5.19) holds. \square

Remark. In the derivation of the proof of Theorem 2.5.11, we have used the Lyapunov function: $V(x_k, \sigma_k) = (x_k^T P(\sigma_k) x_k)^{\delta/2}$. It is easy to see that x_k is measurable with respect to the σ -algebra $\mathcal{F}(\sigma_{k-1}, \dots, \sigma_0)$ generated by $\sigma_0, \sigma_1, \dots, \sigma_{k-1}$. It is reasonable to construct the Lyapunov function: $V(x_k, \sigma_{k-1}) = (x_k^T P(\sigma_{k-1}) x_k)^{\delta/2}$, as we observed before. Then we can obtain the following:

$$\begin{aligned} \Delta V(x, i) &= E\{V(x_{k+1}, \sigma_k) | x_k = x, \sigma_{k-1} = i\} - V(x, i) \\ &= E\{[x_k^T H^T(\sigma_k) P(\sigma_k) H(\sigma_k) x_k]^{\delta/2} - (x^T P(i) x)^{\delta/2}\} \\ &= \sum_{j=1}^N p_{ij} [x^T H^T(j) P(j) H(j) x]^{\delta/2} - (x^T P(i) x)^{\delta/2}. \end{aligned}$$

From this consideration, the following results can be achieved.

Theorem 2.5.15:

If there exist pd matrices $P(1), P(2), \dots, P(N)$ such that

$$\max_{\|x\|_2=1} \prod_{j=1}^N \left(\frac{x^T H^T(j) P(j) H(j) x}{x^T P(i) x} \right)^{p_{ij}} < 1, \quad \forall i \in \underline{N},$$

then (2.1.1) is almost surely stable. \square

Remark: If we choose $P(1) = P(2) = \dots = P(N) = I$, and assume that $\{\sigma_k\}$ is a finite state iid chain, then Theorem 2.5.11 gives Theorem 2.5.9.

A easier testable condition is given by

Corollary 2.5.16:

If there exist pd matrices $P(1), P(2), \dots, P(N)$ such that

$$\prod_{j=1}^N (\lambda_{\max}(H^T(j) P(j) H(j) P^{-1}(i)))^{p_{ij}} < 1, \quad \forall i \in \underline{N},$$

then (2.1.1) is almost surely stable. \square

We conjecture that the sufficient condition in either Theorem 2.5.11 or Theorem 2.5.15 is also necessary condition for a finite state irreducible Markov chain $\{\sigma_k\}$. Up to date, we are able to find a rigorous proof for this. Obviously, to use the sufficient condition, we need to solve a minimax problem (2.5.19), the feasibility study in solving this problem constitutes our future research.

2.6. Estimation of Top δ -moment Lyapunov Exponent

It is known that top δ -moment Lyapunov exponent may give a sufficient condition for δ -moment stability and at the same time for almost sure stability. However, the exact expression and the computation of top Lyapunov exponent becomes horribly difficult and hardly possible ([75],[79],[124]). Thus, a good approach to obtain estimates of top Lyapunov exponent becomes attractive. In this section, we only present some basic new results for jump linear systems. For the convenience, we only discuss the case that $\{\sigma_k\}$ is a finite state Markov chain with probability transition matrix P and with unique invariant measure π . For this case, it can be easily seen that

$$g(\delta) = \overline{\lim}_{k \rightarrow \infty} \frac{1}{k} \log E_\pi \|H(\sigma_k) \cdots H(\sigma_0)\|^\delta,$$

where $g(\delta) = \overline{\beta}(\delta, \pi)$ is the top δ -moment Lyapunov exponent.

Theorem 2.6.1:

Suppose that $\{\sigma_k\}$ is a finite state Markov chain with probability transition matrix $P = (p_{ij})$. Then for any matrix norm $\|\cdot\|$ we have

$$g(\delta) \leq \log \rho(DP),$$

where $D = \text{diag}\{\|H(1)\|^\delta, \|H(2)\|^\delta, \dots, \|H(N)\|^\delta\}$.

Proof: From the proof of Theorem 2.4.2, we have

$$g(\delta) \leq \overline{\lim}_{k \rightarrow \infty} \frac{1}{k} \log \pi (DP)^k b = \log \rho(DP),$$

where we have used a fact in the proof of large deviation theorem (see Lemma 2.3.9). This completes the proof. \square

Remark:

- (1). It can be easy to observe that for one dimensional case, $g(\delta) = \log \rho(DP)$ for nonzero $H(1), H(2), \dots, H(N)$.
- (2). Since D depends on the choice of matrix norm, from the study of matrix norms in Appendix A, we can see that appropriate choice of matrix norms, we can obtain many easily testable criteria. The reader is also refered to [174] for the discussions how to make such choices.

From the discussion about the second moment stability in section 2.2, we can modify the procedure to prove the following result.

Theorem 2.6.2:

Suppose that $\{\sigma_k\}$ is a finite state irreducible Markov chain with probability transition matrix P , then we have

$$g(2) = \rho \left(\text{diag} \{ H(1) \otimes H(1), H(2) \otimes H(2), \dots, H(N) \otimes H(N) \} (P^T \otimes I) \right).$$

Proof: This can be proved from the Coupled Lyapunov Equation (2.2.1) or (2.2.2). □

For $\delta \geq 2$, we can extend the above two results to obtain a more general estimates for $g(\delta)$.

Theorem 2.6.3:

Suppose that $\{\sigma_k\}$ is a finite state irreducible Markov chain, then for $\delta \geq 2$ and any matrix norm $\|\cdot\|$, we have

$$g(\delta) \leq \rho \left(\text{diag} \{ \|H(1)\|^{\delta-2} H(1) \otimes H(1), \dots, \|H(N)\|^{\delta-2} H(N) \otimes H(N) \} (P^T \otimes I) \right).$$

Proof: For matrix norm $\|\cdot\|$, we have

$$\|x_k\| \leq \|H(\sigma_{k-1})\| \cdots \|H(\sigma_0)\| \|x_0\|.$$

Let $z_k = \|H(\sigma_{k-1})\|^{(\delta-2)/2} \cdots \|H(\sigma_0)\|^{(\delta-2)/2} \|x_0\|^{(\delta-2)/2}$, then $z_{k+1} = \|H(\sigma_k)\|^{(\delta-2)/2} z_k$. Since any norms in finite dimensional space are equivalent, so there exists an $M > 0$ such that $\|\cdot\| \leq M\|\cdot\|_2$, thus we have

$$\begin{aligned} E\|x_k\| &= E\|x_k\|^2 \|x_k\|^{\delta-2} \\ &= M^2 E\|x_k\|^2 [\|H(\sigma_{k-1})\| \cdots \|H(\sigma_0)\| \|x_0\|]^{\delta-2} \\ &= M^2 E\|z_k x_k\|_2^2. \end{aligned} \tag{2.6.1}$$

Let $y_k = z_k x_k$ and $B(\sigma_k) = H(\sigma_k) \|H(\sigma_k)\|^{(\delta-2)/2}$, then we have

$$y_{k+1} = z_{k+1} x_{k+1} = \|H(\sigma_k)\|^{(\delta-2)/2} z_k H(\sigma_k) x_k = B(\sigma_k) y_k.$$

Then using (2.6.1) and Theorem 2.6.2, we obtain

$$\begin{aligned} g(\delta) &= \overline{\lim}_{k \rightarrow \infty} \frac{1}{k} \log E\|x_k\|^\delta \\ &\leq \overline{\lim}_{k \rightarrow \infty} \frac{1}{k} \log E\|y_k\|_2^2 \\ &= \rho(\text{diag}\{B(1) \otimes B(1), \dots, B(N) \otimes B(N)\} (P^T \otimes I)) \\ &= \rho(\text{diag}\{\|H(1)\|^{\delta-2} H(1) \otimes H(1), \dots, \|H(N)\|^{\delta-2} H(N) \otimes H(N)\} \\ &\quad \times (P^T \otimes I)) \end{aligned}$$

This proves the theorem. □

2.7. Almost Sure Stabilization

Second moment stabilization problem has been studied by Ji et al ([73]) and it has been shown that controllability in each mode can not guarantee the second moment stabilizability. What happens to almost sure stabilizability? In this section, we briefly discuss the almost sure stabilizability problem.

To proceed, we first formulate the stabilizability problem. Consider

$$x_{k+1} = A(\sigma_k)x_k + B(\sigma_k)u_k \quad (2.7.1)$$

where $\{\sigma_k\}$ is a finite state Markov chain with probability transition matrix P . If there exist matrices $K(1), K(2), \dots, K(N)$ such that the closed-loop system (resulting from the control $u(k) = -K(\sigma_k)x_k$)

$$x_{k+1} = (A(\sigma_k) - B(\sigma_k)K(\sigma_k))x_k \quad (2.7.2)$$

is almost surely stable, then we say that the system (2.7.1) is almost surely stabilizable. If (2.7.2) is second moment stable, then (2.7.1) is said to be second moment stabilizable.

Surprisingly, different from second moment stabilizability, almost sure stabilizability is much weaker than second moment stabilizability. We have

Theorem 2.7.1:

Suppose that $\{\sigma_k\}$ is a finite state time-homogenous irreducible and aperiodic Markov chain. If $(A(i), B(i))$ ($i \in \underline{N}$) are controllable, then the system (2.7.1) is almost surely stabilizable.

Proof: Since $(A(i), B(i))$ is controllable, there exists a feedback matrix $K(i)$ such that

$$A(i) - B(i)K(i) = T(i)^{-1} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} T(i), \quad (i \in \underline{N})$$

where $T(i)$ is a nonsingular matrix. It is easy to verify that $(A(i) - B(i)K(i))^k = 0$ for $k \geq n$, i.e., the closed-loop system matrix for each mode is dead beat system. From the *Case (a)* in the proof of Theorem 2.3.7, we can prove that (2.7.2) is almost surely stable. This completes the proof. \square

Remarks:

- (1). It can be seen that this theorem is valid for more general Markovian case, for example, as long as the following condition: $P(A) = 1$, where

$$A = \{\omega : \exists i = i(\omega) \in \underline{N}, \exists k = k(\omega) \geq 0, \text{ s.t. } \sigma_{k+1} = \cdots = \sigma_{k+n} = i\},$$

then Theorem 2.7.1 is valid.

- (2). ergodicity (or irreducibility) without aperiodicity is not enough to guarantee the almost sure stabilizability. This can be illustrated by almost sure stability property. Let $H(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $H(2) = H^T(1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is evident that $H(1)^2 = H(2)^2 = 0$ and the given Markov chain is ergodic and irreducible, and has a single cyclically ergodic class. However, the system (2.1.1) with the given structure is not almost surely stable.
- (3). From the proof, the controllability of all $(A(i), B(i))$ ($i \in \underline{N}$) can be relaxed that there exists an $i \in \underline{N}$ such that $(A(i), B(i))$ is controllable. To

avoid repetition, the reader is referred to the techniques developed in the study of almost sure stabilization of continuous-time jump linear systems presented in chapter three.

As we know that any almost sure stability criteria can be used to study the almost sure stabilization problem. This will involve a set of inequalities for feedback matrices $K(1), K(2), \dots, K(N)$. As an illustration, we use Theorem 2.5.1 or Theorem 2.5.3 as example.

Theorem 2.7.2:

Suppose that $\{\sigma_k\}$ is a finite state irreducible Markov chain with the unique invariant measure π . If there exists a matrix $K(1), K(2), \dots, K(N)$ and a matrix norm $\|\cdot\|$ such that

$$\|A(1) - B(1)K(1)\|_2^{\pi_1} \|A(2) - B(2)K(2)\|_2^{\pi_2} \cdots \|A(N) - B(N)K(N)\|_2^{\pi_N} < 1, \quad (2.7.3)$$

then (2.7.1) is almost surely stabilizable.

Proof: This is obvious from Theorem 2.5.3. □

It is clear from this theorem that if we want to solve the almost sure stabilization problem, we need to solve the inequality (2.7.3). When we choose spectral norm, a new problem arise: how to assign the singular values for linear system. From the above theorem, we can see that if we can assign singular values less than unity for each individual mode, then the system (2.7.1) is almost surely stabilizable. Thus singular value assignment becomes very important. It is well-known that if (A, B) is controllable, then we can assign eigenvalues anywhere in the complex plane. Unfortunately, this is not true for singular value assignment.

Example 2.7.4:

Consider the controllable pair (A, B) , where

$$A = \begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is the canonical controllability form. We will show that no matter what linear feedback we use, the singular values of the closed-loop system can not be strictly less than unity.

Let $K = (k_1, k_2)$ be any linear feedback matrix, the closed-loop system matrix is given by

$$A - BK = \begin{pmatrix} 0 & 1 \\ a_1 - k_1 & a_2 - k_2 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ x_1 & x_2 \end{pmatrix}.$$

It is easily compute that the largest singular value is given by

$$f(x_1, x_2) = \|A - Bk\|_2^2 = \frac{1}{2} \left(1 + x_1^2 + x_2^2 + \sqrt{(1 + x_1^2 + x_2^2)^2 - 4x_1^2} \right).$$

Simple computation leads to

$$\min_{x \in \mathbf{R}^2} f(x_1, x_2) = 1.$$

From this, we can conclude that for any feedback, the largest singular value of the closed-loop system is not less than unity.

The reason for this phenomena is that the eigenvalues are invariant under similarity transformation, however, singular values are really depends on the representation of the state space model. The assignment of singular values and the solutions of the related inequalities for stabilization problem forms another future research direction.

For the second moment stabilizability problem, Ji et al ([73]) have already obtained some necessary and sufficient conditions. They proved that the system (2.7.1) is stochastically stabilizable if and only if a set of matrix sequences converges to a set of positive semidefinite matrices. Although they claimed that this condition is testable, it is in fact not so easy! It needs a lot of computations. Interestingly, from Theorem 2.3.1, we can obtain the following easily testable condition.

Theorem 2.7.5:

The system (2.7.1) is second moment stochastically stabilizable if and only if the following JLQ optimal problem of (2.7.1) with the objective function

$$J(x_0, \sigma_0) = E \left\{ \sum_{k=0}^{\infty} x_k^T x_k \mid x_0, \sigma_0 \right\}$$

has a finite minimal cost.

Proof: Suppose that (2.7.1) is stochastically stabilizable, then there exists control law $u(k) = -K(\sigma_k)x_k$ such that the closed-loop system is stochastically (or second moment exponentially) stable, i.e., for the given control law, $J(x_0, \sigma_0) < +\infty$, thus minimal cost of JLQ problem will be definitely finite. Conversely, suppose that the JLQ problem has a finite optimal cost, when we use the optimal control law, $J(x_0, \sigma_0) < \infty$, the closed-loop system of (2.7.1) will be stochastically stable, therefore, (2.7.1) is second moment stochastically stabilizable. \square

From the first look, we may say that we complicate the problem. However, when we notice that there may be simple solution for JLQ problem in current literature, the tedious stabilization problem will be reduced to some algebraic

condition, then Theorem 2.7.5 becomes important. To illustrate this point, we study a simple example. But first, we quote a theorem for JLQ problem.

Theorem 2.7.6:(Chizeck et al [56])

Consider the system

$$x_{k+1} = A(\sigma_k)x_k + B(\sigma_k)u_k,$$

where $\{\sigma_k\}$ is a finite state Markov chain with probability transition matrix $P = (p_{ij})$ and the initial condition is $x(k_0) = x_0$ and $\sigma(k_0) = \sigma_0$. The cost to be minimized is

$$J_L(x_0, \sigma_0) = E \left[\sum_{k=k_0}^{L-1} \{u_k^T R(\sigma_k)u_k + x_{k+1}^T Q(\sigma_{k+1})x_{k+1}\} + x_L^T K_T(\sigma_L)x_L \right],$$

where $R(j), Q(j)$ and $K_T(j)$ are positive semidefinite for each j and k satisfying

$$R(i) + B^T(i) \left[\sum_{j=1}^N p_{ij} Q(j) \right] B(i) > 0, \quad i \in \underline{N}.$$

Then the optimal control law is given by

$$u_k = -L_k(\sigma_k)x_k, \quad k \in \underline{L-1}$$

where

$$\begin{aligned} L_k(i) &= [R(i) + B^T(i)Q_{k+1}^*(i)B(i)]^{-1}B^T(i)Q_{k+1}^*(i)A(i) \\ Q_{k+1}^*(i) &= \sum_{j=1}^N p_{ij}[Q(j) + K_{k+1}(j)] \\ K_k(i) &= A^T(i)Q_{k+1}^*(i)[A(i) - B(i)L_k(i)], \quad K_L(i) = K_T(i), \quad i \in \underline{N} \end{aligned}$$

and the optimal cost is given by $x_0^T K_{k_0}(\sigma_0)x_0$. □

Now we use Theorem 2.7.5 and Theorem 2.7.6 to show that although individual modes are controllable, the jump linear system may not be second moment stabilizable.

Example 2.7.7:

Let $A(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $A(2) = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, $B(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $B(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The form process $\{\sigma_k\}$ is a two-state iid chain with probability measure $(\pi_1, \pi_2) = (0.5, 0.5)$. It is easy to check that $(A(i), B(i))$ is controllable. In Theorem 2.7.6, let $K_T(i) = 0$, $Q(i) = I$ and $R(i) = 0$ which does satisfy the required condition. We can see that $p_{ij} = 0.5$, thus $Q_L^*(i) = Q(i) = I$ and $L_{L-1}(0) = L_{L-1}(2) = 0$. It is easy to show by induction from Theorem 2.7.6 that $Q_k^*(i) = (N - k + 1)I$ and $L_k(i) = 0$. Thus the minimal cost of JLQ problem is given by

$$x_0^T K_0(\sigma_0) x_0 = L x_0^T A^T(\sigma_0) A(\sigma_0) x_0 \xrightarrow{L \rightarrow \infty} \infty.$$

Thus for any control $u_k = -K(\sigma_k)x_k$, we have

$$E \left\{ \sum_{k=k_0}^L x_k^T x_k \right\} \geq J_L(x_0, \sigma_0) = L x_0^T A^T(\sigma_0) A(\sigma_0) x_0$$

for any $L > k_0$, hence

$$E \left\{ \sum_{k=0}^{\infty} x_k^T x_k \right\} = \infty.$$

From Theorem 2.7.5, we conclude that the system is not second moment stabilizable.

2.8. Robust Stability Consideration

It may be also interesting to consider the robust stability of jump linear systems against any randomness, sometimes this is also refer to as absolute stability problem. This observation is very important, because if the system is absolute stable, then we do not need to worry about the statistics of the form process. The following results are motivated from the robust stability analysis of interval dynamical systems (see Fang et al [174]). In this section, we only consider the robust almost sure stability problem. In fact, the conditions we have obtained also guarantee the robust moment stability, and the robust stability problem is reduced to a deterministic stability problem.

Definition 2.8.1:

The system (2.1.1) is robustly almost surely stable if it is almost surely stable for any finite state form process $\{\sigma_k\}$.

The following result is immediate from matrix norm property (Appendix A).

Theorem 2.8.2.

If there exists a matrix norm $\|\cdot\|$ such that $\|H(i)\| < 1$ for any $i \in \underline{N} \stackrel{\text{def}}{=} \{1, 2, \dots, N\}$, then the system (2.1.1) is robustly almost surely stable. \square

Proof: Let $r = \max_{1 \leq i \leq N} \|H(i)\|$, then $r < 1$ from the assumption. Thus, we have

$$\begin{aligned} \|x_k\| &= \|H(\sigma_{k-1}) \cdots H(\sigma_0)x_0\| \leq \|H(\sigma_{k-1})\| \cdots \|H(\sigma_0)\| \|x_0\| \\ &\leq r^k \|x_0\| \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Therefore, (2.1.1) is almost surely stable for any finite state form process. This completes the proof. \square

Although this is a very simple result with a simple proof, this is in fact a very general sufficient condition for robust almost sure stability. This is because that by specifying a matrix norm in Theorem 2.8.2 appropriately, many nice results can be obtained. To illustrate the generality of Theorem 2.8.2, we derive some simple testable sufficient conditions for robust almost sure stability as corollaries.

Corollary 2.8.3:

If $H(1), H(2), \dots, H(N)$ are normal matrices, then (2.1.1) is robustly almost sure stable if and only if $H(1), H(2), \dots, H(N)$ are Schur stable. \square

Proof. *Necessity:* Let we choose $\{\sigma_k\}$ is N state iid process with probability distribution $\{p_1, p_2, \dots, p_N\} = \{1, 0, \dots, 0\}$. Since (2.1.1) is robustly almost sure stable, it is almost surely stable for this form process, from which we obtain that $H(1)$ is Schur stable. Similar arguments leads to the stability of other matrices.

Sufficiency: Before we do this, we need the following: suppose that A is a normal matrix, i.e., $A^*A = AA^*$, from [148], there exists a unitary matrix U such that $A = U^* \text{diag}\{\mu_1, \mu_2, \dots, \mu_n\}U$, thus

$$A^*A = U^* \text{diag}\{\overline{\mu_1}\mu_1, \overline{\mu_2}\mu_2, \dots, \overline{\mu_n}\mu_n\}U.$$

From this, we have

$$\lambda_{\max}(A^*A) = \max_{1 \leq i \leq n} \{\overline{\mu_i}\mu_i\} = \left(\max_{1 \leq i \leq n} |\mu_i| \right)^2 = \rho(A)^2.$$

So, $\|A\|_2 = \rho(A)$. Suppose that $H(1), H(2), \dots, H(N)$ are stable, then $\rho(H(i)) < 1$ for $i \in \underline{N}$. Hence, for 2–norm, we have $\|H(i)\|_2 = \rho(H(i)) < 1$ for $i \in \underline{N}$. From Theorem 2.8.2, (2.1.1) is robustly almost surely stable. This concludes the proof. \square

Since symmetric matrix or skew-symmetric matrix is normal matrix, if $H(1), H(2), \dots, H(N)$ are either symmetric or skew-symmetric matrices, then (2.1.1) is robustly almost surely stable if and only if $H(1), H(2), \dots, H(N)$ are Schur stable.

Corollary 2.8.4:

The system (2.1.1) is robustly almost surely stable if one of the following conditions holds:

- (a). $\max_i \left\{ \sum_{j=1}^n |h_{ij}(l)| \right\} < 1$ for $l \in \underline{N}$;
- (b). $\max_j \left\{ \sum_{i=1}^n |h_{ij}(l)| \right\} < 1$ for $l \in \underline{N}$;
- (c). $\sum_i \sum_j h_{ij}^2(l) < 1$ for $l \in \underline{N}$.

\square

Proof: Using 1–norm, ∞ –norm and Frobenius norm, the proof can be obtained directly from Theorem 2.8.2. \square

Using generalized Gershgorin typed norm, we can obtain

Corollary 2.8.5:

If there exists positive numbers r_1, r_2, \dots, r_n such that one of the following conditions holds:

- a). $\max_i \left\{ \sum_{j=1}^n \frac{r_j}{r_i} |h_{ij}(l)| \right\} < 1$ for any $l \in \underline{N}$;
- b). $\max_j \left\{ \sum_{i=1}^n \frac{r_i}{r_j} |h_{ij}(l)| \right\} < 1$ for any $l \in \underline{N}$;
- c). $\sum_{i,j} \frac{r_i}{r_j} h_{ij}^2(l) < 1$ for any $l \in \underline{N}$,

then (2.1.1) is robustly almost surely stable. \square

This is a result similar to the one obtained from the Gershgorin's Circle Theorem. The problem here is how to choose the positive numbers r_1, r_2, \dots, r_n . In what follows, we derive an equivalent result which is independent of the choice of r_1, r_2, \dots, r_n . To accomplish this, we need the following results from matrix theory.

Lemma 2.8.6: ([149],[173])

If $A = (a_{ij})$ satisfies: $a_{ij} \leq 0$ ($i \neq j$) for $i, j = 1, 2, \dots, n$, then the following statements are equivalent:

- (1). A is positive Hurwitz stable (all eigenvalues of A have positive real parts), i.e., A is an M -matrix;
- (2). All principal minors of A are positive;
- (3). All leading principal minors of A are positive;
- (4). There exists a positive vector x such that $Ax > 0$;
- (5). There exists a positive diagonal matrix D such that $DA + A^T D$ is positive definite;

(6). There exists positive numbers r_1, r_2, \dots, r_n such that

$$a_{ii}r_i > \sum_{j \neq i} r_j |a_{ij}|, \quad i = 1, 2, \dots, n.$$

(7). $-A$ is Hurwitz stable. \square

Corollary 2.8.7:

Define $Q = (q_{ij})$ and $H(l) = (h_{ij}(l))$, where $q_{ij} = \max_{1 \leq l \leq N} \{|h_{ij}(l)|\}$, then (2.1.1) is robustly almost surely stable if $Q - I$ is Hurwitz stable or all leading principal minors of $I - Q$ are positive. \square

Proof: Let $U = I - Q = (u_{ij})$, then $u_{ij} \leq 0$ ($i \neq j$). Since $-U$ is Hurwitz stable, from Lemma 2.8.6, there exist positive numbers r_1, r_2, \dots, r_n such that $u_{ii}r_i > -\sum_{j \neq i} r_j u_{ij}$, i.e.,

$$\sum_{j=1}^n \frac{r_j}{r_i} q_{ij} < 1, \quad i = 1, 2, \dots, n.$$

This is equivalent to

$$\max_i \left\{ \sum_{j=1}^n \frac{r_j}{r_i} \max_{1 \leq l \leq N} |h_{ij}(l)| \right\} < 1.$$

From this, we obtain that

$$\max_i \left\{ \sum_{j=1}^n \frac{r_j}{r_i} |h_{ij}(l)| \right\} < 1.$$

From a) of Corollary 2.8.5, we conclude that (2.1.1) is robustly almost surely stable. \square

From this corollary, we can observe that the robust almost sure stability of (2.1.1) can be tested by the Hurwitz stability of one special matrix. From (3)

of Lemma 2.8.6, the following easy test result can be obtained: If the leading principal minors of matrix $I - Q$ are positive, then (2.1.1) is robustly almost surely stable.

The following example illustrates the usefulness of Corollary 2.8.7.

Example 2.8.8:

Let

$$H(1) = \begin{pmatrix} -0.5 & 0 \\ -0.25 & 0 \end{pmatrix}, \quad H(2) = \begin{pmatrix} 0.5 & 0.6 \\ 0.75 & 0 \end{pmatrix}.$$

Since

$$\begin{aligned} \max_i \left\{ \sum_{j=1}^n |h_{ij}(2)| \right\} &= 1.1 > 1, \\ \max_j \left\{ \sum_{i=1}^n |h_{ij}(2)| \right\} &= 1.25 > 1, \\ \sum_{i,j} h_{ij}^2(2) &= 1.1725 > 1, \end{aligned}$$

we can not use Corollary 2.8.4. However, we have

$$Q = (\max_l |h_{ij}(l)|) = \begin{pmatrix} 0.5 & 0.6 \\ 0.75 & 0 \end{pmatrix}, \quad U = I - Q = \begin{pmatrix} 0.5 & -0.6 \\ -0.75 & 1 \end{pmatrix},$$

and the leading principal minors of U is 0.5 and 0.05, which are all positive, from Corollary 2.8.7, we conclude that (2.1.1) is robustly almost surely stable.

□

Corollary 2.8.9:

Let $H(1), H(2), \dots, H(N)$ be upper triangular matrices, then (2.1.1) is robustly almost surely stable if and only if $H(1), H(2), \dots, H(N)$ are Schur stable. □

Proof: The necessity is trivial. We only need to prove the sufficiency. Let $H(l) = (h_{ij}(l))$ with $h_{ij}(l) = 0$ for $i > j$. Let $q_{ij} = \max_{1 \leq l \leq N} \{|h_{ij}(l)|\}$. It is easy to show that $q_{ij} = 0$ for $i > j$. Suppose that $H(1), H(2), \dots, H(N)$ are Schur stable. Because they are triangular, we obtain that $q_{ii} < 1$. Hence Q is Schur stable and $Q - I$ is Hurwitz stable. From Corollary 2.8.7, (2.1.1) is robustly almost surely stable. \square

Corollary 2.8.10:

If there exists a positive definite matrix S such that $H(l)^*SH(l) - S$ is negative definite for $1 \leq l \leq N$, then (2.1.1) is robustly almost surely stable. \square

Proof: Since S is positive definite, there exists a nonsingular matrix T such that $S = T^*T$, we use the norm $\|A\|_T = \|TAT^{-1}\|_2$, which is induced by the vector norm $\|x\|_T = \|Tx\|_2$. Let $H(l)^*SH(l) - S = -Q_l$ for $l \in \underline{N} = \{1, 2, \dots, N\}$. Where Q_l is positive definite. From this equation, we obtain

$$(TH(l)T^{-1})^*(TH(l)T^{-1}) = I - (T^{-1})^*Q_lT^{-1}$$

and

$$\begin{aligned} \|H(l)\|_T &= \sqrt{\lambda_{\max}((TH(l)T^{-1})^*(TH(l)T^{-1}))} \\ &= \sqrt{1 - \lambda_{\min}((T^{-1})^*Q_lT^{-1})} < 1, \quad l \in \underline{N}. \end{aligned}$$

From Theorem 2.8.2, we obtain that (2.1.1) is robustly almost surely stable. \square

In this section, we have seen that we can use the matrix norm to study the robust almost sure stability as we observed in the almost sure stability study. However, there exists a difficulty in using this approach, that is, how to choose an appropriate matrix norm such that we can obtain the most powerful

result. One solution for this problem is to solve the following multi-objective optimization problem:

$$\min_{T \in \mathcal{T}} J(T) = \begin{pmatrix} \|H(1)\|_l^T \\ \|H(2)\|_l^T \\ \vdots \\ \|H(N)\|_l^T \end{pmatrix} = \begin{pmatrix} \|TH(1)T^{-1}\|_l \\ \|TH(2)T^{-1}\|_l \\ \vdots \\ \|TH(N)T^{-1}\|_l \end{pmatrix}$$

where $l = 1, 2, +\infty$. Let T_* be one solution with optimal objective $J(T_*)$, then we can conclude that (2.1.1) is robustly almost surely stable if $J(T_*) < c$, where $c = (1, 1, \dots, 1)'$. From the matrix norm properties we presented in Appendix A, we may conclude that the system (2.1.1) is robustly almost surely stable if and only if there exists an optimal solution T_* for the above optimization problem such that $J(T_*) < c$. This issue will be explored in the future.

Notice that as we show in [174] that when the parameters of the individual mode system matrix varies in the convex hull of $H(1), H(2), \dots, H(N)$, (2.1.1) is still robustly almost surely stable. To be precise, we formulate it into the following theorem:

Theorem 2.8.11:

If there exists a matrix norm such that $\|A_l\| < 1$ ($l \in \underline{N}$) then (2.1.1) is robustly almost surely stable for any $H(1), H(2), \dots, H(N)$ satisfying

$$H(j) \in \left\{ A \mid A = \sum_{l=1}^N \alpha_l A_l, \quad \alpha_l \geq 0, \quad \sum_{l=1}^N \alpha_l = 1 \right\}, \forall j \in \underline{N}.$$

This theorem implies that the robust stability results we presented in this section is very strong results. The robustness is not only against the randomness, but also against parameter uncertainty in individual modes. This may be good tool for the practical designs.

Example 2.8.12:

Let

$$H(1) = \begin{pmatrix} 0.4 & 1 \\ -0.2 & 1.3 \end{pmatrix}, \quad H(2) = \begin{pmatrix} -7.5 & 20 \\ -3.2 & 8.5 \end{pmatrix}.$$

We first try to use Corollary 2.8.7, we have

$$Q = \begin{pmatrix} 7.5 & 20 \\ 3.2 & 8.5 \end{pmatrix}, \quad I - Q = \begin{pmatrix} -6.5 & -20 \\ -3.2 & -7.5 \end{pmatrix},$$

the leading principal minors of $I - Q$ are not all positive, we can not use Corollary 2.8.7, this means that Gershgorin-like criteria can not be used in this example.

We choose $T = \begin{pmatrix} 10 & 2 \\ 4 & 1 \end{pmatrix}$ and use the vector norm $\|Tx\|_\infty$, the induced matrix norm is given by $\|A\|_T = \|TAT^{-1}\|_\infty$, then we have

$$\begin{aligned} \|H(1)\|_T &= \|TH(1)T^{-1}\|_\infty = \left\| \begin{pmatrix} 0.8 & 0 \\ 0 & 0.9 \end{pmatrix} \right\|_\infty = 0.9 < 1 \\ \|H(2)\|_T &= \|TH(2)T^{-1}\|_\infty = \left\| \begin{pmatrix} 0.5 & 0.4 \\ 0 & 0.5 \end{pmatrix} \right\|_\infty = 0.9 < 1. \end{aligned}$$

From Theorem 2.8.2, we conclude that (2.1.1) is robustly almost surely stable.

□

2.9. Illustrative Examples

In this section, some examples will be given to illustrate previous results up to this moment. Some relationship between almost sure stability and moment stability is concretely demonstrated by examples.

Example 2.9.1:

Suppose that $\{\sigma_k\}$ is a finite state iid chain. Take $H(1) = 2.5$, $H(2) = 0.1$, and $p_1 = p_2 = 0.5$, then $\lambda_1 = 6.25$ and $\lambda_2 = 0.01$. Thus

$$\begin{aligned}\lambda_1^{p_1} \lambda_2^{p_2} &= \sqrt{6.25 * 0.01} = 0.25 < 1 \\ p_1 \lambda_1 + p_2 \lambda_2 &= \frac{6.25 + 0.01}{2} = 3.13 > 1,\end{aligned}$$

and from Theorem 2.4.1 and Theorem 2.5.1, we conclude that the system (2.1.1) is almost surely stable, but not second moment stable. \square

Example 2.9.2:

Let $\{\sigma_k\}$ be a two-state i.i.d. sequence, and let $H(1) = \alpha$, and $H(2) = \beta$, and $p_1 = p_2 = 0.5$. Then according to Theorem 2.4.1, the system (2.1.1) is δ -moment stable if and only if

$$\frac{|\alpha|^\delta + |\beta|^\delta}{2} < 1.$$

and it is almost surely stable if and only if

$$|\alpha||\beta| < 1.$$

The following graph illustrates the stability regions:

Fig.1. Almost sure and δ -moment stability regions

In Fig.1, R_0 denotes the almost sure stability region, which is the open connected region enclosed by the four (disconnected) hyperbolic curves. R_δ is the δ -moment stability region with a $\delta < 1$, which is the open connected bounded region enclosed by the next four (connected) hyperbolic curves. The diamond-shaped region denoted by R_1 is the first moment stability region. The open connected region R_2 enclosed by the ellipse is the second moment stability region. Finally, the open connected square $R_{+\infty}$ is the δ -moment stability region for $\delta = +\infty$. Indeed, we have $R_\infty \subset R_2 \subset R_1 \subset R_\delta \subset R_0$, and as δ decreases to 0^+ , R_δ tends to R_0 monotonically. This is consistent with our previous analysis.

Example 2.9.3: (Almost sure stability does not imply that the individual modes are stable)

Let $H(1) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $H(2) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ for systems of type (2.1.1) with an iid form process σ_k with $p_1 = p_2 = 0.5$.

For any $m, n \geq 1$, $H(1)^m H(2)^n = 0$, and the system is almost surely stable. But $H(1)$ and $H(2)$ are not stable. \square

Example 2.9.4: (A general illustrative example)

Let $H(1) = \tilde{\alpha} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $H(2) = \tilde{\alpha} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ for systems of type (2.1.1) with an iid form process σ_k with $p_1 = p_2 = 0.5$.

In this example, we study the stability properties of the system (2.1.1) as $\tilde{\alpha}$ varies in the interval $(0, 1]$.

(i). $\|H(1)\|_2 = \|H(2)\|_2 = \left(\frac{\sqrt{5}+1}{2}\right) \tilde{\alpha}$, using Theorem 2.5.1 we obtain that if $\tilde{\alpha} < \frac{\sqrt{5}-1}{2}$, the system (2.1.1) is almost surely stable.

(ii). Next we use Theorem 2.5.9 to study the almost sure stability:

Let $x = (\cos \theta, \sin \theta)^T$, and let $f(\theta) = \|H(1)x\|_2^2 \|H(2)x\|_2^2$. Then

$$\begin{aligned} f(\theta) &= \tilde{\alpha}^4 \left\| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right\|_2^2 \left\| \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right\|_2^2 \\ &= \tilde{\alpha}^4 \left((\cos \theta + \sin \theta)^2 + \sin^2 \theta \right) \left(\cos^2 \theta + (\cos \theta + \sin \theta)^2 \right), \\ &= \tilde{\alpha}^4 \left(2 + 3 \sin 2\theta + \frac{5}{4} \sin^2 2\theta \right) \end{aligned}$$

and $\max f(\theta) = \frac{25}{4} \tilde{\alpha}^4$, so $\sigma_{max} = \sqrt{\frac{5}{2}} \tilde{\alpha}$. From Theorem 2.5.9, (2.1.1) is almost surely stable if $\tilde{\alpha} < \sqrt{\frac{2}{5}}$. This is an improved estimate of the almost sure stability region.

(iii). Since $\{\sigma_k\}$ is iid, from $x_{k+1} = H(\sigma_k)x_k$ we have

$$\begin{aligned} Ex_{k+1} &= EH(\sigma_k)Ex_k = \cdots = (EH(\sigma_0))^k x_0 \\ &= (p_1H(1) + p_2H(2) + \cdots + p_NH(N))^k x_0. \end{aligned}$$

From this, we obtain that (2.1.1) is mean value stable if and only if $\sum_{i=1}^N p_i H(i)$ is schur stable. From this, we know that (2.1.1) is mean value stable if and only if $p_1H(1) + p_2H(2)$ is a schur stable matrix. Thus we obtain that if $\tilde{\alpha} < 2/3$, (2.1.1) is mean value stable. By the remark given after Lemma 2.5.6, we obtain that (2.1.1) is almost surely stable if $\tilde{\alpha} < 2/3$.

(iv). Although the proof in (iii) is simple, the following approach seems to be applicable to more general cases.

Define

$$G_k = \begin{pmatrix} G_k^1 & G_k^2 \\ G_k^3 & G_k^4 \end{pmatrix} = H(\sigma_k) \cdots H(\sigma_0), H(\sigma_k) = \tilde{\alpha} \begin{pmatrix} 1 & \delta_k \\ 1 - \delta_k & 1 \end{pmatrix}$$

where $\delta_k = 1$ if $\sigma_k = 1$ and $\delta_k = 0$ if $\sigma_k = 2$.

