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ORIGINAL ARTICLE

Stability analysis of linear fractional differential system with multiple time delays

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Abstract In this paper, we study the stability of n-dimensional linear fractional differential equation with time delays, where the delay matrix is defined in $(R^+)^{n\times n}$. By using the Laplace transform, we introduce a characteristic equation for the above system with multiple time delays. We discover that *if all roots of the characteristic equation have negative parts*, *then the equilibrium of the above linear system with fractional order is Lyapunov globally asymptotical stable if the equilibrium exist* that is almost the same as that of classical differential equations. As its an application, we apply our theorem to the delayed system in one spatial dimension studied by Chen and Moore [*Nonlinear Dynamics* **29**, 2002, 191] and determine the asymptotically stable region of the system. We also deal with synchronization between the coupled Duffing oscillators with time delays by the linear feedback control method and the aid of our theorem, where the domain of the control-synchronization parameters is determined.

Keywords Delay · Duffing oscillator · Linear fractional differential system · Stability · Synchronization

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1 Introduction

Delayed differential equations have been abundantly studied in [1, 2], and references cited therein. Recently, time delays and multiple time delays are introduced to complex dynamical networks, e.g., see [3, 4]. All these publications are for (integer-order or typical) differential equations.

Although they have almost the same history as those of typical differential equations, the fractional(-order) calculus and differential equations did not attract much attention till recent decades [5–7]. It was lately found that many systems can be modelled via using fractional derivatives. These systems display fractional-order dynamics, such as heat transfer, viscoelasticity, electrical circuit, electro-chemistry, dynamics, economics, polymer physics, and control, see recent two volumes of *Nonlinear Dynamics* [8, 9]. Similar to classical differential systems, the study of stability is always a central task for fractional differential systems. In 1996, Matignon [10] studied stability of $n-$dimensional linear fractional systems from a point of view of control.

Recently, Chen and Moore [11] studied stability of 1-dimensional fractional systems with retard time.

Similar to [3] and [4], in this paper, we introduce multiple time delays to the fractional differential equations. Then we study the (asymptotical) stability of such systems. In details, we briefly introduce the definitions of the fractional derivatives and the fractional equations in Section 2. In Section 3, the main stability theorems are proved. The examples are included in Section 4. These illustrative examples support our theoretical analysis. And the conclusions are given in the last section.

2 Preliminaries

Generally speaking, three fractional derivative definitions, i.e., Grünwald–Letnikov fractional derivative, Riemann–Liouville fractional derivative, and Caputo's fractional derivative, are mostly used. The former two definitions are often used by pure mathematicians, while the last one is adopted by applied scientists, since it is more convenient in engineering applications. Here we only discuss Caputo derivative:

$$\frac{d^q x(t)}{dt^q} = J^{m-q} x^{(m)}(t), \quad \alpha > 0,$$

where $m = \lceil q \rceil$, i.e., m is the first integer that is not less than q, $x^{(m)}$ is a conventional mth order derivative, J^β is the βth order Riemann–Liouville integral operator, which is expressed as follows:

$$J^\beta x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s)\, ds, \quad \beta > 0.$$

In engineering, the fractional order q often lies in $(0, 1)$, so we always suppose that the 'order' q is a positive number but less than 1 in this paper.

In the present article, we study the following n-dimensional linear fractional differential system with multiple time delays:

$$\begin{cases} \dfrac{d^{q_1} x_1(t)}{d^{q_1} t} = a_{11} x_1(t-\tau_{11}) + a_{12} x_2(t-\tau_{12}) \\ \qquad + \cdots + a_{1n} x_n(t-\tau_{1n}), \\ \dfrac{d^{q_2} x_2(t)}{d^{q_2} t} = a_{21} x_1(t-\tau_{21}) + a_{22} x_2(t-\tau_{22}) \\ \qquad + \cdots + a_{2n} x_n(t-\tau_{2n}), \\ \qquad \cdots \\ \dfrac{d^{q_n} x_n(t)}{d^{q_n} t} = a_{n1} x_1(t-\tau_{n1}) + a_{n2} x_2(t-\tau_{n2}) \\ \qquad + \cdots + a_{nn} x_n(t-\tau_{nn}), \end{cases} \tag{1}$$