Let Σ_k is the sum of all the entries of G_k . Then it is easy to show that

$$\Sigma_k = \tilde{\alpha} (\Sigma_{k-1} + \delta_k(G_{k-1}^3 + G_{k-1}^4) + (1 - \delta_k)(G_{k-1}^1 + G_{k-1}^2)) \quad (2.9.1)$$

Since $\{\sigma_k\}$ is an iid sequence, δ_k and $1 - \delta_k$ are independent of G_{k-1}^j ($1 \leq j \leq 4$) and $E\delta_k = E(1 - \delta_k) = 0.5$. Thus from (2.9.1), let $m_k = E\Sigma_k$, we have

$$\begin{aligned} m_k &= \tilde{\alpha} (E\Sigma_{k-1} + E\delta_k E(G_{k-1}^3 + G_{k-1}^4) E(1 - \delta_k) E(G_{k-1}^1 + G_{k-1}^2)) \\ &= \tilde{\alpha} \left(m_{k-1} + \frac{1}{2} (E(G_{k-1}^3 + G_{k-1}^4) + E(G_{k-1}^1 + G_{k-1}^2)) \right) \\ &= \tilde{\alpha} (m_{k-1} + \frac{1}{2} E\Sigma_{k-1}) = \frac{3}{2} \tilde{\alpha} m_{k-1} = \left(\frac{3}{2} \tilde{\alpha} \right)^k m_0 \end{aligned}$$

Let \mathcal{F}_k be the smallest σ -algebra generated by $\sigma_k, \dots, \sigma_0$. Then it is easy to show that

$$E(\Sigma_{k+1}|\mathcal{F}_k) = \left(\frac{3}{2}\tilde{\alpha}\right)\Sigma_k$$

Thus, if $\tilde{\alpha} < 2/3$, then $E(\Sigma_{k+1}|\mathcal{F}_k) \leq \Sigma_k$, which implies that $\{\Sigma_k, \mathcal{F}_k\}$ is a supermartingale. From the Martingale Convergence Theorem ([141]), there exists a random variable Σ such that $\lim_{k \rightarrow \infty} \Sigma_k = \Sigma$, almost surely.

Next, we want to prove that $\Sigma = 0$. In fact, for $c > 0$, we have

$$\begin{aligned} P(\Sigma > 0) &\leq P(\cup_{k=m}^{\infty} (\Sigma_k > c)) \leq \sum_{k=m}^{\infty} P(\sigma_k > c) \\ &\leq \sum_{k=m}^{\infty} \frac{1}{c} E\Sigma_k \leq \frac{1}{c} \sum_{k=m}^{\infty} \left(\frac{3}{2}\tilde{\alpha}\right)^k m_0 \end{aligned} \quad (2.9.2)$$

If $\tilde{\alpha} < 2/3$, then $\sum_{k=1}^{\infty} (\frac{3}{2}\tilde{\alpha})^k$ is a convergent series. In (2.9.2), let m go to infinity, we obtain $P(\Sigma > c) = 0$. Thus

$$P(\Sigma > 0) \leq \sum_{k=1}^{\infty} P(\Sigma > \frac{1}{k}) = 0$$

so $P(\Sigma > 0) = 0$. Since $\Sigma \geq 0$, we therefore have

$$P(\Sigma = 0) = 1$$

From which we obtain that if $\tilde{\alpha} < 2/3$, then (2.1.1) is almost surely stable.

(v). From Theorem 2.2.13, we have proved that if $EH(\sigma_1)^T H(\sigma_1)$ is stable, then (2.1.1) is second moment stable. Since

$$EH(\sigma_1)^T H(\sigma_1) = \frac{1}{2}(H(1)^T H(1) + H(2)^T H(2)) = \frac{\tilde{\alpha}^2}{2} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

and its eigenvalues are $5\tilde{\alpha}^2/2$ and $\tilde{\alpha}^2/2$, if $\tilde{\alpha}^2/2 < 1$, i.e. $\tilde{\alpha} < \sqrt{0.4}$, (2.1.1) is second moment stable, so it is almost sure stable (same as (ii)).

(vi). The Kronecker product is good tool for studying the second moment stability. From Corollary 2.2.12, we obtain that (2.1.1) with an iid form process is second moment stable if and only if $p_1 H(1) \otimes H(1) + \cdots + p_N H(N) \otimes H(N)$ is stable. A variation of this is the following:

Let $P_k = E x_k x_k^T$, (2.1.1) is second moment stable if and only if P_k is a matrix sequence which converges to zero. For the present example we have been studying, we have

$$\begin{aligned} P_{k+1} &= \frac{1}{2} (H(1)P_k H(1)^T + H(2)P_k H(2)^T) \\ &= \frac{\tilde{\alpha}^2}{2} \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} P_k \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} P_k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \end{aligned}$$

Let $P_k = \begin{pmatrix} p_k^1 & p_k^2 \\ p_k^2 & p_k^3 \end{pmatrix}$ and $y_k = (p_k^1, p_k^2, p_k^3)^T$, then

$$y_{k+1} = \frac{\tilde{\alpha}^2}{2} \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix} y_k \triangleq R y_k.$$

(2.1.1) is second moment stable if and only if R is a stable matrix, i.e., all eigenvalues of R have modulus less than unity. The eigenvalues of R are $-\tilde{\alpha}^2/2$ and $\frac{5 \pm \sqrt{17}}{4} \tilde{\alpha}^2$. Thus if $\frac{5 + \sqrt{17}}{4} \tilde{\alpha}^2 < 1$, i.e., $\tilde{\alpha} < \sqrt{\frac{5 - \sqrt{17}}{2}}$, (2.1.1) is second moment stable.

Remark: Notice that $\sqrt{\frac{5 - \sqrt{17}}{2}} < \frac{2}{3}$, and we know that $\tilde{\alpha} < \frac{2}{3}$ is sufficient for almost sure stability. Hence, criteria for almost sure stability and second moment stability can differ greatly, and almost sure stability does not necessarily imply second moment stability.

(vii). Let $M(G)$ be the largest entry of a matrix G . Notice that $H(1)^m = \tilde{\alpha}^m \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $H(2)^m = \tilde{\alpha}^m \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$, then $M(H(1)^m H(2)^n) \geq \tilde{\alpha}^{m+n} mn$

and $M(H(2)^m H(2)^n) \geq \tilde{\alpha}^{m+n} mn$. Let

$$H(\sigma_n) \cdots H(\sigma_1) = \tilde{\alpha}^n \begin{pmatrix} 1 & l_m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ l_{m-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix}$$

or the variations with l_m or l_1 at the opposite positions in the corresponding matrices, where $l_i > 0$ and $\sum_{i=1}^m l_i = n$, and $m = m(n)$ is a random sequence and $m \rightarrow \infty$ if $n \rightarrow \infty$. Since $\{\sigma_n\}$ are an iid sequence with the probability distribution $(0.5, 0.5)$, $\{l_i\}$ is also an iid sequence with the distribution $P\{l_i = k\} = 0.5^k$ ($k > 0$). It is also easy to show that

$$M(H(\sigma_n) \cdots H(\sigma_1)) \geq \tilde{\alpha}^n l_1 l_2 \cdots l_m$$

Thus

$$\frac{1}{m} \log M(H(\sigma_n) \cdots H(\sigma_1)) \geq \frac{1}{m} \sum_{i=1}^m \log l_i + \left(\frac{1}{m} \sum_{i=1}^m l_i \right) \log \tilde{\alpha}$$

From the Law of Large Numbers, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{i=1}^m \log l_i &= E \log l_1 = \sum_{k=1}^{\infty} \frac{\log k}{2^k} \\ \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m l_i &= E l_1 = \sum_{k=1}^{\infty} \frac{k}{2^k} \end{aligned}$$

then, noting that $m = m(n)$, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{m} \log M(H(\sigma_n) \cdots H(\sigma_1)) \geq \sum_{k=1}^{\infty} \frac{\log k}{2^k} + \left(\sum_{k=1}^{\infty} \frac{k}{2^k} \right) \log \tilde{\alpha}$$

So if it is positive, i.e.,

$$\tilde{\alpha} > \exp \left(- \frac{\sum_{k=1}^{\infty} \frac{\log k}{2^k}}{\sum_{k=1}^{\infty} \frac{k}{2^k}} \right) = 0.7758$$

then (2.1.1) is almost surely unstable.

This example shows that although the individual modes are stable, i.e., $H(1)$ and $H(2)$ are stable matrices (with $\tilde{\alpha} < 1$), for $\tilde{\alpha} > 0.7758$, the jump linear system (2.1.1) is not almost surely stable, let alone second moment stable. In the current literature, it is known that it is very easy to give an example of a finite state Markov chain jump linear system whose individual modes are stable, but the system is not almost surely stable. But it is very difficult to give an example for the case with an iid form process, the above example provides such an example.

(viii). Let $\Pi(A)$ denote the product of all the entries of matrix A and let $G_k = H(\sigma_k) \cdots H(\sigma_0)$, where the notation for entries of G_k is the same as before, let $\Pi_k = \Pi(G_k)$. As before, we can obtain

$$\Pi_{k+1} = \tilde{\alpha}^4 (G_k^1 + G_k^3)(G_k^2 + G_k^4)((1 - \delta_k)G_k^1 + \delta_k G_k^3)((1 - \delta_k)G_k^2 + \delta_k G_k^4)$$

Let \mathcal{F}_k be the smallest σ -algebra generated by $\{\sigma_{k-1}, \dots, \sigma_0\}$, then from the inequality $a + b \geq 2\sqrt{ab}$ for $a, b \geq 0$, we obtain

$$\begin{aligned} E(\Pi_{k+1} | \mathcal{F}_k) &= \tilde{\alpha}^4 (G_k^1 + G_k^3)(G_k^2 + G_k^4) \left[\frac{1}{2} (G_k^1 G_k^2 + G_k^3 G_k^4) \right] \\ &\geq \tilde{\alpha}^4 (2\sqrt{G_k^1 G_k^3})(2\sqrt{G_k^2 G_k^4}) \sqrt{G_k^1 G_k^2 G_k^3 G_k^4} \\ &= 4\tilde{\alpha}^4 G_k^1 G_k^2 G_k^3 G_k^4 = 4\alpha^4 \Pi_k \end{aligned}$$

From a similar argument, we also have

$$E\Pi_{k+1}^\delta \geq (4\tilde{\alpha}^4)^\delta (E\Pi_k^\delta)^\frac{1}{\delta}$$

Thus, if $4\tilde{\alpha}^4 > 1$, i.e., $\tilde{\alpha} > \sqrt{0.5} = 0.7071$, $\lim_{k \rightarrow \infty} E\Pi_k^\delta = +\infty$ for any $\delta > 0$.

If we use the 2-norm, then $\|A\| = \sqrt{\lambda_{max}(A^T A)}$. Thus for any $x \in R^n$, we have $x^T A^T A x \leq \|A\|^2 x^T x$. Choose $x = e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$,

we have $\sum_{k=1}^n a_{ik}^2 \leq \|A\|^2$, from which we obtain that $\max_{i,j} |a_{ij}| \leq \|A\|$. For nonnegative matrix A , $\Pi(A) \leq \|A\|^{n^2}$. Thus, for our problem, we have $\Pi_k \leq \|H(\sigma_k) \cdots H(\sigma_1)\|^4$. Therefore, we obtain

$$\left(E\|H(\sigma_k) \cdots H(\sigma_1)\|^{4\delta}\right)^{1/\delta} \geq (E\Pi_k^\delta)^{1/\delta} \geq \cdots \geq C(4\tilde{\alpha}^4)^k$$

from which we have

$$\frac{1}{k} \log E\|H(\sigma_k) \cdots H(\sigma_1)\|^{4\delta} \geq \frac{1}{k} \delta \log C + \delta \log(4\tilde{\alpha}^4)$$

Taking the limit supremum, we obtain

$$\bar{\beta}(4\delta, \pi) \geq \delta \log(4\tilde{\alpha}^4)$$

where $\bar{\beta}(\delta, \pi)$ is the top δ -moment Lyapunov exponent. From Theorem 2.3.14, we know that $\dot{\bar{\beta}}(0, \pi) = \gamma$, where γ is the top Lyapunov exponent. Thus, if $4\tilde{\alpha}^4 > 1$, i.e., $\tilde{\alpha} > 0.7071$, $\gamma > 0$, which means that (2.1.1) is exponentially unstable almost surely.

This result is better than (vii). This is due to the fact that we have used the relationship between almost sure stability and δ -moment stability presented in this chapter.

Summarizing the above for this example, we have

- (a). (2.1.1) is second moment stable if and only if $0 \leq \tilde{\alpha} < \sqrt{\frac{5-\sqrt{17}}{2}}$;
- (b). (2.1.1) is mean value stable if and only if $0 \leq \tilde{\alpha} < \frac{2}{3}$;
- (c). (2.1.1) is almost surely stable if $0 \leq \tilde{\alpha} < \frac{2}{3}$;
- (d). (2.1.1) is almost surely unstable if $\tilde{\alpha} > 0.7071$.

For $\frac{2}{3} \leq \tilde{\alpha} \leq \frac{\sqrt{2}}{2}$, we have not found a rigorous method to determine the stability property of the system, which will be studied in the future.

CHAPTER THREE

STABILITY OF CONTINUOUS-TIME JUMP LINEAR SYSTEMS

Continuous-time jump linear systems can not be directly studied by the discretized version, this is because that when the continuous-time jump linear system is discretized, the resulting discrete-time system is no longer jump linear system with a finite state Markov from process, rather a jump linear system with a infinite dimensional Markov form process. This will complicate the study considerably, and we can not use the result we have developed in Chapter Two. This chapter will study the stability of continuous-time jump linear system with a finite state Markov form process. Section 1 gives a brief introduction to this research area, followed by a detailed study of δ -moment stability properties in section 2. Tetable conditions for almost sure stability will be presented in third section, where matrix measure techniques have been used to obtain some simple but useful criteria. In section 4, we deal with estimation and analytic properties of (moment) Lyapunov exponents, robust stability issue will be briefly examined in section 5. Finally, we will study almost sure and moment stabilization in some details in section 6.

3.1. A Brief Introduction

Consider the continuous-time jump linear system in the form

$$\dot{x}(t) = A(\sigma_t)x(t) \tag{3.1.1}$$

where σ_t is a finite state random process (step process), usually a finite state, time homogeneous, Markov process. Stability analysis of systems of this type

can be traced back to the work Bergen ([135]) who generalized Bellman's ([134]) idea for discrete-time jump linear systems to study the moment stability properties of the continuous time systems (3.1.1) with piecewise constant form process $\{\sigma_t\}$. Later, Bhuracha ([136]) used Bellman's idea developed in [134] to generalize Bergen's results and studied both asymptotic stability of the mean and exponential stability of the mean. Darkhovskii and Leibovich ([166]) investigated second moment stability of system (3.1.1) where σ_t is a step process and the time intervals between jumps are iid and the modes of the system are governed by a finite state time homogeneous Markov chain. They obtained a necessary and sufficient conditions for second moment stability in terms of the Kronecker matrix product for the second moment stability, which is an extension of Bhuracha's result.

There is an alternative approach to the study of stochastic stability. Kats and Krasovskii ([36]) and Bertram and Sarachik ([37]) used a stochastic version of Lyapunov's second method to study almost sure stability and moment stability. Unfortunately, constructing an appropriate Lyapunov functions is difficult in general, this is a common disadvantage of Lyapunov's second method. Also, in many cases, the criteria obtained from this method are similar to moment stability criteria, are often too conservative. For certain classes of systems, such as (3.1.1), it is possible to obtain testable stability conditions. Kats and Krasovskii ([36]) and Feng et al ([74],[75]) used Lyapunov's second method to study the stability of (3.1.1) where $\{\sigma_t\}$ is a finite state Markov chain. Necessary and sufficient conditions are obtained for second moment stability of continuous time (3.1.1) jump linear systems.

As Kozin ([38]) pointed out, moment stability implies almost sure stability under fairly general conditions, but the converse is not true. In practical

applications, almost sure stability is more than often the more desirable properties because we can only observe sample path of the system and the moment stability criteria can sometimes be too conservative to be useful.

Although Lyapunov exponent techniques may provides necessary and sufficient conditions for almost sure stability, it is very difficult to compute the top Lyapunov exponent or to obtain good estimates of the top Lyapunov exponent for almost sure stability. Testable conditions are difficult to obtain from this theory.

Arnold et al ([116],[117]) studied the relationship between the top Lyapunov exponent and the δ -moment top Lyapunov exponent for a diffusion process. Using a similar idea, Leizarowitz ([153]) obtained similar results for (3.1.1). A general conclusions was that δ -moment stability implies the almost sure stability. Thus sufficient conditions for almost sure stability can be obtained through δ -moment stability, which is one of the motivations for study of δ -moment stability. There are many definitions for moment stability: δ -moment stability, exponential δ -moment stability and stochastic δ - moment stability. Feng et al ([74],[75]) showed that all the second moment stability concepts are equivalent for the system (3.1.1), and also proved that for one dimensional system, the region for δ -moment stability is monotonically converging to the region for almost sure stability as $\delta \downarrow 0^+$. This is tantamount to concluding that almost sure stability is equivalent to δ -moment stability for sufficient small δ . This is a significant result because the study of almost sure stability can be reduced to the study of δ -moment stability.

3.2. δ -moment Stability

Consider the continuous-time jump linear system (3.1.1) where $\{\sigma_t : t \geq 0\}$ is a finite state time homogeneous Markov process (which is referred to as form process) with a state space $S = \{1, 2, \dots, N\}$. Let $Q = (q_{ij})_{N \times N}$ be the infinitesimal matrix of $\{\sigma_t\}$. For simplicity, we assume that x_0 be a fixed constant vector in \mathcal{R}^n . In order to make clarity of stochastic stability we will study in this chapter, we use the following definition similar to the discrete case.

Definition 3.2.1:

For the system (3.1.1), let Ξ denote the collection of probability measures on S and $\Psi \subset \Xi$ be a nonempty subset of Ξ . The system is said to be

- (I). *asymptotically δ -moment stable* with respect to (w.r.t.) Ψ , if for any $x_0 \in \mathcal{R}^n$ and any initial probability distribution $\psi \in \Psi$ of σ_t ,

$$\lim_{t \rightarrow \infty} E \{ \|x(t, x_0)\|^\delta \} = 0,$$

where $x(t, x_0)$ is a sample solution of (3.1.1) initial from $x_0 \in \mathcal{R}^n$. If $\delta = 2$, we say that the system (3.1.1) is *asymptotically mean square stable w.r.t.* Ψ ; if $\delta = 1$, we say that the system (3.1.1) is *asymptotically mean stable w.r.t.* Ψ . If $\Psi = \Xi$, we say simply that (3.1.1) is *asymptotically mean square stable*. Similar statements apply to the following definitions.

- (II). *exponentially δ -moment stable* with respect to Ψ , if for any $x_0 \in \mathcal{R}^n$ and any initial distribution $\psi \in \Xi$ of σ_t , there exist constants $\alpha, \beta > 0$ independent of x_0 and ψ such that

$$E \{ \|x(t, x_0)\|^\delta \} \leq \alpha \|x_0\|^\delta e^{-\beta t}, \quad \forall t \geq 0.$$

(III). *stochastically δ -moment stable* with respect to Ψ , if for any $x_0 \in \mathcal{R}^n$ and any initial distribution $\psi \in \Psi$ of σ_t ,

$$\int_{t=0}^{\infty} E \{ \|x(t, x_0)\|^\delta \} < +\infty.$$

(IV). *almost surely (asymptotically) stable* with respect to Ψ , if for any $x_0 \in \mathcal{R}^n$ and any initial distribution $\psi \in \Psi$ of σ_t ,

$$P \left\{ \lim_{t \rightarrow \infty} \|x(t, x_0)\| = 0 \right\} = 1.$$

(V). *Almost surely exponentially stable* with respect to Ψ , if for any $x_0 \in \mathcal{R}^n$ and any initial distribution $\psi \in \Psi$ of σ_t ,

$$\lim_{t \rightarrow \infty} \|x(t, x_0)\| = 0$$

at an exponential rate almost surely, i.e., there exist $M(\omega) > 0$ and $\lambda(\omega) > 0$ such that

$$\|x(t, x_0)\| \leq M(\omega) \|x_0\| e^{-\lambda(\omega)t}, \text{ a.s..}$$

□

In the above definition, the initial probability distribution of $\{\sigma_t\}$ plays a very important role. The stochastic stability definitions as given above means the robust stability against (Ψ -structured) uncertainty of the initial distributions of the form process. Since $(x(t), \sigma_t)$ is the state of the system and in practice, the initial probability distribution of the form process $\{\sigma_t\}$ is usually not exactly known, this is a reasonable requirement. Also, as illustrated in [75], stability with respect to a single initial distribution, say, the ergodic invariant

distribution π , may not be a good stability criterion and an arbitrarily small perturbation to π can actually destroy the stability.

Because of the above consideration on the initial distribution of σ_t , we actually deal with a family of Markovian form processes parameterized by the initial distribution $\psi \in \Psi$. To signify the dependence of a quantity Q on ψ , sometimes we use a subscript Q_ψ . For example, P_ψ will denote the probability measure induced by ψ for σ_t and E_ψ will be the expectation with respect to P_ψ , etc..

Before we go further, we first establish some preliminaries for the finite state Markov process $\{\sigma_t\}$. For all $i, j \in S$, define

$$\begin{aligned} p_{ii} &= 0 \\ q_i &= -q_{ii} = \sum_{l \neq i} q_{il} \\ p_{ij} &= \frac{q_{ij}}{q_i} \quad (i \neq j) \end{aligned}$$

Let $\{r_k; k = 1, 2, \dots\}$ be the (discrete-time) Markov chain defined on the state space S with the one-step transition matrix $(p_{ij})_{N \times N}$ and initial distribution ψ . This chain is referred to as the *embedded Markov chain* of $\{\sigma_t\}$. We have the following sojourn time description of the process $\{\sigma_t\}$ ([138, p.254]).

Let $\tau_k, k = 0, 1, \dots$ be the successive sojourn times between jumps. Let $t_k = \sum_{l=0}^{k-1} \tau_l$ for $k = 1, 2, \dots$ be the waiting time for k -th jump and $t_0 = 0$. Starting in state $\sigma_0 = i$, the process sojourns there for a duration of time that is exponentially distributed with parameter q_i . The process then jumps to the state $j \neq i$ with probability p_{ij} ; the sojourn time in the state j is exponentially distributed with parameter q_j ; and so on. The sequence of the states visited by the process $\{\sigma_t\}$, denoted by i_1, i_2, \dots is

the embedded Markov chain $\{r_k; k = 1, 2, \dots\}$. Conditioning on i_1, i_2, \dots , the successive sojourn times denoted by $\{\tau^{(i_k)}\}$ are independent exponentially distributed random variables with parameters q_{i_k} . Clearly, the joint process $\{(r_k, \tau_k) : k = 0, 1, \dots\}$ is a time homogeneous Markov process and it fully describes the form process $\{\sigma_t\}$. From this description, we can see that if we use discretization to get the discrete systems from (3.1.1) will be determined by the joint process $\{r_k, \tau_k\}$ which is not a type of jump linear system (2.1.1), hence we need to modify the analysis method in chapter two to handle the continuous-time jump linear system (3.1.1). The following notations will be used throughout this chapter. Let $A_i = A(i)$ and $\kappa_i = \|A_i\|$ for all $i \in S$. Let $\kappa = \max_{i \in S} \kappa_i$. Let $\mathcal{F}^n = \sigma\{(r_k, \tau_k) : 0 \leq k \leq n\}$ be the σ -algebra generated by $\{(r_k, \tau_k) : 0 \leq k \leq n\}$. Let $\Phi(t, s)$ denote the transition matrix of (2.1). For each $i \in S$, let e_i denote the initial distribution of σ_t concentrated at i -th state. If σ_t has a single ergodic class, let π denote the unique invariant distribution of σ_t . For a matrix B , let $\lambda_i(B)$ denote one of the eigenvalues of B , and let $\lambda_{\max}(B) = \max_i(\operatorname{Re} \lambda_i(B))$ and $\lambda_{\min}(B) = \min_i(\operatorname{Re} \lambda_i(B))$ denote the largest and smallest real parts of eigenvalues of B , respectively.

In [74], Feng et al proved that for the system (3.1.1), the second moment stability, the second moment exponential stability and the second moment stochastic stability are equivalent, and all of them imply the almost sure stability. A natural question is if this relationship can be generalized to the δ -moment stability. The answer to this question is positive, we have

Theorem 3.2.2:

For any $\delta > 0$, the δ -moment stability, the exponential δ -moment stability and the stochastic δ -moment stability are equivalent for system (3.1.1), and they all imply the almost sure stability. \square

Before we prove the theorem, we need the following lemma:

Lemma 3.2.3:

- (i) Let $\kappa_i = \|A_i\|$ for all $i \in S$, and let $\kappa = \max_i \kappa_i$. Then, for any $t'_2 \geq t'_1 \geq 0$, $\|\Phi(t'_2, t'_1)\| \leq \exp(\kappa(t'_2 - t'_1))$.
- (ii) For any $\delta > 0$, if (3.1.1) is δ -moment stable, then for any $\psi \in \Xi$ and $k \geq 0$ integer,

$$E\{\|\Phi(t, t_k)\|^\delta | \mathcal{F}^k\} \leq \eta, \quad P_\psi - \text{a.s.} \quad \forall t \in [t_k, t_{k+1}], \quad (3.2.1)$$

where in (2.2), $\eta = \max_{i \in S} \eta_i$ and

$$\eta_i = E_{e_i}\{\|\Phi(t_1, 0)\|^\delta\} = \int_0^{+\infty} q_i \|\exp(A_i t)\|^\delta e^{-q_i t} dt < +\infty \quad (3.2.2)$$

for all $i \in S$. It follows that for any $\psi \in \Xi$ and $k \geq i \geq 0$ integers, if $t \in [t_k, t_{k+1})$, then

$$E\{\|\Phi(t, t_i)\|^\delta | \mathcal{F}^i\} \leq \eta^{(k+1-i)}, \quad P_\psi - \text{a.s.} \quad (3.2.3)$$

\square

Proof of lemma: For any matrix A , it is easy to verify that $\|\exp(At)\| \leq \exp(\|A\|t)$ for all $t \geq 0$. Since using the sojourn time description of the form process σ_t , the transition matrix $\Phi(t'_2, t'_1)$ can be written as a product of the

exponential matrices (transition matrices) of the systems $\dot{x}(t) = A_i x(t)$ for $i \in S$, the result (i) follows directly.

For (ii), note that the δ -moment stability of (3.1.1) implies that for each $i \in S$, (3.2.2) holds. To show (3.2.1), by the Markovian and time homogeneous properties of the system, it is enough to show that for any $\psi = (\psi_1, \dots, \psi_N) \in \Xi$,

$$E_\psi \{ \|\Phi(t, 0)\|^\delta \} \leq \eta, \quad \forall t \in [t_1, 0], \quad \psi \in \Xi.$$

However,

$$\begin{aligned} E_\psi \{ \|\Phi(t, 0)\|^\delta \} &= \sum_{i=1}^N \psi_i E_{e_i} \{ \|\Phi(t, 0)\|^\delta \} \\ &= \sum_{i=1}^N \psi_i \int_0^t q_i \|\exp(A_i \alpha_i)\|^\delta \exp(-q_i \alpha_i) d\alpha_i \\ &\leq \sum_{i=1}^N \psi_i \int_0^{+\infty} q_i \|\exp(A_i \alpha_i)\|^\delta \exp(-q_i \alpha_i) d\alpha_i \\ &= \sum_{i=1}^N \psi_i \eta_i \leq \eta < +\infty. \end{aligned}$$

This shows (3.2.1). To show (3.2.3), we apply (3.2.1) in the following way:

$$\begin{aligned} E \{ \|\Phi(t, t_i)\|^\delta | \mathcal{F}^i \} &\leq E \{ \|\Phi(t, t_k)\|^\delta \dots \|\Phi(t_{i+1}, t_i)\|^\delta | \mathcal{F}^i \} \\ &= E \{ E \{ \dots E \{ E \{ \|\Phi(t, t_k)\|^\delta | \mathcal{F}^k \} \|\Phi(t_k, t_{k-1})\|^\delta | \mathcal{F}^{k-1} \} \\ &\quad \dots | \mathcal{F}^{i+1} \} \|\Phi(t_{i+1}, t_i)\|^\delta | \mathcal{F}^i \} \\ &\stackrel{(3.2.1)}{\leq} \eta^{k+1-i}, \quad P_\psi - \text{a.s.} \end{aligned}$$

□

Proof of Theorem 3.2.2. It is very easy to show that the exponential δ -moment stability implies the stochastic δ -moment stability. So it is sufficient to prove that the stochastic δ -moment stability implies the δ -moment

stability, the δ -moment stability implies the exponential δ -moment stability, and the exponential δ -moment stability implies almost sure stability.

Stochastic δ -moment stability implies the δ -moment stability: If (3.1.1) is stochastically δ -moment stable, then $\int_0^\infty E\|x(t)\|^\delta dt < +\infty$. It follows that

$$\int_0^{+\infty} E\|x(t)\|^\delta dt = \sum_{k=0}^{\infty} \int_k^{k+1} E\|x(t)\|^\delta dt = \sum_{k=0}^{+\infty} \int_0^1 E\|x(k+t)\|^\delta dt < +\infty \quad (3.2.4)$$

Let $\beta_k(t) = E\|x(k+t)\|^\delta \geq 0$, and let $\beta(t) = \sum_{k=0}^{+\infty} \beta_k(t)$. From (3.2.4), we obtain that $\lambda(\beta(t) = +\infty) = 0$ with λ the Lebesgue measure on $[0,1]$ and henceforth $\beta(t) < \infty$ almost everywhere on $[0,1]$ with respect to λ . Therefore, there exists a $t^* \in [0,1]$ such that $\beta(t^*) < +\infty$, i.e., $\sum_0^\infty \beta_k(t^*) < +\infty$, so we have $\lim_{k \rightarrow \infty} \beta_k(t^*) = 0$. Now, consider that for any t large enough, we have ($[t]$ denotes the integral part of t) from (i) of Lemma 3.2.3 that $\|\Phi(t, [t] - 1 + t^*)\|^\delta \leq \exp(\kappa(t - [t] + 1 - t^*)) \leq \exp(2\kappa) \stackrel{\text{def}}{=} M$. Then

$$\begin{aligned} E\|x(t)\|^\delta &= E\|\Phi(t, 0)x_0\|^\delta = E\|\Phi(t, [t] - 1 + t^*)\Phi([t] - 1 + t^*, 0)x_0\|^\delta \\ &\leq E\{\|\Phi(t, [t] - 1 + t^*)\|^\delta \|\Phi([t] - 1 + t^*, 0)x_0\|^\delta\} \\ &\leq ME\|\Phi([t] - 1 + t^*, 0)x_0\|^\delta \\ &= ME\|x([t] - 1 + t^*)\|^\delta = M\beta_{[t]-1}(t^*) \longrightarrow 0, \quad (t \rightarrow \infty) \end{aligned}$$

This means that (3.1.1) is δ -moment stable.

δ -moment stability implies exponential δ -moment stability: Suppose that (3.1.1) is δ -moment stable. Then, for any $\psi \in \Xi$ $\lim_{t \rightarrow +\infty} E_\psi\{\|\Phi(t, 0)\|^\delta\} = 0$. This implies that there exist sufficiently large $m > 0$ so that

$$\max_{i \in S} E_{e_i}\{\|\Phi(t_m, 0)\|^\delta\} \leq \rho < 1$$

and it follows that

$$\max_{\zeta \in \Xi} E_{\zeta} \{ \|\Phi(t_m, 0)\|^\delta \} \leq \rho < 1. \quad (3.2.5)$$

From (3.2.5), we have for any $\xi \in \Xi$ and $l \geq 0$,

$$E_{\xi} \{ \|\Phi(t_{l+m}, t_l)\|^\delta | \mathcal{F}^l \} \leq \rho, \quad P_{\xi} - \text{a.s.} \quad (3.2.6)$$

For large k , let $k = pm + q$ with $0 \leq q \leq m - 1$. For $t \in [t_k, t_{k+1})$, we have

$$\begin{aligned} E_{\psi} \{ \|\Phi(t, 0)\|^\delta \} &\leq E_{\psi} \{ \|\Phi(t, t_{pm})\|^\delta \prod_{i=1}^p \|\Phi(t_{im}, t_{(i-1)m})\|^\delta \} \\ &= E_{\psi} \{ E_{\psi} \dots E_{\psi} \{ E_{\psi} \{ \|\Phi(t, t_{pm})\|^\delta | \mathcal{F}^{pm} \} \|\Phi(t_{pm}, t_{(p-1)m})\|^\delta | \mathcal{F}^{(p-1)m} \} \\ &\quad \dots | \mathcal{F}^m \} \|\Phi(t_m, t_0)\|^\delta \} \\ &\stackrel{(3.2.3)}{\leq} \eta^{q+1} E_{\psi} \{ E_{\psi} \dots E_{\psi} \{ \|\Phi(t_{pm}, t_{(p-1)m})\|^\delta | \mathcal{F}^{(p-1)m} \} \\ &\quad \dots | \mathcal{F}^m \} \|\Phi(t_m, t_0)\|^\delta \} \\ &\stackrel{(3.2.6)}{\leq} \eta^{q+1} \rho^p = \eta^{q+1} \rho^{-m/q} (\rho^{1/m})^k \stackrel{\text{def}}{=} M' \rho_1^k. \end{aligned} \quad (3.2.7)$$

where $0 \leq M' < +\infty$ and $0 \leq \rho_1 < 1$. Since there exist constants $0 < \alpha \leq \beta < +\infty$ such that

$$\alpha \leq \lim_{k \rightarrow +\infty} \frac{t_k}{k} = \lim_{k \rightarrow +\infty} \frac{1}{k} \sum_{i=0}^{k-1} \tau_i \leq \beta, \quad \text{a.s.}$$

we have from (3.2.7) that

$$\frac{1}{t} \log E_{\psi} \{ \|\Phi(t, 0)\|^\delta \} \leq \frac{1}{t} \log M' + \frac{k}{t} \log \rho_1 \leq \frac{1}{t} \log M' + \frac{k}{t} \log \rho_1.$$

Taking the limit $t \rightarrow +\infty$, we arrive at

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log E_{\psi} \{ \|\Phi(t, 0)\|^\delta \} \leq \alpha^{-1} \log \rho_1 < 0.$$

This implies that the system (3.1.1) is exponentially δ -moment stable. Summarizing the above, we have established the equivalence of the δ -moment

stability, the exponentially δ -moment stability and the stochastic δ -moment stability.

Exponential δ -moment stability implies the almost sure stability. Suppose that (3.1.1) is exponentially δ -moment stable, i.e., for any $\psi \in \Xi$, there exist $M'' > 0$ and $0 < \rho_0$ such that

$$E_\psi\{\|\Phi(t, 0)\|^\delta\} \leq M'' \exp(-\rho_0 t), \quad \forall t \geq 0. \quad (3.2.8)$$

To show almost sure stability, by (i) of Lemma 3.2.3, it is enough to show that for $n \in \mathcal{Z}$ integer,

$$\lim_{n \rightarrow +\infty} \|\Phi(n, 0)\| = 0, \quad P_\psi - \text{a.s.} \quad (3.2.9)$$

Let $g \stackrel{\text{def}}{=} \overline{\lim}_{n \rightarrow +\infty} \|\Phi(n, 0)\|$. For any $\varepsilon > 0$, by Markov inequality we have

$$\begin{aligned} P_\psi\{g > \varepsilon\} &= P_\psi\left\{\bigcap_{m \geq 0} \bigcup_{n \geq m} \{\|\Phi(n, 0)\| > \varepsilon\}\right\} \\ &\leq P_\psi\left\{\bigcup_{n \geq m} \{\|\Phi(n, 0)\| > c\}\right\} \leq \sum_{n=m}^{+\infty} P_\psi\{\|\Phi(n, 0)\| > \varepsilon\} \\ &\leq \sum_{n=m}^{+\infty} \frac{1}{\varepsilon^\delta} E_\psi\{\|\Phi(n, 0)\|^\delta\} \\ &\leq \frac{1}{\varepsilon^\delta} \sum_{n=m}^{+\infty} M'' (\exp(-\rho_0))^n \rightarrow 0, \quad m \rightarrow +\infty. \end{aligned}$$

Therefore, $P_\psi\{g = 0\} = 1$ and (3.2.9) holds. \square

This theorem is very important because the moment stability of jump linear system with a finite state Markov form process has special features, from this theorem we can obtain the following characterization for the δ -moment stability, define the top δ -moment Lyapunov exponent:

$$g(\delta) = \overline{\lim}_{t \rightarrow +\infty} \frac{1}{t} \log E\|\Phi(t, t_0)\|^\delta$$

then the system (3.1.1) is δ -moment stable if and only if $g(\delta) < 0$. This result is not true in general. For certain systems, even though the system is δ -moment stable, $g(\delta) = 0$ is possible, Theorem 3.2.2 excludes this case.

In [74], Feng et al studied the relationship between almost sure and δ -moment stability and showed that for one-dimensional jump linear system (3.1.1) with a Markov form process which has a single ergodic class, the δ -moment stability regions is monotonically increasing and tend to the almost sure stability region as $\delta \downarrow 0^+$. They further conjectured that this is also true for the high dimensional cases. In the following, we will verify this conjecture. In the rest of this section, we assume that σ_t has a single ergodic class $E \subset S$ without specification. Let $\pi \in \Xi$ denote the unique invariant (ergodic) distribution of σ_t supported on E . Define the top (sample) Lyapunov exponent as

$$\alpha_\psi(\omega) = \overline{\lim}_{t \rightarrow +\infty} \log \|\Phi(t, 0)\|, \quad (3.2.10)$$

where in (3.2.10), ψ signifies the dependence on the initial distribution $\psi \in \Xi$. Let $\alpha \in \mathcal{R}$ be the top Oseledec's exponent [100], i.e.,

$$\alpha = \lim_{t \rightarrow +\infty} \log \|\Phi(t, 0)\|, \quad P_\pi - \text{a.s.} \quad (3.2.11)$$

which is a nonrandom constant. We have the following basic result:

Proposition 3.2.4:

Suppose that σ_t has a single ergodic class. Then, For any $\psi \in \Xi$,

- (i) $\alpha_\psi(\omega) = \alpha, \quad P_\psi\text{-a.s.}$

$$(ii) \alpha = a^{-1} \overline{\lim}_{k \rightarrow +\infty} \frac{1}{k} \log \|\Phi(t_k, 0)\| = a^{-1} \lim_{k \rightarrow +\infty} \frac{1}{k} E_\pi \{\log \|\Phi(t_k, 0)\|\},$$

P_ψ -a.s. with

$$a \stackrel{\text{def}}{=} \lim_{k \rightarrow +\infty} \frac{1}{k} t_k > 0, \quad P_\pi - \text{a.s.} \quad (3.2.12)$$

Proof of Proposition: (i). It is enough to show that $\alpha_{e_i}(\omega) = \alpha$, P_{e_i} -a.s. for all $i \in S$. In the case when $i \in E$, the result follows from the nonrandom spectrum theorem [76]. Suppose now $i \in S \setminus E$. Since $|S|$ is finite and E is the only ergodic class, the first entrance time τ of σ_t to E is finite P_{e_i} -a.s.. Since $\Phi(t, \tau)$ is invertible, we have

$$\begin{aligned} \alpha_{e_i}(\omega) &= \overline{\lim}_{t \rightarrow +\infty} \frac{1}{t} \log \|\Phi(t, 0)\| = \overline{\lim}_{t \rightarrow +\infty} \frac{1}{t} \log \|\Phi(t, \tau)\Phi(\tau, 0)\| \\ &= \overline{\lim}_{t \rightarrow +\infty} \frac{1}{t} \log \|\Phi(t, \tau)\|, \quad P_{e_i} - \text{a.s.} \end{aligned} \quad (3.2.13)$$

However, by the time homogeneous property and the preceding argument

$$\overline{\lim}_{t \rightarrow +\infty} \log \|\Phi(t, \tau)\| = \alpha, \quad P_{e_i} - \text{a.s.}, \quad \forall i \in S \quad (3.2.14)$$

(3.2.13) and (3.2.14) yield the result for the case $i \in S \setminus E$.

(ii). Since σ_t has single ergodic class, (3.2.12) holds. The first equality holds trivially, because the sojourn time $\tau_k = t_{k+1} - t_k$ is almost surely bounded and then for $t \in [t_k, t_{k+1})$,

$$\begin{aligned} \alpha &= \overline{\lim}_{t \rightarrow +\infty} \frac{1}{t} \log \|\Phi(t, 0)\| = \overline{\lim}_{t \rightarrow +\infty} \frac{1}{t} \log \|\Phi(t, t_k)\Phi(t_k, 0)\| \\ &= \overline{\lim}_{k \rightarrow +\infty} \left(\frac{t_k}{k}\right)^{-1} \frac{t_k}{t} \frac{1}{k} \log \|\Phi(t_k, 0)\|, \quad P_\psi - \text{a.s.} \end{aligned}$$

We show the second equality next, i.e.,

$$\alpha = a^{-1} \lim_{k \rightarrow +\infty} \frac{1}{k} E_\pi \{\log \|\Phi(t_k, 0)\|\}, \quad P_\psi - \text{a.s.} \quad (3.2.15)$$

First of all, the existence of the limit in (3.2.15) follows from the fact that $a_k \stackrel{\text{def}}{=} E_\pi \{\log \|\Phi(t_k, 0)\|\}$ is a subadditive sequence by the stationarity. Let

$k = pm + q$ with $0 \leq q < m$, then using the sojourn time description of σ_t , we have

$$\begin{aligned} \frac{1}{k} \log \|\Phi(t_k, 0)\| &= \frac{1}{k} \log \|\exp(A_{r_k} \tau_k) \cdots \exp(A_{r_0} \tau_0)\| \\ &\leq \frac{1}{k} \log \|\exp(A_{r_k} \tau_k) \cdots \exp(A_{r_{pm}} \tau_{pm})\| \\ &+ \frac{1}{pm+q} \sum_{j=1}^p \log \|\exp(A_{r_{jm-1}} \tau_{jm-1}) \cdots \exp(A_{r_{(j-1)m}} \tau_{(j-1)m})\| \end{aligned} \quad (3.2.16)$$

Since $\|\exp(A_j t)\| \leq \exp(\kappa t)$ for $t \geq 0$, we have

$$\begin{aligned} &\frac{1}{k} \log \|\exp(A_{r_k} \tau^{(r_k)}) \cdots \exp(A_{r_{pm-1}} \tau^{(r_{pm-1})})\| \\ &\leq \frac{1}{k} \log \|\exp(A_{r_k} \tau^{(r_k)})\| \cdots \|\exp(A_{r_{pm-1}} \tau^{(r_{pm-1})})\| \\ &\leq M \frac{\tau^{(r_{pm+q})} + \cdots + \tau^{(r_{pm-1})}}{k} \xrightarrow{k \rightarrow \infty} 0 \quad P_\psi - \text{a.s.} \end{aligned} \quad (3.2.17)$$

Moreover, since σ_t has a single ergodic class implies that the Markov chain $\{\zeta_k = (r_k, \tau_k) : k \geq 0\}$ has also a single ergodic class, and so does the Markov chain $\{(\zeta_{(j+1)m-1}, \dots, \zeta_{jm}) : j \geq 0\}$. From the Law of Large Numbers, we have

$$\begin{aligned} &\lim_{p \rightarrow +\infty} \frac{1}{p} \sum_{j=1}^p \log \|\exp(A_{r_{jm-1}} \tau_{jm-1}) \cdots \exp(A_{r_{(j-1)m}} \tau_{(j-1)m})\| \\ &= E_\pi \log \|\exp(A_{r_m} \tau_m) \cdots \exp(A_{r_0} \tau_0)\|. \end{aligned} \quad (3.2.18)$$

From (3.2.16)-(3.2.18), we obtain

$$\begin{aligned} &\lim_{k \rightarrow \infty} \frac{1}{k} \log \|\exp(A_{r_k} \tau_k) \cdots \exp(A_{r_0} \tau_0)\| \\ &\leq E_\pi \log \|\exp(A_{r_{m-1}} \tau_{m-1}) \cdots \exp(A_{r_0} \tau_0)\| \end{aligned} \quad (3.2.19)$$

Let

$$\begin{aligned} \gamma &= \lim_{m \rightarrow +\infty} \frac{1}{m} E_\pi \log \|\exp(A_{r_{m-1}} \tau_{m-1}) \cdots \exp(A_{r_0} \tau_0)\| \\ \beta_k &= \frac{1}{k} \log \|\exp(A_{r_{k-1}} \tau_{k-1}) \cdots \exp(A_{r_0} \tau_0)\| \\ w_k &= \frac{1}{k} \sum_{i=0}^{k-1} \log \|\exp(A_{r_i} \tau_i)\| - \beta_k \end{aligned}$$

then $w_k \geq 0$. From the Law of Large Numbers, we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} w_k &= E_\pi \log \|\exp(A_{r_0} \tau_0)\| - \overline{\lim}_{k \rightarrow \infty} \frac{1}{k} \beta_k \\ &\stackrel{(i)}{=} E_\pi \log \|\exp(A_{r_0} \tau_0)\| - a\alpha \quad P_\psi - \text{a.s.} \end{aligned} \quad (3.2.20)$$

On the other hand, by stationarity, we have

$$\begin{aligned} E_\pi w_k &= E_\pi \log \|\exp(A_{r_0} \tau_0)\| - \frac{1}{k} E_\pi \{\log \|\exp(A_{r_{k-1}} \tau_{k-1}) \cdots \exp(A_{r_0} \tau_0)\|\} \\ &\xrightarrow{k \rightarrow \infty} E_\pi \log \|\exp(A_{r_0} \tau_0)\| - \gamma \end{aligned} \quad (3.2.21)$$

From Fatou's Lemma, (3.2.20) and (3.2.21), we arrive at $a\alpha \geq \gamma$. This together with (3.2.19) implies that $a\alpha = \gamma$. The proof of the proposition is completed.

□

Note that $\alpha < 0$ is equivalent to the almost sure exponential stability. Now, we begin to study the relationship between δ -moment stability and almost sure stability. The δ -moment stability region Σ^δ and the almost sure stability region Σ^a in the parameter space are defined by

$$\Sigma^\delta = \{(A(1), \dots, A(N)) : (3.1.1) \text{ is } \delta\text{-moment stable.}\}$$

and

$$\Sigma^a = \{(A(1), \dots, A(N)) : (3.1.1) \text{ is almost surely stable.}\}$$

respectively. We can decompose Σ^a into disjoint union in the form

$$\Sigma^a = \Sigma_-^a \cup \Sigma_0^a$$

with

$$\Sigma_-^a \stackrel{\text{def}}{=} \Sigma^a \cap \{(A(1), \dots, A(N)) : \alpha < 0\}$$

and

$$\Sigma_0^a \stackrel{\text{def}}{=} \Sigma^a \cap \{(A(1), \dots, A(N)) : \alpha = 0\}.$$

Then, Σ_-^a is the almost sure exponential stability region. It is very intuitive that for the “linear” system (3.1.1), the (asymptotic) almost sure stability is equivalent to exponential almost sure stability, i.e., Σ_0^a should be an empty set. However, a rigorous justification seems to be hard to find at the moment. The following is one of our main results:

Theorem 3.2.5:

For the system (3.1.1) with a (general) Markov form process σ_t ,

- (i). For any $0 < \delta_1 \leq \delta_2$, $\Sigma^{\delta_2} \subset \Sigma^{\delta_1} \subset \Sigma^a$ and Σ^{δ_1} is open.
- (ii). If σ_t is irreducible, i.e., $E = S$, then

$$\Sigma_-^a = \lim_{\delta \downarrow 0^+} \Sigma^\delta = \bigcup_{\delta > 0} \Sigma^\delta \subset \Sigma^a. \quad (3.2.22)$$

□

Before we prove this theorem, we would like to comment on the result. This result states that the δ –moment stability region in parameter space becomes larger and larger as δ becomes smaller and smaller and when δ is sufficiently small, the δ –moment stability region is coincident with the almost sure exponential stability region, so the almost sure stability can be studied by investigating the δ –moment stability for sufficiently small δ . To the author’s knowledge, traditionally, the almost sure stability of the system is inferred from the second moment stability. However, as observed in many researches, for example ([74]), the second moment stability is usually conservative for stability, compared with the almost sure stability. It seems to be much more difficult to deal with almost sure (sample) stability properties directly, for example, to compute or estimate the (top) Lyapunov exponent α . This theorem provides

an alternative approach to study the almost sure stability. Indeed, as we will see later, δ -moment stability can be used to give better testable criteria than second moment stability for almost sure (sample) stability.

To prove Theorem 3.2.5, we need a few basic results. The key result used is the following Large Deviation Theorem:

Lemma 3.2.6: (Large Deviation Theorem)

Suppose that σ_t is irreducible with a unique invariant distribution π . For any fixed $m > 1$, let $\Delta = E_\pi \{\log \|\Phi(t_m, t_0)\|\}$. Then, for any $\varepsilon > 0$, there exist $M, \delta > 0$ such that

$$P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq \Delta + \varepsilon \right) \leq M \exp(-\delta p). \quad (3.2.23)$$

□

Proof of Lemma. Define

$$Y_p = \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\|, \quad (3.2.24)$$

$$c_p(\delta) = \frac{1}{p} \log E_\pi \{ \exp(\delta Y_p) \}.$$

Then we have

$$c_p(\delta) = \log E_\pi \prod_{j=0}^{p-1} \|\Phi(t_{(j+1)m}, t_{jm})\|^\delta.$$

We first show that $c(\delta) = \lim_{p \rightarrow +\infty} c_p(\delta)$ exists for all $\delta \in \mathcal{R}$ (possibly be $+\infty$).

For any sequence $l = (i_0, i_1, \dots, i_{m-1}) \in S^m$ and $\delta \in \mathcal{R}$, write

$$\begin{aligned} \Lambda(l; \delta) &= \Lambda(i_0, i_1, \dots, i_{m-1}; \delta) \\ &= E_\pi \{ \|\Phi(t_m, t_0)\|^\delta \mid r_0 = i_0, r_1 = i_1, \dots, r_{m-1} = i_{m-1} \} \end{aligned}$$

Then, using the sojourn time description of σ_t , we have

$$\begin{aligned}\Lambda(l; \delta) &= \Lambda(i_0, i_1, \dots, i_{m-1}; \delta) \\ &= \int \|e^{A_{i_{m-1}}\beta_{m-1}} e^{A_{i_{m-2}}\beta_{m-2}} \dots e^{A_{i_0}\beta_0}\|^\delta \\ &\quad \mu_{i_{m-1}}(d\beta_{m-1})\mu_{i_{m-2}}(d\beta_{m-2}) \dots \mu_{i_0}(d\beta_0) > 0\end{aligned}\tag{3.2.25}$$

where in (3.2.25), μ_i denotes the measure induced by the exponential distribution function with a parameter q_i . Clearly, as a function of δ , $\Lambda(l; \delta)$ is differentiable function on its domain of definition, which is a nonempty open interval contains $\delta = 0$. Since σ_t is irreducible, the embedded chain r_k is therefore irreducible. It follows that the chain $\tilde{r}_j = (r_{(j+1)m-1}, \dots, r_{jm})$ for $j = 0, 1, \dots$ is also irreducible with a state space

$$\tilde{S} = \{(i_0, \dots, i_{m-1}) \in S^m : P_\pi(r_{k+m-1} = i_{m-1}, \dots, r_k = i_0) > 0, \exists k \geq 0\}.$$

Suppose that $|\tilde{S}| = \tilde{N}$ and that we have ordered the states in a certain way so that for $k \in \{1, 2, \dots, \tilde{N}\}$ $\Lambda(k; \delta)$ is defined obviously. Now, consider that

$$\begin{aligned}E_\pi \left\{ \prod_{j=0}^{p-1} \|\Phi(t_{(j+1)m}, t_{jm})\|^\delta \right\} &= \sum_{i_0, \dots, i_{pm-1}} \pi_{i_0} p_{i_0 i_1} \dots p_{i_{pm-2} i_{pm-1}} \\ &\quad \times E_\pi \left\{ \prod_{j=0}^{p-1} \|\Phi(t_{(j+1)m}, t_{jm})\|^\delta \mid r_{pm-1} = i_{pm-1}, \dots, r_0 = i_0 \right\} \\ &= \sum_{i_0, \dots, i_{pm-1}} \pi_{i_0} p_{i_0 i_1} \dots p_{i_{pm-2} i_{pm-1}} \Lambda(i_0, i_1, \dots, i_{m-1}; \delta) \\ &\quad \times \Lambda(i_m, i_{m+1}, \dots, i_{2m-1}; \delta) \dots \Lambda(i_{(p-1)m}, i_{(p-1)m+1}, \dots, i_{pm-1}; \delta) \\ &= E_{\hat{\pi}} \left\{ \prod_{j=0}^{p-1} \Lambda(\tilde{r}_j; \delta) \right\} = \sum_{l_0, \dots, l_{p-1}} \hat{\pi}_{l_0} \hat{p}_{l_0 l_1} \dots \hat{p}_{l_{p-2} l_{p-1}} \prod_{l=0}^{p-1} \Lambda(l; \delta) \\ &= \sum_{l_0, \dots, l_{p-1}} (\hat{\pi}_{j_0} \Lambda(j_0; \delta)) (\hat{p}_{j_0 j_1} \Lambda(j_1, \delta)) \dots (\hat{p}_{j_{p-2} j_{p-1}} \Lambda(j_{p-1}; \delta)) \\ &= x^T (B(\delta))^{p-2} y,\end{aligned}\tag{3.2.26}$$

where in (3.2.26) $\hat{P} = (\hat{p}_{jl})_{\tilde{N} \times \tilde{N}}$ is the transition matrix of \tilde{r}_k and $\hat{\pi} = (\hat{\pi}_1, \dots, \hat{\pi}_{\tilde{N}})$ is its initial distribution induced by π , and

$$\begin{aligned} 0 \neq x^T &\stackrel{\text{def}}{=} (\hat{\pi}_1 \Lambda(1; \delta), \dots, \hat{\pi}_{\tilde{N}} \Lambda(\tilde{N}; \delta)) \geq 0 \\ y^T &= (1, 1, \dots, 1) > 0 \\ B(\delta) &= (\hat{p}_{jl} \Lambda(l; \delta))_{\tilde{N} \times \tilde{N}} \geq 0. \end{aligned} \tag{3.2.27}$$

Since the transition matrix \hat{P} is irreducible and $\Lambda(l; \delta) > 0$, we see that $B(\delta)$ is an irreducible nonnegative matrix. Next, we show that with $c_p(\delta) = \log x^T B(\delta)^{p-2} y$, $c(\delta) = \lim_{p \rightarrow \infty} \frac{1}{p} c_p(\delta)$ exists and is differentiable at any point where it is defined. Since $B(\delta)$ is irreducible nonnegative matrix, from matrix theory [148], there exists a positive vector v such that $B(\delta)v = \rho(B(\delta))v$. Since v and y are positive vectors, there exists positive numbers $L > 0$ and $U > 0$ such that $Lv \leq y \leq Uv$ and $x^T v > 0$ because of the nonnegativity of $x \neq 0$. Thus, we have

$$\begin{aligned} c_p(\delta) &\leq \log x^T B^{p-2}(Uv) = \log \{U x^T \rho(B(\delta))^{p-2} v\} \\ &= \log U + (p-2) \log \rho(B(\delta)) + \log x^T v, \end{aligned}$$

from which we obtain that

$$\overline{\lim}_{p \rightarrow \infty} \frac{1}{p} c_p(\delta) \leq \log \rho(B(\delta)). \tag{3.2.28}$$

In a similar way, we have

$$c_p(\delta) \geq \log L + (p-2) \log \rho(B(\delta)) + \log x^T v,$$

from which we have

$$\underline{\lim}_{p \rightarrow \infty} \frac{1}{p} c_p(\delta) \geq \log \rho(B(\delta)). \tag{3.2.29}$$

(3.2.28) and (3.2.29) yields

$$c(\delta) \stackrel{\text{def}}{=} \lim_{p \rightarrow \infty} \frac{1}{p} c_p(\delta) = \log \rho(B(\delta)).$$

Due to the continuity of $B(\delta)$, $\mathcal{D}(c) \stackrel{\text{def}}{=} \{\delta \in \mathcal{R} : c(\delta) < +\infty\}$ is a nonempty open interval containing 0, and c is a closed convex function. Furthermore, because $B(\delta)$ is irreducible, $\rho(B(\delta))$ is simple [148]. Due to the differentiability of $B(\delta)$, it follows that $c(\delta)$ is differentiable [160]. Therefore, From Theorem IV.1 of [155], we obtain that there exists γ such that for any $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that

$$P_\pi \left(\left| \frac{1}{p} Y_p - \gamma \right| \geq \varepsilon \right) \leq \exp(-\eta p) \quad (3.2.30)$$

for sufficiently large p . However, by the Law of Large Numbers, we should have $\gamma = \Delta$. Thus, (3.2.23) follows from (3.2.30) directly. \square

Lemma 2.7:

Suppose that σ_t is irreducible. If the top Lyapunov (Oseledec) exponent α is negative, then there exists a $\delta > 0$, such that $\lim_{t \rightarrow +\infty} E_\psi \|\Phi(t, 0)\|^\delta = 0$ for any $\psi \in \Xi$ \square

Proof of lemma: Since σ_t is irreducible, we have $\pi = (\pi_1, \dots, \pi_N) > 0$. Since $E_\psi \{\cdot\} = \sum_{i=1}^N \psi_i E_{e_i} \{\cdot\}$, it is enough to show $\lim_{t \rightarrow +\infty} E_\pi \{\|\Phi(t, t_0)\|^\delta\} = 0$.

From Proposition 3.2.4, there exists $\varepsilon_1 > 0$ and large $m > 1$ such that

$$E_\pi \{\log \|\Phi(t_k, t_0)\|\} \leq m(a\alpha + \varepsilon_1) < 0. \quad (3.2.31)$$

For $d < 0$ to be determined, let $k = pm + q$ for sufficiently large k with $0 \leq q < m$, consider that

$$\begin{aligned} & P_\pi \left(\frac{1}{k} \log \|\Phi(t_k, t_1)\| \geq d \right) \\ &= P_\pi \left(\frac{1}{k} \log \|\Phi(t_k, t_{pm}) \Phi(t_{pm}, t_{(p-1)m+1}) \cdots \Phi(t_m, t_1)\| \geq d \right) \end{aligned}$$

$$\begin{aligned}
&\leq P_\pi \left(\frac{1}{k} \log \|\Phi(t_k, t_{pm})\| \|\Phi(t_{pm}, t_{(p-1)m+1})\| \cdots \|\Phi(t_m, t_1)\| \geq d \right) \\
&= P_\pi \left(\frac{1}{k} \log \|\Phi(t_k, t_{pm})\| + \frac{1}{k} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq d \right) \\
&= P_\pi \left(\frac{1}{k} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq -\frac{1}{p} \log \|\Phi(t_k, t_{pm})\| + \frac{k}{p} d \right)
\end{aligned} \tag{3.2.32}$$

Since $\|\Phi(t, t_{pm})\| \leq \exp(\kappa(t_k - t_{pm})) \leq \exp(\kappa \sum_{j=1}^q \tau_{k-j})$, we have

$$\begin{aligned}
E_\pi \{ \|\Phi(t_k, t_{pm})\|^{\delta_1} \} &\leq E_\pi \{ \exp(\delta_1 \kappa \sum_{j=1}^q \tau_{k-j}) \} \\
&\leq \left(\int_0^{+\infty} \exp(\delta_1 \kappa t) \tilde{q} \exp(-\tilde{q}t) dt \right)^q < M_1 < +\infty
\end{aligned}$$

for sufficiently small $\delta_1 > 0$, where $\tilde{q} = \min_{i \in S} q_i > 0$. It follows from Chebyshev's inequality that for $\varepsilon_2 > 0$ small,

$$\begin{aligned}
P_\pi \left(\frac{1}{p} \log \|\Phi(t_k, t_{pm})\| \geq \varepsilon_2 \right) &= P_\pi \left(\|\Phi(t_k, t_{pm})\| \geq e^{p\varepsilon_2} \right) \\
&\leq \frac{E_\pi \|\Phi(t_k, t_{pm})\|^{\delta_1}}{e^{p\varepsilon_2 \delta_1}} \leq M_1 (e^{-\delta_1 \varepsilon_2})^p.
\end{aligned} \tag{3.2.33}$$

From (3.2.32), we obtain

$$\begin{aligned}
&P_\pi \left(\frac{1}{k} \log \|\Phi(t_k, t_1)\| \geq d \right) \\
&\leq P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq -\frac{1}{p} \log \|\Phi(t_k, t_{pm})\| + \frac{k}{p} d \right)
\end{aligned}$$

$$\begin{aligned}
&= P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq -\frac{1}{p} \log \|\Phi(t_k, t_{pm})\| + \frac{k}{p}d \right. \\
&\quad \left. \mid -\frac{1}{p} \log \|\Phi(t_k, t_{pm})\| \geq -\varepsilon_2 \right) \times P_\pi \left(-\frac{1}{p} \log \|\Phi(t_k, t_{pm})\| \geq -\varepsilon_2 \right) \\
&+ P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq -\frac{1}{p} \log \|\Phi(t_k, t_{pm})\| + \frac{k}{p}d \right. \\
&\quad \left. \mid -\frac{1}{p} \log \|\Phi(t_k, t_{pm})\| < -\varepsilon_2 \right) \times P_\pi \left(-\frac{1}{p} \log \|\Phi(t_k, t_{pm})\| < -\varepsilon_2 \right) \\
&\leq P_\pi \left(\frac{1}{p} \log \|\Phi(t_k, t_{pm})\| \geq \varepsilon_2 \right) \\
&\quad + P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq -\varepsilon_2 + \frac{k}{p}d \right) \\
&\stackrel{(3.2.33)}{\leq} M (e^{-\varepsilon_2 \delta_1})^p + P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq -\varepsilon_2 + \frac{k}{p}d \right) \tag{3.2.34}
\end{aligned}$$

Note that $kd/p = md + qd/p$ and $(a\alpha + \varepsilon_1) < 0$. Select ε_2 above small and $\varepsilon_3 > 0$ small and sufficiently large $p_0 > 0$ such that

$$\begin{aligned}
d &\stackrel{\text{def}}{=} (a\alpha + \varepsilon_1) + \frac{\varepsilon_2 + \varepsilon_3}{m} < 0 \\
\varepsilon_4 &\stackrel{\text{def}}{=} \varepsilon_3 + \frac{q}{p}d > 0 \quad \forall p \geq p_0.
\end{aligned} \tag{3.2.35}$$

Then, from (3.2.31) and (3.2.34), we have

$$\begin{aligned}
&P_\pi \left(\frac{1}{k} \log \|\Phi(t_k, t_1)\| \geq d \right) \\
&\stackrel{(3.2.34)}{\leq} M (e^{-\delta_1 \varepsilon_2})^p + P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq -\varepsilon_2 + \frac{q}{p}d \right) \\
&= P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq m(a\alpha + \varepsilon_1) + \varepsilon_3 + \frac{q}{p}d \right) \\
&\quad + M (e^{-\varepsilon_2 \delta_1})^p
\end{aligned}$$

$$\begin{aligned}
(3.2.31) \quad & \leq P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq E_\pi \{ \log \|\Phi(t_m, t_0)\| \} + \varepsilon_4 \right) \\
& + M(e^{-\varepsilon_2 \delta_1})^p, \quad \forall p \geq p_0.
\end{aligned} \tag{3.2.36}$$

Applying Lemma 3.2.6, we have that there exist $M', \varepsilon' > 0$ such that

$$P_\pi \left(\frac{1}{k} \log \|\Phi(t_k, t_0)\| \geq d \right) \leq M' e^{-\varepsilon' k}, \quad \forall k \geq 0 \tag{3.2.37}$$

Next, we want to prove that there exists $\delta_2 > 0$ small such that

$$\lim_{k \rightarrow +\infty} \int_{(t_k \geq k)} e^{\delta_2 \kappa t_k} P_\pi(d\omega) = 0. \tag{3.2.38}$$

In fact, for $\delta_2 > 0$ sufficiently small,

$$\begin{aligned}
& \int_{(t_k \geq k)} e^{\delta_2 \kappa t_k} P_\pi(d\omega) = \int_{(t_k \geq k)} \exp(\delta_2 \kappa \sum_{j=0}^{k-1} \tau_j) P_\pi(d\omega) \\
& = \sum_{i_0, \dots, i_{k-1}} \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{k-2} i_{k-1}} \\
& \int_{(x_1 + \dots + x_k \geq k, x_j \geq 0)} e^{\delta_2 \kappa (x_0 + \dots + x_{k-1})} q_{i_0} e^{-q_{i_0} x_1} \cdots q_{i_{k-1}} e^{-q_{i_{k-1}} x_{k-1}} dx_0 \cdots dx_{k-1} \\
& \leq \sum_{i_0, \dots, i_{k-1}} \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{k-2} i_{k-1}} \int_k^{+\infty} q_{i_0} e^{-(q_{i_0} - \delta_2 \kappa) x_0} dx_0 \\
& \quad \times \prod_{j=1}^{k-1} \int_0^{+\infty} q_{i_j} e^{-(q_{i_j} - \delta_2 \kappa) x_j} dx_j \\
& = \sum_{i_0, \dots, i_{k-1}} \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{k-2} i_{k-1}} e^{-(q_{i_0} - \delta_2 \kappa) k} \prod_{j=0}^{k-1} \frac{q_{i_j}}{q_{i_j} - \delta_2 \kappa} \longrightarrow 0, \quad k \rightarrow +\infty.
\end{aligned}$$

This shows (3.2.38).

Now we are in the position to prove that $\lim_{t \rightarrow +\infty} E_\pi \|\Phi(t, t_0)\|^{\delta'} = 0$ for small $\delta' > 0$. We first show that for small $\delta > 0$,

$$\lim_{k \rightarrow +\infty} E_\pi \|\Phi(t_k, t_0)\|^\delta = 0. \tag{3.2.29}$$

Let

$$\mathcal{A} = \left(\frac{1}{k} \log \|\Phi(t_k, t_0)\| \geq d \right), \quad \mathcal{A}^c = \left(\frac{1}{k} \log \|\Phi(t_k, t_0)\| < d \right)$$

where $d < 0$ is chosen previously. For $\delta > 0$ to be determined, consider that

$$\begin{aligned} E_\pi \|\Phi(t_k, t_0)\|^\delta &= \int_{\mathcal{A}} \|\Phi(t_k, t_0)\|^\delta P_\pi(d\omega) + \int_{\mathcal{A}^c} \|\Phi(t_k, t_0)\|^\delta P_\pi(d\omega) \\ &\leq \int_{\mathcal{A}} e^{\delta\kappa t_k} P_\pi(d\omega) + e^{k\delta d} \\ &= \int_{\mathcal{A} \cap (t_k \leq k)} e^{\delta\kappa t_k} P_\pi(d\omega) + \int_{\mathcal{A} \cap (t_k \geq k)} e^{\delta\kappa t_k} P_\pi(d\omega) + e^{k\delta d} \\ &\leq \int_{\mathcal{A} \cap (t_k \leq k)} e^{\delta\kappa t_k} P_\pi(d\omega) + \int_{(t_k \geq k)} e^{\delta\kappa t_k} P_\pi(d\omega) + e^{k\delta d} \\ &\leq e^{\delta\kappa k} P_\pi(\mathcal{A}) + \int_{(t_k \geq k)} e^{\delta\kappa t_k} P_\pi(d\omega) + e^{k\delta d} \\ &\stackrel{(3.2.37)}{\leq} M' e^{-(\varepsilon' - \delta\kappa)k} + \int_{(t_k \geq k)} e^{\delta\kappa t_k} P_\pi(d\omega) + e^{k\delta d} \end{aligned} \tag{3.2.40}$$

Select $\delta > 0$ such that $\varepsilon' - \delta\kappa > 0$. Then, as $k \rightarrow +\infty$ the first and the last terms go to zero exponentially and the second term goes to zero as well from (3.2.38). This establishes (3.2.39). Now, for $t \in [t_k, t_{k+1})$,

$$\begin{aligned} E_\pi \{\|\Phi(t, t_0)\|^{\delta'}\} &= E_\pi \{\|\Phi(t, t_k)\Phi(t_k, t_0)\|^{\delta'}\} \leq E_\pi \{\|\Phi(t, t_k)\|^{\delta'} \|\Phi(t_k, t_0)\|^{\delta'}\} \\ &\leq E_\pi \{e^{\delta'\kappa\tau_k} \|\Phi(t_k, t_0)\|^{\delta'}\} = E_\pi \{E_\pi \{e^{\delta'\kappa\tau_k} | \mathcal{F}^k\} \|\Phi(t_k, t_0)\|^{\delta'}\} \end{aligned} \tag{3.2.41}$$

Select $0 < \delta' < \delta$ satisfying $\delta'\kappa - \tilde{q} < 0$ such that from the Markov property, we have

$$E_\pi \{e^{\delta'\kappa\tau_k} | \mathcal{F}^k\} \leq \int_0^{+\infty} e^{\delta'\kappa t} \tilde{q} e^{-\tilde{q}t} dt < M'' < +\infty \quad (\text{a.s.}) \tag{3.2.42}$$

Note that by Lyapunov inequality, $0 < \delta' < \delta$ implies that

$$(E_\pi \{\|\Phi(t_k, t_0)\|^{\delta'}\})^{1/\delta'} \leq (E_\pi \{\|\Phi(t_k, t_0)\|^\delta\})^{1/\delta}. \tag{3.2.43}$$

Then, (3.2.41)-(3.2.43) yields

$$E_\pi\{\|\Phi(t, t_0)\|^{\delta'}\} \leq M'' E_\pi\{\|\Phi(t_k, t_0)\|^{\delta'}\} \leq M'' (E_\pi\{\|\Phi(t_k, t_0)\|^\delta\})^{\delta'/\delta}$$

Taking the limit $k \rightarrow +\infty$ (or $t \rightarrow +\infty$), $\lim_{t \rightarrow +\infty} E_\pi\{\|\Phi(t, t_0)\|^{\delta'}\} = 0$ follows from (3.2.39). \square

Now, we are ready to prove Theorem 3.2.5.

Proof of Theorem 2.5: For (i), by Lyapunov inequality, for $0 < \delta_1 < \delta_2$, we have

$$(E_\psi\{\|\Phi(t, 0)\|^{\delta_1}\})^{1/\delta_1} \leq (E_\psi\{\|\Phi(t, 0)\|^{\delta_2}\})^{1/\delta_2}.$$

Thus, $\Sigma^{\delta_2} \subset \Sigma^{\delta_1}$. By Theorem 3.2.2, we know that $\Sigma^{\delta_1} \subset \Sigma^a$. We need to show that Σ^δ is open for any $\delta > 0$. Suppose $(A(1), \dots, A(N)) \in \Sigma^\delta$. Then, for system (3.1.1) with mode matrices $(A(1), \dots, A(N))$, the system is δ -moment stable. Then, for any $0 < \rho < 1$ given, there is a sufficiently large m such that

$$\zeta_i(A(1), \dots, A(N)) \stackrel{\text{def}}{=} E_{e_i}\{\|\Phi(t_m, t_0)\|^\delta\} < \rho, \quad \forall 1 \leq i \leq N. \quad (3.2.44)$$

Since

$$\begin{aligned} \zeta_i(A(1), \dots, A(N)) &= \sum_{i_1, \dots, i_{m-1}} p_{i_1} \cdots p_{i_{m-2} i_{m-1}} \\ &\int \|e^{A_{i_{m-1}} t_{m-1}} \cdots e^{A_{i_1} t_1}\|^\delta \mu_{i_{m-1}}(dt_{m-1}) \cdots \mu_{i_1}(dt_1) \end{aligned}$$

is continuous, for $(\hat{A}(1), \dots, \hat{A}(N))$ in a neighbourhood of $(A(1), \dots, A(N))$, (3.2.44) still holds (for (3.1.1) with mode matrices $(\hat{A}(1), \dots, \hat{A}(N))$). Consider that for $t \in [t_k, t_{k+1})$ with sufficiently large k and $k = pm + q$ with $0 \leq q < m$, from Lemma 3.2.3, we have for any $\psi \in \Xi$,

$$\begin{aligned}
E_\psi\{\|\Phi(t, t_0)\|^\delta\} &\leq E_\psi\{\|\Phi(t, t_{pm})\|^\delta \prod_{j=0}^{p-1} \|\Phi(t_{(j+1)m}, t_{jm})\|^\delta\} \\
&= E_\psi\{E_\psi\{\|\Phi(t, t_{pm})\|^\delta | \mathcal{F}^{pm}\} \prod_{j=0}^{p-1} \|\Phi(t_{(j+1)m}, t_{jm})\|^\delta\} \\
&\leq \eta^{(q+1)} E_\psi\{E_\psi\{\|\Phi(t_{pm}, t_{(p-1)m})\|^\delta | \mathcal{F}^{(p-1)m}\} \prod_{j=0}^{p-2} \|\Phi(t_{(j+1)m}, t_{jm})\|^\delta\} \\
&\leq \eta^{(q+1)} \rho E_\psi\{\prod_{j=0}^{p-2} \|\Phi(t_{(j+1)m}, t_{jm})\|^\delta\} \leq \dots \leq \eta^{(q+1)} \rho^p
\end{aligned} \tag{3.2.45}$$

In arriving at (3.2.45), we have used the time homogeneous and Markov properties and where in (3.2.45) η is given in Lemma 3.2.3. It follows from (3.2.45) that $\lim_{t \rightarrow +\infty} E_\psi\{\|\Phi(t, t_0)\|^\delta\} = 0$. This shows that (3.1.1) with mode matrices $(\hat{A}(1), \dots, \hat{A}(N))$ is also δ -moment stable. Thus, Σ^δ is open.

For (ii), note that $\Sigma_-^a \subset \cup_{\delta>0} \Sigma^\delta$ by Lemma 3.2.7 and $\cup_{\delta>0} \Sigma^\delta \subset \Sigma^a$ by Theorem 3.2.2. Hence, we have

$$\Sigma_-^a = \lim_{\delta \downarrow 0^+} \Sigma^\delta = \cup_{\delta>0} \Sigma^\delta \subset \Sigma^a.$$

□

Theorem 3.2.5 characterizes the relationship between almost sure stability and δ -moment stability for continuous-time jump linear system (3.1.1). This result is similar to the result presented in last section. The following example is adopted from [74], which illustrates Theorem 3.2.5 graphically.

Example 3.2.8:

Consider the two form scalar jump linear system:

$$\begin{cases} \dot{x}_t = a(\sigma_t)x_t, & t \geq 0 \\ x_0 \in \mathcal{R} \end{cases} \quad (3.2.46)$$

where $\sigma_t \in \{1, 2\}$ has the infinitesimal matrix

$$Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix}, \quad 0 < q < +\infty \quad (3.2.47)$$

with unique stationary distribution $\pi = (1/2, 1/2)$. Let $a_i = a(i)$ for $i = 1, 2$. We can directly compute the δ -moment of the solution process using the sojourn time description of σ_t given before. For any $\delta > 0$, we have

$$E\{|x_{t_k}(x_0, \omega)|^\delta\} = |x_0|^\delta \sum_{(i_1, \dots, i_k)} p_{i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \prod_{j=1}^k q_{i_j} \int_0^{+\infty} e^{\delta a(i_j)t} e^{-q_{i_j}t} dt \quad (3.2.48)$$

Since

$$p_i = \frac{1}{2}, \quad p_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i \neq j \end{cases}, \quad q_j = q, \quad \forall i, j \in \{1, 2\},$$

it follows that

$$E\{|x_{t_{2l}}(x_0, \omega)|^\delta\} = \left(\frac{q}{q - \delta a_1}\right)^l \left(\frac{q}{q - \delta a_2}\right)^l |x_0|^\delta. \quad (3.2.49)$$

Together (3.2.48) and (3.2.49) imply that a necessary condition for δ -moment stability is

$$\begin{aligned} \delta a_1 &< q \\ \delta a_2 &< q \\ \left(\frac{q}{q - \delta a_1}\right) \left(\frac{q}{q - \delta a_2}\right) &< 1 \end{aligned} \quad (3.2.50)$$

It is easy to show that (3.2.50) is also sufficient for δ -moment stability.

It can be easily proved that a necessary and sufficient condition for almost sure stability is $a_1 + a_2 < 0$. Figure 2 illustrates the stability regions in the (a_1, a_2) parameter space for almost sure and δ -moment stability. The stability region for δ -moment stability increases as δ decreases and it tends to the region for almost sure stability as δ goes to 0^+ .

□

Fig. 2. Stability regions for system (3.2.46)

3.3. Testable Criteria for Almost Sure Stability

In the above section, we have shown that the almost sure stability of (3.1.1) can be studied by the δ -moment stability. It is expected that δ -moment stability criteria may give refined almost sure stability criteria as $\delta > 0$ becomes sufficiently small. In this section, we develop some testable conditions for almost sure stability via the relationship between δ -moment stability and almost sure stability we established in the previous section. The following theorem is our main result of this section.

Theorem 3.3.1:

If there exists positive definite matrices $P(1), \dots, P(N)$ such that for any i with $1 \leq i \leq N$,

$$\sup_{\|x\|=1} \left(\frac{x'(P(i)A(i) + A'(i)P(i))x}{x'P(i)x} + \sum_{j \neq i} q_{ij} \log \left(\frac{x'P(j)x}{x'P(i)x} \right) \right) < 0, \quad (3.3.1)$$

then there exists a $\delta > 0$ such that (3.1.1) is δ -moment stable and thus, (3.1.1) is almost surely stable.