where q_i is real and lies in (0,1), the initial values $x_i(t) = \phi_i(t)$ are given for $-\max_{i,j}\ \tau_{ij} = -\tau_{\max} \le t \le 0$ and $i = 1, \ldots, n$. In this system, time-delay matrix $T = (\tau_{ij})_{n\times n} \in (R^+)^{n\times n}$, coefficient matrix $A = (a_{ij})_{n\times n}$, state variables $x_i(t),\ x_i(t-\tau_{ij}) \in R$, and initial values $\phi_i(t) \in C^0[-\tau_{\max},\ 0]$. Its fractional order is defined as $q = (q_1, q_2, \ldots, q_n)$. If $q_i = q_j$ and $\tau_{ij} = 0$, $i, j = 1, 2, \ldots, n$, then system (1) is actually the one considered in [10]. If $n = 1$, then (1) is reduced to the system studied in [11].

Next, we study the stability of system (1). Taking Laplace transform [12] on both sides of (1) gives

$$\begin{cases} s^{q_1} X_1(s) - s^{q_1-1}\phi_1(0) = a_{11} e^{-s\tau_{11}} \big(X_1(s) \\ \quad + \int_{-\tau_{11}}^0 e^{-st}\phi_1(t)\, dt \big) + a_{12} e^{-s\tau_{12}} \big(X_2(s) \\ \quad + \int_{-\tau_{12}}^0 e^{-st}\phi_2(t)\, dt \big) + \cdots + a_{1n} e^{-s\tau_{1n}} \big(X_n(s) \\ \quad + \int_{-\tau_{1n}}^0 e^{-st}\phi_n(t)\, dt \big) \\ s^{q_2} X_2(s) - s^{q_2-1}\phi_2(0) = a_{21} e^{-s\tau_{21}} \big(X_1(s) \\ \quad + \int_{-\tau_{21}}^0 e^{-st}\phi_1(t)\, dt \big) + a_{22} e^{-s\tau_{22}} \big(X_2(s) \\ \quad + \int_{-\tau_{22}}^0 e^{-st}\phi_2(t)\, dt \big) + \cdots + a_{2n} e^{-s\tau_{2n}} \big(X_n(s) \\ \quad + \int_{-\tau_{2n}}^0 e^{-st}\phi_n(t)\, dt \big) \\ \qquad \cdots \\ s^{q_n} X_n(s) - s^{q_n-1}\phi_n(0) = a_{n1} e^{-s\tau_{n1}} \big(X_1(s) \\ \quad + \int_{-\tau_{n1}}^0 e^{-st}\phi_1(t)\, dt \big) + a_{n2} e^{-s\tau_{n2}} \big(X_2(s) \\ \quad + \int_{-\tau_{n2}}^0 e^{-st}\phi_2(t)\, dt \big) + \cdots + a_{nn} e^{-s\tau_{nn}} \big(X_n(s) \\ \quad + \int_{-\tau_{nn}}^0 e^{-st}\phi_n(t)\, dt \Big), \end{cases} \tag{2}$$

where $X_i(s)$ is the Laplace transform of $x_i(t)$ with $X_i(s) = \mathcal{L}(x_i(t))$, $1 \le i \le n$.

We can rewrite (2) as follows:

$$\Delta(s)\cdot\begin{pmatrix}X_1(s)\\X_2(s)\\\vdots\\X_n(s)\end{pmatrix}=\begin{pmatrix}b_1(s)\\b_2(s)\\\vdots\\b_n(s)\end{pmatrix}, \tag{3}$$

in which

$$\begin{cases}b_1(s)=a_{11}e^{-s\tau_{11}}\int_{-\tau_{11}}^{0}e^{-st}\phi_1(t)\,dt+a_{12}e^{-s\tau_{12}}\int_{-\tau_{12}}^{0}e^{-st}\phi_2(t)\,dt\\ \qquad +\cdots+a_{1n}e^{-s\tau_{1n}}\int_{-\tau_{1n}}^{0}e^{-st}\phi_n(t)dt+s^{q_1-1}\phi_1(0)\\ b_2(s)=a_{21}e^{-s\tau_{21}}\int_{-\tau_{21}}^{0}e^{-st}\phi_1(t)dt+a_{22}e^{-s\tau_{22}}\int_{-\tau_{22}}^{0}e^{-st}\phi_2(t)\,dt\\ \qquad +\cdots+a_{2n}e^{-s\tau_{2n}}\int_{-\tau_{2n}}^{0}e^{-st}\phi_n(t)\,dt+s^{q_2-1}\phi_2(0)\\ \cdots \qquad \cdots\\ b_n(s)=a_{n1}e^{-s\tau_{n1}}\int_{-\tau_{n1}}^{0}e^{-st}\phi_1(t)dt+a_{n2}e^{-s\tau_{n2}}\int_{-\tau_{n2}}^{0}e^{-st}\phi_2(t)\,dt\\ \qquad +\cdots+a_{nn}e^{-s\tau_{nn}}\int_{-\tau_{nn}}^{0}e^{-st}\phi_n(t)dt+s^{q_n-1}\phi_n(0),\end{cases}$$