□

Proof of Theorem. Let $x_t = x(t, x_0)$ and $\sigma_t = \sigma(t)$. The joint process $\{(x_t, \sigma_t) : t \geq 0\}$ is a strong Markov process with right continuous sample path and the (weak) infinitesimal generator

$$\mathcal{L} = Q + \text{diag}\{x'A'(1)\frac{\partial}{\partial x}, \dots, x'A'(N)\frac{\partial}{\partial x}\}. \quad (3.3.2)$$

Consider a Lyapunov function in the form $V(x, i) = (x'P(i)x)^{\delta/2}$ for $x \in \mathcal{R}^n$ and $i \in S$. Then, we have

$$\begin{aligned}
(\mathcal{L}V)(x, i) &= \sum_{j=1}^N q_{ij} (x'P(i)x)^{\delta/2} + x'A'(i) \frac{\delta}{2} (x'A'(i)x)^{\delta/2-1} (2P(i)x) \\
&= (x'P(i)x)^{\delta/2} \left(\frac{\delta}{2} \frac{x'(A^T(i)P(i) + P(i)A(i))x}{x'P(i)x} + \sum_{j=1}^N q_{ij} \left(\frac{x'P(j)x}{x'P(i)x} \right)^{\delta/2} \right) \\
&\stackrel{\text{def}}{=} (x'P(i)x)^{\delta/2} \Theta(x, i) = V(x, i) \Theta(x, i, \delta).
\end{aligned} \tag{3.3.3}$$

Next, we want to prove that if (3.3.1) holds, then there exists a $\delta > 0$ such that for any $i \in S$,

$$\begin{aligned}
&\sup_{\|x\|=1} \Theta(x, i, \delta) \\
&= \sup_{\|x\|=1} \left\{ \frac{\delta}{2} \frac{x'(A^T(i)P(i) + P(i)A(i))x}{x'P(i)x} + \sum_{j=1}^N q_{ij} \left(\frac{x'P(j)x}{x'P(i)x} \right)^{\delta/2} \right\} < 0
\end{aligned} \tag{3.3.4}$$

For notational simplicity, let

$$M_i(x) = \frac{x'(A^T(i)P(i) + P(i)A(i))x}{x'P(i)x}, \quad G_j(x) = \frac{x'P(j)x}{x'P(i)x}.$$

Suppose that (3.3.4) is not true. Then for some $i \in S$ and integer k , with $\delta = 1/k$, there exists x_k such that $\|x_k\| = 1$ and

$$\frac{1}{2k} M_i(x_k) + \sum_{j=1}^N q_{ij} G_j(x_k)^{1/(2k)} \geq 0 \tag{3.3.5}$$

Since $S^n \stackrel{\text{def}}{=} \{x : \|x\| = 1\}$ is compact, there exists a convergence subsequence of x_k . With loss of generality, assume that there exists a sequence $\{\delta_k\}$ and

sequence $\{x_k\}$ satisfying $\delta_k \downarrow 0^+$ and $x_k \rightarrow x_0 \in S^n$ as $k \rightarrow \infty$ (where $\|x_k\| = 1$) such that

$$\frac{\delta_k}{2} M_i(x_k) + \sum_{j=1}^N q_{ij} G_j(x_k)^{\delta_k/2} \geq 0 \quad (3.3.6)$$

By continuity, for any $\varepsilon > 0$, there exists a k_0 such that for $k \geq k_0$,

$$0 \leq M_i(x_k) \leq M_i(x_0) + \varepsilon, \quad 0 \leq G_j(x_k) \leq G_j(x_0) + \varepsilon.$$

From (3.3.6), we have

$$-q_i + \frac{\delta_k}{2} (M_i(x_0) + \varepsilon) + \sum_{j \neq i} q_{ij} (G_j(x_0) + \varepsilon)^{\delta_k/2} \geq 0 \quad (3.3.7)$$

Since $q_i = \sum_{j \neq i} q_{ij}$, we have $\sum_{j \neq i} q_{ij}/q_i = 1$, and

$$\lim_{k \rightarrow \infty} \left(\sum_{j \neq i} \frac{q_{ij}}{q_i} (G_j(x_0) + \varepsilon)^{\delta_k/2} \right)^{2/\delta_k} = \prod_{j \neq i} (G_j(x_0) + \varepsilon)^{q_{ij}/q_i}.$$

It follows that for any $\varepsilon_1 > 0$, there exists a $k_1 > k_0$ such that for $k \geq k_1$,

$$\left(\sum_{j \neq i} \frac{q_{ij}}{q_i} (G_j(x_0) + \varepsilon)^{\delta_k/2} \right)^{2/\delta_k} \leq \prod_{j \neq i} (G_j(x_0) + \varepsilon)^{q_{ij}/q_i} + \varepsilon_1,$$

i.e.,

$$0 \leq \sum_{j \neq i} \frac{q_{ij}}{q_i} (G_j(x_0) + \varepsilon)^{\delta_k/2} \leq \left(\prod_{j \neq i} (G_j(x_0) + \varepsilon)^{q_{ij}/q_i} + \varepsilon_1 \right)^{\delta_k/2}.$$

Taking this into (3.3.7), we obtain

$$\frac{\delta_k}{2} (M_i(x_0) + \varepsilon) - q_i + q_i \left(\prod_{j \neq i} (G_j(x_0) + \varepsilon)^{q_{ij}/q_i} + \varepsilon_1 \right)^{\delta_k/2} \geq 0$$

Rewriting this in the following form:

$$M_i(x_0) + \varepsilon - q_i \frac{1 - \left(\prod_{j \neq i} (G_j(x_0) + \varepsilon)^{q_{ij}/q_i} + \varepsilon_1 \right)^{\delta_k/2}}{\delta_k/2} \geq 0.$$

Letting $k \rightarrow \infty$ and using the fact that $\lim_{x \rightarrow 0} (1 - a^x)/x = -\log a$ for $a > 0$, we have

$$M_i(x_0) + \varepsilon + q_i \log \left(\prod_{j \neq i} (G_j(x_0) + \varepsilon)^{q_{ij}/q_i} + \varepsilon_1 \right) \geq 0$$

Since ε and ε_1 are arbitrarily chosen, the following can be obtained from the above:

$$M_i(x_0) + \sum_{j \neq i} q_{ij} \log G_j(x_0) \geq 0.$$

Therefore,

$$\sup_{\|x\|=1} \left(M_i(x) + \sum_{j \neq i} q_{ij} \log G_j(x) \right) \geq 0,$$

this contradicts to (3.3.2). So we have proved (3.3.4).

Now, for the choice of $\delta > 0$ in (3.3.4), let $-\gamma = \sup_{\|x\|=1} \Theta(x, i, \delta) = \sup_{x \neq 0} \Theta(x, i, \delta) < 0$. From (3.3.3), we obtain

$$(\mathcal{L}V)(x, i) \leq -\gamma(x'P(i)x)^{\delta/2} = -\gamma V(x, i). \quad (3.3.8)$$

Applying Dynkin's formula and Fubini's theorem ([94]), we have

$$\begin{aligned} E\{V(x_t, \sigma_t) | x_0 = x, \sigma_0 = i\} - V(x, i) &= E\left\{ \int_0^t (\mathcal{L}V)(x_s, \sigma_s) ds | x_0 = x, \sigma_0 = i \right\} \\ &\leq E\left\{ \int_0^t -\gamma V(x_s, \sigma_s) ds | x_0 = x, \sigma_0 = i \right\} \\ &= -\gamma \int_0^t E\{V(x_s, \sigma_s) | x_0 = x, \sigma_0 = i\} ds. \end{aligned} \quad (3.3.9)$$

Applying the Gronwall-Bellman Lemma, we have for any $x \neq 0$ and $i \in S$,

$$E\{V(x_t, \sigma_t) | x_0 = x, \sigma_0 = i\} \leq V(x_0, i) \exp(-\gamma t), \quad \forall t \geq 0.$$

Let $\hat{\lambda}_{\min} = \min_{1 \leq i \leq N} \lambda_{\min}(P(i))$, we have

$$\begin{aligned} E\{\|x_t\|^\delta | x_0 = x, \sigma_0 = i\} &= E\{(x'_t x_t)^{\delta/2} | x_0 = x, \sigma_0 = i\} \\ &\leq \frac{1}{\hat{\lambda}_{\min}} E\{(x'_t P(\sigma_t) x_t)^{\delta/2} | x_0 = x, \sigma_0 = i\} \leq \frac{V(x_0, i)}{\hat{\lambda}_{\min}} e^{-\gamma t}. \end{aligned}$$

This implies that the system (3.1.1) is exponentially δ -moment stable and thus almost surely stable. This completes the proof of Theorem 3.3.1. \square

Corollary 3.3.2:

If there exists pd matrices $P(1), \dots, P(N)$ such that for any $i \in S$,

$$\lambda_{\max}((P(i)A(i) + A^T(i)P(i))P(i)^{-1}) + \sum_{j \neq i} q_{ij} \log \lambda_{\max}(P(j)P(i)^{-1}) < 0.$$

then, (3.1.1) is δ -moment stable for some $\delta > 0$ and thus almost surely stable. \square

Proof of corollary. Using the fact that

$$\sup_{\|x\|=1} \frac{x^T Q x}{x^T P x} = \lambda_{\max}(QP^{-1}) \quad (3.3.10)$$

where Q is symmetric and P is positive definite, the corollary is trivial when the maximization is done for each terms in (3.3.2). \square

In the proof of the theorem, we use the Lyapunov function $V(x, i) = (x'P(i)x)^{\delta/2}$ to obtain a sufficient condition for δ -moment stability. In [74], Feng et al established a necessary and sufficient condition for second moment stability using a quadratic Lyapunov function (V with $\delta = 2$). It is very

tempting to conjecture that V is also “necessary” for δ -moment stability, i.e., if (3.1.1) is δ -moment stable, there exist $P(i)$ such that $V(x, i) = (x'P(i)x)^{\delta/2}$ is a Lyapunov function in the sense that $\mathcal{L}V < 0$. Put in another words, we want to claim that (3.3.4) is also necessary for δ -moment stability for any $\delta > 0$. However, we have not obtained a rigorous proof for the result so far. The second moment case and the one-dimensional case illustrate the intuitive idea.

Proposition 3.3.3:

- (i) (3.1.1) is second moment stable if and only if with $\delta = 2$, (3.3.4) holds, i.e., for $i \in S$, $\sup_{\|x\|=1} \Theta(x, i, \delta) < 0$ for some positive definite $P(i)$'s. Furthermore, (3.3.1) is satisfied.
- (ii) If $n = 1$, (3.1.1) is δ -moment stable for some $\delta > 0$ if and only if (3.3.4) holds if and only if (3.3.1) holds. It follows that if σ_t is irreducible, $(A(1), \dots, A(N)) \in \Sigma_-^a$ if and only if (3.3.1) holds. \square

Proof of Proposition. All the “if” parts are established via Theorem 3.3.1 and Theorem 3.2.5. We show the “only if” part(s) using a similar procedure as given in [74]. For $\delta = 2$ or $n = 1$, suppose that (3.1.1) is δ -moment stable. Define the function $V(T - t, x, i)$ for $0 \leq t < T < +\infty$,

$$\begin{aligned}
 V(T - t, x, i) &= E\left\{ \int_t^T \|x_\tau\|^\delta d\tau \mid x_t = x, \sigma_t = i \right\} \\
 &= E\left\{ \int_0^{T-t} \|x_\tau\|^\delta d\tau \mid x_0 = x, \sigma_0 = i \right\} \\
 &= (x'(E\left\{ \int_0^{T-t} (\Phi'(\tau, 0)\Phi(\tau, 0))^{\delta/2} d\tau \mid \sigma_0 = i \right\})^{2/\delta} x)^{\delta/2} \\
 &\stackrel{\text{def}}{=} (x'M(T - t, i)x)^{\delta/2}.
 \end{aligned} \tag{3.3.11}$$

Note that since (3.1.1) is δ -moment stable, then $P(i) \stackrel{\text{def}}{=} \lim_{T \rightarrow +\infty} M(T-t, i)$ and thus $V(x, i) = \lim_{T \rightarrow +\infty} V(T-t, x, i)$ are well-defined. consider that for any $0 \leq t < s < T$,

$$\begin{aligned} & \frac{d}{dt} E\{V(T-t, x_t, \sigma_t) | x_t = x, \sigma_t = i\} \\ &= \lim_{s \downarrow t} \frac{1}{s-t} [E\{V(T-t, x_s, \sigma_s) | x_t = x, \sigma_t = i\} - V(T-t, x, i)] \quad (3.3.12) \\ &= \frac{\partial}{\partial t} V(T-t, x, i) + (\mathcal{L}V)(T-t, x, i). \end{aligned}$$

Note that $\lim_{T \rightarrow +\infty} \frac{\partial}{\partial t} V(T-t, x, i) = 0$ and also

$$\begin{aligned} & \frac{1}{s-t} [E\{V(T-s, x_s, \sigma_s) | x_t = x, \sigma_t = i\} - V(T-t, x, i)] \\ &= \frac{1}{s-t} [E\{V(T-s, x_s, \sigma_s) - V(T-t, x, i) | x_t = x, \sigma_t = i\}] \\ &= \frac{1}{s-t} E\{E\{\int_s^T (x'_\tau x_\tau)^{\delta/2} d\tau | x_s, \sigma_s\} \\ & \quad - E\{\int_t^T (x'_\tau x_\tau)^{\delta/2} d\tau | x_t, \sigma_t\} | x_t = x, \sigma_t = i\} \\ &= \frac{1}{s-t} E\{\int_s^T (x'_\tau x_\tau)^{\delta/2} d\tau - \int_t^T (x'_\tau x_\tau)^{\delta/2} d\tau | x_t = x, \sigma_t = i\} \\ &= -\frac{1}{s-t} E\{\int_0^{s-t} (x'_\tau x_\tau)^{\delta/2} d\tau | x_0 = x, \sigma_0 = i\} \\ &= -(x' (E\{\frac{1}{s-t} \int_0^{s-t} (\Phi'(\tau, 0)\Phi(\tau, 0))^{\delta/2} d\tau | \sigma_0 = i\})^{2/\delta} x)^{\delta/2} \xrightarrow{s \downarrow t} -(x'x)^{\delta/2}. \end{aligned} \quad (3.3.13)$$

Note also that with $P(i)$ defined as above,

$$\lim_{T \rightarrow +\infty} (\mathcal{L}V)(T-t, x, i) = V(x, i)\Theta(x, i, \delta). \quad (3.3.14)$$

From (3.3.12)-(3.3.14), we have

$$-x'x = V(x, i)\Theta(x, i, \delta)$$

and

$$\sup_{\|x\|=1} \Theta(x, i, \delta) \leq - \inf_{\|x\|=1} \frac{x'x}{(x'P(i)x)^{\delta/2}} \stackrel{\text{def}}{=} -\gamma < 0.$$

This shows that δ -moment stability implies that (3.3.4) holds. It remains to show that (3.3.4) holds implies that (3.3.1) holds. Actually, (3.3.4) holds for some $\delta > 0$ gives that

$$\frac{\delta}{2}M_i(x) + \sum_{j=1}^N q_{ij}(G_j(x))^{\delta/2} < 0$$

for all x with $\|x\| = 1$ and $i, j \in S$. Using the fact that $-q_{ii} = \sum_{j \neq i} q_{ij}$ and $G_i(x) = 1$, and the inequality $y - 1 \geq \log y$ for all $y > 0$, we have

$$\begin{aligned} 0 &> \frac{\delta}{2}M_i(x) + \sum_{j=1}^N q_{ij}(G_j(x))^{\delta/2} \geq \frac{\delta}{2}M_i(x) + \sum_{j=1}^N q_{ij}(\log(G_j(x))^{\delta/2} - 1) \\ &= \frac{\delta}{2}[M_i(x) + \sum_{j \neq i} q_{ij} \log G_j(x)]. \end{aligned}$$

Taking the supremum, we have (3.3.1). □

From this proposition, we see that the testable condition (3.3.1) for δ -moment and almost sure stability is a stronger result than second moment stability criteria, i.e., if (3.1.1) is second moment stable, then (3.3.1) holds. For scalar system, the condition is necessary and sufficient for almost sure exponential stability. It seems to us that that (3.3.1) is also necessary for almost sure stability for higher dimensional systems. However, so far we have not obtained a rigorous proof. The proof given above for scalar case illustrates the idea as well as the difficulty. For scalar systems, the necessity of (3.3.2) for

almost sure exponential stability can be justified algebraically. We know from [74] that for one-dimensional system, if σ_t has a single ergodic class, then the top exponent is given by

$$\alpha = \pi_1 A(1) + \dots + \pi_N A(N) = \pi a \quad (3.3.15)$$

where $a = (A(1), \dots, A(N))'$.

Theorem 3.3.4:

For one dimensional system (3.1.1), suppose that σ_t is irreducible. The system is almost surely exponentially stable if and only if $\pi a < 0$ if and only if (3.3.1) holds for some $P(i)$'s if and only if there exists a $y \in \mathcal{R}^N$ such that $a + Qy < 0$. \square

Proof: We only need to establish the equivalence of those three algebraic conditions.

Consider that for any $P(i) > 0$, we have for any i and x

$$\begin{aligned} M_i(x) &+ \sum_{j \neq i} q_{ij} \log G_j(x) \\ &= 2A(i) + \sum_{j \neq i} q_{ij} \log \frac{P(j)}{P(i)} = 2a_i + \sum_{j \neq i} q_{ij} \log P(j) - \left(\sum_{j \neq i} q_{ij} \right) \log P(i) \\ &= 2a_i + \sum_{j=1}^N q_{ij} \log P(j) = 2(a + Q(\log P(1), \dots, \log P(N)))'. \end{aligned}$$

Thus, (3.3.1) holds for some $P(i)$'s if and only if $a + Qy < 0$ for some $y \in \mathcal{R}^N$. Next, we show that $a + Qy < 0$ for some $y \in \mathcal{R}^N$ if and only if $\pi a < 0$. If $a + Qy < 0$, then $\pi(a + Qy) = \pi a < 0$, since $\pi > 0$ due to the irreducibility. On the other hand, assume $\pi a < 0$. We first show that $\text{Im}(Q) = \{y \in \mathcal{R}^N : \pi y = 0\}$. In fact, if $y \in \text{Im}(Q)$, there exists $z \in \mathcal{R}^N$ such that $y = Qz$. Since

$\pi Q = 0$, we have $\pi y = (\pi Q)z = 0$ and thus $\text{Im}(Q) \subset \{y : \pi y = 0\}$. However, $\dim \text{Im}(Q) = \text{rank}(Q) = N - 1$ due to the irreducibility (or uniqueness of π). Thus, $\text{Im}(Q) = \{y : \pi y = 0\}$ because $\dim \text{Im}(Q) = \dim\{y : \pi y = 0\} = N - 1$. Now, let

$$y = -a + \frac{\pi a}{\pi \pi'} \pi' \in \mathcal{R}^N,$$

since $\pi y = -\pi a + \frac{\pi a}{\pi \pi'} \pi \pi' = 0$, we have $y \in \{y : \pi y = 0\} = \text{Im}(Q)$, thus there exists a $z \in \mathcal{R}^N$, such that $Qz = y$, i.e.,

$$Qz + a = \frac{\pi a}{\pi \pi'} \pi' < 0,$$

since $\pi a < 0$ and $\pi' > 0$. This completes the proof. \square

From the above proof, we can observe that $\text{Im}(Q)$ is just the boundary of the almost sure stability region in the system parameter space. Since Theorem 3.3.1 is very general result for almost sure stability, by specifying the positive definite matrices $P(1), \dots, P(N)$, we can obtain some simple test criteria for almost sure stability.

Corollary 3.3.5:

Define $\mu_P(A) = \lambda_{\max}(PAP^{-1} + A')/2$. Then

- (i) If there exists a positive definite matrix P and $y \in \mathcal{R}^N$ such that $u + Qy < 0$ with $u = (\mu_P(A(1)), \dots, \mu_P(A(N)))'$, then (3.1.1) is almost surely stable.
- (ii) If there exists a positive definite matrix P such that

$$\pi_1 \mu_P(A(1)) + \pi_2 \mu_P(A(2)) + \dots + \pi_N \mu_P(A(N)) < 0,$$

then (3.1.1) is almost surely stable. \square

Proof of Corollary: (i). In Theorem 3.3.1, let $P(i) = \alpha_i P$ for some $\alpha_i > 0$.

Then

$$\begin{aligned}
& \sup_{\|x\|=1} \left\{ \frac{x'(P(i)A(i) + A'(i)P(i))x}{x'P(i)x} + \sum_{j \neq i} q_{ij} \log \left(\frac{x'P(j)x}{x'P(i)x} \right) \right\} \\
&= \sup_{\|x\|=1} \left\{ \frac{x'(PA(i) + A'(i)P)x}{x'Px} + \sum_{j \neq i} q_{ij} \log \frac{\alpha_j}{\alpha_i} \right\} \\
&= \sup_{\|x\|=1} \left\{ \frac{x'(PA(i) + A'(i)P)x}{x'Px} \right\} + \sum_{j=1}^N q_{ij} \log \alpha_j \\
&= \lambda_{\max}(PA(i)P^{-1} + A'(i)) + \sum_{j=1}^N q_{ij} \log \alpha_j = 2\mu_P(A(i)) + \sum_{j=1}^N q_{ij} \log \alpha_j
\end{aligned}$$

From proof of Theorem 3.3.4, we know that

$$\mu_P(A(i)) + \sum_{j=1}^N q_{ij} \log \sqrt{\alpha_j} < 0, \quad i = 1, 2, \dots, N$$

if and only if there exists an $y \in \mathcal{R}^N$ such that $u + Qy < 0$. From this and Theorem 3.3.1, we obtain proof of (i). Note also that in the proof of Theorem 3.4, we have shown that $u + Qy < 0$ for some y if and only if

$$\pi_1 \mu_P(A(1)) + \pi_2 \mu_P(A(2)) + \dots + \pi_N \mu_P(A(N)) < 0.$$

Thus, (ii) follows. This completes the proof. \square

Though Theorem 3.3.1 is theoretically attractive, so far we have not found a rigorous procedure to select the positive definite matrices $P(1), \dots, P(N)$ so that (3.3.1) holds. Now, we give some necessary conditions for (3.3.1) to hold. Define

$$h_i(x) = \frac{x'(P(i)A(i) + A'(i)P(i))x}{x'P(i)x} + \sum_{j \neq i} q_{ij} \log \left(\frac{x'P(j)x}{x'P(i)x} \right),$$

which are continuous functions on the sphere S^n for all $i \in S$. (3.3.1) is therefore equivalent to

$$\frac{x'(P(i)A(i) + A'(i)P(i))x}{x'P(i)x} + \sum_{j \neq i} q_{ij} \log \left(\frac{x'P(j)x}{x'P(i)x} \right) < 0, \quad \forall x \in S^n, i \in S. \quad (3.3.16)$$

From this we obtain

$$\frac{x'(P(i)A(i) + A'(i)P(i))x}{x'P(i)x} + \sum_{j=1}^N q_{ij} \log(x'P(i)x) < 0, \quad \forall x \in S^n, i \in S.$$

or

$$\begin{pmatrix} \frac{x'(P(1)A(1) + A'(1)P(1))x}{x'P(1)x} \\ \vdots \\ \frac{x'(P(N)A(N) + A'(N)P(N))x}{x'P(N)x} \end{pmatrix} + Q \begin{pmatrix} \log(x'P(1)x) \\ \vdots \\ \log(x'P(N)x) \end{pmatrix} < 0 \quad (3.3.17)$$

Multiplying both sides of (3.3.17) by π and using $\pi Q = 0$, we obtain the following necessary condition for (3.3.1) to hold.

Theorem 3.3.6:

A necessary condition for (3.3.1) to hold is

$$\pi_1 \frac{x'(P(1)A(1) + A'(1)P(1))x}{x'P(1)x} + \cdots + \pi_N \frac{x'(P(N)A(N) + A'(N)P(N))x}{x'P(N)x} < 0 \quad (3.3.18)$$

for any $x \in S^n$. □

Since the second moment stability implies (3.3.1), we see that (3.3.18) is also a necessary condition for second moment stability. □

It is tempted to conjecture that (3.3.18) is also a sufficient condition for almost sure stability. Unfortunately, this is not true. The reason for this is the following: Suppose the conjecture is true. If $A(1), \dots, A(N)$ are stable matrices, from Lyapunov's equations, there exist positive definite matrices

$P(i)$ for $i = 1, 2, \dots, N$ such that $A^T(i)P(i) + P(i)A(i) = -I$. Taking this into (3.3.18), we can verify that (3.3.18) is satisfied. But as we will see in the following examples that the stability of $A(1), \dots, A(N)$ does not guarantee the almost sure stability of the system (3.1.1) in general.

Example 3.3.7:

(i). Consider the harmonic oscillator:

$$u_t'' + k^2(1 + b\xi_t)u_t = 0, \quad t \geq 0$$

where $\{\xi_t\}$ is a random telegraph process with state space $\{-1, +1\}$, i.e., a two state time-homogeneous Markov chain with the infinitesimal generator $Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix}$. From Feng and Loparo ([78]), we know that for any $k > 0$ and $b \in (-1, +1)$, the Lyapunov exponent exists almost surely:

$$\alpha = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \sqrt{\dot{u}^2(t) + k^2 u^2(t)}$$

regardless of the initial distribution of ξ_t and the initial state $(u(0), \dot{u}(0))$. Furthermore, the number α is strictly positive.

Now, let $k = 1$ and $b = 0.5$. We have the following state space representation for the oscillator: $\dot{x}(t) = A_0(\xi_t)x(t)$ with $x_1 = u, x_2 = \dot{u}$ and

$$A_0(-1) = \begin{pmatrix} 0 & 1 \\ -1.5 & 0 \end{pmatrix}, \quad A_0(+1) = \begin{pmatrix} 0 & 1 \\ -0.5 & 0 \end{pmatrix}.$$

Define a new system $\dot{x}(t) = A(\xi_t)x(t)$, where $A(-1) = -0.5\alpha I + A_0(-1)$ and $A(+1) = -0.5\alpha I + A_0(+1)$, it is easy to show that $A(1)$ and $A(2)$ are stable matrices. However, the top Lyapunov exponent for this system is

$$\hat{\alpha} = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|x(t)\| = -0.5\alpha + \alpha = 0.5\alpha > 0. \quad \text{a.s.}$$

Therefore, the system is almost surely unstable.

(ii). We give another example to show that the individual modes stability does not guarantee the almost sure stability. The proof is elementary.

Let

$$A(1) = \begin{pmatrix} -a & 1 \\ 0 & -a \end{pmatrix}, \quad A(2) = \begin{pmatrix} -a & 0 \\ 1 & -a \end{pmatrix}, \quad Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix},$$

where $a > 0$ and $q > 0$ are to be determined. Using the same notations as before, we have

$$\begin{aligned} \Phi(t_{k+1}, 0) &= \exp(A(r_k)\tau_k) \cdots \exp(A(r_0)\tau_0) \\ &= \exp\left(-a \sum_{i=0}^k \tau_i\right) \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tau_{k-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \tau_0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Let $M(A)$ denote the largest element of A , then it is easy to verify that

$$M(\Phi(t_{k+1}, 0)) \geq \tau_k \cdots \tau_0 \exp\left(-a \sum_{i=0}^k \tau_i\right).$$

Thus from this, we obtain

$$\frac{1}{k} \log M(\Phi(t_{k+1}, 0)) \geq \frac{1}{k} \sum_{i=0}^k \log \tau_i - a \frac{1}{k} \sum_{i=0}^k \tau_i.$$

From the Law of Large Numbers, we have

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log M(\Phi(t_{k+1}, 0)) \geq E_{\tilde{\pi}}\{\log \tau_0\} - a E_{\tilde{\pi}}\{\tau_0\}$$

where $\tilde{\pi}$ is the unique invariant measure of the joint process $\{(\tau_i, r_i)\}$. Compute

$$E_{\tilde{\pi}}\{\log \tau_0\} = -\log q + \int_0^{+\infty} (\log x) e^{-x} dx, \quad E_{\tilde{\pi}}\{\tau_0\} = \frac{1}{q}.$$

It follows that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log M(\Phi(t_{k+1}, 0)) \geq -\log q + \int_0^{+\infty} (\log x) e^{-x} dx - \frac{a}{q}.$$

Choose q so small that $-\log q + \int_0^\infty (\log x)e^{-x} dx - 1 \geq 0$, and choose a so small that $1 - a/q \geq 0.5$, then we finally arrive at

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log M(\Phi(t_{k+1}, 0)) \geq 1 - \frac{a}{q} \geq \frac{1}{2}$$

almost surely, i.e., $\lim_{k \rightarrow \infty} M(\Phi(t_k, 0)) = +\infty$ almost surely. Therefore, the system (3.1.1) is almost surely unstable with the choice of a and q . However, $A(1)$ and $A(2)$ are obviously stable. \square

Remark: Mariton ([86]) remarked that a necessary and sufficient condition for higher order moment stability is $\max\{\sigma(A(1)), \sigma(A(2)), \dots, \sigma(A(N))\} < 0$, where $\sigma(A)$ the largest real part of eigenvalues of A . The above examples show that this statement is not true, in fact, it is even not true for almost sure stability.

Although (3.3.18) is not in general a sufficient condition for (3.1.1) to be almost surely stable, for certain cases, it is also sufficient. Define $\phi(t) = x(t)/\|x(t)\| \in S^n$, then we have

Theorem 3.3.8:

Suppose that $\{(\phi(t), \sigma_t)\}$ is ergodic with a unique invariant measure $p(\phi, \sigma)d\phi\pi(d\sigma)$. If there exists a positive definite matrix P such that

$$\max_{\|x\|=1} \left\{ \sum_{i=1}^N \pi_i \frac{x'(PA(i) + A'(i)P)x}{x'Px} \right\} < 0,$$

then the system (3.1.1) is almost surely stable. \square

Proof of theorem: Let

$$\tau = \max_{\|x\|=1} \left\{ \sum_{i=1}^N \pi_i \frac{x'(PA(i) + A'(i)P)x}{x'Px} \right\}.$$

Then from the assumption, We have

$$\sum_{i=1}^N \pi_i \frac{x'(PA(i) + A'(i)P)x}{x'Px} \leq \tau < 0, \quad \forall x \in S^n \quad (3.3.19)$$

Define the Lyapunov function $V(t) = \frac{1}{2}x'(t)Px(t)$ and let $\bar{A}(\sigma_t) = (A'(\sigma_t)P + PA(\sigma_t))/2$. We obtain

$$\dot{V}(t) = x'(t)\bar{A}(\sigma_t)x(t) = \frac{\phi'(t)\bar{A}(\sigma_t)\phi(t)}{\phi'(t)P\phi(t)}V(t).$$

Thus, we have

$$V(T) = V(0) \exp \left(\int_0^T \frac{\phi'(t)\bar{A}(\sigma_t)\phi(t)}{\phi'(t)P\phi(t)} dt \right).$$

This implies that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log V(t) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \frac{\phi'(t)\bar{A}(\sigma_t)\phi(t)}{\phi'(t)P\phi(t)} dt \quad (3.3.20)$$

Since the process $\{(\phi(t), \sigma_t)\}$ is ergodic, then from the Law of Large Numbers and (3.3.20), we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{1}{t} \log V(t) &= E \left\{ \frac{\phi'(t)\bar{A}(\sigma_t)\phi(t)}{\phi'(t)P\phi(t)} \right\} \\ &= \int_{S^n \times S} \frac{x'\bar{A}(y)x}{x'Px} p(x, y) dx \pi(dy) \\ &= \sum_{j=1}^N \pi_j \int_{S^n} \frac{x'\bar{A}(j)x}{x'Px} p(x, j) dx \\ &= \frac{1}{2} \int_{S^n} \sum_{j=1}^N \pi_j \frac{x'(A'(j)P + P(j)A(j))x}{x'Px} p(x, j) dx \leq \frac{1}{2} \tau < 0 \end{aligned}$$

Since P is positive definite, we can easily obtain

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|x\| = \frac{1}{2} \lim_{t \rightarrow \infty} \frac{1}{t} \log V(t) \leq \frac{1}{4} \tau < 0$$

almost surely, this implies that (3.1.1) is almost surely stable. \square

An interesting application of the above theorem is when we have a diagonal system. Let $A(j) = \text{diag}\{a_1(j), a_2(j), \dots, a_n(j)\}$ for $j = 1, 2, \dots, N$. Let $P = I$, for any $\|x\| = 1$, we then have

$$\begin{aligned} \sum_{i=1}^N \pi_i x' (A^T(i) + A(i))x &= 2 \sum_{i=1}^N \pi_i x' A(i)x \\ &= 2 \left[\left(\sum_{i=1}^N \pi_i a_1(i) \right) x_1^2 + \dots + \left(\sum_{i=1}^N \pi_i a_n(i) \right) x_n^2 \right] \\ &\leq 2 \left\{ \max_j \left(\sum_{i=1}^N \pi_i a_j(i) \right) \right\} (x_1^2 + \dots + x_n^2) = 2 \max_j \left(\sum_{i=1}^N \pi_i a_j(i) \right) \end{aligned}$$

Thus, if $\max_j \left(\sum_{i=1}^N \pi_i a_j(i) \right) < 0$, then (3.1.1) is almost surely stable. In fact, it is easy to see that this condition is also necessary for the diagonal system (3.1.1) to be almost surely exponentially stable.

The above procedure can be extended to some more general cases. As an illustration, let $P = I$ such that $\bar{A}(i) = (A'(i) + A(i))/2$ which is symmetric. Then there exists a unitary matrix $U(i)$ such that

$$U(i)\bar{A}(i)U^T(i) = \text{diag}\{\lambda_1(i), \dots, \lambda_n(i)\}.$$

Define a new process $\psi(t) = U(\sigma_t)\phi(t) = (\psi_1(t), \dots, \psi_n(t))' \in S^n$ with $\phi(t)$ defined previously. Then we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \log \|x(t)\| &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \phi(s)^T \bar{A}(\sigma_s) \phi(s) ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \phi^T(s) U^T(\sigma_s) \text{diag}\{\lambda_1(\sigma_s), \dots, \lambda_n(\sigma_s)\} U(\sigma_s) \phi(s) ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \psi^T(s) \text{diag}\{\lambda_1(\sigma_s), \dots, \lambda_n(\sigma_s)\} \psi(s) ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sum_{i=1}^n \lambda_i(\sigma_s) \psi_i^2(s) ds \end{aligned} \tag{3.3.21}$$

If $\{(\psi(t), \sigma_t)\}$ is ergodic with an unique invariant measure $f(\psi, \sigma)d\psi\pi(d\sigma)$, from (3.3.21) and the Law of Large Numbers, we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \log \|x\| &= E\left\{\sum_{i=1}^n \lambda_i(\sigma_0) \psi_i^2(0)\right\} \\ &= \int_{S^n} \sum_{i=1}^n \sum_{j=1}^N \pi_j \lambda_i(j) \psi_i^2 f(\psi, j) d\psi \\ &\leq \int_{S^n} \sum_{j=1}^N \pi_j \left(\max_{1 \leq i \leq n} \lambda_i(j)\right) \left(\sum_{i=1}^n \psi_i^2\right) f(\psi, j) d\psi \\ &= \sum_{j=1}^N \pi_j \left(\max_{1 \leq i \leq n} \lambda_i(j)\right). \end{aligned}$$

Therefore, if $\sum_{j=1}^N \pi_j (\max_{1 \leq i \leq n} \lambda_i(j)) < 0$, the system (3.1.1) is almost surely stable. Clearly, by making more general choice of $P(1), \dots, P(N)$, we may obtain stronger result. However, it is not trivial to verify the ergodicity of the angular process of the system (3.1.1).

It is noted that all conditions we developed before, except for the scalar case, are sufficient conditions for almost sure stability. However, for some special cases, a necessary and sufficient condition can be obtained, for example, as we noted before, for the diagonal systems, the necessary and sufficient condition for almost sure exponential stability is easy to obtained. In chapter two, we gave some necessary and sufficient conditions for almost sure stability of discrete-time jump linear systems with special structures, the commuting structure, for instance. In the following, we will present the parallel results for continuous-time jump linear system (3.1.1).

Theorem 3.3.9:

Suppose that $A(1), \dots, A(N)$ are given by

$$A(\sigma_t) = \begin{pmatrix} \lambda_1(\sigma_t) & a_{12}(\sigma_t) & \dots & a_{1n}(\sigma_t) \\ & \lambda_2(\sigma_t) & \dots & a_{2n}(\sigma_t) \\ & & \ddots & \vdots \\ & & & \lambda_n(\sigma_t) \end{pmatrix}$$

and suppose that σ_t has a single ergodic class. Then the system (3.1.1) is almost surely exponentially stable if and only if

$$E_\pi \lambda_i(\sigma_0) = \pi_1 \lambda_i(1) + \pi_2 \lambda_i(2) + \dots + \pi_N \lambda_i(N) < 0, \quad i = 1, 2, \dots, n$$

where $\pi = \{\pi_1, \dots, \pi_N\}$ is the unique invariant measure of the form process $\{\sigma_t\}$.

Proof of Theorem. *Necessity.* The fundamental matrix of the system (3.1.1) is given by

$$\Phi(t, 0) = \exp(A(r_{k+1})(t - t_k)) \exp(A(r_k)\tau_k) \cdots \exp(A(r_0)\tau_0).$$

Since $A(i)$ is upper triangular for any i , $\Phi(t, 0)$ is also upper triangular. Suppose that (3.1.1) is almost surely exponentially stable. Then $\lim_{t \rightarrow \infty} \Phi(t, 0) = 0$ almost surely exponentially fast. Observing the diagonal elements of $\Phi(t, 0)$, we obtain

$$\lim_{t \rightarrow \infty} \exp(\lambda_i(r_{k+1})(t - t_k)) \exp(\lambda_i(r_k)\tau_k) \cdots \exp(\lambda_i(r_0)\tau_0) = 0$$

almost surely exponentially fast. This is equivalent to the almost sure exponential stability of the one dimensional system $\dot{x}(t) = \lambda_i(\sigma_t)x(t)$. Thus,

$$E_\pi \lambda_i(\sigma_0) = \pi_1 \lambda_i(1) + \pi_2 \lambda_i(2) + \dots + \pi_N \lambda_i(N) < 0, \quad i = 1, 2, \dots, n \quad (3.3.22)$$

Sufficiency. For upper triangular structured system, (3.1.1) can be expressed explicitly as

$$\dot{x}_1(t) = \lambda_1(\sigma_t)x_1(t) + a_{12}(\sigma_t)x_2(t) + \cdots + a_{1n}(\sigma_t) \quad (e.1)$$

$$\vdots$$

$$\dot{x}_{n-1}(t) = \lambda_{n-1}(\sigma_t)x_{n-1}(t) + a_{n-1,n}(\sigma_t)x_n(t) \quad (e.n-1)$$

$$\dot{x}_n(t) = \lambda_n(\sigma_t)x_n(t) \quad (e.n)$$

Since $\{\sigma_t\}$ is a finite state Markov chain, there exists an $M > 0$ such that $|a_{ij}(\sigma_t)| \leq M$ ($1 \leq i, j \leq n$). From (3.3.15) and the equation (e.n), we obtain that (e.n) is almost surely stable with a nonrandom exponential convergence rate, i.e., there exists an $M_n(\omega) > 0$ and a nonrandom constant $\beta_n > 0$ such that

$$|x_n(t)| \leq M_n(\omega)e^{-\beta_n t}, \quad \forall t \geq 0 \quad (3.3.23)$$

From (e.n-1), we have

$$\begin{aligned} x_{n-1}(t) &= \exp\left(\int_0^t \lambda_{n-1}(\sigma_s) ds\right) x_{n-1}(0) \\ &\quad + \int_0^t \exp\left(\int_\tau^t \lambda_{n-1}(\sigma_s) ds\right) a_{n-1,n}(\sigma_\tau) x_n(\tau) d\tau \end{aligned}$$

from which we obtain

$$\begin{aligned} |x_{n-1}(t)| &\leq \exp\left(\int_0^t \lambda_{n-1}(\sigma_s) ds\right) |x_{n-1}(0)| \\ &\quad + MM_n(\omega) \int_0^t \exp\left(\int_\tau^t \lambda_{n-1}(\sigma_s) ds\right) e^{-\beta_n \tau} d\tau \end{aligned} \quad (3.3.24)$$

Since $\{\sigma_t\}$ is ergodic, from ergodic theory, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lambda_{n-1}(\sigma_s) ds &= E_\pi \lambda_{n-1}(\sigma_0) \\ &= \pi_1 \lambda_{n-1}(1) + \cdots + \pi_N \lambda_{n-1}(N) \triangleq \alpha_{n-1} < 0. \end{aligned}$$

and

$$\begin{aligned} \lim_{t-\tau \rightarrow \infty} \frac{1}{t-\tau} \int_{\tau}^t \lambda_{n-1}(\sigma_s) ds &= \lim_{t-\tau \rightarrow \infty} \frac{1}{t-\tau} \int_0^{t-\tau} \lambda_{n-1}(\sigma_{s+\tau}) ds \\ &= E_{\pi} \lambda_{n-1}(\sigma_{\tau}) = \alpha_{n-1} < 0. \end{aligned}$$

Thus, there exists $T = T(\omega) > 0$ such that for any t and τ , $t - \tau \geq T$, we have

$$\int_{\tau}^t \lambda_{n-1}(\sigma_s) ds < \frac{\alpha_{n-1}}{2}(t - \tau).$$

From (3.3.24), we obtain that for large t

$$\begin{aligned} |x_{n-1}(t)| &\leq e^{\alpha_{n-1}t/2} |x_{n-1}(0)| \\ &\quad + MM_n(\omega) \int_0^{t-T} \exp\left(\int_{\tau}^t \lambda_{n-1}(\sigma_s) ds\right) e^{-\beta_n \tau} d\tau \\ &\quad + MM_n(\omega) \int_{t-T}^t \exp\left(\int_{\tau}^t \lambda_{n-1}(\sigma_s) ds\right) e^{-\beta_n \tau} d\tau \\ &\leq e^{\alpha_{n-1}t/2} |x_{n-1}(0)| \\ &\quad + MM_n(\omega) \left(\int_0^{t-T} e^{\alpha_{n-1}(t-\tau)/2} e^{-\beta_n \tau} d\tau + \int_{t-T}^t e^{M(t-\tau)} e^{-\beta_n \tau} d\tau \right) \\ &\leq e^{\alpha_{n-1}t/2} |x_{n-1}(0)| + MM_n(\omega) \left[\frac{1}{\beta_n + \alpha_{n-1}/2} \right. \\ &\quad \left. \times \left(e^{\alpha_{n-1}t/2} - e^{\alpha_{n-1}T/2} e^{-\beta_n(t-T)} \right) + \frac{e^{MT}}{\beta_n} \left(e^{-\beta_n(t-T)} - e^{-\beta_n t} \right) \right] \\ &\leq M_{n-1}(\omega) e^{-\beta_{n-1}t} \end{aligned}$$

where β_{n-1} is a nonrandom positive numbers and $M_{n-1}(\omega) > 0$, the existence of such numbers is guaranteed by the fact that $\beta_n > 0$ and $\alpha_{n-1} < 0$. Therefore, there exists $M_n(\omega)$, $M_{n-1}(\omega)$, $\beta_n > 0$ and $\beta_{n-1} > 0$, such that

$$|x_n(t)| \leq M_n(\omega) e^{-\beta_n t}, \quad |x_{n-1}(t)| \leq M_{n-1}(\omega) e^{-\beta_{n-1} t}, \quad \forall t \geq 0.$$

Using the similar procedure, we can prove that there exists $M_i(\omega) > 0$ and $\beta_i > 0$ for $i = 1, 2, \dots, n$, such that

$$|x_i(t)| \leq M_i(\omega) e^{-\beta_i t}, \quad \forall t \geq 0, 1 \leq i \leq n.$$

This implies that (3.1.1) is almost surely exponentially stable. This completes the proof. \square

Corollary 3.3.10:

Suppose that σ_t has a single ergodic class. If $A(1), \dots, A(N)$ can be simultaneously transformed into the following upper triangular forms

$$\begin{pmatrix} \lambda_1(i) & a_{12}(i) & \dots & a_{1n}(i) \\ & \lambda_2(i) & \dots & a_{2n}(i) \\ & & \ddots & \vdots \\ & & & \lambda_n(i) \end{pmatrix}, \quad i = 1, 2, \dots, N,$$

by the same similarity transformation, then the system (3.1.1) is almost surely exponentially stable if and only if

$$E_\pi \lambda_i(\sigma_0) = \pi_1 \lambda_i(1) + \pi_2 \lambda_i(2) + \dots + \pi_N \lambda_i(N) < 0, \quad i = 1, 2, \dots, n$$

where $\pi = \{\pi_1, \dots, \pi_N\}$ is the unique invariant measure of the form process $\{\sigma_t\}$.

It is well-known ([148]) that if $A(1), \dots, A(N)$ are commute, then there exists a unitary matrix U such that $U' A(j) U$ for $j = 1, 2, \dots, N$ are upper triangular. Thus, using Theorem 3.3.9, we can completely solve the almost sure exponential stability problem for the jump linear system (3.1.1) with commuting structure. We have

Corollary 3.3.11:

If σ_t has a single ergodic class, suppose that $A(1), \dots, A(N)$ commute, then there exists a unitary matrix U such that $U' A(j) U$ for $j = 1, 2, \dots, N$ are

of the following form:

$$U' A(i) U = \begin{pmatrix} \lambda_1(i) & a_{12}(i) & \dots & a_{1n}(i) \\ & \lambda_2(i) & \dots & a_{2n}(i) \\ & & \ddots & \vdots \\ & & & \lambda_n(i) \end{pmatrix}, \quad i = 1, 2, \dots, N.$$

It follows that the system (3.1.1) is almost surely exponentially stable if and only if

$$E_\pi \lambda_i(\sigma_0) = \pi_1 \lambda_i(1) + \pi_2 \lambda_i(2) + \dots + \pi_N \lambda_i(N) < 0 \quad (i = 1, 2, \dots, n)$$

where $\pi = \{\pi_1, \dots, \pi_N\}$ is the unique invariant measure of the form process $\{\sigma_t\}$. □

3.4. ESTIMATES OF LYAPUNOV EXPONENTS

It is well-known that the (top) Lyapunov exponent of the jump linear system (3.1.1) can be used to study the almost sure stability of the system, and (top) moment Lyapunov exponent can be used to study δ -moment stability ([74],[75], [96],[153]). Unfortunately, the computation of Lyapunov exponents is very difficult and usually impossible ([78]). Therefore, it is more practical to obtain some estimates for the Lyapunov exponents. In [153], Leizarowitz used the backward partial differential equations to obtain some estimates for the moment Lyapunov exponent. In this section, we will use the matrix measure to give estimates for the top Lyapunov exponent and moment Lyapunov exponents and a set of almost sure and δ -moment stability criteria is obtained as well. Moreover, an improvement over Leizarowitz's result is obtained and the proof is substantially simplified. For some cases, the differentiability of the δ -moment Lyapunov exponent is also discussed. Throughout this section, we only consider the system (3.1.1) with a stationary and ergodic form process σ_t , and all stability are with respect to $\Psi = \{\pi\}$ with π the unique invariant distribution of σ_t .

Let $|\cdot|$ be a vector norm on C^n and $\|\cdot\|$ be the induced matrix norm on $C^{n \times n}$, the (induced) matrix measure is defined as

$$\mu(A) = \lim_{\theta \downarrow 0^+} \frac{\|I + \theta A\| - 1}{\theta}$$

where I is the identity matrix.

The following development crucially depends on the Coppel's Inequality for linear systems of differential equations: Let $A(t)$ be a locally integrable function on $[t_0, +\infty)$, then the solution $x(t)$ of the linear system $\dot{x}(t) = A(t)x(t)$

with initial condition $x(t_0) = x_0$ satisfies the so-called Coppel's Inequality ([158]):

$$|x_0| \exp \left(- \int_{t_0}^t \mu(-A(s)) ds \right) \leq |x| \leq |x_0| \exp \left(\int_{t_0}^t \mu(A(s)) ds \right). \quad (3.4.1)$$

As before, the *top Lyapunov exponent* for the system (3.1.1) is defined as

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|\Phi(t, t_0)\|,$$

and the *top δ -moment Lyapunov exponent* is defined as

$$g(\delta) = \lim_{t \rightarrow \infty} \frac{1}{t} \log E \|\Phi(t, t_0)\|^\delta.$$

The indicated limits exist for the jump linear system with stationary ergodic form process and λ is a nonrandom constant almost surely. It is easy to show that the values of λ and $g(\delta)$ does not depend on the choice of the norm and the value of t_0 (we take $t_0 = 0$) and the above definition is consistent with [153].

From Coppel's Inequality, we can obtain the following estimates for the (top) Lyapunov exponent:

Theorem 3.4.1:

If $\{\sigma_t\}$ is a stationary ergodic finite state Markov chain with the unique invariant measure $\pi = (\pi_1, \dots, \pi_N)$, then we have

$$-[\pi_1 \mu(-A(1)) + \dots + \pi_N \mu(-A(N))] \leq \lambda \leq \pi_1 \mu(A(1)) + \dots + \pi_N \mu(A(N)).$$

Thus, if there exists a matrix measure μ such that

$$\pi_1 \mu(A(1)) + \pi_2 \mu(A(2)) + \dots + \pi_N \mu(A(N)) < 0,$$

then (3.1.1) is almost surely stable (w.r.t. $\{\pi\}$), where $\pi = (\pi_1, \dots, \pi_N)$ is the unique invariant distribution of σ_t . \square

Proof of Theorem: Notice that if $\{\sigma_t\}$ is ergodic, using Coppel's Inequality and the ergodic theorem, we can easily obtain the proof. \square

Remark: In [7], Mariton proved a similar result (Theorem 2.6 on page 44) stating that the system with an ergodic Markov chain is almost surely stable if $\sum_{i=1}^N \pi_i \sigma(A(i)) < 0$, where $\sigma(A)$ is the real part of the dominant eigenvalue of A . This is incorrect because from this we can conclude that if $A(1), \dots, A(N)$ are Hurwitz stable, then the system will be almost surely stable, however Example 3.3.7 gives counterexample to this statement. The reason is that Coppel's inequality is misinterpreted.

From Theorem 3.4.1, some simple stability criteria can be obtained by the suitable choice of matrix measure.

Corollary 3.4.2:

(i) If there exists a positive definite matrix P such that

$$\sum_{i=1}^N \pi_i \lambda_{\max}(PA(i)P^{-1} + A'(i)) < 0,$$

then the system (3.1.1) is almost surely stable.

(ii) If there exists positive numbers $\rho_1, \rho_2, \dots, \rho_n$ such that

$$\sum_{p=1}^N \pi_p \max_i \left\{ a_{ii}(p) + \sum_{j \neq i} \frac{\rho_j}{\rho_i} |a_{ij}(p)| \right\} < 0$$

or,

$$\sum_{p=1}^N \pi_p \max_j \left\{ a_{jj}(p) + \sum_{i \neq j} \frac{\rho_i}{\rho_j} |a_{ij}(p)| \right\} < 0,$$

then (3.1.1) is almost surely stable, where $A(p) = (a_{ij}(p))_{n \times n}$ for $p = 1, 2, \dots, N$. \square

Proof of Corollary: For (i), choose the norm $|x| = \sqrt{x'Px}$, then the induced matrix measure is given by

$$\mu_P(A) = \frac{1}{2} \lambda_{\max}(PAP^{-1} + A') \quad (3.4.2)$$

applying Theorem 3.4.1, we obtain the proof.

For (ii) Let $R = \text{diag}\{\rho_1, \rho_2, \dots, \rho_n\}$. If the first condition holds, we choose the norm $|x| = |R^{-1}x|_{\infty}$, where ∞ denotes ∞ -norm, then the induced matrix measure is given by

$$\mu(A) = \max_i \left\{ a_{ii} + \sum_{j \neq i} \frac{\rho_j}{\rho_i} |a_{ij}| \right\}.$$

If the second condition holds, we choose the norm $|x| = |Rx|_1$, where 1 denotes the 1-norm. The induced matrix measure is given by

$$\mu(A) = \max_j \left\{ a_{jj} + \sum_{i \neq j} \frac{\rho_i}{\rho_j} |a_{ij}| \right\}.$$

The result then follows from Theorem 3.4.1. \square

In [159] and Appendix B, we have studied many properties of matrix measure and discussed how to select appropriate matrix measure to improve the stability results. For large scale systems, we used matrix measure to obtain the Gershgorin typed circle theorem which can be applied to obtain sharp stability condition via the above results. The following example illustrates how to use the matrix measure to improve the stability result.

Example 3.4.3:

Let

$$A(1) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad A(2) = \begin{pmatrix} 0.5 & 1 \\ 0 & 0.5 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The problem is to study the stability of (3.1.1) with such structure. This example was studied by Feng et al ([74]), it has been proved that (3.1.1) is not second moment stable and the Lyapunov exponent technique has been used to study the almost sure stability. Here, we use the results developed above to study the almost sure stability. Note that $\{\sigma_t\}$ is ergodic with the unique invariant measure $\pi = (0.5, 0.5)$. Choose $P = \text{diag}\{1, 2\}$, from (3.4.2), it is easy to compute that

$$\mu_P(A(1)) = -1, \quad \mu_P(A(2)) = \frac{1}{2} + \frac{1}{2\sqrt{2}}.$$

Thus

$$\pi_1 \mu_P(A(1)) + \pi_2 \mu_P(A(2)) = \frac{1}{2} \left(-\frac{1}{2} + \frac{1}{2\sqrt{2}} \right) < 0.$$

From Theorem 3.4.1, we obtain that (3.1.1) is almost surely stable. In fact, for this example, we can use this method to approach the top Lyapunov exponent. Choose $P = \text{diag}\{1, b\}$, where b is a positive number to be determined. Then it is easy to verify that

$$\pi_1 \mu_P(A(1)) + \pi_2 \mu_P(A(2)) = \frac{1}{2} \left(-\frac{1}{2} + \frac{1}{2\sqrt{b}} \right).$$

Let $b \rightarrow +\infty$, then $\pi_1 \mu_P(A(1)) + \pi_2 \mu_P(A(2)) \rightarrow -1/4$, this is the top Lyapunov exponent as noted in [74].

In [153], Leizarowitz obtained the exact expression for the second moment top Lyapunov exponent and estimates for δ -moment top Lyapunov exponent.

The results in [153] involved a linear operator which is implicitly represented. Here, we first give a explicit representation for the linear operator and comment on the estimates obtained in [153], and then present some new estimates for δ -moment Lyapunov exponent $g(\delta)$.

Let \otimes denote the Kronecker product and \oplus be the Kronecker sum $A \oplus B = A \otimes B + B \otimes A$, and let $\text{vec}(A)$ denote the vector expansion of the matrix A (Refer to [149]).

Theorem 3.4.4:

For the jump linear system (3.1.1), define

$$H = \text{diag}\{I \oplus A'(1), I \oplus A'(2), \dots, I \oplus A'(N)\} + Q \otimes I$$

where I is the identity matrix with appropriate dimension. Then $g(2)$ is the largest real part of the eigenvalues of H . \square

Proof of Theorem: Let $S_i(t) = E\{\Phi'(t,0)\Phi(t,0)|\sigma_0 = i\}$, then it satisfies the backward (linear partial differential) equations, as presented in [153]:

$$\frac{dS_i(t)}{dt} = A'(i)S_i(t) + S_i(t)A(i) + \sum_{j=1}^N q_{ij}S_j(t), \quad S_i(0) = I, \quad \forall 1 \leq i \leq N, \quad t \geq 0.$$

This yields

$$\frac{d\text{vec}(S_i(t))}{dt} = (I \otimes A'(i))\text{vec}(S_i(t)) + (A' \otimes I)\text{vec}(S_i(t)) + \sum_{j=1}^N q_{ij}\text{vec}(S_j(t)).$$

Let $Y(t) = (\text{vec}'(S_1(t)), \dots, \text{vec}'(S_N(t)))'$, then the above equations can be written as the following compact form:

$$\frac{dY(t)}{dt} = H Y(t)$$

where H is given by

$$\begin{pmatrix} I \oplus A^T(1) + q_{11}I & q_{12}I & \dots & q_{1N}I \\ q_{21}I & I \oplus A^T(2) + q_{22}I & \dots & q_{2N}I \\ \vdots & \vdots & \ddots & \vdots \\ q_{N1}I & q_{N2}I & \dots & I \oplus A^T(N)I + q_{NN}I \end{pmatrix}$$

which can be written as

$$H = \text{diag}\{I \oplus A^T(1), I \oplus A^T(2), \dots, I \oplus A^T(N)\} + Q \otimes I.$$

Then it is easy to prove that

$$g(2) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|Y(t)\| = \max_{1 \leq i \leq n} \text{Re} \lambda_i(H).$$

This completes the proof. \square

For the general $\delta > 0$, Leizarowitz ([153]) used the backward equation with the aid of the comparison principle to obtain the following result:

Proposition 3.4.5: (Leizarowitz [153])

Choose a_1, a_2, \dots, a_N satisfying

$$\begin{cases} 2a_i I < A'(i) + A(i), & \text{if } 0 < \delta < 2; \\ 2a_i I > A'(i) + A(i), & \text{if } \delta > 2. \end{cases} \quad (3.4.4)$$

Let L be a linear operator defined on $M = (M_1, \dots, M_N) \in (\mathcal{R}^{n \times n})^N$,

$$(LM)_i = [A(i) + \frac{1}{2}(\delta - 2)a_i I]' M_i + M_i [A(i) + \frac{1}{2}(\delta - 2)a_i I] + \sum_{j=1}^N q_{ij} M_j.$$

Let $\zeta(\delta)$ be the largest real part of eigenvalues of L , then $g(\delta) \leq \zeta(\delta)$. \square

In his original paper ([153]), Leizarowitz missed the $1/2$ before $\delta - 2$. It is easy to show that if $0 < \delta < 2$, then $g(\delta) \leq g(2)$, i.e., $g(2)$ is an upper bound of $g(\delta)$. One may wonder if the proposition gives a better upper bound than

this. Unfortunately, for a large class of systems, Proposition 3.4.5 does not give improved result better than $g(2)$. We give the reason for this:

Assume that $\delta < 2$. We say A is positive stable if all real parts of eigenvalues of A are positive (i.e. $-A$ is stable). If $A(1), A(2), \dots, A(N)$ are not positive stable, we can prove that $a_i \leq 0$ for $i = 1, 2, \dots, N$. In fact, if $a_i > 0$, then according to the choice of a_i in Proposition 3.4.5, $A(i) + A'(i)$ is positive definite and thus for any $x \in C^n$, we have $x^*(A(i) + A'(i))x \geq \lambda_{\min}(A(i) + A'(i))x^*x \geq 0$. For any eigenvalue λ of $A(i)$, there exists a nonzero x such that $A(i)x = \lambda x$, so we have $x^*Ax = \lambda x^*x$ and $x^*A^*x = \bar{\lambda}x^*x$. Hence

$$(\lambda + \bar{\lambda})x^*x = x^*(A(i) + A'(i))x \geq \lambda_{\min}(A(i) + A'(i))x^*x > 0.$$

So $\text{Re}\lambda > 0$ and $A(i)$ is positive stable, which contradicts the assumption. Now, Let H denote the matrix defined previously and \bar{H} denote the matrix representation of the operator L . It is easy to show that \bar{H} has the similar structure as H . And we have

$$\begin{aligned} \bar{H} &= \text{diag}\{I \oplus (A(1) + \frac{1}{2}(\delta - 2)a_1I)', \dots, I \oplus (A(N) + \frac{1}{2}(\delta - 2)a_NI)'\} \\ &\quad + Q \otimes I \\ &= \text{diag}\{I \oplus A'(1), \dots, I \oplus A'(N)\} + Q \otimes I \\ &\quad + \frac{\delta - 2}{2} \text{diag}\{I \oplus (a_1I), \dots, I \oplus (a_NI)\} \\ &= H + \frac{\delta - 2}{2} \text{diag}\{I \oplus (a_1I), \dots, I \oplus (a_NI)\} \triangleq H + \frac{\delta - 2}{2}H_0. \end{aligned}$$

Since $a_i \leq 0$, $H_0 \leq 0$ elementwise, and $\delta < 2$, we have $(\delta - 2)H_0/2 \geq 0$ elementwise. Also notice that the off-diagonal elements of \bar{H} and H are nonnegative. Let $\sigma > 0$ be a large number such that $H + \sigma I$ and $\bar{H} + \sigma I$

are nonnegative. We have $\bar{H} + \sigma I \geq H + \sigma I$ and from the nonnegative matrix theory [148], we also have

$$\lambda_{\max}(\bar{H} + \sigma I) \geq \lambda_{\max}(H + \sigma I).$$

It follows that $\lambda_{\max}(\bar{H}) + \sigma \geq \lambda_{\max}(H) + \sigma$. Therefore, $\lambda_{\max}(H) \leq \lambda_{\max}(\bar{H}) = \zeta(\delta)$, i.e., $g(2) \leq \zeta(\delta)$. $g(\delta) \leq g(2)$ is thus a better estimate for this case. It is trivial that if A is stable, then it is not positive stable. Hence the above argument applies to a very large class of jump linear systems. The lower bound estimates for $g(\delta)$ given in [153] suffer a similar problem.

Despite the above dilemma, the proposition is still interesting for $\delta > 2$. In the following, we want to use the matrix measure technique to improve the above result and at the same time give a more direct proof for the Proposition 3.4.5. More important, the procedure used here is much more revealing and is of great potentiality in future development.

Theorem 3.4.6:

Let μ be an induced matrix measure by a norm $\|\cdot\|$. Define

$$\begin{aligned} H(\delta) = & \text{diag}\{I \oplus (A(i) + \frac{\delta-2}{2}\mu(A(i))I)', \dots, I \oplus (A(N) + \frac{\delta-2}{2}\mu(A(N)))'\} \\ & + Q \otimes I, \end{aligned}$$

for $\delta \geq 2$ and

$$\begin{aligned} H(\delta) = & \text{diag}\left\{I \oplus (A(i) - \frac{\delta-2}{2}\mu(-A(i))I)', \dots, I \oplus (A(N) \right. \\ & \left. - \frac{\delta-2}{2}\mu(-A(N)))'\right\} + Q \otimes I \end{aligned}$$

for $\delta < 2$. Let $\zeta(\delta)$ is the largest real part of the eigenvalues of $H(\delta)$. Then $g(\delta) \leq \zeta(\delta)$.

Proof of Theorem: Since the vector norm over \mathcal{R}^n are equivalent, for the vector norm $\|\cdot\|$ which induces the matrix measure μ and the 2–norm $\|\cdot\|_2$, there exists a constant $M > 0$ such that for any $x \in \mathcal{R}^n$, $\|x\| \leq M\|x\|_2$. From Coppel’s Inequality, we have

$$\|x(t)\| \leq \|x_0\| \exp\left(\int_0^t \mu(A(\sigma_s)) ds\right) \leq M\|x_0\|_2 \exp\left(\int_0^t \mu(A(\sigma_s)) ds\right) \quad (3.4.5)$$

For $\delta \geq 2$, we have

$$\begin{aligned} E\{\|x(t)\|^\delta\} &= E\{\|x(t)\|^2 \|x(t)\|^{\delta-2}\} \\ &\leq M^\delta E\{\|x(t)\|_2^2 \|x_0\|_2^{\delta-2} \exp\left((\delta-2) \int_0^t \mu(A(\sigma_s)) ds\right)\} \\ &\leq M^\delta \|x_0\|_2^{\delta-2} E\{\|x(t) \exp\left(\frac{\delta-2}{2} \int_0^t \mu(A(\sigma_s)) ds\right)\|_2^2\} \end{aligned} \quad (3.4.6)$$

Let $y(t) = x(t) \exp\left(\frac{\delta-2}{2} \int_0^t \mu(A(\sigma_s)) ds\right)$. then

$$\dot{y}(t) = (A(\sigma_t) + \frac{\delta-2}{2} \mu(A(\sigma_t))I)y(t), \quad y(0) = x_0 \quad (3.4.7)$$

Thus, from (3.4.6), we have

$$g(\delta) = \lim_{t \rightarrow \infty} \frac{1}{t} \log E\{\|x(t)\|^\delta\} \leq \lim_{t \rightarrow \infty} \frac{1}{t} \log E\{\|y(t)\|_2^2\} = \zeta(\delta).$$

In arriving the last equality, we have used (3.4.7) and Theorem 3.4.4. For $\delta < 2$, we use the other half of Coppel’s inequality to obtain the proof. This completes the proof of the theorem. \square

Since different matrix measure will give different estimate, the above result is a very general estimation for δ –moment top Lyapunov exponent. Indeed, when we use the matrix measure induced by the 2–norm, we can recover Leizarowitz’s result, the proposition 3.4.5. This can be seen as follows: As we

know that for any matrix A , $\mu_2(A) = \lambda_{\max}(A + A')/2$. According to the choice of a_i , for $\delta \geq 2$, $a_i \geq \mu_2(A(i))$, we have

$$E\{\|x(t)\|^\delta\} \leq E\{\|x(t)\|^2 \|x_0\|^{\delta-2} \exp((\delta-2)a_i t)\}$$

Following the same procedure as in the proof of Theorem 3.4.6, we obtain the result for the case when $\delta \geq 2$. Using the similar argument, we can obtain the result for the case when $\delta < 2$.

In a similar fashion, we can obtain the lower bound estimate for the δ -moment top Lyapunov exponent. The proof of the following result is completely similar to the upper bound case. Again, when applying the following result with the matrix measure induced by 2-norm, we recover the lower bound estimate of $g(\delta)$ given in Theorem 3.4.5 of [153].

Theorem 3.4.7:

Let μ be an induced matrix measure by a norm $\|\cdot\|$, define

$$L(\delta) = \text{diag} \left\{ I \oplus \left(A(i) - \frac{\delta-2}{2} \mu(-A(i)) I \right)', \dots, I \oplus \left(A(N) - \frac{\delta-2}{2} \mu(-A(N)) \right)' \right\} + Q \otimes I$$

for $\delta \geq 2$ and

$$L(\delta) = \text{diag} \left\{ I \oplus \left(A(i) + \frac{\delta-2}{2} \mu(A(i)) I \right)', \dots, I \oplus \left(A(N) + \frac{\delta-2}{2} \mu(A(N)) \right)' \right\} + Q \otimes I$$

for $\delta < 2$. Let $l(\delta)$ be the largest real part of the eigenvalues of $L(\delta)$. Then $g(\delta) \geq l(\delta)$. □

Remark:

It is well-known ([134]) that when δ is an integer, especially an even integer, the δ -moment Lyapunov can be represented by the real part of a large matrix obtained by the Kronecker product in terms of $A(1), A(2), \dots, A(N)$. The above procedure can be used to estimate the δ -moment Lyapunov exponent when δ is not an integer. For example, if we use $[\delta]$ to denote the integral part of δ and suppose that we have a way to compute the $[\delta]$ -moment Lyapunov exponent, then the δ -moment Lyapunov exponent with $A(1), A(2), \dots, A(N)$ has the upper bound which is the $[\delta]$ -Lyapunov exponent with

$$A(1) + \frac{\delta - [\delta]}{[\delta]} \mu(A(1))I, A(2) + \frac{\delta - [\delta]}{[\delta]} \mu(A(2))I, \dots, A(N) + \frac{\delta - [\delta]}{[\delta]} \mu(A(N))I.$$

Similar argument applies to the lower bound estimation. In [7] and [86], Mariton proved (Theorem 2.5 on page 43 in [7]) that (3.1.1) p -th moment stable iff F_p is Hurwitz stable, where F_p is a matrix in terms of Kronecker product, it is easy to see that this is only true for $p = 2$. From [86], we can easy to observe that the procedure used by Mariton is only valid for p -th mean value stability, not for the p -th moment stability. The mistake lies in the confusion between vector norm and the product of vector components. \square

Since δ -moment Lyapunov is an upper bound for $\delta\lambda$, here $\delta > 0$ and λ is the Lyapunov exponent. It follows that

$$\lambda \leq \frac{g(\delta)}{\delta}, \forall \delta > 0. \quad (3.4.8)$$

It is easy to observe that the above estimates for δ -moment Lyapunov exponent is in fact the exact expression for $g(2)$ when $\delta = 2$. Moreover, for

one dimensional system (3.1.1), the estimates are also the exact expression for $g(\delta)$ as stated in next proposition.

Proposition 3.4.8:

For one dimensional system (3.1.1), let $\tilde{a}_i = A(i)$. Define

$$H(\delta) = \delta \text{diag}\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_N\} + Q.$$

Then $g(\delta) = \lambda_{\max}(H(\delta))$. Moreover, (3.1.1) is δ -moment stable if and only if $H(\delta)$ is stable. \square

Proof of Proposition: Since for any matrix measure μ , $\mu(\pm A(i)) = \mu(\pm a_i) = \pm a_i$, using Theorem 3.4.6 and Theorem 3.4.7, we obtain $g(\delta) = \zeta(\delta) = l(\delta)$. The sufficiency for the δ -moment stability is trivial, and the necessity follows from the fact that δ -moment stability is equivalent to the δ -moment exponential stability, as proved in section 3.2. This completes the proof. \square

Remark: Like for the discrete-time case, we can give another proof for this. For one dimensional system, let $a(\sigma_k) = A(\sigma_k)$, then we can solve the equation (3.1.1), we have

$$\|x(t)\|^\delta = \left(\|x_0\|^\delta e^{\int_0^t \frac{\delta}{2} a(\sigma_\tau) d\tau} \right)^2.$$

Thus, the top δ -moment Lyapunov exponent of (3.1.1) is equal to the top second moment Lyapunov exponent of the following system:

$$\dot{y}(t) = \frac{\delta}{2} a(\sigma_t) y(t), \quad y(0) = \|x_0\|^{\delta/2}.$$

Then from Theorem 3.4.4 we can obtain Proposition 3.4.8.

From Coppel's Inequality and the above proposition, we can obtain the following result for higher dimensional system (3.1.1).

Theorem 3.4.9:

For any given matrix measure μ , let

$$L(\delta) = \delta \text{diag}\{-\mu(-A(1)), -\mu(-A(2)), \dots, -\mu(-A(N))\} + Q$$

$$U(\delta) = \delta \text{diag}\{\mu(A(1)), \mu(A(2)), \dots, \mu(A(N))\} + Q$$

then we have

$$\lambda_{\max}(L(\delta)) \leq g(\delta) \leq \lambda_{\max}(U(\delta)).$$

Moreover, (3.1.1) is δ -moment stable if $U(\delta)$ is stable. \square

Proof of Theorem: From Coppel's Inequality, we have

$$\|x_0\| \exp\left(\int_0^T -\mu(-A(\sigma_s))ds\right) \leq \|x(t)\| \leq \|x_0\| \exp\left(\int_0^t \mu(A(\sigma_s))ds\right).$$

Then we consider the one dimensional system

$$\dot{y}(t) = -\mu(-A(\sigma_t))y(t), \quad \dot{z}(t) = \mu(A(\sigma_t))z(t)$$

and using Proposition 3.4.8, we obtain the proof. \square

We mention here that since $H(\delta), L(\delta)$ and $U(\delta)$ are matrices whose off-diagonal elements are nonnegative, their eigenvalues with the largest real parts are actually real.

In the rest of this section, we study the analytic properties of the top δ -moment Lyapunov exponent and the top Lyapunov exponent. From Arnold et al's results ([96]), one may conjecture that $g(\delta)$ is an analytic function in a neighborhood of $\delta = 0$. Leizarowitz ([153]) heuristically showed that this can not be established for jump linear systems using the approach given by Arnold et al ([96]). However, Leizarowitz ([153]) did not provide a definite answer as to whether the conjecture was true or false. In what follows, we show that this

conjecture is in general not true, and more surprisingly, $g(\delta)$ is not generally differentiable in a neighborhood of the origin. Consequently, Arnold's formula connecting the sample path and moment Lyapunov exponents is not valid in general and a modification of Arnold's formula will be given here.

First, we study the one dimensional system (3.1.1). We have

Theorem 3.4.10:

For the one dimensional system (3.1.1) with an irreducible and stationary form process σ_t , let $\tilde{a}_i = A(i)$ for $i = 1, 2, \dots, N$. $g(\delta)$ is differentiable at $\delta = 0$ and

$$g'(0) = \alpha = \pi_1 a_1 + \pi_2 a_2 + \dots + \pi_N a_N.$$

□

Proof of theorem. Since the off-diagonal elements of $H(\delta)$ are nonnegative, the eigenvalue of $H(\delta)$ with the largest real part is real. From Theorem 3.4.8, we have $g(\delta) = \lambda_{\max}(H(\delta))$. Since Q is an irreducible matrix, there is a $K > 0$ sufficiently large such that $KI + H(\delta)$ is a nonnegative irreducible matrix and $g(\delta) = -K + \lambda_{\max}(KI + H(\delta))$. It follows from ([148]) that $\lambda_{\max}(KI + H(\delta))$ is a simple eigenvalue of $KI + H(\delta)$ for all δ with $|\delta|$ sufficiently small. Since $KI + H(\delta)$ is differentiable with respect to δ at $\delta = 0$, from Theorem 3.2.3 or Theorem 5.4 of ([160]), $\lambda_{\max}(KI + H(\delta))$ is differentiable at $\delta = 0$. Hence $g(\delta)$ is differentiable at $\delta = 0$. To show $g'(0) = \alpha$, we prove next that $g'(0^+) = \alpha$. For $\delta > 0$ with $\delta\alpha \leq g(\delta)$, we have $\alpha \leq g'(0^+)$. For any $\varepsilon > 0$, consider the following modified system

$$\dot{y}(t) = (A(\sigma_t) - (\alpha + \varepsilon))y(t) \tag{3.4.9}$$

Then it is easy to verify that the top Lyapunov exponent of (3.4.9) is $\alpha + (-(\alpha + \varepsilon)) = -\varepsilon < 0$. From Theorem 3.2.5, there exists $\delta_0 > 0$ such that (3.4.9) is exponentially δ_0 -moment stable and therefore exponentially δ -moment stable for all $0 < \delta \leq \delta_0$. Hence the δ -moment Lyapunov exponent of (3.4.9) is negative for $0 < \delta \leq \delta_0$. However, the δ -moment Lyapunov exponent of (3.4.9) is $-(\alpha + \varepsilon)\delta + g(\delta)$. Thus, we have $-(\alpha + \varepsilon)\delta + g(\delta) < 0$, i.e., $g(\delta) < (\alpha + \varepsilon)\delta$ for all $0 < \delta \leq \delta_0$. It follows that $g(\delta)/\delta \leq \alpha + \varepsilon$ for $0 < \delta \leq \delta_0$. From this we obtain that $g'(0^+) \leq \alpha$ and therefore $g'(0^+) = \alpha$. This completes the proof. \square

Remark:

We expect that for the one dimensional case, $g(\delta)$ is analytic. We use the following example to illustrate this point. For $N = 2$, $A(1) = a_1$ and $A(2) = a_2$ and $Q = \begin{pmatrix} -p & p \\ q & -q \end{pmatrix}$, then

$$H(\delta) = \begin{pmatrix} \delta a_1 - p & p \\ q & \delta a_2 - q \end{pmatrix}.$$

It is easy to compute $g(\delta)$, in fact,

$$\begin{aligned} g(\delta) &= \lambda_{\max}(H(\delta)) \\ &= \frac{\delta(a_1 + a_2) - p - q + \sqrt{\delta^2(a_1 - a_2)^2 + 2\delta(a_1 - a_2)(q - p) + (p + q)^2}}{2}, \end{aligned}$$

For small δ , $g(\delta)$ is indeed analytic.

For higher dimensional systems (3.1.1), one question remains: what happens to the analyticity of the top moment Lyapunov exponent $g(\delta)$? The following example shows that $g(\delta)$ is not even differentiable, and therefore is not analytic.

Example 3.4.11:

Consider the jump linear system (3.1.1) with a two-state Markov chain with infinitesimal operator $Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$. Assume that

$$A(\sigma_k) = \begin{pmatrix} a(\sigma_k) & 0 \\ 0 & b(\sigma_k) \end{pmatrix}, \quad a(1) = 1, \quad a(2) = b(1) = b(2) = 0.$$

Let $g_1(\delta)$ and $g_2(\delta)$ denote the top δ -moment Lyapunov exponents for the scalar jump linear systems $\dot{x}(t) = a(\sigma_t)x(t)$ and $\dot{x}(t) = b(\sigma_t)x(t)$, respectively. It can be easily proved that the top δ -moment Lyapunov exponent of (3.1.1) is given by

$$g(\delta) = \max\{g_1(\delta), g_2(\delta)\}.$$

It is easy to see that $g_2(\delta) = 0$, and from the formula for the one-dimensional case, we have

$$g_1(\delta) = \lambda_{\max}(H(\delta)) = \frac{\delta - 1 + \sqrt{1 + \delta^2}}{2}.$$

Therefore, we obtain

$$g(\delta) = \max\{g_1(\delta), g_2(\delta)\} = \begin{cases} \frac{\delta - 1 + \sqrt{1 + \delta^2}}{2}, & \delta \geq 0 \\ 0, & \delta < 0 \end{cases}.$$

Thus, it is easy to verify that $g'(0^+) = 0.5$ and $g'(0^-) = 0$, hence $g(\delta)$ is not differentiable at $\delta = 0$.

Notice however that we can compute the top Lyapunov exponent as follows: because the form process is ergodic with the unique invariant measure $\pi_1 = \pi_2 = 0.5$, if we let α_1 and α_2 denote the top Lyapunov exponents of the system $\dot{x}(t) = a(\sigma_t)x(t)$ and $\dot{x}(t) = b(\sigma_t)x(t)$, respectively, then from Theorem 3.9, the top Lyapunov exponent of (3.1.1) is given by

$$\alpha = \max\{\alpha_1, \alpha_2\} = \max\{\pi_1 a(1) + \pi_2 a(2), \pi_1 b(1) + \pi_2 b(2)\} = \frac{1}{2}.$$

We still have $g'(0^+) = \alpha$. From this observation, we may conjecture that this result holds for the general system (3.1.1). The following result for higher dimensional continuous-time jump linear systems is a modification of Arnold's formula ([96]).

Theorem 3.4.12:

Suppose that $\{\sigma_t\}$ is a finite state irreducible ergodic Markov chain, then $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0^+) = \alpha$.