$$\Delta(s)=\begin{pmatrix}s^{q_1}-a_{11}e^{-s\tau_{11}} & -a_{12}e^{-s\tau_{12}} & \cdots & -a_{1n}e^{-s\tau_{1n}}\\ -a_{21}e^{-s\tau_{21}} & s^{q_2}-a_{22}e^{-s\tau_{22}} & \cdots & -a_{2n}e^{-s\tau_{2n}}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{n1}e^{-s\tau_{n1}} & -a_{n2}e^{-s\tau_{n2}} & \cdots & s^{q_n}-a_{nn}e^{-s\tau_{nn}}\end{pmatrix}.$$

We call $\Delta(s)$ a *characteristic matrix* of system (1) for simplicity and $\det(\Delta(s))$ a *characteristic polynomial* of (1). The distribution of $\det(\Delta(s))$'s eigenvalues totally determines the stability of system (1). This can be seen from the following discussion.

Obviously, if a linear fractional differential equation has a non-zero equilibrium, we can move this equilibrium to the origin by the translation transform. *Throughout the paper, we always suppose that* (1) *has a zero solution and all complex computations are done in the branch of the principle value of argument.*

3 Main theorems

In this section, we establish several stability theorems.

Multiplying s on both sides of (3) gives

$$\Delta(s)\cdot\begin{pmatrix}sX_1(s)\\sX_2(s)\\\vdots\\sX_n(s)\end{pmatrix}=\begin{pmatrix}sb_1(s)\\sb_2(s)\\\vdots\\sb_n(s)\end{pmatrix}. \tag{4}$$

If all roots of the transcendental equation $\det(\Delta(s))=0$ lie in open left half complex plane, i.e., $\mathrm{Re}(s)<0$, then we consider (4) in $\mathrm{Re}(s)\geq 0$. In this restricted area, (4) has a unique solution $(sX_1(s),\ldots,sX_n(s))$. So, we have

$$\lim_{s\to 0,\,\mathrm{Re}(s)\geq 0} sX_i(s)=0,\quad i=1,\ldots,n.$$

From the assumption of all roots of the characteristic equation $\det(\Delta(s))=0$ and the final-value theorem of Laplace transform [12], we get

$$\lim_{t\to+\infty}x_i(t)=\lim_{s\to 0,\,\mathrm{Re}(s)\geq 0} sX_i(s)=0,\quad i=1,\ldots,n.$$

It immediately follows the theorem below.

Theorem 1. *If all the roots of the characteristic equation* $\det(\Delta(s))=0$ *have negative real parts, then the zero solution of system (1) is Lyapunov globally asymptotically stable.*

Remark 1. If $\tau_{ij}=\tau>0$ for $i,j=1,\ldots,n$ and $q_1=q_2=\cdots=q_n=1$, then the characteristic matrix and characteristic equation of (1) are reduced to $sI-Ae^{-s\tau}$ and $\det(sI-Ae^{-s\tau})=0$, respectively. They coincide with the usual definitions of the characteristic matrix and characteristic equation of delayed equations. Especially, if $\tau=0$, then the characteristic matrix and characteristic equation of (1) are respectively reduced to $sI-A$ and $\det(sI-A)=0$, which agree with the typical definitions for typical differential equations.

Corollary 1. *Suppose* $\tau_{ij}=0$ *for* $i,j=1,\ldots,n$ *and* $q_1=q_2=\cdots=q_n=\alpha\in(0,1)$. *If all the roots of the equation* $\det(\lambda I-A)=0$ *satisfy* $|\arg(\lambda)|>\frac{\alpha\pi}{2}$, *then the zero solution of system (1) is Lyapunov globally asymptotically stable.*

This result is Theorem 2 of [10]. Here, we can very easily prove it by using Theorem 1 of the present paper.

Proof: For this case, (1)'s characteristic equation becomes $\det(s^\alpha I-A)=0$. Let λ be s^α, then $s=\lambda^{\frac{1}{\alpha}}$. Because all the roots λs of equation $\det(\lambda I-A)=0$ satisfy $|\arg(\lambda)|>\frac{\alpha\pi}{2}$, it follows that $|\arg(s)|=|\arg(\lambda^{\frac{1}{\alpha}})|>\frac{\pi}{2}$. So all the characteristic roots of system (1) have negative real parts. This completes the proof. □

Corollary 2. *Suppose that $\tau_{ij} = 0$, and all q_is are rational numbers between 0 and 1, for $i, j = 1, \ldots, n$. Let M be the lowest common multiple of the denominators u_is of q_is, where $q_i = v_i/u_i$, $(u_i, v_i) = 1$, $u_i, v_i \in Z^+$, $i = 1, \ldots, n$, and set $\gamma = 1/M$. Then the zero solution of system (1) is Lyapunov globally asymptotically stable if all the roots λs of the equation*

$$\det \begin{pmatrix} \lambda^{Mq_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{Mq_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{Mq_n} - a_{nn} \end{pmatrix} = 0 \tag{5}$$

satisfies $|\arg(\lambda)| > \gamma\pi/2$.

From Theorem 1, the characteristic equation of (1) are of fractional powers of s. This corollary tells that the characteristics equation of (1) can be transformed to an integer-order polynomial equation if all q_is are rational numbers, simplifying the computations.

Proof: Obviously, the characteristic equation is

$$\det \begin{pmatrix} s^{q_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s^{q_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s^{q_n} - a_{nn} \end{pmatrix} = 0. \tag{6}$$

Denote λ by s^γ, then $s = \lambda^{\frac{1}{\gamma}}$, hence (6) is changed to (5). $| \arg(s) | = | \arg(\lambda^{\frac{1}{\gamma}}) | > \frac{\pi}{2}$ due to the argument assumption of Equation (5). The conclusion holds. □

Corollary 3. *If $q_1 = q_2 = \cdots = q_n = \alpha \in (0, 1)$, all the eigenvalues λs of A satisfy $| \arg(\lambda) | > \frac{\alpha\pi}{2}$ and the characteristic equation $\det(\Delta(s)) = 0$ has no purely imaginary roots for any $\tau_{ij} > 0$, $i, j = 1, \ldots, n$, then the zero solution of system (1) is Lyapunov globally asymptotically stable.*

Since the characteristic equation of (1) is a transcendent, finding the its eigenvalues is often difficult. However, to find the eigenvalues of the coefficient matrix A of (1) is comparatively easier. This result simplifies computations of the original problem.

Proof: Assume that $l(\lambda) = \det(\lambda I - A)$ and $\phi(\lambda) = \det(\Delta(\lambda^{\frac{1}{\alpha}})) - \det(\lambda I - A)$, then $l(\lambda) + \phi(\lambda) = \det(\Delta(\lambda^{\frac{1}{\alpha}}))$. Obviously, $l(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots$ and $\phi(\lambda) = (\sum_{i=1}^n d_{ii} e^{-\lambda^{\frac{1}{\alpha}}\tau_{ii}} + d_0)\lambda^{n-1} + (\sum_{i,j=1}^n h_{ij} e^{-\lambda^{\frac{1}{\alpha}}\tau_{ij}} + h_0)\lambda^{n-2} + \cdots$, where $d_0, h_0, c_1, d_{ii}, h_{ij}, \ldots$ are constants.

Clearly, there exists an area $\{\lambda \mid | \lambda | > r, | \arg(\lambda) | \le \frac{\alpha\pi}{2}\}$ such that $| l(\lambda) | > | \phi(\lambda) |$ and there are no roots to $l(\lambda) = 0$. According to Rouché Theorem [13], there is no root of $l(\lambda) + \phi(\lambda) = 0$ in the above same area.