Proof: From Proposition 3.2.4, we have

$$\lim_{k \rightarrow \infty} \frac{1}{k} E_\pi \{ \log \|\Phi(t_k, t_0)\| \} = a\alpha.$$

Thus, for any $\varepsilon > 0$, there exists an $m > 0$ such that

$$E_\pi \{ \log \|\Phi(t_m, t_0)\| \} < m(a\alpha + \varepsilon). \quad (3.4.10)$$

From the following inequality:

$$\frac{1}{pm} \log \|\Phi(t_{pm}, t_0)\| \leq \frac{1}{pm} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\|,$$

we obtain, for any $\varepsilon_1 > 0$,

$$\begin{aligned} & P_\pi \left(\frac{1}{pm} \log \|\Phi(t_{pm}, t_0)\| \geq a\alpha + \varepsilon + \varepsilon_1 \right) \\ & \leq P_\pi \left(\frac{1}{pm} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq a\alpha + \varepsilon + \varepsilon_1 \right) \\ & = P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq m(a\alpha + \varepsilon) + m\varepsilon_1 \right) \\ & \leq P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq E_\pi \{ \log \|\Phi(t_m, t_0)\| \} + m\varepsilon_1 \right), \end{aligned} \quad (3.4.11)$$

where in the last inequality, we have used (3.4.10). From the large deviation theorem (Lemma 3.2.6), there exists a $\delta_1 > 0$ and an $M_1 > 0$ such that

$$P_\pi \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi(t_{(j+1)m}, t_{jm})\| \geq E_\pi \{\log \|\Phi(t_m, t_0)\|\} + m\varepsilon_1 \right) \leq M_1 e^{-\delta_1 p}.$$

From this and (3.4.11), we have

$$P_\pi \left(\frac{1}{pm} \log \|\Phi(t_{pm}, t_0)\| \geq a\alpha + \varepsilon + \varepsilon_1 \right) \leq M_1 \left(e^{-\delta/m} \right)^{pm}. \quad (3.4.12)$$

Since $\{\sigma_t\}$ is a finite state Markov chain, there exists a $\kappa > 0$ such that

$$\|e^{A(j)t}\| \leq M_2 e^{\kappa t} \quad (\forall t \geq 0). \quad (3.4.13)$$

Let

$$A = \left(\omega : \frac{1}{pm} \log \|\Phi(t_{pm}, t_0)\| \geq a\alpha + \varepsilon + \varepsilon_1 \right),$$

$$A^c = \left(\omega : \frac{1}{pm} \log \|\Phi(t_{pm}, t_0)\| < a\alpha + \varepsilon + \varepsilon_1 \right).$$

Then, from (3.4.13), for all $\omega \in A$, we have

$$\|\Phi(t_{pm}, t_0)\| \leq e^{\kappa t_{pm}},$$

and for all $\omega \in A^c$, we have

$$\|\Phi(t_{pm}, t_0)\| \leq e^{pm(a\alpha + \varepsilon + \varepsilon_1)}.$$

Thus, we obtain

$$\begin{aligned} E_\pi \|\phi(t_{pm}, t_0)\|^\delta &= \int_A \|\Phi(t_{pm}, t_0)\|^\delta P_\pi(d\omega) + \int_{A^c} \|\Phi(t_{pm}, t_0)\|^\delta P_\pi(d\omega) \\ &\leq \int_A e^{\kappa t_{pm} \delta} P_\pi(d\omega) + e^{pm(a\alpha + \varepsilon + \varepsilon_1) \delta}. \end{aligned} \quad (3.4.14)$$

Following a similar procedure as in the proof of Lemma 3.2.7 and using (3.4.12), we can prove that there exists a $\delta_2 > 0$, such that for $0 < \delta \leq \min\{\delta_1, \delta_2\}$ we have

$$\lim_{p \rightarrow \infty} \int_A e^{\kappa t_{pm} \delta} P_\pi(d\omega) = 0.$$

Hence for sufficiently large p , we have

$$\int_A e^{\kappa t_{pm} \delta} P_\pi(d\omega) < 1. \quad (3.4.15)$$

If $\alpha \geq 0$, then from (3.4.14) and (3.4.15), we have

$$\frac{1}{pm} \log E_\pi \|\Phi(t_{pm}, t_0)\|^\delta \leq \frac{1}{pm} \log \left(1 + e^{pm(a\alpha + \varepsilon + \varepsilon_1)\delta} \right) \leq \frac{\log 2}{pm} + (a\alpha + \varepsilon + \varepsilon_1)\delta.$$

Letting $p \rightarrow +\infty$ and using the fact:

$$\lim_{p \rightarrow \infty} \frac{1}{pm} \log E_\pi \|\Phi(t_{pm}, t_0)\|^\delta = \lim_{p \rightarrow \infty} \frac{t_{pm}}{pm} \cdot \frac{1}{t_{pm}} \log E_\pi \|\Phi(t_{pm}, t_0)\|^\delta = ag(\delta),$$

we obtain $ag(\delta) \leq (a\alpha + \varepsilon\varepsilon_1)\delta$, i.e.,

$$\frac{g(\delta)}{\delta} \leq \alpha + \frac{\varepsilon + \varepsilon_1}{a}.$$

Letting $\delta \downarrow 0^+$, we have $g'(0^+) \leq \alpha + (\varepsilon + \varepsilon_1)/a$. Because of the arbitrary choice of ε and ε_1 , we obtain $g'(0^+) \leq \alpha$. On the other hand, it is very easy to show that $g'(0^+) \geq \alpha$. This proves that $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0^+) = \alpha$ for $\alpha \geq 0$.

If $\alpha < 0$, then we choose $\beta > -\alpha$, and define $B(j) = A(j) + \beta I$ ($j \in \underline{N}$).

Consider the new system

$$\dot{x}(t) = B(\sigma_t)x(t), \quad (3.4.16)$$

let $\tilde{g}(\delta)$ and $\tilde{\alpha}$ denote the top δ -moment Lyapunov exponent and the top Lyapunov exponent, respectively, for the system (3.4.16). It is easy to show

that $\tilde{\alpha} = \beta + \alpha$ and $\tilde{g}(\delta) = g(\delta) + \beta\delta$. Now, since $\tilde{\alpha} > 0$, $\tilde{g}(\delta)$ is differentiable from the right at $\delta = 0$ and $\tilde{g}'(0^+) = \tilde{\alpha}$, from which we conclude that $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0^+) = \alpha$. This completes the proof. \square

One may ask under what conditions will $g(\delta)$ be differentiable at $\delta = 0$? A careful study shows that in Example 3.4.11, the system (3.1.1) is not regular in the sense of Arnold ([96]) or in the sense of the following *weak regularity condition*:

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log \|\Phi^{-1}(t, t_0)\| = -\alpha, \quad (\text{a. s.}) \quad (\text{WRC})$$

If the above weak regularity condition holds, then $g(\delta)$ is indeed differentiable at $\delta = 0$. The following result can be proved:

Proposition 3.4.11:

Suppose that $\{\sigma_t\}$ is a finite state irreducible ergodic Markov chain and the system (3.1.1) satisfies the weak regularity condition (WRC), then $g(\delta)$ is differentiable at $\delta = 0$ and $g'(0) = \alpha$.

Schetch of the Proof: It suffices to prove that $g'(0^-) = \alpha$. First, from the weak regularity condition (WRC), the following can be obtained:

$$\lim_{t \rightarrow \infty} \frac{1}{t} E_\pi \log \|\Phi^{-1}(t, t_0)\| = -\alpha. \quad (3.4.17)$$

From the inequality:

$$1 = \|I\| = \|\Phi(t, t_0) \cdot \Phi^{-1}(t, t_0)\| \leq \|\Phi(t, t_0)\| \cdot \|\Phi^{-1}(t, t_0)\|,$$

we can obtain the following:

$$\frac{1}{pm} \log \|\Phi(t_{pm}, t_0)\| \geq -\frac{1}{m} \left(\frac{1}{p} \sum_{j=0}^{p-1} \log \|\Phi^{-1}(t_{(j+1)m}, t_{jm})\| \right). \quad (3.4.18)$$

From (3.4.17) and (3.4.18), a large deviation result can be obtained, i.e., for any $\varepsilon > 0$, there exists a $\delta_1 > 0$ and an $M_1 > 0$ such that

$$P_\pi \left(\frac{1}{pm} \log \|\Phi(t_{pm}, t_0)\| < \alpha - \varepsilon \right) \leq M_1 e^{-\delta_1 p}.$$

then the procedure in the proof of Theorem 5.2 can be modified to prove $g'(0^-) = \alpha$. □

3.5. Robust Stability Analysis

As we remarked in Section 2.8 for discrete-time case, we consider the robust stability against any randomness and possible parameter variations. If we can design the system so that the resulting closed-loop jump linear system is robust against any randomness, then the system performance is much more desirable because in practice the statistics of the form process may not be exactly determined or hard to collect. In this section, we will use the matrix measure techniques to study this issue, for illustration, we only discuss the almost sure stability, in fact, all conditions we obtained also guarantee the robust moment stability and the robust stability problem is reduced to a deterministic stability problem. For the details of the matrix measure, the reader is referred to Appendix B.

Definition 3.5.1:

The system (3.1.1) is robustly almost surely stable if it is almost surely stable for any finite state form process.

From the Coppel's inequality (3.4.1), we can easily obtain the following result.

Theorem 3.5.2:

If there exists a matrix measure $\mu(\cdot)$ such that $\mu(A(i)) < 0$ for any $i \in \underline{N}$, then (3.1.1) is robustly almost surely stable.

Proof: Let $h = \max_{1 \leq i \leq N} \mu(A(i))$, then from the assumption, we have $h < 0$. From the Coppel's inequality, we have

$$\|x(t)\| \leq \|x_0\| e^{\int_0^t \mu(\sigma_\tau) d\tau} \leq \|x_0\| e^{ht} \xrightarrow{t \rightarrow \infty} 0.$$

Therefore, (3.1.1) is robustly almost surely stable. \square

As we observed in the discrete-time case, this result, though simple, is very general, which we can see next.

Let $A(k) = (a_{ij}^{(k)})$ ($k = 1, 2, \dots, N$), then, we have

Theorem 3.5.3.

Let $m_{ij} = \max_{1 \leq k \leq N} |a_{ij}^{(k)}|$ ($i \neq j$), and $m_{ii} = \max_{1 \leq k \leq p} a_{ii}^{(k)}$ and $M = (m_{ij})$, then (3.1.1) is robustly almost surely stable if M is Hurwitz stable or all the leading principal minors of $-M$ are positive.

Proof: Since $m_{ij} \geq 0$ ($i \neq j$) and M is stable, we know that $-M$ is M -matrix, i.e., $-M$ satisfying (1) of Lemma 3.8.6, from which we obtain from Lemma 3.8.6 that there exists positive numbers r_1, r_2, \dots, r_n such that $\sum_{j \neq i} r_j m_{ij} < r_i(-m_{ii})$, that is,

$$\max_{1 \leq k \leq p} a_{ii}^{(k)} + \sum_{j \neq i} \frac{r_j}{r_i} \max_{1 \leq k \leq p} |a_{ij}^{(k)}| < 0 \quad (1 \leq i \leq n). \quad (3.5.1)$$

Let $R = \text{diag}\{r_1, r_2, \dots, r_n\}$, from Appendix B, the matrix measure induced by the vector norm $\|Rx\|_\infty$ is given by

$$\mu_R^\infty(A) = \max_i \left[a_{ii} + \sum_{j \neq i} \frac{r_j}{r_i} |a_{ij}| \right].$$

From (3.5.1), it is easy to verify that $\mu_R^\infty(A(k)) < 0$ ($1 \leq k \leq N$). From Theorem 2.5.2, we conclude that (3.1.1) is robustly almost surely stable. This completes the proof. \square

Remarks:

- (a). From Lemma 2.8.6, we can see that the sufficiency condition can be replaced by any one of equivalent conditions in Lemma 2.8.6, in particular, the most easiest one to check is that (3.1.1) is robustly almost surely stable if the leading principal minors of $-M$ are positive.
- (b). Theorem 3.5.3 implies that we do not need to find the exact positive numbers r_1, r_2, \dots, r_n for the stability test, this is great improvement over the Gershgorin-type criteria, in which suitable choice of the scaling numbers r_1, r_2, \dots, r_n has to be determined, which is not an easy task. Moreover, to check the robust almost sure stability, we only need to check the stability of one matrix.

For some special class of jump linear systems, Theorem 3.5.3 may give necessary and sufficient condition for the robust stability. We have

Corollary 3.5.4:

Suppose that $a_{ij}^{(k)} \geq 0$ ($i \neq j$), and there exists a $r \in \{1, 2, \dots, N\}$ such that $A(k) \leq A(r)$ elementwise, then (3.1.1) is robustly almost surely stable if and only if A_r is Hurwitz stable. \square

Proof: The necessity are trivial. Now, we prove the sufficiency. Because $a_{ij}^{(k)} \geq 0$ ($i \neq j$), and $A(k) \leq A(r)$, we have

$$\max_{1 \leq k \leq N} |a_{ij}^{(k)}| = \max_{1 \leq k \leq N} a_{ij}^{(k)} = a_{ij}^{(r)}, \quad (i \neq j),$$

and $\max_{1 \leq k \leq N} a_{ii}^{(k)} = a_{ii}^{(r)}$, so $M = A(r)$. From Theorem 3.5.3, we obtain that (3.1.1) is robustly almost surely stable. This completes the proof. \square

The following results can also be easily obtained:

Corollary 3.5.5:

- (a). Suppose that $A(1), A(2), \dots, A(N)$ are normal matrices, then (3.1.1) is robustly almost surely stable if and only if $A(1), A(2), \dots, A(N)$ are Hurwitz stable;
- (b). Suppose that $A(1), A(2), \dots, A(N)$ are upper triangular matrices, then (3.1.1) is robustly almost surely stable if and only if $A(1), A(2), \dots, A(N)$ are Hurwitz stable;
- (c). Suppose there exists a positive definite matrix S such that $SA(i) + A^T(i)S$ is negative definite for any $i \in \underline{N}$, then (3.1.1) is robustly almost surely stable. □

As we remarked in last chapter, the robustness results are very strong, the robustness we obtained is not only against any randomness, but also against parameter perturbation. In fact, we have

Theorem 3.5.6:

Suppose there exists a matrix measure $\mu(\cdot)$ such that $\mu(\bar{A}(i)) < 0$ for any $i \in \underline{N}$, then the system (3.1.1) is robustly almost sure stable for any $A(1), A(2), \dots, A(N)$ satisfying

$$A(j) \in \left\{ A \mid A = \sum_{k=1}^N \beta_k \bar{A}(k), \beta_k \geq 0, \sum_{k=1}^N \beta_k = 1 \right\}, \quad \forall j \in \underline{N}.$$

□

Next, we present some illustrative examples how the criteria developed in this paper can be used to study the robust stability.

Example 3.5.7

Let

$$A(1) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad A(2) = \begin{pmatrix} -12 & -3 \\ 4 & 5 \end{pmatrix}.$$

Let $T = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, then

$$TA(1)T^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad TA(2)T^{-1} = \begin{pmatrix} -9 & 2 \\ 0 & -8 \end{pmatrix}.$$

Define $|x| = |Tx|_2$, the matrix measure induced by this norm is denoted by μ , then $\mu(A) = \mu_2(TAT^{-1})$ and $\mu(A(1)) = -1 < 0$ and $\mu(A(2)) = -7.382 < 0$. From Theorem 3.5.2, we obtain that (3.1.1) is robustly almost surely stable.

Remark: Note that for any $R = \text{diag}\{r_1, r_2\}$, $\mu_R(A) = \mu_\infty(R^{-1}AR)$, then for any positive numbers r_1 and r_2 , from Appendix B,

$$\mu_R(A_2) = \mu_\infty(R^{-1}A_2R) = 4\frac{r_1}{r_2} + 5 > 0$$

the Gershgorin like criteria can not be used. This can also be verified by Theorem 3.5.3. In fact, $M = \begin{pmatrix} 0 & 3 \\ 4 & 5 \end{pmatrix}$, which is not Hurwitz stable, hence we can not use Theorem 3.5.3.

Example 3.5.8:

$$A(1) = \begin{pmatrix} -5 & 1 \\ 4 & 6 \end{pmatrix}, \quad A(2) = \begin{pmatrix} -3 & 2 \\ 5 & -4 \end{pmatrix}.$$

We want to use Theorem 3.5.3. Since we have $M = \begin{pmatrix} -3 & 2 \\ 5 & -4 \end{pmatrix}$, and $-M = \begin{pmatrix} 3 & -2 \\ -5 & -4 \end{pmatrix}$, whose principal minors are 3, 4 and 2 which are all positive, from Theorem 3.5.3 we obtain that (3.1.1) is robustly almost surely stable.

3.6. Almost Sure and Moment Stabilization

Second moment (mean square) stabilization problems have been studied by many researchers in the current literature. Ji et al ([71]) and Morozan ([53]) reduced the second moment stabilization problem to solving a coupled system of Riccati equations. Mariton ([86]) applied the Homotopy theory to give a numerical procedure for second moment stabilization. In this section, we present some preliminary research results for both almost sure and δ -moment stabilization.

For clarity, we first give the following definition.

Definition 3.6.1:

Consider the jump linear control system

$$\dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))u(t). \quad (3.6.1)$$

If there exists a feedback control $u(t) = -K(\sigma(t))x(t)$ such that the resulting closed-loop control system is stochastically stable in the sense of Definition 3.2.1, then the control system (3.6.1) is said to be stochastically stabilizable in the corresponding sense. If the resulting closed-loop system is absolutely stable, then the system (3.6.1) is absolutely stabilizable. If the feedback control $K(\sigma(t)) = K$ is independent of the form process, then the system (3.6.1) is simultaneously stochastically stabilizable in the corresponding sense.

Similar to Theorem 3.2.2, we obtain the following result.

Theorem 3.6.2:

For the system (3.6.1) with a finite state Markov form process $\{\sigma(t)\}$ and with any $\delta > 0$, δ -moment stabilizability, δ -moment exponential stabilizability and stochastic δ -moment stabilizability are equivalent, and each implies almost sure stabilizability. \square

Remark: Due to this theorem, from now on, we use δ -moment stabilizability to denote any one of the above three δ -moment stabilizability concepts.

It is easy to see that absolute stabilizability implies stochastic stabilizability in any sense, and simultaneous stochastic stabilizability implies stochastic stabilizability in the corresponding sense. However, absolute stabilizability is too conservative to be useful in applications. The simultaneous stochastic stabilizability problem has been studied in the current literature, however, simultaneous stabilizability is also often too conservative, the next example is illustrative.

Example 3.6.3: (δ -moment stabilizability does not imply simultaneous δ -moment stabilizability, and almost sure stabilizability does not guarantee simultaneous almost sure stabilizability)

Let $A(1) = a > 0$, $A(2) = b > 0$, $B(1) = 1$ and $B(2) = -1$, the form process $\{\sigma(t)\}$ has the infinitesimal generator $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, with the unique invariant measure $\pi = (0.5, 0.5)$. If we choose $K(1) = a + 1$ and $K(2) = -b - 1$, then $A(\sigma(t)) - B(\sigma(t))K(\sigma(t)) = -1$, hence the closed-loop system is deterministic and stable. Then the jump linear system (3.6.1) is absolutely stabilizable, δ -moment stabilizable and almost surely stabilizable.

However, we want to show that the system can not be simultaneously almost surely stabilized, and from Theorem 3.6.2 this implies that the system can not be δ -moment stabilized. For any K , using the feedback control $u(t) = -Kx(t)$, the closed-loop system becomes

$$\dot{x}(t) = (A(\sigma(t)) - B(\sigma(t))K)x(t),$$

and its solution is given by

$$x(t) = x_0 \exp \left(\int_0^t (A(\sigma(\tau)) - B(\sigma(\tau))K) d\tau \right).$$

From this and the ergodic theorem, we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \log \|x(t)\| &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (A(\sigma(\tau)) - B(\sigma(\tau))K) d\tau \\ &= \pi_1(A(1) - B(1)K) + \pi_2(A(2) - B(2)K) \\ &= \frac{1}{2}(a - K + b + K) = \frac{1}{2}(a + b) > 0. \end{aligned}$$

This implies that (3.6.1) can not be simultaneously almost surely stabilized.

From Theorem 3.6.2, we can see that δ -moment stabilizability is equivalent to δ -moment stochastic stabilizability which involves a cost functional similar to linear quadratic optimal control system design. Therefore, we may easily reduce the δ -moment stabilization problem to an appropriate optimal control problem.

Theorem 3.6.4:

Given any positive definite matrices $Q(1), Q(2), \dots, Q(N)$ and $R(1), R(2), \dots, R(N)$, the system (3.6.1) is δ -moment stabilizable if and only if the following minimization problem:

$$\min J[u] = \int_0^\infty E (x^T(t)Q(\sigma(t))x(t) + u^T(t)R(\sigma(t))u(t))^{\delta/2} dt$$

subject to (3.6.1) has a finite cost. In particular, the system (3.6.1) is δ -moment stabilizable if and only if the following optimal control problem:

$$\min J[u] = \int_0^\infty E (x^T(t)x(t) + u^T(t)u(t))^{\delta/2} dt$$

subject to (3.6.1) has a finite cost.

Proof: Suppose that (3.6.1) is δ -moment stabilizable, then there exist matrices $K(1), K(2), \dots, K(N)$ such that the system

$$\dot{x}(t) = (A(\sigma(t)) - B(\sigma(t))K(\sigma(t)))x(t), \quad (3.6.2)$$

is δ -moment stabilizable, i.e., for the feedback control:

$$u(t) = -K(\sigma(t))x(t),$$

we have (from Theorem 3.2.2)

$$\int_0^\infty E \|x(t)\|^\delta dt = \int_0^\infty E (x^T(t)x(t))^{\delta/2} dt < \infty.$$

Thus for the given control, we have

$$\begin{aligned} J[u] &= \int_0^\infty E (x^T(t)Q(\sigma(t))x(t) + u^T(t)R(\sigma(t))u(t))^{\delta/2} dt \\ &= \int_0^\infty E (x^T(t)(Q(\sigma(t)) + K^T(\sigma(t))R(\sigma(t))K(\sigma(t)))x(t))^{\delta/2} dt \\ &\leq M^{\delta/2} \int_0^\infty E (x^T(t)x(t))^{\delta/2} dt < \infty, \end{aligned}$$

where

$$M = \max_{1 \leq i \leq N} \lambda_{\max} (Q(i) + K^T(i)R(i)K(i)).$$

Therefore, the optimal control problem has a finite cost.

Conversely, suppose that the given optimal control problem has a finite cost, then there exist matrices $K(1), K(2), \dots, K(N)$ such that the solution of

the closed-loop system (3.6.2) with the feedback control $u(t) = -K(\sigma(t))x(t)$ satisfies the following:

$$J[u] = \int_0^\infty E(x^T(t)Q(\sigma(t))x(t) + u^T(t)R(\sigma(t))u(t))^{\delta/2} dt < \infty,$$

therefore, we have

$$\int_0^\infty E(x^T(t)x(t))^{\delta/2} dt \leq M_1^{\delta/2} \int_0^\infty E(x^T(t)Q(\sigma(t))x(t))^{\delta/2} dt < \infty,$$

where

$$M_1 = \max_{1 \leq i \leq N} \frac{1}{\lambda_{\min}(Q(i))}.$$

From Theorem 3.2.1, (3.6.2) is δ -moment stable, hence (3.6.1) is δ -moment stabilizable. This completes the proof. \square

It may seem that we have complicated the matter by reducing the stabilizability problem to an optimal control problem which seems to be much more difficult to solve. However, since the optimal control problem has been studied for a long time and many numerical algorithms have been developed in the current literature, for some cases, this consideration may lead to some tractable criteria for moment stabilization. This is certainly the case for second moment stabilizability. We have

Theorem 3.6.5:

Given any positive definite matrices $Q(1), Q(2), \dots, Q(N)$ and $R(1), R(2), \dots, R(N)$, (3.6.1) is mean square (second moment) stabilizable if and only if the following coupled systems of algebraic Riccati equations: for $i \in \underline{N}$)

$$A^T(i)P(i) + P(i)A(i) - P(i)B(i)R^{-1}(i)B^T(i)P(i) + \sum_{j=1}^N q_{ij}P(j) = -Q(i), \quad (3.6.3)$$

has a positive definite solution $P(1), P(2), \dots, P(N)$.

In particular, the system (3.6.1) is mean square stabilizable if and only if the following coupled system of algebraic Riccati equations:

$$A^T(i)P(i) + P(i)A(i) - P(i)B(i)B^T(i)P(i) + \sum_{j=1}^N q_{ij}P(j) = -I, \quad (i = 1, 2, \dots, N) \quad (3.6.4)$$

has a positive definite solution $P(1), P(2), \dots, P(N)$.

Proof: This result can be obtained by using Theorem 3.6.5 and the result for linear quadratic optimal control problem (see Ji and Chizeck ([71]) or Wonham ([35])). \square

Corollary 3.6.6:

If (3.6.1) is mean square stabilizable, then there exist positive definite matrices $P(1), P(2), \dots, P(N)$ such that

$$A(i) - B(i)B^T(i)P(i) - \frac{1}{2}q_i I \quad (i = 1, 2, \dots, N)$$

are stable.

Proof: Suppose that (3.6.1) is mean square stabilizable, from Theorem 3.6.5, there exist positive definite matrices $P(1), P(2), \dots, P(N)$ such that (3.3) holds. From (3.3), we can easily obtain

$$\begin{aligned} & (A(i) - B(i)B^T(i)P(i) - \frac{1}{2}q_i I)^T + P(i)(A(i) - B(i)B^T(i)P(i) - \frac{1}{2}q_i I) \\ &= -I - \sum_{j=1}^N q_{ij}P(j) - P(i)B(i)B^T(i)P(i). \end{aligned}$$

Since $P(i)$ and $I + \sum_{j=1}^N q_{ij}P(j) + P(i)B(i)B^T(i)P(i)$ are positive definite, from Lyapunov equation theory, we conclude that $(A(i) - B(i)B^T(i)P(i) - \frac{1}{2}q_i I)$ is stable. \square

It is obvious that the mean square stabilizability problem is equivalent to the existence of a positive definite solution of the coupled system of Riccati equations, this does not, however, reduce the complexity of the problem considerably. For a linear time-invariant system, we know that controllability implies stabilizability. For convenience, we introduce the following definition.

Definition 3.6.7:

For the system (3.6.1) with an N state Markov chain $\{\sigma(t)\}$, (3.6.1) is individual mode controllable (stabilizable) if $(A(i), B(i))$ ($i = 1, 2, \dots, N$) are controllable (stabilizable) for each i .

One natural question to ask is: does the individual mode controllability imply δ -moment stabilizability? For $\delta = 2$, Corollary 3.6.6 can be used to construct simple example to show that the answer to this question is negative.

Example 3.6.8: (*Individual mode controllability does not imply mean square stabilizability*)

Let

$$A(1) = \begin{pmatrix} 0.5 & 10 \\ 0 & 0.5 \end{pmatrix}, \quad B(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$A(2) = \begin{pmatrix} 0.5 & 0 \\ 10 & 0.5 \end{pmatrix}, \quad B(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(Without specification, we always use Q to denote the infinitesimal generator of the finite state Markov chain $\{\sigma_t\}$). It is obvious that $(A(1), B(1))$ and

$(A(2), B(2))$ are controllable, hence (3.6.1) is individual mode controllable. However, for any positive matrix $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, we have

$$A(1) - B(1)B^T(1)P - \frac{1}{2}q_1I = \begin{pmatrix} -p_{11} & 10 - p_{12} \\ 0 & 0 \end{pmatrix},$$

which is not stable. From Corollary 3.6.6, we can conclude that (3.6.1) is not mean square stabilizable. Notice also that the form process $\{\sigma_t\}$ with the given infinitesimal generator Q is an ergodic Markov chain!

From Theorem 3.6.5, we can also observe that although (3.6.1) is individual mode controllable, the infinite horizon linear quadratic optimal control problem does not have a solution.

Although the mean square stabilizability problem has been reduced to the solvability of a coupled system of Riccati equations (3.6.3) or (3.6.4), it is not very difficult to solve (3.6.3) or (3.6.4) analytically. Wonham ([35]) gave a recursive procedure for solving the coupled system of Riccati equations under a very restrictive condition. This algorithm involved integration over an infinite horizon, which made the algorithm impractical. In order to give some qualitative properties about the solution of the coupled Riccati equation, we first consider the coupled Lyapunov equation, which plays a key role in the study of mean square stability.

Consider the coupled system of Lyapunov equations:

$$A^T(i)P(i) + P(i)A(i) + \sum_{j=1}^N q_{ij}P(j) = -Q(i), \quad i \in \underline{N}, \quad (3.6.5)$$

where $\underline{N} = \{1, 2, \dots, N\}$ (which will be used for the rest of the paper). Let $\text{vec}(X)$ denote the column vector expansion of a matrix X , \otimes denotes

the Kronecker product and \oplus denotes the Kronecker sum, i.e., $A \oplus B = A \otimes B + B \otimes A$. We have

Theorem 3.6.9:

For any matrices $Q(1), Q(2), \dots, Q(N)$, (3.6.5) has a unique solution if and only if the following matrix F is nonsingular:

$$F = \begin{pmatrix} I \oplus A^T(1) - q_1 I & q_{12} I & \dots & q_{1N} I \\ q_{21} I & I \oplus A^T(2) - q_2 I & \dots & q_{2N} I \\ \vdots & \vdots & \ddots & \vdots \\ q_{N1} I & q_{N2} I & \dots & I \oplus A^T(N) - q_N I \end{pmatrix} \\ = \text{diag}\{I \oplus A^T(1), \dots, I \oplus A^T(N)\} + Q \otimes I,$$

where Q is the infinitesimal generator of the finite state Markov chain $\{\sigma_t\}$.

Proof: From (3.6.5), applying the vector expansion operator vec on both sides of (3.6.5) and the property $\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X)$ (refer to Horn and Johnson [149]), we obtain

$$(I \otimes A^T(i))\text{vec}(P(i)) + (A^T(i) \otimes I)\text{vec}(P(i)) + \sum_{j=1}^N q_{ij}\text{vec}(P(j)) = -\text{vec}(Q(i)),$$

hence, we obtain

$$F \begin{pmatrix} \text{vec}(P(1)) \\ \text{vec}(P(2)) \\ \vdots \\ \text{vec}(P(N)) \end{pmatrix} = - \begin{pmatrix} \text{vec}(Q(1)) \\ \text{vec}(Q(2)) \\ \vdots \\ \text{vec}(Q(N)) \end{pmatrix}.$$

From this, we conclude that (3.6.5) has a unique solution if and only if F is nonsingular. \square

From Theorem 3.4.4 and Theorem 3.6.9, we obtain the following:

Theorem 3.6.10:

The jump linear system $\dot{x}(t) = A(\sigma(t))x(t)$ is mean square stable if and only if for any positive definite matrices $Q(1), Q(2), \dots, Q(N)$, the coupled system of Lyapunov equations (3.6.5) has a positive definite solution, equivalently, if and only if F is Hurwitz stable. \square

Corollary 3.6.11:

Given positive definite matrices $Q(1), Q(2), \dots, Q(N)$, then the coupled system of Lyapunov equations (3.6.5) has a positive definite solution $P(1), P(2), \dots, P(N)$ if and only if F is Hurwitz stable. \square

Corollary 3.6.12:

Given positive semidefinite matrices $Q(1), Q(2), \dots, Q(N)$, if F is Hurwitz stable, then the coupled system of Lyapunov equations (3.6.5) has a positive semidefinite solution $P(1), P(2), \dots, P(N)$.

Proof: Since $Q(i) \geq 0$, for any $\beta > 0$, $Q(i) + \beta I > 0$. If F is Hurwitz stable, then from Corollary 3.6.11, the coupled Lyapunov equation:

$$A^T(i)P(i) + P(i)A(i) + \sum_{j=1}^N q_{ij}P(j) = -(Q(i) + \beta I), \quad i \in \underline{N} \quad (3.6.6)$$

has a unique solution, say, $P(1, \beta), \dots, P(N, \beta)$, and

$$\begin{pmatrix} \text{vec}(P(1, \beta)) \\ \text{vec}(P(2, \beta)) \\ \vdots \\ \text{vec}(P(N, \beta)) \end{pmatrix} = -F^{-1} \begin{pmatrix} \text{vec}(Q(1) + \beta I) \\ \text{vec}(Q(2) + \beta I) \\ \vdots \\ \text{vec}(Q(N) + \beta I) \end{pmatrix}. \quad (3.6.7)$$

Because F^{-1} is a constant matrix, the right hand side of (3.6.7) is continuous in β , so is the left hand side of (3.6.7), hence $P(i, \beta)$, $\lambda_{\min}(P(i, \beta))$ are continuous

in β . From $\lambda_{\min}(P(i, \beta)) > 0$, we obtain $\lambda_{\min}(P(i, 0)) \geq 0$, i.e., $P(i, 0)$ is a positive semidefinite solution of (3.6.5), This completes the proof. \square

Next, we study the properties of the solutions of a coupled system of Riccati equations. We will concentrate on the equation (3.6.4), we have

Theorem 3.6.13:

If the coupled system of Riccati equations (3.6.4) has a positive definite solution, then it is unique. That is, (3.6.4) has at most one positive definite solution.

Proof: Let $P(i)$ and $\bar{P}(i)$ ($i \in \underline{N}$) be two positive definite solution of (3.6.4), let $K(i) = -B^T(i)P(i)$ and $\bar{K}(i) = -B^T(i)\bar{P}(i)$, then from (3.6.4), we have

$$\begin{aligned} & (A(i) - B(i)K(i))^T P(i) + P(i)(A(i) - B(i)K(i)) \\ & + K^T(i)K(i) + \sum_{j=1}^N q_{ij}P(j) = -I, \end{aligned} \quad (3.6.8)$$

$$\begin{aligned} & (A(i) - B(i)\bar{K}(i))^T \bar{P}(i) + \bar{P}(i)(A(i) - B(i)\bar{K}(i)) \\ & + \bar{K}^T(i)\bar{K}(i) + \sum_{j=1}^N q_{ij}\bar{P}(j) = -I. \end{aligned} \quad (3.6.9)$$

Let $E(i) = P(i) - \bar{P}(i)$, subtracting (3.6.8) from (3.6.9) and using the following identity:

$$\begin{aligned} & (A(i) - B(i)K(i))^T P(i) + P(i)(A(i) - B(i)K(i)) + K^T(i)K(i) \\ & = (A(i) - B(i)\bar{K}(i))^T P(i) + P(i)(A(i) - B(i)\bar{K}(i)) + \bar{K}(i)\bar{K}(i) \\ & \quad - (K(i) - \bar{K}(i))^T (K(i) - \bar{K}(i)), \end{aligned}$$

we can obtain

$$\begin{aligned} & (A(i) - B(i)\bar{K}(i))^T E(i) + E(i)(A(i) - B(i)\bar{K}(i)) + \sum_{j=1}^N q_{ij}E(j) \\ & = (K(i) - \bar{K}(i))^T (K(i) - \bar{K}(i)). \end{aligned} \quad (3.6.10)$$

Since $I + \bar{K}^T(i)\bar{K}(i)$ and $\bar{P}(i)$ are positive definite matrices, from (3.6.9) and Corollary 3.6.11, we conclude that the matrix

$$\bar{F} = \text{diag}\{I \oplus (A(1) - B(1)\bar{K}(1)), \dots, I \oplus (A(N) - B(N)\bar{K}(N))\} + Q \otimes I$$

is Hurwitz stable. From (3.6.10) and Corollary 3.6.12, we have $E(i) \leq 0$, i.e., $P(i) \leq \bar{P}(i)$. Switching the roles of $P(i)$ and $\bar{P}(i)$, we conclude that $\bar{P}(i) \leq P(i)$, hence $P(i) = \bar{P}(i)$. This concludes the proof. \square

Now, we are ready to give a recursive procedure to solve the coupled system of Riccati equation (3.6.4).

Algorithm:

Step 1: Suppose that there are no positive definite matrices $P(1), P(2), \dots, P(N)$ such that the matrices

$$A(i) - B(i)B^T(i)P(i) - \frac{1}{2}q_i I, \quad i \in \underline{N} \quad (A1)$$

are all stable, then (3.6.1) is not mean square stabilizable, and the algorithm terminates. Otherwise, we can find a set of positive definite matrices, denoted by $P_0(i)$ ($i \in \underline{N}$) and let $P(i) = P_0(i)$ in (A1);

Step 2: Suppose that at the k th step in the algorithm we have found positive definite matrices $P_k(i)$ ($i \in \underline{N}$), we solve the following Lyapunov equation for the positive definite matrix $P(i)$:

$$\begin{aligned} & (A(i) - B(i)B^T(i)P_k(i) - \frac{1}{2})^T P(i) + P(i)(A(i) - B(i)B^T(i)P_k(i) - \frac{1}{2}) \\ & = -I - \sum_{j \neq i} q_{ij} P_k(j) - P_k(i)B(i)B^T(i)P_k(i), \end{aligned}$$

we let $P_{k+1}(i) = P(i)$ ($i \in \underline{N}$);

Step 3: Return to Step 2 with $k \longrightarrow k + 1$ and solve for $P_{k+2}(i)$ ($i \in \underline{N}$).

In order to establish the validity of this algorithm, we first need to show that in Step 2 a positive definite solution $P(i)$ exists. From Lyapunov equation theory, we need to prove that if the solution $P_k(i)$ at k -th iteration is positive definite, then $A(i) - B(i)B^T(i)P_k(i) - 0.5q_iI$ is stable. This then guarantees the existence of a positive definite solution $P_{k+1}(i)$. In fact, suppose that $P_k(i)$ is the positive definite solution at the k -th iteration, i.e.,

$$\begin{aligned} & (A(i) - B(i)B^T(i)P_{k-1}(i) - \frac{1}{2}q_iI)^T P_k(i) \\ & + P_k(i)(A(i) - B(i)B^T(i)P_{k-1}(i) - \frac{1}{2}q_iI) \\ & = -I - \sum_{j \neq i} q_{ij}P_k(j) - P_{k-1}(i)B(i)B^T(i)P_{k-1}(i), \end{aligned}$$

from which we obtain the following:

$$\begin{aligned} & (A(i) - B(i)B^T(i)P_k(i) - \frac{1}{2}q_iI)^T P_k(i) \\ & + P_k(i)(A(i) - B(i)B^T(i)P_k(i) - \frac{1}{2}q_iI) \\ & = -(I + \sum_{j \neq i} q_{ij}P_k(j) + P_k(i)B(i)B^T(i)P_k(i) + \Delta P_k(i)B(i)B^T(i)\Delta P_k(i)), \end{aligned} \tag{3.6.11}$$

where $\Delta P_k(i) = P_k(i) - P_{k-1}(i)$. Since $P_k(i)$ is positive definite, from (3.6.11) and Lyapunov equation theory, the matrix $(A(i) - B(i)B^T(i)P_k(i) - \frac{1}{2}q_iI)$ is stable, hence the Lyapunov equation

$$\begin{aligned} & (A(i) - B(i)B^T(i)P_k(i) - \frac{1}{2}q_iI)^T P_{k+1}(i) \\ & + P_{k+1}(i)(A(i) - B(i)B^T(i)P_k(i) - \frac{1}{2}q_iI) \\ & = -I - \sum_{j \neq i} q_{ij}P_{k+1}(j) - P_k B(i)B^T(i)P_k(i), \end{aligned} \tag{3.6.12}$$

has a positive definite solution $P_{k+1}(i)$.

It is easy to see that if the algorithm is convergent, then the limit of $P_k(i)$ is the solution of (3.6.4), from which we can conclude that the system (3.6.1) is mean square stabilizable. The next question is when does the algorithm converge.

From Corollary 3.6.6, if the system (3.6.1) is mean square stabilizable, then $(A(i) - 0.5q_i I, B(i))$ is stabilizable, and for any positive definite matrices $Q(1), \dots, Q(N)$, the following Riccati equations have unique positive definite solutions $P_0(1), \dots, P_0(N)$: ($\forall i \in \underline{N}$)

$$(A(i) - 0.5q_i I)^T P_0(i) + P_0(i)(A(i) - 0.5q_i I) - P_0(i)B(i)B^T(i)P_0(i) = -Q(i), \quad (3.6.13)$$

Using this to initialize the algorithm, we can obtain the following result.

Theorem 3.6.14:

If there exists positive definite matrices $Q(1), \dots, Q(N)$ such that the positive definite solution $P_0(1), P_0(2), \dots, P_0(N)$ of (3.6.13) satisfy

$$\int_0^\infty e^{(A(i)-B(i)B^T(i)P_0(i)-0.5q_i I)^T t} \left\{ \sum_{j \neq i} q_{ij} P_0(j) - Q(i) \right\} \times e^{(A(i)-B(i)B^T(i)P_0(i)-0.5q_i I)t} dt \leq 0$$

for any $i \in \underline{N}$, then the algorithm initialized with the solution of (3.6.13) is convergent and the coupled system of Riccati equations (3.6.4) has a unique positive definite solution, hence the system (3.6.1) is mean square stabilizable.

Proof: We only need to prove that the algorithm converges. Subtracting (3.6.11) from (3.6.12), we obtain ($k > 0$)

$$\begin{aligned}
& (A(i) - B(i)B^T(i)P_k(i) - \frac{1}{2}q_i I)^T \Delta P_{k+1}(i) \\
& \quad + \Delta P_{k+1}(i)(A(i) - B(i)B^T(i)P_k(i) - \frac{1}{2}q_i I) \\
& = - \sum_{j \neq i} q_{ij} \Delta P_{k+1}(j) + \Delta P_k B(i)B^T(i) \Delta P_k(i),
\end{aligned} \tag{3.6.14}$$

from which we arrive at

$$\begin{aligned}
\Delta P_{k+1}(i) & = \int_0^\infty e^{(A(i) - B(i)B^T(i)P_k(i) - 0.5q_i I)^T t} \\
& \quad \times \left\{ \sum_{j \neq i} q_{ij} \Delta P_k(j) - \Delta P_k(i)B(i)B^T(i) \Delta P_k(i) \right\} \\
& \quad \times e^{(A(i) - B(i)B^T(i)P_k(i) - 0.5q_i I)t} dt \\
& \leq \int_0^\infty e^{(A(i) - B(i)B^T(i)P_k(i) - 0.5q_i I)^T t} \left\{ \sum_{j \neq i} q_{ij} \Delta P_k(j) \right\} \\
& \quad \times e^{(A(i) - B(i)B^T(i)P_k(i) - 0.5q_i I)t} dt.
\end{aligned}$$

Thus, if $P_0(i)$ satisfies the given condition, then we have $\Delta P_1(i) \leq 0$ ($i \in \underline{N}$). Applying induction to (3.6.14), we obtain $\Delta P_{k+1}(i) \leq 0$, i.e., $0 < P_{k+1}(i) \leq P_k(i)$, which implies that $P_k(i)$ converges, and the algorithm is convergent. \square

From this theorem, the following simpler criterion can be easily obtained.

Theorem 3.6.15:

If there exist positive definite matrices $Q(1), Q(2), \dots, Q(N)$ such that the solution of (3.6.13), say, $P_0(1), P_0(2), \dots, P_0(N)$, satisfies

$$\sum_{j \neq i} q_{ij} P_0(j) \leq Q(i), \quad (i = 1, 2, \dots, N)$$

then the system (3.6.1) is mean square stabilizable.

In particular, if there exist positive numbers $\beta(1), \dots, \beta(N)$ such that the solution of (3.6.13) with $Q(i) = \beta(i)I$ satisfies

$$\sum_{j \neq i} q_{ij} P_0(j) \leq \beta(i)I, \quad (i = 1, 2, \dots, N)$$

then (3.6.1) is mean square stabilizable. It is simpler to choose a sufficiently large number $\beta > 0$ and to set $\beta(i) = \beta$, which may lead to the desired result. \square

Up to now, we have only discussed the mean square stabilization problem, which has been a central topic in the current literature. There are essentially no results for δ -moment stabilization for arbitrary $\delta > 0$. Even the mean square stabilization results are complicated and difficult to use. In section 3.4, some δ -moment stability criteria are given, and can be used to study the δ -moment stabilization problem. In the following, we study this approach.

We first give a result for mean square stabilization.

Theorem 3.6.16:

The system (3.6.1) is mean square stabilizable if and only if there exist matrices $K(1), \dots, K(N)$ such that the matrix

$$H = \text{diag}\{I \oplus (A(1) - B(1)K(1))^T, \dots, \oplus (A(N) - B(N)K(N))^T\} + Q \otimes I$$

is Hurwitz stable, where I is the identity matrix with appropriate dimension.

Proof: This can be proved by the mean square stability result obtained by Theorem 3.4.4. \square

Thus, the mean square stabilization problem is reduced to choosing feedback matrices to stabilize one “larger” matrix. Mariton ([86]) applied homotopy theory to numerically compute the feedback matrices $K(1), \dots, K(N)$, although this procedure is also complicated.

For general $\delta > 0$, we have the following similar result.

Theorem 3.6.17:

Let $\mu(\cdot)$ be an induced matrix measure. Let

$$\begin{aligned}\bar{A}(i) &= A(i) - B(i)K(i) + \frac{\delta - 2}{2}\mu(A(i) - B(i)K(i))I, \\ \underline{A}(i) &= A(i) - B(i)K(i) - \frac{\delta - 2}{2}\mu(-A(i) + B(i)K(i))I, \quad (i \in \underline{N}).\end{aligned}$$

Define

$$H(\delta) = \begin{cases} \text{diag}\{I \oplus \bar{A}(1), \dots, I \oplus \bar{A}^T(N)\} + Q \otimes I, & \delta \geq 2; \\ \text{diag}\{I \oplus \underline{A}(1), \dots, I \oplus \underline{A}^T(N)\} + Q \otimes I, & \delta < 2. \end{cases}$$

If there exist matrices $K(1), \dots, K(N)$ such that $H(\delta)$ is Hurwitz stable, then the system (3.6.1) is δ -moment stabilizable.

Proof: It has been proved (Theorem 3.4.6) that the δ -moment top Lyapunov exponent of the system (3.6.1) with the feedback control $u(t) = -K(\sigma(t))x(t)$ is less than and equal to the largest real part of the eigenvalues of the matrix $H(\delta)$, the proof of Theorem 3.6.17 is then straightforward. \square

Remark: It is easy to see that when $\delta = 2$, Theorem 3.6.17 reduces to Theorem 3.6.16, hence Theorem 3.6.17 is a general sufficient condition for δ -moment stabilizability. The homotopy procedure developed in Mariton ([86]) can be used to numerically find $K(1), \dots, K(N)$.

We observe that when the dimension of the system and the number of states of the finite state Markov chain increase, the dimension of the matrix $H(\delta)$ or H increases, so the above criteria for δ -moment stabilization becomes increasingly more complicated. The following result gives a simpler and possibly more useful result for δ -moment stabilization.

Theorem 3.6.18:

Let $\mu(\cdot)$ denote any induced matrix measure, define

$$U(\delta) = \delta \text{diag}\{\mu(A(1) - B(1)K(1)), \dots, \mu(A(N) - B(N)K(N))\} + Q.$$

If there exist matrices $K(1), \dots, K(N)$ such that the matrix $U(\delta)$ is Hurwitz stable, then the system (3.6.1) is δ -moment stabilizable. In particular, for a one-dimensional system, (3.6.1) is δ -moment stabilizable if and only if there exists matrices $K(1), \dots, K(N)$ such that $U(\delta)$ is Hurwitz stable. In this case, $U(\delta) = \text{diag}\{A(1) - B(1)K(1), \dots, A(N) - B(N)K(N)\} + Q$.

Proof: This can be proved using the δ -moment stability result (Proposition 3.4.8). □

This criterion generally depends on the choice of matrix measure. Different choices of the induced matrix measure can lead to more or less conservative testable conditions for δ -moment stabilization. This has already been observed for δ -moment stability in section 3.4. The following matrix measures $\mu_1(A)$ ($\mu_R^1(A)$), $\mu_2(A)$, $\mu_\infty(A)$ ($\mu_R^\infty(A)$) and $\mu_P(A)$ (P is positive definite matrix) defined in Theorem B.1 in Appendix B can be used to give different test criteria for δ -moment stabilization. How to choose an appropriate matrix measure to improve the testable condition is a challenging problem which requires further investigation.

Using the matrix measure, we can obtain some criteria for absolute stabilization, keeping in mind that absolute stabilizability itself is a very conservative concept for stabilization studies. If the system is absolutely stabilizable, then we do not need to study the properties of the form process. The following is our preliminary result for absolute stabilizability.

Theorem 3.6.19:

- (1). If there exists a matrix measure $\mu(\cdot)$ and matrices $K(1), \dots, K(N)$ such that $\mu(A(i) - B(i)K(i)) < 0$, then the system (3.6.1) is absolutely stabilizable;
- (2). If (3.6.1) is absolutely stabilizable, then (3.6.1) is individual mode stabilizable;
- (3). For a one dimensional system (3.6.1), it is absolutely stabilizable if and only if it is individual mode stabilizable.

Proof: (1). This can be proved by using the well-known Coppel's inequality; (2). For any $i \in \underline{N}$, choosing an N state Markov chain such that the i -th state is absorbing and the rest of the states are transient, we can obtain the result directly; (3). This can be proved by using (1) and (2). \square

Almost sure stabilizability is a new topic in this area and in general, it is much more difficult to obtain general criterion for almost sure stabilizability than for moment stabilizability. Ezzine and Haddad ([182]) briefly discussed this issue, and pointed out the difficulty of this problem.

In linear system theory, we know that controllability implies stabilizability. However, as we discussed earlier, individual mode controllability does not

imply mean square stabilizability. What happens to almost sure stabilizability? Surprisingly, we will show that individual mode controllability implies almost sure stabilizability under a very fairly general condition. We have

Theorem 3.6.20:

Suppose that $\{\sigma(t)\}$ is a finite state ergodic Markov chain, then individual mode controllability implies almost sure stabilizability. \square

To prove this, we need the following lemma.

Lemma 3.6.21:

Consider a companion matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \end{pmatrix}$$

which has distinct real eigenvalues $\lambda_1, \dots, \lambda_n$ with $|\lambda_i - \lambda_j| \geq 1$ ($i \neq j$), then there exists a constant $M > 0$ and a positive integer k which are independent of $\lambda_1, \dots, \lambda_n$, and a nonsingular matrix T satisfying

$$\|T\| \leq M \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^k, \quad \|T^{-1}\| \leq M \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^k,$$

such that

$$T^{-1}AT = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}.$$

Proof of Lemma 3.6.21: Since A has distinct real eigenvalues, A can be diagonalized over the real field. After some algebraic manipulation, we can

find the transformation matrix

$$T = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix}$$

such that $T^{-1}AT = \text{diag}\{\lambda_1, \dots, \lambda_n\}$. To prove that T satisfies the required condition, we can use the 1–norm because all matrix norms over the real field are equivalent. Notice that

$$T^{-1} = \frac{\text{adj}(T)}{\det(T)} = \frac{\text{adj}(T)}{\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)},$$

hence from $|\lambda_i - \lambda_j| \geq 1$, we have $\|T^{-1}\| \leq \|\text{adj}(T)\|$. Since all entries of T and $\text{adj}(T)$ are polynomials of $\lambda_1, \dots, \lambda_n$, it is easy to show that there exists an $M > 0$ and a positive integer $k > 0$, which are independent of $\lambda_1, \dots, \lambda_n$, such that

$$\|T\| \leq M(\max_{1 \leq i \leq n} |\lambda_i|)^k, \quad \|T^{-1}\| \leq M(\max_{1 \leq i \leq n} |\lambda_i|)^k.$$

This completes the proof. □

Now, we are ready to prove Theorem 3.6.20.

Proof of Theorem 3.6.20: Without loss of generality, we only prove the single input case. For any $j \in \underline{N}$, from the individual mode controllability assumption, $(A(j), B(j))$ is controllable, then there exists a nonsingular matrix $T_1(j)$ such that

$$T_1(j)A(j)T_1^{-1}(j) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ x_1(j) & x_2(j) & x_3(j) & \dots & x_n(j) \end{pmatrix} \stackrel{\text{def}}{=} A_1(j),$$

$$T_1(j) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \stackrel{\text{def}}{=} B_1(j).$$

Let $\lambda_1, \dots, \lambda_n$ be negative real numbers satisfying: $2n \geq |\lambda_i - \lambda_j| \geq 1$ ($i \neq j$) (say, $-m-1, -m-2, \dots, -m-n$ for sufficiently large m , for example), then we can choose a matrix $K_1(j)$ such that

$$A_1(j) - B_1(j)K_1(j) \stackrel{\text{def}}{=} \bar{A}(j)$$

has eigenvalues $\lambda_1, \dots, \lambda_n$ for any $j \in \underline{N}$. Now $\bar{A}(j)$ is in companion form, from Lemma 3.6.21, there exists $M_1 > 0$, $l > 0$, which are independent of $\lambda_1, \dots, \lambda_n$ and j , and nonsingular matrices $T_2(j)$ ($j \in \underline{N}$) satisfying

$$\|T_2(j)\| \leq M_1 \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^l, \quad \|T_2^{-1}(j)\| \leq M_1 \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^l,$$

such that

$$T_2^{-1}(j)\bar{A}(j)T_2(j) = \text{diag}\{\lambda_1, \dots, \lambda_n\} \stackrel{\text{def}}{=} D, \quad (j \in \underline{N}).$$

Now, choose the feedback control $u(t) = -K(\sigma(t))x(t)$, where

$$K(j) = K_1(j)T_1(j), \quad T(j) = T_1^{-1}(j)T_2(j), \quad (j \in \underline{N}).$$

Then the closed-loop system becomes

$$\dot{x}(t) = T(\sigma(t))DT^{-1}(\sigma(t))x(t), \quad (3.6.15)$$

From the choice of $T_1(j)$ and $T_2(j)$, there exists an $M_2 > 0$ and $l > 0$, which are independent of j and $\lambda_1, \dots, \lambda_n$, such that

$$\|T(j)\| \leq M_2 \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^l, \quad \|T^{-1}(j)\| \leq M \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^l.$$

Since $\lambda_j < 0$, let $\lambda = \max_{1 \leq i \leq n} \lambda_i$, then it is easy to show that there exists an $M_3 > 0$, which is independent of $\lambda_1, \dots, \lambda_n$ and j , such that

$$\|e^{Dt}\| \leq M_3 e^{\lambda t}, \quad (t \geq 0).$$

From the sojourn time description of a finite state Markov chain, (3.6.15) is almost surely stable if and only if the state transition matrix

$$\Phi(t, 0) = e^{\tilde{A}(r_k)(t-t_k)} e^{\tilde{A}(r_{k-1})\tau_{k-1}} \dots e^{\tilde{A}(r_0)\tau_0} \xrightarrow{t \rightarrow \infty} 0$$

almost surely, where $\tilde{A}(j) = T(j)DT^{-1}(j)$ ($j \in \underline{N}$). However, we have

$$\begin{aligned} & \|\Phi(t, 0)\| \\ &= \|T(r_k)e^{D(t-t_k)}T^{-1}(r_k)T(r_{k-1})e^{D\tau_{k-1}}T^{-1}(r_{k-1}) \dots T(r_0)e^{D\tau_0}T^{-1}(r_0)\| \\ &\leq \|T(r_k)\| \|e^{D(t-t_k)}\| \|T^{-1}(r_k)\| \|T(r_{k-1})\| \|e^{D\tau_{k-1}}\| \times \dots \\ &\quad \times \|T(r_0)\| \|e^{D\tau_0}\| \|T^{-1}(r_0)\| \\ &\leq \left[M_2 (\max_{1 \leq i \leq n} |\lambda_i|)^l \right]^{2(k+1)} M_3^{k+1} e^{\lambda(\tau_k + \tau_{k-1} + \dots + \tau_0)} \\ &\stackrel{\text{def}}{=} \left[M e^{\lambda \frac{\tau_k + \tau_{k-1} + \dots + \tau_0}{k+1}} \right]^{k+1} \end{aligned} \quad (3.6.16)$$

where $M = (M_2 \max_{1 \leq i \leq n} |\lambda_i|)^2 M_3$. Since $\{\sigma_t\}$ is a finite state ergodic Markov chain, from the Law of Large Numbers, there exists a nonrandom constant $a > 0$, which is the average sojourn time, such that

$$\lim_{k \rightarrow \infty} \frac{\tau_k + \tau_{k-1} + \dots + \tau_0}{k+1} = a \quad (a.s.).$$

Hence, almost surely, we have

$$\lim_{k \rightarrow \infty} M e^{\lambda \frac{\tau_k + \tau_{k-1} + \dots + \tau_0}{k+1}} = M e^{\lambda a} \leq [M_2(|\lambda| + 2n)^l]^2 M_3 e^{\lambda a} \xrightarrow{\lambda \rightarrow -\infty} 0.$$

So we can choose $|\lambda|$ sufficiently large so that $Me^{\lambda a} < 1$, from (3.6.16), almost surely

$$\lim_{t \rightarrow \infty} \Phi(t, 0) = 0,$$

i.e., (3.6.15) is almost surely stable, hence the system (3.6.1) is almost surely stabilizable. This completes the proof. \square

Remark: From the proof, we can observe that it is possible to relax the ergodicity assumption. In fact, all that is required is that the average sojourn time is positive.

One may wonder if we can relax the individual mode controllability to the individual stabilizability in Theorem 3.6.20. The answer is negative, the following example show that although the system is individual mode stabilizable, the jump linear control system is not almost surely stabilizable.

Example 3.6.22: (*Individual mode stabilizability does not guarantee δ -moment stabilizability and almost sure stabilizability*)

Let

$$A(1) = \begin{pmatrix} -a & 1 \\ 0 & -a \end{pmatrix}, \quad A(2) = \begin{pmatrix} -a & 0 \\ 1 & -a \end{pmatrix}, \quad Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix},$$

$$B(1) = B(2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $a > 0$ and $q > 0$ satisfying $1 - a/q \geq 0.5$. The system (3.6.1) is individual mode stabilizable, however, from Example 3.3.7, we know that the top Lyapunov exponent for this system is positive, hence the system (3.6.1) with any control is almost surely unstable.

Example 3.6.23: (*Almost sure stabilizability does not imply individual mode stabilizability*)

Let $A(1) = 1$, $A(2) = 2$, $B(1) = 0$, $B(2) = 1$ and $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. Obviously, $(A(1), B(1))$ is not stabilizable. However, the system is almost surely stabilizable. In fact, the invariant measure of the form process is $\pi = \{0.5, 0.5\}$. $K(1) = 0$ and $K(2) = 10$ almost surely stabilizes the system.

If we choose $A(2) = -10$ and $B(2) = 0$ in this example, then the system is still almost surely stabilizable, and none of individual modes are controllable.

From Example 3.6.23, we can see that when one of the individual modes is controllable, then the system is almost surely stabilizable. This is true for any one dimensional systems. This is formalized as follows.

Proposition 3.6.24:

Assume that the form process $\{\sigma(t)\}$ is a finite state ergodic Markov chain with invariant measure π . For a one dimensional system (3.6.1), if at least one of the individual modes, say, $(A(1), B(1))$ is controllable and $\pi_1 > 0$, then the system (3.6.1) is almost surely stabilizable.

Proof: Notice that for any scalar c , we have $\mu(c) = c$. Suppose that $(A(1), B(1))$ is controllable, i.e., $B(1) \neq 0$, then we can choose $B(2) = B(2) = \dots = B(N) = 0$, and choose $K(1)$ such that

$$\pi_1(A(1) - B(1)K(1)) < \pi_2A(2) + \dots + \pi_NA(N),$$

then it can be easily shown that the system (3.6.1) is almost surely stabilizable.

□

Proposition 3.6.24 seems to be true also for high dimensional system, that is, if there exists an $i \in \underline{N}$ such that $(A(i), B(i))$ is controllable and $\pi_i > 0$,

then the system (3.6.1) is almost surely stabilizable. This is formalized as the following proposition, which is a generalization of Theorem 3.6.20.

Proposition 3.6.25:

Assume that $\{\sigma(t)\}$ is a finite state ergodic Markov chain with the invariant measure π . If there exists an $i \in \underline{N}$ such that $(A(i), B(i))$ is controllable and $\pi_i > 0$, then the system (3.6.1) is almost surely stabilizable.

Proof: We do not want to give a detailed proof because a similar idea as used in the proof of Theorem 3.6.20 can be used here with some mathematical modifications. A sketch of the proof is given as follows. Without loss of generality, assume $i = 1$. Choose $B(2) = B(3) = \dots = B(N) = 0$, and choose $K(1)$ and λ as in the proof of Theorem 3.6.20, then there exists an $M > 0$ and α , which are independent of λ , such that

$$\begin{aligned} \|e^{A(i)t}\| &\leq M e^{\alpha t} \quad (i \neq 1), \\ \|e^{(A(1)-B(1)K(1))t}\| &\leq p(\lambda)e^{\lambda t}, \quad \forall t \geq 0. \end{aligned}$$

Here $p(\lambda)$ is a polynomial with degree independent of λ . Let γ_k^1 denote the time occupied by the state 1 during the time interval $(0, t_k)$ and let γ_k^2 denote the time occupied by the states $2, 3, \dots, N$ during the interval $(0, t_k)$. From the ergodicity of $\{\sigma(t)\}$, we can obtain

$$\lim_{k \rightarrow \infty} \frac{\gamma_k^1}{t_k} = \pi_1, \quad \lim_{k \rightarrow \infty} \frac{\gamma_k^2}{t_k} = 1 - \pi_1.$$

Similar to the proof of Theorem 3.6.20, we can obtain

$$\|\Phi(t_k, 0)\| \leq \left[(Mp(\lambda)0^{(k+1)/t_k} e^{\lambda\gamma_k^1/t_k + \alpha\gamma_k^2/t_k} \right]^{t_k},$$

and the term inside $[\dots]$ has a limit

$$(Mp(\lambda))^{1/a} e^{\pi_1\lambda + (1-\pi_1)\alpha} \longrightarrow 0 \quad (\lambda \rightarrow -\infty).$$

From this argument, we can complete the proof. \square

As we observed before that matrix measure can be used to study δ -moment stabilization, we can use the matrix measure to derive a testable result for almost sure stabilization. We have

Theorem 3.6.26:

Suppose that $\{\sigma(t)\}$ is a finite state ergodic Markov chain with invariant measure $\pi = \{\pi_1, \pi_2, \dots, \pi_N\}$. For any matrix measure $\mu(\cdot)$, if there exists matrices $K(1), K(2), \dots, K(N)$ such that

$$\pi_1 \mu(A(1) - B(1)K(1)) + \dots + \pi_N \mu(A(N) - B(N)K(N)) < 0,$$

then the system (3.6.1) is almost surely stabilizable. Moreover, for the one-dimensional system, the above condition is also necessary.

Proof: This can be proved by the almost sure stability result(refer to section 3.3). \square

By specifying the matrix measure $\mu(\cdot)$ in Theorem 3.6.26, we can obtain many easy-to-use results for almost sure stabilization. Application of Theorem B.1 in Appendix B gives the following result.

Corollary 3.6.27:

Suppose that $\{\sigma(t)\}$ is a finite state ergodic Markov chain with invariant measure π , let $\bar{A}(i) = A(i) - B(i)K(i)$ ($i \in \underline{N}$). The system (3.6.1) is almost surely stabilizable if there exists matrices $K(1), K(2), \dots, K(N)$ such that one of the following conditions holds:

(1). There exists a positive definite matrix P such that

$$\sum_{i=1}^N \pi_i \lambda_{\max}[P\bar{A}(i)P^{-1} + \bar{A}(i)^T] < 0;$$

(2). There exists positive numbers r_1, r_2, \dots, r_N such that

$$\sum_{p=1}^N \pi_p \max_i \left\{ \bar{a}_{ii}(p) + \sum_{j \neq i} \frac{r_j}{r_i} |\bar{a}_{ij}(p)| \right\} < 0,$$

or

$$\sum_{p=1}^N \pi_p \max_j \left\{ \bar{a}_{jj}(p) + \sum_{i \neq j} \frac{r_i}{r_j} |\bar{a}_{ij}(p)| \right\} < 0,$$

where $\bar{A}(i) = (\bar{a}_{ij})$;

(3).

$$\sum_{p=1}^N \pi_p \max_i \left\{ \bar{a}_{ii}(p) + \sum_{j \neq i} |\bar{a}_{ij}(p)| \right\} < 0,$$

or

$$\sum_{p=1}^N \pi_p \max_j \left\{ \bar{a}_{jj}(p) + \sum_{i \neq j} |\bar{a}_{ij}(p)| \right\} < 0;$$

(4).

$$\sum_{i=1}^N \pi_i \lambda_{\max}[\bar{A}(i) + \bar{A}(i)^T] < 0.$$

□

Remarks:

(a). (3) and (4) are just special cases of (2) and (1), respectively. Although they are easy to use, sometimes they yield conservative results. As we remarked

previously, usually, a similarity transformation is necessary before the results of Corollary 3.6.27 can be applied.

- (b). In order to use (2), the positive numbers have to be appropriately chosen. Using the following fact from M -matrix theory, we can obtain a necessary condition for (2) to be applicable: If $A = (a_{ij})$ satisfying $a_{ij} \leq 0$ ($i \neq j$), then there exists positive numbers r_1, r_2, \dots, r_n such that $a_{ii}r_i > \sum_{j \neq i} r_j |a_{ij}|$ ($i = 1, 2, \dots, n$) if and only if A is Hurwitz stable or equivalently, all principal minors of A are positive. Let $U = (u_{ij})_{n \times n}$, where

$$u_{ii} = \sum_{p=1}^N \pi_p \bar{a}_{ii}(p), \quad u_{ij} = \sum_{p=1}^N \pi_p |\bar{a}_{ij}(p)| \quad (j \neq i).$$

Then, if (2) is satisfied, then U is Hurwitz stable and all principal minors of $-U$ are positive. From this, we can see that if we want to use (2), then we need to check if U is Hurwitz stable. If not, then (2) can not be satisfied. We conjecture that the stability of U is also a sufficient condition for almost sure stabilizability.

In section 3.2, we have shown that in the parameter space of the system, the domain for δ -moment stability monotonically increases and converges, roughly speaking, to the domain of almost sure stability as $\delta > 0$ decreases to zero. This implies that almost sure stability is equivalent to δ -moment stability for sufficiently small $\delta > 0$. From this, we can also say that almost sure stabilizability is equivalent to δ -moment stabilizability for sufficiently small $\delta > 0$, that is, the system (3.6.1) is almost surely stabilizable if and only if there exists a $\delta > 0$ such that the system (3.6.1) is δ -moment stabilizable. Thus, almost stabilizability can be studied by δ -moment stabilizability. From

this idea, we can obtain the following general sufficient condition for almost sure stabilizability.

Theorem 3.6.28:

Let $\bar{A}(i) = A(i) - B(i)K(i)$ ($i \in \underline{N}$). If there exists matrices $K(1), K(2), \dots, K(N)$ and positive definite matrices $P(1), P(2), \dots, P(N)$ such that for any $i \in \underline{N}$

$$\max_{\|x\|_2=1} \left(\frac{x^T [P(i)\bar{A}(i) + \bar{A}^T(i)P(i)]x}{x^T P(i)x} + \sum_{j \neq i} q_{ij} \log \left(\frac{x^T P(j)x}{x^T P(i)x} \right) \right) < 0, \quad (3.6.17)$$

then there exists a $\delta > 0$ such that (3.6.1) is δ -moment stabilizable, hence it is also almost surely stabilizable.

Proof: This can be proved in a manner similar to the proof of the almost sure stability result Theorem 3.3.1 . □

This result does not require that the form process is ergodic, thus Theorem 3.6.28 is more general and is likely to have more applications in practice. The following result shows that Theorem 3.6.28 is very general sufficient condition for almost sure stabilizability.

Corollary 3.6.29:

- (1). If the system (3.6.1) is second moment stabilizable, then there exists matrices $K(1), \dots, K(N)$ and positive definite matrices $P(1), \dots, P(N)$ such that (3.6.17) is satisfied;
- (2). For a one-dimensional system (3.6.1), it is almost sure stabilizable if and only if there exists $K(1), \dots, K(N)$ and positive numbers $P(1), \dots, P(N)$ such that (3.6.17) holds;

(3). If there exists matrices $K(1), \dots, K(N)$ and positive definite matrices $P(1), \dots, P(N)$ such that for $i \in \underline{N}$,

$$\lambda_{\max}\{[P(i)\bar{A}(i) + \bar{A}^T(i)P(i)]P^{-1}(i)\} + \sum_{j \neq i} q_{ij} \log[P(j)P^{-1}(i)] < 0,$$

then (3.6.1) is almost surely stabilizable with feedback control $u(t) = -K(\sigma(t))x(t)$.

Proof: (1) can be proved by the second moment stabilizability result, (2) can be proved by calculating the explicit solution and (3) is straightforward from (3.6.17). \square

The necessary and sufficient condition (2) in Corollary 3.6.29 for the one dimensional system is very interesting and can be used to obtain some sufficient conditions for almost sure stabilization for higher dimensional systems. The idea is to use Coppel's inequality to reduce a higher dimensional system to a one dimensional system for the purpose of almost sure stabilizability. Theorem 3.6.26 can be applied only for the case where the form process is ergodic, condition (2) in Corollary 3.6.29 may provide a more general sufficient condition for almost sure stabilizability. First, we consider the one dimensional system (3.6.1). From (2) of Corollary 3.6.29, (3.6.1) is almost surely stabilizable if and only if there exists matrices $K(1), \dots, K(N)$ such that the following series of inequalities hold: ($A \leq_e B$ or $A <_e B$ means elementwise inequalities of the matrices A and B)

$$\begin{aligned} \exists P(i) > 0, \quad 2\bar{A}(i) + \sum_{j \neq i} q_{ij} \log \frac{P(j)}{P(i)} < 0, \quad (i \in \underline{N}) \\ \iff \exists P(i) > 0, \quad 2\bar{A}(i) + \sum_{j=1}^N q_{ij} \log P(j) < 0, \quad (i \in \underline{N}) \end{aligned}$$

$$\iff \exists y \in R^N, y > 0, \begin{pmatrix} \bar{A}(1) \\ \bar{A}(2) \\ \vdots \\ \bar{A}(N) \end{pmatrix} + Q \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} <_e 0.$$

From this, we can obtain the following result.

Theorem 3.6.30:

Let $\mu(\cdot)$ denote any induced matrix measure, and let $a = (\mu(A(1) - B(1)K(1)), \dots, \mu(A(N) - B(N)K(N)))^T$. If there exists matrices $K(1), \dots, K(N)$ such that the inequality $a + Qy <_e 0$ has a solution $y \in R^N$, then the system (3.6.1) is almost surely stabilizable. Moreover, the solvability of the inequality $a + Qy <_e 0$ is also a necessary condition for almost sure stabilizability for a one dimensional system.

Proof: Let $\mu(\cdot)$ be the matrix measure induced by the vector norm $\|\cdot\|$ and let $x(t)$ denote the sample solution of the closed-loop system $\dot{x}(t) = [A(\sigma(t)) - B(\sigma(t))K(\sigma(t))]x(t)$. From Coppel's inequality, we have

$$\|x(t)\| \leq \|x_0\| \exp\left[\int_0^t \mu(\bar{A}(\sigma(\tau)))d\tau\right]. \quad (3.6.18)$$

Consider the system $\dot{z}(t) = \mu[\bar{A}(\sigma(t))]z(t)$ with initial condition $z(0) = \|x_0\|$. Then the sample solution $z(t)$ can be represented on the right hand side of the inequality (3.6.18). It is easy to show that if $\dot{z}(t) = \mu[\bar{A}(\sigma(t))]z(t)$ is almost surely stable, then from (3.6.18), the system (3.6.1) is almost surely stabilizable with the feedback control $u(t) = -K(\sigma(t))x(t)$. Using the result for one dimensional systems, we can complete the proof. \square

As we observed earlier, by specifying the matrix measure, we can obtain some useful easy-to-use criteria for almost sure stabilizability, this is left to the reader. Next, we want to show that Theorem 3.6.30 is more general than

Theorem 3.6.26. In fact, we have proved that if Q and π are the infinitesimal generator and invariant measure, respectively, of a finite state ergodic Markov chain, then for any vector a , the inequality $a + Qy <_e 0$ has a solution y if and only if $\pi a < 0$. Suppose that $\{\sigma(t)\}$ is a finite state ergodic Markov chain, from the above fact, it follows that Theorem 3.6.26 and Theorem 3.6.30 are equivalent. However, when the form process $\{\sigma(t)\}$ is not ergodic, then Theorem 3.6.30 can not be used, however, Theorem 3.6.30 is still applicable. This is illustrated in the next example.

Example 3.6.31:

Let $A(1) = a_1$ and $A(2) = a_2$ denote two real numbers and $B(1) = B(2) = 0$. Assume that the form process $\{\sigma(t)\}$ is a two state Markov chain with infinitesimal generator $Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. It is obvious that the form process is not ergodic, and Theorem 3.6.26 can not be used. However, from Theorem 3.6.30, the system (3.6.1) is almost surely stabilizable if and only if $a + Qy = a <_e 0$, i.e., $a_1 < 0$ and $a_2 < 0$. From Q , we see that the only uncertainty about the form process is the initial probability distribution.

In the rest of this section, we present some examples to show how the criteria developed in this section can be used to study stochastic stabilizability. We first begin with an example motivated by the study of dynamic reliability of multiplexed control systems ([91]).

Example 3.6.32:

Let

$$A(1) = \begin{pmatrix} 0 & 0 & 0 \\ 1.5 & 0 & 1.5 \\ 0 & 0 & 0 \end{pmatrix}, \quad B(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad K(1) = \begin{pmatrix} k_1 & k_2 & 0 \\ 0 & k_2 & k_1 \end{pmatrix},$$

$$\begin{aligned}
A(2) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1.5 \\ 0 & 0 & 0 \end{pmatrix}, & B(2) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, & K(2) &= \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & k_1 \end{pmatrix}, \\
A(3) &= \begin{pmatrix} 0 & 0 & 0 \\ 1.5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & B(3) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, & K(3) &= \begin{pmatrix} k_1 & k_2 & 0 \\ 0 & 0 & k_1 \end{pmatrix}, \\
A(4) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & B(4) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, & K(4) &= \begin{pmatrix} k_1 & 0 & 0 \\ 0 & 0 & k_1 \end{pmatrix}.
\end{aligned}$$

This models a first order system with two controllers (incorporating the redundancy principle for reliability) (see [91] for details). The first mode (state 1) corresponds to the case where both controllers are good, and the second and third modes (the states 2 and 3) correspond to the case where one of the controllers fails, and the fourth mode (the state 4) corresponds to the case where both controllers fail. We assume that whenever a controller fails, it will be repaired. Suppose that the failure rate is λ and the repair rate is μ , and the failure process and the repair process are both exponentially distributed. Then the form process is a finite state Markov chain with infinitesimal generator

$$Q = \begin{pmatrix} -2\lambda & \lambda & \lambda & 0 \\ \mu & -(\lambda + \mu) & 0 & \lambda \\ \mu & 0 & -(\lambda + \mu) & \lambda \\ 0 & \mu & \mu & -2\mu \end{pmatrix}.$$

In [91], Ladde and Siljak developed a sufficient condition for second moment (mean square) stabilizability and used this to show that when $\lambda = 0.4$ and $\mu = 0.55$, $k_1 = 2.85$ and $k_2 = 0.33$, the controller $u(t) = -K(\sigma(t))x(t)$ stabilizes the jump linear system (3.6.1). In this approach, an appropriate choice of positive definite matrices should be sought, which is very difficult. However, using Theorem 3.6.16, we can easily check that in this case, the eigenvalues of the matrix H in Theorem 3.6.16 have negative real parts, hence H is Hurwitz stable. It is also easy by computation to show that for the failure

rate $\lambda = 0.4$ and the repair rate $\mu = 0.55$, any controller with the parameters k_1 and k_2 satisfying $0.1 \leq k_1 \leq 5$ and $0.1 \leq k_2 \leq 5$ can second moment stabilize the jump linear system (3.6.1). Similarly, using the controller with $k_1 = 4$ and $k_2 = 0.55$, for any failure rate $0.4 \leq \lambda \leq 0.6$ and repair rate $0.4 \leq \mu \leq 0.6$, the system (3.6.1) can be second moment stabilized by such controller. One important fact is that even if the failure rate is greater than the repair rate, this controller still stabilizes the system, i.e., the multiplexed control system is reliable. This result is not readily apparent from [91]. From the computer computations, it seems that whenever the repair rate is greater than the failure rate, then this controller can second moment stabilize the system (3.6.1).

Example 3.6.33:

In this example, we study the δ -moment stabilization problem for general $\delta > 0$. Consider the one-dimensional jump linear system (3.6.1) with

$$A(1) = a_1, \quad B(1) = b_1, \quad A(2) = a_2, \quad B(2) = b_2, \quad Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix}.$$

Theorem 3.6.18, we can easily obtain that the system (3.6.1) is δ -moment stabilizable if and only if there exists k_1 and k_2 such that

$$\begin{aligned} \delta(a_1 - b_1 k_1) &< q \\ \delta(a_2 - b_2 k_2) &< q \\ \frac{q}{q - \delta(a_1 - b_1 k_1)} \cdot \frac{q}{q - \delta(a_2 - b_2 k_2)} &< 1 \end{aligned} \tag{3.6.19}$$

- (1). If $b_1 \neq 0$ and $b_2 \neq 0$, i.e., the system is individual mode controllable, then we can choose k_1 and k_2 such that $a_1 - b_1 k_1 < 0$ and $a_2 - b_2 k_2 < 0$, then (3.6.19) is satisfied, hence (3.6.1) is δ -moment stabilized by such a controller;

(2). If $b_1 \neq 0$ and $b_2 = 0$, then we can prove that (3.6.1) is δ -moment stabilizable if and only if $a_2 < q/\delta$. The necessity can be proved by the second inequality in (3.6.19). Suppose that $a_2 < q/\delta$, choosing k_1 such that

$$b_1 k_1 > \frac{q(a_1 + a_2) - \delta a_1 a_2}{q - \delta a_2},$$

we can easily verify that (3.6.19) is satisfied with such a k_1 and for any k_2 , hence (3.6.1) can be stabilized by such controller;

(3). If $b_1 = 0$ and $b_2 \neq 0$, then (3.6.1) is δ -moment stabilizable if and only if $a_1 < q/\delta$;

(4). If $b_1 = b_2 = 0$, then (3.6.1) is δ -moment stabilizable if and only if

$$\begin{aligned} \delta a_1 &< q \\ \delta a_2 &< q \\ \frac{q}{q - \delta a_1} \cdot \frac{q}{q - \delta a_2} &< 1. \end{aligned}$$

The domain of (a_1, a_2) for which the system (3.6.1) is δ -moment stabilizable is illustrated in [74].