Note that all the zero points of $l(\lambda)$ are in the area $| \arg(\lambda) | \ge \frac{\alpha\pi}{2}$. If there is a zero point of $l(\lambda) + \phi(\lambda) = 0$ in the area $| \arg(\lambda) | < \frac{\alpha\pi}{2}$, then there exist a set of parameters τ_{ij} $(i, j = 1, \ldots, n)$ such that λ passes through one of the two lines: $\arg(\lambda) = \frac{\alpha\pi}{2}$ and $\arg(\lambda) = \frac{-\alpha\pi}{2}$, between $-r$ and r, this contradicts the assumptions of this theorem, since the assumption that there is no purely imaginary roots to $\det(\Delta(s)) = 0$, which is to say there are no roots λ, where $| \arg(\lambda) | = \frac{\alpha\pi}{2}$, for $\det(\Delta(\lambda^{\frac{1}{\alpha}})) = 0$. Thus, all the roots of $\det(\Delta(\lambda^{\frac{1}{\alpha}})) = 0$ satisfy $| \arg(\lambda) | > \frac{\alpha\pi}{2}$, i.e. $\det(\Delta(s)) \neq 0$ for $\mathrm{Re}(s) > 0$. The proof is finished. □

It immediately follows that we have the result below.

Corollary 4. *If $q_1 = q_2 = \cdots = q_n = \alpha \in (0.5, 1)$, all the eigenvalues λs of A satisfy $| \arg(\lambda) | > \frac{\pi}{2}$, and the equation $\det(\Delta(\lambda^{\frac{1}{\alpha}})) = 0$ has no purely imaginary roots for any $\tau_{ij} > 0$, $i, j = 1, \ldots, n$, then the zero solution of system (1) is Lyapunov globally asymptotically stable.*

Corollary 5. *Suppose that all q_is are rational numbers between 0 and 1, for $i, j = 1, \ldots, n$. Let M be the lowest common multiple of the denominators u_is of q_is, where $q_i = v_i/u_i$, $(u_i, v_i) = 1$, $u_i, v_i \in Z^+$, $i = 1, \ldots, n$, and set $\gamma = 1/M$. If all the roots λ of Equation (5) satisfies $|\arg(\lambda)| > \frac{\gamma\pi}{2}$ and the characteristic equation $\det(\Delta(s)) = 0$ has no purely imaginary roots for any $\tau_{ij} > 0$, $i, j = 1, \ldots, n$, then the zero solution of system (1) is Lyapunov globally asymptotically stable.*

Proof: Let

$$l(\lambda) = \det \begin{pmatrix} \lambda^{Mq_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{Mq_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{Mq_n} - a_{nn} \end{pmatrix},$$

and $\phi(\lambda) = \det(\Delta(\lambda^{\frac{1}{\gamma}})) - l(\lambda)$. The rest proof is the same as that of Corollary 3, so it is omitted here. □

4 Applications

Chen and Moore [11] considered the following delayed fractional equation:

$$\frac{d^q y(t)}{dt^q} = K_p y(t-\tau), \tag{7}$$

where q and K_p are real numbers and $0 < q < 1$, time delay τ is a positive constant.

They got a stability condition of (7), given by a transcendent inequality via the Lambert function [11, 14]. Here we derive another stability condition, which is very convenient for application.

The characteristic equation of (7) is

$$s^q - K_p e^{-\tau s} = 0. \tag{8}$$

Assume that $s = \omega i = |\omega|(\cos\frac{\pi}{2} + i\sin(\pm\frac{\pi}{2}))$ is a root of (8), where ω is real number and when $\omega > 0$, we take $i\sin\frac{\pi}{2}$, while when $\omega < 0$, we take $i\sin(-\frac{\pi}{2})$. Then one gets

$$|\omega|^q\left(\cos\frac{q\pi}{2} + i\sin\left(\pm\frac{q\pi}{2}\right)\right) - K_p(\cos\omega\tau - i\sin(\pm\omega\tau)) = 0.$$

That is,

$$|\omega|^q \cos\frac{q\pi}{2} - K_p\cos\omega\tau = 0$$

and

$$|\omega|^q \sin\frac{q\pi}{2} + K_p\sin\omega\tau = 0.$$

From the above two equations, one obtains

$$(|\omega|^q)^2 + K_p^2 - 2|\omega|^q K_p \cos\left(\frac{q\pi}{2} + \omega\tau\right) = 0.$$

Thus, we can easily get that when

$$\frac{q\pi}{2} \pm (-K_p)^{1/q}\tau \neq (2k+1)\pi,$$

where $k \in Z$, Equation (8) has no purely imaginary roots.