Example 3.6.34:

Let

$$\begin{aligned} A(1) &= \begin{pmatrix} 0 & 1 \\ -4 & 10 \end{pmatrix}, \quad A(2) = \begin{pmatrix} 0 & 0 \\ -100 & 27 \end{pmatrix}, \quad B(1) = B(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ Q &= \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

It is obvious that the system (3.6.1) with this data is individual mode controllable. From Theorem 3.6.20, we can obtain that the system (3.6.1) is almost surely stabilizable. We want to find the feedback matrices $K(1)$ and

$K(2)$ such that the control $u(t) = -K(\sigma(t))x(t)$ almost surely stabilizes the system (3.6.1). We use Theorem 3.6.26 to solve this problem. First notice that the invariant measure for the form process is $\pi = (0.5, 0.5)$. Choose $K(2) = (-100, 27)$, then $\bar{A}(2) = A(2) - B(2)K(2) = 0$, and for any matrix measure $\mu(\cdot)$, $\mu(\bar{A}(2)) = 0$. For the first mode, we can choose a $K(1)$ such that the eigenvalues of $A(1) - B(1)K(1)$ can be assigned, for example, to -1 and -3 . This can be achieved by setting $K(1) = (-2, 13)$, then $\bar{A}(1) = A(1) - B(1)K(1) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$. From the proof of Lemma 3.6.21, we know that

$$TAT^{-1} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad \text{where } T = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}^{-1}.$$

Define the vector norm $\|x\|_T = \|Tx\|_2$, from Appendix B, the induced matrix measure is given by $\mu_T(A) = \mu_2(TAT^{-1})$. From this, we can easily compute

$$\mu_T(\bar{A}(1)) = \mu_2\left(\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}\right) = -1, \quad \mu_T(\bar{A}(2)) = \mu_T(0) = 0.$$

We have $\pi_1\mu_T(\bar{A}(1)) + \pi_2\mu_T(\bar{A}(2)) = 0.5 \times (-1) + 0.5 \times 0 = -0.5 < 0$, and from Theorem 3.6.26 the controller $u(t) = -K(\sigma(t))x(t)$ with $K(1) = (-2, 13)$ and $K(2) = (-100, 27)$ almost surely stabilizes the system (3.6.1).

CHAPTER FOUR

CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

In this dissertation, we have studied stochastic stability of both discrete-time and continuous-time jump linear systems with a finite state Markov chain form process. We first study the various moment stability properties and deliberately illustrate the relationship between almost sure stability and moment stability, and between the regions of almost sure stability and moment stability. An important tool so-called large deviation theorem is first introduced to solve the stochastic stability problems. This work is first one in current literature systematically studying almost sure (sample path) stability of jump linear systems, and it is our hope that this work could pave the way for the future research on sample path stability of more general stochastic systems, e.g. the nonlinear jump systems with a finite state Markov chain form process.

The following few research directions are suggested for near future.

(1). Computational Procedure for Almost Sure Stability

In Chapter two and three, we have obtained some very general sufficient conditions for almost sure stability, e.g., Theorem 2.5.11–2.5.16 and Theorem 3.3.1–3.3.2. We conjectured that the conditions in Theorem 2.5.11 and Theorem 3.3.1 are also necessary, respectively. One research topic is to find a vigorous proof or to seek counterexamples. It is witnessed that to effectively apply these general sufficient conditions, we have to find an appropriate positive definite matrices $P(1), P(2), \dots, P(N)$. However, at present, we are not able to find a good procedure. One possibility is to solve the corresponding

minimax problem. Unfortunately, the admissible domain for minimum problem in the minimaxiation is all positive definite matrices, which really complicates the matter. One practically important research topic will be to seek a feasible and practical procedure to solve or approximately solve such minimaxiation problems.

As we also noticed that matrix norm and matrix measure also give some simple but useful sufficient conditions for almost sure stability (e.g. Theorem 2.4.1 or Theorem 3.4.1). As we show that different choice of matrix norm or matrix measure may provide different results. It is worthwhile finding some appropriate procedure to obtain a “good” matrix norm or matrix measure so that the “best” results can be obtained for almost sure stability.

Another research topic is to continue to look for verifiable δ –moment stability criteria for sufficiently small $\delta > 0$. Some computational test procedures for δ –moment stability are also desirable. One possible procedure is to obtain some good estimates for top δ –moment Lyapunov exponent of jump linear systems.

(2). Stabilization Problems

Almost sure stabilization problem has been briefly studied in Chapter two and substantially investigated in Chapter three. As we remarked there that any almost sure stability criteria can be used to obtain almost sure stabilizability criteria, which involve solving a set of inequalities with matrix variables. This is not easy problem and is usually reduced to multiobjective optimization problems. One possible solution for this may lie in the nonlinear optimization algorithm development whose variables are matrices.

It is also interesting to consider the δ -moment stabilization problem for jump linear systems. Second moment stabilization problem has been studied by Ji et al ([71]-[74]), however, their criteria are reduced to test the convergence of matrix sequence which may be very difficult. We also suggested in Chapter two and three that second moment stabilization problem can be reduced to a linear quadratic optimization problem, which is hoped to reduce the complexity of the problem. This is still an open problem in the current literature. When $\delta \neq 2$, the δ -moment stabilization problem is totally new and is a very important research area considering its relationship to almost sure stabilizability.

As we noted earlier, all the stabilization results were developed under the assumption that $\{\sigma_k\}$ or $\{\sigma_t\}$ is observable at the moment k or t . If this is not true, then all results failed and the corresponding linear quadratic optimal problem has dual effect ([80]). Another rich research area is to find some physically realizable feedback control to stabilize the jump linear systems. Up to now, there is no results yet except the robust stabilizability criteria obtained from the robust stability criteria (see section 2.8 and section 3.5).

(3). Optimal Control of Jump Linear Systems

The Linear quadratic optimal control has been studied extensively in the literature ([7],[49],[67]-[73]). When the form process is not observable, this is reduced to dual control problem ([80]), it is interesting to study this dual control problem because jump linear control systems with a finite state Markov chain form process. Another important problem is to consider the case when the form process is a controlled finite state Markov chain. This problem, which has a practical background, is also intensively studied by many researchers ([180],[181]).

It may be also interesting to consider the optimal control problem whose cost function is given by

$$J(u) = E \left\{ \sum_{k=0}^L [x(k)^T Q(\sigma_k) x(k) + u^T(k) R(\sigma_k) u(k)]^\delta \right\},$$

or

$$J(u) = E \left\{ \int_0^T [x^T(t) Q(\sigma_t) x(t) + u^T(t) R(\sigma_t) u(t)]^\delta dt \right\}.$$

This cost functional or its variants may be helpful, especially for the suboptimal control designs. This cost functional is never be used in current literature and may be worthwhile studying.

(4). Stability Analysis of Nonlinear Jump Systems

In this dissertation, we only study the stochastic stability of jump linear systems with a finite state Markov chain form process. For some practical problems, the resulting closed-loop control system may be a nonlinear jump system, therefore, it is useful to study the stochastic stability of such systems. It may be possible to generalize the results in this dissertation to the case of nonlinear systems with a finite state Markov chain form process.

(5). Applications of Large Deviation Theory

As we witnessed in our research, the large deviation theorem is very important tool to study the relationship between almost sure stability and moment stability. It is possible to apply this theory to more general systems for stochastic stability study. This forms another new rich research area for the future.

(6). Stability of Jump Linear Systems with Noise

The linear quadratic optimal designs of jump linear systems with additive Gaussian noise have been studied by many authors ([7],[72],[73]). It is also very important research topic to generalize our results in this dissertation to such systems. This seems to be a fairly easy task to be accomplished.

APPENDIX A

SOME PROPERTIES OF MATRIX NORMS

To study the stability of discrete-time jump linear systems, we need some properties of (induced) matrix norms. Some of the properties are known, and some of them are new. Let M_n denote the $n \times n$ matrix set on real or complex field, a function $\|\cdot\|: M_n \rightarrow R$ is called a matrix norm if for all $A, B \in M_n$, it satisfies the following conditions:

- (1). $\|A\| \geq 0$, and $\|A\| = 0$ iff $A = 0$;
- (2). $\|cA\| = |c|\|A\| \forall c \in C$;
- (3). $\|A + B\| \leq \|A\| + \|B\|$;
- (4). $\|AB\| \leq \|A\|\|B\|$.

It is not necessary to assume (4) for the definition of matrix norms, but for the stability study purpose, it is essential to assume this, which is satisfied by a norm induced by a vector norm. A matrix norm $\|\cdot\|$ is called an induced matrix norm by a vector norm $|\cdot|$ if

$$\|A\| = \sup_{x \neq 0} \frac{|Ax|}{|x|}.$$

In [172], the matrix norm defined in the above fashion is called consistent matrix norm and the induced norm is called the operator norm and it can be proved that for any matrix norm $\|\cdot\|$ defined above, there exists a vector norm ν such that $\nu(Ax) \leq \|A\|\nu(x)$, if we use $\|\cdot\|_\nu$ to denote the induced matrix norm by ν , then we have $\|A\|_\nu \leq \|A\|$. For the stability purpose as we will

see, the induced matrix norm is enough. we summarize some of the useful properties in the following lemma.

Theorem A.1:

For any matrix norm,

- (a). $\|\cdot\|$ is convex, i.e., for any $A_1, A_2, \dots, A_p \in M_n$, and $\lambda_1, \lambda_2, \dots, \lambda_p \geq 0$, where $\sum_{1 \leq i \leq p} \lambda_i = 1$, we have

$$\left\| \sum_{1 \leq i \leq p} \lambda_i A_i \right\| \leq \sum_{1 \leq i \leq p} \lambda_i \|A_i\|;$$

- (b). The following are the special matrix norms:

$$\|A\|_s = \sum_{i,j=1}^n |a_{ij}|$$

$$\|A\|_F = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2} \quad (\text{Frobenius norm})$$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| \quad (\text{induced by 1-norm})$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| \quad (\text{induced by } \infty\text{-norm})$$

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)} \quad (\text{induced by 2-norm})$$

- (c). $\|A\|_2^2 \leq \|A\|_1 \|A\|_\infty$;

- (d). If T is nonsingular, and $\|\cdot\|$ is a vector norm, then $\|Tx\|$ is also a vector norm and its induced matrix norm is given by $\|A\|_T = \|TAT^{-1}\|$, where we still use $\|\cdot\|$ to denote the matrix norm induced by the vector norm $\|\cdot\|$. In particular, let $R = \text{diag}\{r_1, \dots, r_n\}$, where r_1, \dots, r_n are positive

numbers, then the induced matrix norm by the vector norm $\|Rx\|_p$, where $p = 1, +\infty, F$, are given, respectively, by

$$\begin{aligned}\|A\|_1^R &= \max_{1 \leq j \leq n} \sum_{i=1}^n \frac{r_i}{r_j} |a_{ij}| \\ \|A\|_\infty^R &= \max_{1 \leq i \leq n} \sum_{j=1}^n \frac{r_i}{r_j} |a_{ij}| \\ \|A\|_F^R &= \sqrt{\sum_{i,j=1}^n \frac{r_i^2}{r_j^2} |a_{ij}|^2}\end{aligned}$$

(e). For any matrix norm $\|\cdot\|$, we have $\rho(A) \leq \|A\|$. Here $\rho(A)$ denotes the spectral radius of A .

(f). Let \mathcal{T} denote the set of nonsingular matrices, and $\|\cdot\|_p^T$ denote the matrix norm induced by the vector norm $\|Tx\|_p$ where $p = 1, 2, +\infty$, then we have

$$\inf_{T \in \mathcal{T}} \|A\|_1^T = \inf_{T \in \mathcal{T}} \|A\|_2^T = \inf_{T \in \mathcal{T}} \|A\|_\infty^T = \rho(A).$$

(g). A is (Schur) stable if and only if there exists an induced matrix norm $\|\cdot\|$ such that $\|A\| < 1$.

Proof: (a),(b),(d) can be deduced easily from the definition of matrix norm. (c) can be found in [172].

(e). Suppose that λ is an eigenvalue of A with spectral radius $\rho(A) = |\lambda|$, then there exists a vector $x \neq 0$ such that $Ax = \lambda x$. Define $X = (x, x, \dots, x)$, then we have $AX = \lambda X$, hence from the multiplicative property of the matrix norm, we obtain

$$|\lambda| \|X\| = \|\lambda X\| = \|AX\| \leq \|A\| \|X\|,$$

and since $X \neq 0$, we have $|\lambda| \leq \|A\|$, i.e., $\rho(A) \leq \|A\|$.

(f). In (f), for 2–norm case, the proof can be found in [145]. Following a similar procedure, we can prove the rest. For example, consider the ∞ –norm case. From Jordan’s Theorem, there exists a nonsingular matrix P such that $PAP^{-1} = D + U$, where $D = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ is diagonal matrix whose diagonal elements λ_i are the eigenvalues of the matrix A and $U = (u_{ij})$ is a matrix satisfying $u_{ij} = 0$ if $j \neq i + 1$ and $u_{ij} = 0$ or 1 if $j = i + 1$. Let $R = \text{diag}\{1, \varepsilon^{-1}, \dots, \varepsilon^{-(n-1)}\}$, where $\varepsilon > 0$ is a small positive number to be determined. Let $T = RP$, then T is nonsingular, and it is easy to check that

$$\|A\|_{\infty}^T = \|TAT^{-1}\|_{\infty} \leq \max_{1 \leq i \leq n} \{|\lambda_i| + \varepsilon\} = \rho(A) + \varepsilon.$$

This means that for any $\varepsilon > 0$, there exists a nonsingular matrix T such that $\|A\|_{\infty}^T \leq \rho(A) + \varepsilon$. From (e), it is easy to show that $\inf_{T \in \mathcal{T}} \|A\|_{\infty}^T \geq \rho(A)$, thus we have $\inf_{T \in \mathcal{T}} \|A\|_{\infty}^T = \rho(A)$.

(g). If there exists a matrix norm $\|\cdot\|$ such that $\|A\| < 1$, from (e), we have $\rho(A) \leq \|A\| < 1$, hence A is (schur) stable. Conversely, if A is stable, then $\rho(A) < 1$. From (f), there exists a matrix norm $\|\cdot\|$ such that $\|A\| < 1$, this completes the proof of (g). \square

Remarks:

(1). Property (g) is the crucial one which explains why the matrix norm can be used to study the stability of discrete-time systems. Consider the discrete linear time-invariant system

$$x(k+1) = Ax(k), \quad x_0 = x(0).$$

For the vector norm $\|\cdot\|$ and its induced matrix norm $\|\cdot\|$, we have

$$\|x(k)\| = \|Ax(k-1)\| \leq \|A\|\|x(k-1)\| \leq \cdots \leq \|A\|^k \|x_0\|.$$

And because all norms on M_n are equivalent, there exists an $M > 0$ such that

$$\|x(k)\|_2 \leq M\|x(k)\| \leq M\|x_0\|\|A\|^k.$$

Thus, $\|A\|$ is an estimate for the degree of stability as defined in [4]. From this, we observe that different choice of matrix norm will give a different estimate for the degree of stability. (f) suggests a method to choose a suitable matrix norm so that the best estimate for the degree of stability can be achieved.

- (2). The infimum in (f) may not be achievable, that is, we can not change the infimum to minimum. For example, let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $\rho(A) = 0$. However, there is no matrix norm $\|\cdot\|$ such that $\|A\| = \rho(A) = 0$.

APPENDIX B

PROPERTIES OF MATRIX MEASURE

Matrix measure had been used to study the error bounds in the numerical integration of ordinary differential equations ([164],[165],[167]) and estimation and stability of solutions of differential equations ([158],[169]). Blankenship ([166]) used this technique to study the stochastic stability of linear stochastic differential equations with random coefficients and obtained the best results in this area up to date. Ladde ([168]) generalized the matrix measure concept to random matrix case. Recently, Fang et al ([159]) revitalized this technique to study the robust stability of interval dynamical systems and observed that this may be the best way to deal with the robust stability against convex parameter uncertainty. For convenience, we collect in this section properties of matrix measure, some of which are new to the readers.

Let $|x|$ denote a vector norm of x on C^n , and $\|A\|$ is the induced matrix norm of A given the vector norm $|\cdot|$. $\mu(A)$ is the matrix measure of A defined as

$$\mu(A) \triangleq \lim_{\theta \downarrow 0^+} \frac{\|I + \theta A\| - 1}{\theta}$$

where I is identity matrix.

Theorem B.1.

$\mu(A)$ is well defined for any induced norm and has the following properties:

- (a). $\mu(I) = 1$, $\mu(-I) = -1$, $\mu(0) = 0$;
- (b). $\mu(\alpha A) = \alpha\mu(A)$, $\forall \alpha \geq 0$;

(c). $\mu(A + cI) = \mu(A) + \operatorname{Re}(c)$, $\forall c \in C$;

(d). $\max[\mu(A) - \mu(-B), -\mu(-A) + \mu(B)] \leq \mu(A + B) \leq \mu(A) + \mu(B)$;

(e). μ is convex on $C^{n \times n}$, i.e., for any $\alpha_j \geq 0$ ($1 \leq j \leq k$), and $\sum_{j=1}^k \alpha_j = 1$, and any matrices A_j ($1 \leq j \leq k$), we

$$\mu \left(\sum_{j=1}^k \alpha_j A_j \right) \leq \sum_{j=1}^k \alpha_j \mu(A_j).$$

(f). For any norm, and any A , we have

$$-\|A\| \leq -\mu(-A) \leq \operatorname{Re}\lambda(A) \leq \mu(A) \leq \|A\|.$$

(g). $|\mu(A) - \mu(B)| \leq \max\{|\mu(A - B)|, |\mu(A + B)|\} \leq \|A - B\|$;

(h). $-\mu(A)|x| \leq |Ax|, -\mu(A) \leq |Ax|$;

(i). If A is nonsingular, $\max[-\mu(A), -\mu(-A)] \leq (\|A^{-1}\|)^{-1} \leq \|A\|$;

(j). Let $|\cdot|$ be a norm on C^n and $P \in C^{n \times n}$ be nonsingular. Let μ_P be the matrix measure induced by the vector norm $|x|_P = |Px|$, then

$$\mu_P(A) = \mu(PAP^{-1});$$

(k). For the 1-norm $|x|_1 = \sum_{i=1}^n |x_i|$, the induced matrix measure μ_1 is given by

$$\mu_1(A) = \max_j \left[\operatorname{Re}(a_{jj}) + \sum_{i \neq j} |a_{ij}| \right];$$

For the 2–norm $|x|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$, the induced matrix measure μ_2 is given by

$$\mu_2(A) = \max_i [\lambda_i(A + A^*)/2];$$

For the ∞ –norm $|x|_\infty = \max_{1 \leq i \leq n} |x_i|$, the induced matrix measure is given by

$$\mu_\infty(A) = \max_i \left[\operatorname{Re}(a_{ii}) + \sum_{j \neq i} |a_{ij}| \right].$$

(l). Let H denote a positive definite Hermitian matrix and μ_H denote the matrix measure induced by the vector norm $|x|_H = \sqrt{x^* H x}$, then

$$\mu_H(A) = \frac{1}{2} \lambda_{\max}(H A H^{-1} + A^*);$$

(m). For any positive numbers $\{r_1, r_2, \dots, r_n\}$, let $R = \operatorname{diag}\{r_1^{-1}, r_2^{-1}, \dots, r_n^{-1}\}$, then the matrix measure μ_R^1 induced by the norm $|x|_R = |R x|_1$ is given by

$$\mu_R^1(A) = \max_j \left[\operatorname{Re}(a_{jj}) + \sum_{i \neq j} \frac{r_i}{r_j} |a_{ij}| \right];$$

and the matrix measure μ_R^∞ induced by the vector norm $|x|_R = |R x|_\infty$ is given by

$$\mu_R^\infty(A) = \max_i \left[\operatorname{Re}(a_{ii}) + \sum_{j \neq i} \frac{r_i}{r_j} |a_{ij}| \right];$$

(n). If A is given by the following block form:

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{pmatrix}$$

then for the induced matrix measure μ , we have

$$\operatorname{Re}\lambda(A) \leq \max_{1 \leq i \leq k} \left[\mu(A_{ii}) + \sum_{j \neq i} \|A_{ij}\| \right]$$

or a stronger result for certain choice of norm:

$$\operatorname{Re}\lambda(A) \leq \max_{1 \leq i \leq k} \left[\max \operatorname{Re}\lambda(A_{ii}) + \sum_{j \neq i} \|A_{ij}\| \right]$$

For any positive numbers $\{r_1, r_2, \dots, r_k\}$, we have

$$\operatorname{Re}\lambda(A) \leq \max_{1 \leq i \leq k} \left[\mu(A_{ii}) + \sum_{j \neq i} \frac{r_j}{r_i} \|A_{ij}\| \right]$$

or a stronger form for certain choice of norm:

$$\operatorname{Re}\lambda(A) \leq \max_{1 \leq i \leq k} \left[\max \operatorname{Re}\lambda(A_{ii}) + \sum_{j \neq i} \frac{r_j}{r_i} \|A_{ij}\| \right];$$

- (o). Let \mathcal{N} be the set of all vector norms on C^n , for any $\rho \in \mathcal{N}$, the corresponding matrix measure is denoted as μ_ρ . Then for any matrix on $C^{n \times n}$, we have

$$\max_{1 \leq i \leq n} \operatorname{Re}\lambda_i(A) = \inf_{\rho \in \mathcal{N}} \mu_\rho(A)$$

- (p). Let \mathcal{P} be the set of nonsingular matrices in $C^{n \times n}$, and \mathcal{H} be the set of positive Hermitian matrices in $C^{n \times n}$, and using μ_P^p and μ_H to denote the matrix measures induced by $|x|_P^p = |Px|_p$ ($p = 1, 2, \infty$) and $|x|_H = \sqrt{x^* H x}$, respectively, then

$$\max_{1 \leq i \leq n} \operatorname{Re}\lambda_i(A) = \inf_{P \in \mathcal{P}} \mu_P^1(A) = \inf_{P \in \mathcal{P}} \mu_P^2(A) = \inf_{P \in \mathcal{P}} \mu_P^\infty(A) = \inf_{H \in \mathcal{H}} \mu_H(A);$$

- (q). A is a stable matrix iff there exists a matrix measure μ such that $\mu(A) < 0$;

- (r). $\mu_2(A) < \frac{1}{2}(\mu_1(A) + \mu_\infty(A))$;
- (s). If A is skew Hermitian, then $\mu_2(A) = 0$; if A is Hermitian, then $\mu_2(A) = \max_i \lambda_i(A)$; If A is diagonal, then $\mu_p(A) = \max_i \operatorname{Re} \lambda_i(A)$ ($p = 1, 2, \infty$); If A is singular positive semidefinite, then $\mu_2(-A) = 0$; If A is normal, i.e., $A^*A = AA^*$, then $\mu_2(A) = \max_i \operatorname{Re} \lambda_i(A)$;
- (t). For any induced matrix measure μ , $\mu(A)$ is continuous in A and is convex in its C^{n^2} ; And $\mu_2(A)$ is infinitely differentiable in A .

Proof: The proof of (a), (b),(d)-(f),(h)-(k) can be found in [145]. (c) has been proved only for the case where c is real.

(c). If c is complex, then we have $\mu(A + cI) = \mu(A + biI) + \operatorname{Re}(c)$, where $i = \sqrt{-1}$. So it suffices to show that $\mu(A + biI) = \mu(A)$. In fact,

$$\begin{aligned} \mu(A + biI) &= \lim_{\theta \downarrow 0^+} \frac{\|I + \theta(A + biI)\| - 1}{\theta} = \lim_{\theta \downarrow 0^+} \frac{|1 + b\theta i| \|I + \frac{\theta}{1+bi}A\| - 1}{\theta} \\ &= \lim_{\theta \downarrow 0^+} \frac{\sqrt{1 + b^2\theta^2} \|I + \frac{\theta}{1+b^2\theta^2}A - \frac{bi}{1+b^2\theta^2}A\theta^2\| - 1}{\theta} \\ &= \lim_{\theta \downarrow 0^+} \frac{\sqrt{1 + b^2\theta^2} \|I + \frac{\theta}{1+b^2\theta^2}A\| - 1}{\theta} = \mu(A) \end{aligned}$$

where we have used the fact that $\theta/(1 + b^2\theta^2) \downarrow 0^+$.

(g). We only need to prove the first inequality. If $\mu(A) \geq \mu(B)$, then

$$|\mu(A) - \mu(B)| = \mu(A) - \mu(B) = \mu(B + (A - B)) - \mu(B) \leq \mu(A - B) \leq |\mu(A - B)|.$$

In the similar fashion, we can prove that if $\mu(B) \geq \mu(A)$, then $|\mu(A) - \mu(B)| \leq |\mu(B - A)|$. Therefore, the first inequality holds.

(i). In [145], it has proved that $-\mu(-A) \leq (\|A^{-1}\|)^{-1}$. Replacing A with $-A$ in the above, we obtain $-\mu(A) \leq (\|A^{-1}\|)^{-1}$, this completes the proof of (i).

(l). Since H is positive definite, there exists a nonsingular matrix P , so that $H = P^*P$. Then $|x|_H = |Px|_2$, using (j) and (k), we have

$$\mu_H(A) = \mu_2(PAP^{-1}) = \frac{1}{2}\lambda_{\max}(PAP^{-1} + P^{-*}A^*P^*) = \frac{1}{2}\lambda_{\max}(HAH^{-1} + A^*).$$

(m). Since

$$RAR^{-1} = \begin{pmatrix} a_{11} & \frac{r_2}{r_1}a_{12} & \cdots & \frac{r_n}{r_1}a_{1n} \\ \frac{r_1}{r_2}a_{21} & a_{22} & \cdots & \frac{r_n}{r_2}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{r_1}{r_n}a_{n1} & \frac{r_2}{r_n}a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

using (j) and (k), we obtain the proof of (m).

(n). To prove this, we need the following Lemma due to Feigold and Varga ([170])

Lemma. For the partitioned A , let $\|\cdot\|$ be the corresponding induced norm, if λ is an eigenvalue of A , then for some i

$$\|(\lambda I - A_{ii})^{-1}\|^{-1} \leq \sum_{j \neq i} \|A_{ij}\|.$$

Applying (c) and (i), we obtain for the corresponding induced matrix measure μ

$$\|(\lambda I - A_{ii})^{-1}\|^{-1} \geq -\mu(-(\lambda I - A_{ii})) = \operatorname{Re}(\lambda) - \mu(A_{ii})$$

hence applying the Lemma, we obtain

$$\operatorname{Re}(\lambda) \leq \mu(A_{ii}) + \sum_{j \neq i} \|A_{ij}\|$$

From which, we prove the first inequality in (n).

The second inequality can be proved by observing the fact in (o).

Let $R = \operatorname{diag}\{r_1 I_1, r_2 I_2, \dots, r_k I_k\}$, where I_j is the identity matrix with the same dimension as that of A_{jj} . Applying the above results to $R^{-1}AR$, we can prove the rest of the inequalities in (n).

(o). This has been established in [159].

(p). In fact, we have proved in [159] that $\max_{1 \leq i \leq n} \operatorname{Re} \lambda_i(A) = \inf_{P \in \mathcal{P}} \mu_P^2(A)$ for the 2–norm, i.e., $p = 2$ case.

For any A , there exists nonsingular matrix T such that $TAT^{-1} = J$ where J is Jordan form, let $R = \operatorname{diag}\{r_1, r_2, \dots, r_n\}$, where r_1, r_2, \dots, r_n are positive numbers to be determined, then let $P = RT \in \mathcal{P}$, we have

$$PAP^{-1} = RTAT^{-1}R^{-1} = \begin{pmatrix} \lambda_1 & \frac{r_1}{r_2}e_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \frac{r_2}{r_3}e_2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ddots & \frac{r_{n-1}}{r_n}e_{n-1} \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A and $e_j = 0$ or 1 ($1 \leq j \leq n - 1$).

For $p = 1$, using (k), we have

$$\mu_P^1(A) = \mu_1(PAP^{-1}) = \max_j \left[\operatorname{Re}(\lambda_j) + \frac{r_j}{r_{j+1}}e_j \right]$$

Thus, we choose R in the way that $\frac{r_j}{r_{j+1}} < \varepsilon$, then we have $\mu_P^1(A) < \max_j \operatorname{Re} \lambda_j(A) + \varepsilon$. This proves the case where $p = 1$.

In the similar fashion, we can prove the case where $p = \infty$.

Let $H = (RT)^*RT$, then $H \in \mathcal{H}$, and

$$\begin{aligned} \mu_H(A) &= \frac{1}{2} \lambda_{\max}(HAH^{-1} + A^*) = \frac{1}{2} \lambda_{\max}(RTAT^{-1}R^{-1} + (RTAT^{-1}R^{-1})^*) \\ &\leq \max_j \left[\operatorname{Re}(\lambda_j) + \frac{1}{2} \left(\frac{r_{j-1}}{r_j} + \frac{r_j}{r_{j+1}} \right) \right] \end{aligned}$$

thus using similar choice of R , we can obtain $\mu_H(A) \leq \max_j \operatorname{Re} \lambda_j(A) + \varepsilon$, this proves the last equality in (p). Summarizing the above, we completes the proof of (p).

(q). This is direct result of (o) or (p).

(r). First notice that $\bar{A} = \frac{1}{2}(A + A^*)$ is Hermitian, its eigenvalues are real. From Gershgorin's theorem, we have that for any eigenvalue λ of \bar{A} , there exists a i such that

$$|\lambda - \bar{a}_{ii}| \leq \sum_{j \neq i} |\bar{a}_{ij}|$$

Thus, we obtain

$$\begin{aligned} \lambda &\leq \bar{a}_{ii} + \sum_{j \neq i} |\bar{a}_{ij}| \\ &= \operatorname{Re}(a_{ii}) + \sum_{j \neq i} \frac{1}{2} |a_{ij} + a_{ji}^*| \\ &\leq \frac{1}{2} \left[2\operatorname{Re}(a_{ii}) + \sum_{j \neq i} (|a_{ij}| + |a_{ji}|) \right] = \frac{1}{2} (\mu_1(A) + \mu_\infty(A)) \end{aligned}$$

From this, we conclude the relationship.

(s). The first four claims can be directly verifiable. If A is normal, then from Theorem 2.5.4 in [148], there exists a unitary matrix U such that $U^*AU = \text{diag}\{\lambda_1, \dots, \lambda_n\}$. Thus

$$\begin{aligned}\mu_2(A) &= \frac{1}{2}\lambda_{\max}(A + A^*) = \frac{1}{2}\lambda_{\max}(U^*AU + U^*A^*U) \\ &= \frac{1}{2}\lambda_{\max}(\text{diag}\{\lambda_1 + \lambda_1^*, \dots, \lambda_n + \lambda_n^*\}) = \max_i \text{Re}\lambda_i(A)\end{aligned}$$

This completes the proof of (s).

(t). We first prove the continuity. It is easy to show

$$\mu(A) - \|\Delta A\| \leq \mu(A) - \mu(-\Delta A) \leq \mu(A + \Delta A) \leq \mu(A) + \mu(\Delta A) \leq \mu(A) + \|\Delta A\|.$$

Hence $\forall \varepsilon > 0$, $\exists \delta(\leq \varepsilon) > 0$, such that if $\|\Delta A\| < \delta$, we have $|\mu(A + \Delta A) - \mu(A)| < \varepsilon$, hence $\mu(A)$ is continuous.

The convexity follows directly from (e).

To prove the infinite differentiability, we need the following result:

Lemma: (Kato, [160]) If the matrix family $T(\theta)$, which is infinitely differentiable elementwise, is Hermitian, its eigenvalues $\lambda_i(\theta)$ are infinitely differentiable in θ .

Since $\bar{A} = \frac{1}{2}(A + A^*)$ is always symmetric for any A , and between A and $A + \Delta A$, we can always use smooth curve to connect them so that we can reparametrize the matrix \bar{A} , finally using the Lemma, we can prove (t). \square

Remarks:

(1). In [145], (g) was in the following form:

$$|\mu(A) - \mu(B)| \leq |\mu(A - B)|, \quad |\mu(A) - \mu(B)| \leq |\mu(B - A)|.$$

In general, this is incorrect. For example, $A = 0$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $\mu_2(A) - \mu(B) = -1$, and $\mu_2(A - B) = 0$, thus the first inequality does not hold.

- (2). (f) is the crucial property for the matrix measure to be applicable in stability analysis of control systems. Matrix measure provide a upper bound for the real parts of eigenvalues and if this bound is negative, then the matrix is stable. Moreover, this bound has nice convex properties which provides some information about the real parts of the eigenvalues of matrices in a convex hull. The importance of this property will be seen in the next section;
- (3). In continuous time-invariant system, the norm of system matrix gives us nothing about the stability of the system, and matrix measure contains the stability information. In discrete case, there is no direct counterpart of matrix measure, and induced matrix norm contains the stability information. Thus we have the following corresponding relationship: largest real part of eigenvalues \leftrightarrow spectral radius, matrix measure \leftrightarrow matrix norm;
- (4). It is obvious that matrix measure really depends on the choice of vector norm. Different vector norm will induce different matrix measure. In applying matrix measure technique to the stability analysis, we need to choose the suitable vector norm so that the matrix measure is as small as possible. Properties (o) and (p) are very important in this aspect, because they provide some kind of procedure to minimize the upper bound for the largest real part of eigenvalues. For a set of matrices A_1, \dots, A_N , we may find systematic search method from (o) or (p) for the “best matrix measure” so that we can test the stability of all A_1, \dots, A_N ;

- (5). If matrix A is very large, for example, the system matrix of large scale system, property (n) can be used to study its stability. It can also be applied to the stabilization problems of decentralized control systems;
- (6). From the proof of (r), we can obtain the following easily computable upper bound of $\mu_2(A)$:

$$\mu_2(A) \leq \max_i \left[\operatorname{Re}(a_{ii}) + \sum_{j \neq i} \frac{1}{2} |a_{ij} + a_{ji}| \right].$$

or the following scaling version: for any positive numbers,

$$\mu_2(A) \leq \max_i \left[\operatorname{Re}(a_{ii}) + \frac{1}{2} \sum_{j \neq i} \left| \frac{r_i}{r_j} a_{ij} + \frac{r_j}{r_i} a_{ji} \right| \right];$$

- (7). It is easy to show that (b) and (d) implies the convexity (e). Moreover, (b) and (d) is true if and only if the following is true: for any integer k , any $\alpha_j \geq 0$ and any $A_j \in C^{n \times n}$ ($j = 1, 2, \dots, k$),

$$\mu \left(\sum_{j=1}^k \alpha_j A_j \right) \leq \sum_{j=1}^k \alpha_j \mu(A_j).$$

In general, for any α_j ($j = 1, 2, \dots, k$), the following inequality holds

$$\mu \left(\sum_{j=1}^k \alpha_j A_j \right) \leq \sum_{j=1}^k |\alpha_j| \mu(\operatorname{sgn}(\alpha_j) A_j),$$

where $\operatorname{sgn}(x)$ is the classical sign function defined as follows: $\operatorname{sgn}(x) = 1$ if $x \geq 0$ and $\operatorname{sgn}(x) = -1$ if $x < 0$. This observation will be used in the next section.

- (8). Matrix measure defined above is only for the matrix norm induced by some vector norm. Because of this, (f) can be guaranteed. For general matrix

norm, the above may not be true and the definition of matrix measure should be modified as

$$\eta(A) \stackrel{\text{def}}{=} \lim_{\theta \downarrow 0^+} \frac{\|I + \theta A\| - \|I\|}{\theta}.$$

Follow the same procedure as in [145], we can prove that $\eta(A)$ does exist. In the above definition, we replace 1 with $\|I\|$, because for general matrix norm, $\|I\|$ is not identity. For example, the Frobenius norm defined by

$$\|A\|_F = \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2},$$

we have $\|I\|_F = \sqrt{n} \neq 1$ ($n > 1$)! From this, we can conclude that Frobenius norm can not be induced by some vector norm, because for any induced matrix norm $\|\cdot\|$, $\|I\| = 1$. It is easy to show that

$$\eta_F(A) = \lim_{\theta \downarrow 0^+} \frac{\sqrt{\sum_i |1 + \theta a_{ii}|^2 + \theta^2 \sum_{j \neq i} |a_{ij}|^2} - \sqrt{n}}{\theta} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \text{Re}(a_{ii}).$$

This measure of matrix η can not guarantee the validity of (f). For example, $A = \text{diag}\{-1, -2\}$, then $\lambda_{\max}(A) = -1$ and $\eta_F(A) = \frac{1}{\sqrt{2}}(-1 - 2) < -1$, hence $\lambda_{\max}(A) > \eta_F(A)$. This is why we can not consider this kind measure of matrix for stability analysis. But this may be of interest on its own right;

- (9). For any matrix norm, we have $\|AB\| \leq \|A\|\|B\|$, one may conjecture that this submultiplicativity property also holds for matrix measure, unfortunately, this is not true. For example, $A = \text{diag}\{-1, -2\}$, then $\mu_2(A^2) = 4$ and $\mu_2(A) = -1$, hence $\mu_2(A^2) \leq \mu_2(A)^2$ does not hold;
- (10). From (s), we know that if A is diagonal, then $\mu_p(A) = \max_i \text{Re}\lambda_i(A)$. One may conjecture that for any matrix measure this is true. Unfortunately,

this is not correct. Consider $B = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, this is symmetric, there exists a unitary matrix such that $U^T B U = A = \text{diag}\{(-1 + \sqrt{5})/2, (-1 - \sqrt{5})/2\}$, then $\max_i \text{Re}\lambda_i(A) = (-1 + \sqrt{5})/2$, and $\mu_U^1(A) = \mu_1(U A U^{-1}) = \mu_1(B) = \max\{0 + 1, -1 + 1\} = 1$, which is not the largest real part of the eigenvalues of A . When A is normal, the (s) gives a easy computation of $\mu_2(A)$. Since skew-Hermitian matrices, Hermitian matrices, unitary matrices are all normal, their 2-norm matrix measures can be computed very easily.

- (11). For general matrix measure μ , it may not be differentiable. For example, if $A = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, then $\mu_1(A) = |x|$, which is not differentiable function in x . From the proof of (t), we notice that $|\mu(A + \Delta A) - \mu(A)| \leq \|\Delta A\|$, thus $\mu(A)$ satisfies Lipschitz condition or is a contraction map.
- (12). From (k), we can observe that matrix measure can be used to describe the diagonal dominance property or positive definiteness property for a set of matrices. It is easily verified that $\{A(\omega) | \omega \in \Omega\}$ is uniformly column-sum dominant (or uniformly positive definite, or uniformly row-sum dominant) iff there exists a positive number ε such that $\mu_1(-A(\omega)) \leq -\varepsilon$ (or $\mu_2(-A(\omega)) \leq -\varepsilon$, or $\mu_\infty(A(\omega)) \leq -\varepsilon$) for any $\omega \in \Omega$, respectively. This is already observed in [43]. \square

Although matrix measure is only defined for constant fixed matrix, it applies to any matrix, whether time-invariant or time-varying, deterministic or stochastic. This is why matrix measure can be used to study the stability of linear time-varying systems or stochastic systems. The key idea is that the estimation of solution of linear systems can be obtained by using the matrix

measure technique. The following theorem plays an central role in stability analysis using matrix measure technique.

Theorem B.2 (Coppel's Inequality)

Under fair condition on $A(t)$ (e.g., piecewise continuous, or integrability condition), the solution of the following linear system :

$$\dot{x}(t) = A(t)x(t) \tag{B.1}$$

satisfies the inequalities

$$\|x(t_0)\| \exp \left\{ - \int_{t_0}^t \mu(-A(s)) ds \right\} \leq \|x(t)\| \leq \|x(t_0)\| \exp \left\{ \int_{t_0}^t \mu(A(s)) ds \right\}$$

Proof. The proof can be found either [145] or [158].

□

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