According to Corollary 3, one has the following conclusion.

Stability condition of (7). *If $K_p < 0$, $(-K_p)^{1/q} \neq \frac{1}{\tau}((2k+1)\pi - \frac{q}{2}\pi)$ and $(-K_p)^{1/q} \neq -\frac{1}{\tau}((2k+1)\pi - \frac{q}{2}\pi)$, where $k \in Z$, then the zero solution of system (7) is Lyapunov globally asymptotically stable.*

In what follows, another application is presented.

Recently, chaos synchronization of fractional differential equation attracts increasing interests [15–20] due to its potential applications in control processing and secure communication [10]. Here we take Duffing oscillator as an example. A fractional version of the Duffing oscillator is described as

$$\begin{cases} \dfrac{d^q x_1}{dt^q} = y_1 \\ \dfrac{d^q y_1}{dt^q} = -ky_1 + x_1(t-\tau) - x_1(t-\tau)^3 + B\cos t, \end{cases} \tag{9}$$

where $0 < q < 1$, $\tau > 0$, and k, B are positive numbers.

We choose (9) as a drive system and regard the nonlinear item as a driving signal. The corresponding response system is defined by

$$\begin{cases} \dfrac{d^q x_2}{dt^q} = y_2 - k_1(x_2 - x_1) \\ \dfrac{d^q y_2}{dt^q} = -ky_2 + x_2(t-\tau) - x_1(t-\tau)^3 \\ \qquad +B\cos t - k_2(y_2 - y_1), \end{cases} \tag{10}$$

where k_1, k_2 are control parameters under investigation.

Subtracting (9) from (10), one gets the error system as follows

$$\begin{cases} \dfrac{d^q e_x}{dt^q} = e_y - k_1 e_x \\ \dfrac{d^q e_y}{dt^q} = -(k+k_2)e_y + e_x(t-\tau), \end{cases} \tag{11}$$

where $e_x = x_2 - x_1$, $e_x(t-\tau) = x_2(t-\tau) - x_1(t-\tau)$, and $e_y = y_2 - y_1$.

Obviously, the synchronization between (9) and (10) is equivalent to the globally asymptotical stability of the zero solution to error system (11).

For error system (11), the coefficient matrix A is described by

$$A = \begin{pmatrix} -k_1 & 1 \\ 1 & -(k+k_2) \end{pmatrix}$$

and its characteristic equation $\det(\Delta(s))$ is given as

$$\begin{aligned} \det(\Delta(s)) &= \begin{vmatrix} s^q + k_1 & -1 \\ -e^{-s\tau} & s^q + (k+k_2) \end{vmatrix} \\ &= s^{2q} + (k_1+k+k_2)s^q \\ &\quad + k_1(k+k_2) - e^{-s\tau} = 0. \end{aligned} \tag{12}$$

Suppose that $s = \omega i = |\omega|(\cos\frac{\pi}{2} + i\sin(\pm\frac{\pi}{2}))$ is a root of (12), where ω is real number and when $\omega > 0$, we take $i\sin\frac{\pi}{2}$ whilst when $\omega < 0$, we take $-i\sin\frac{\pi}{2}$. Then one has

$$\begin{aligned} &|\omega|^{2q}(\cos q\pi + i\sin(\pm q\pi)) + |\omega|^q(k_1+k+k_2) \\ &\times\left(\cos\frac{q\pi}{2} + i\sin\left(\pm\frac{q\pi}{2}\right)\right) + k_1(k+k_2) \\ &-\cos\omega\tau + i\sin\omega\tau = 0. \end{aligned}$$

Separating real and imaginary parts gives

$$\begin{aligned} &|\omega|^{2q}\cos q\pi + |\omega|^q(k_1+k+k_2)\cos\frac{q\pi}{2} \\ &+ k_1(k+k_2) = \cos\omega\tau \end{aligned}$$

and

$$\begin{aligned} &|\omega|^{2q}\sin(\pm q\pi) + |\omega|^q(k_1+k+k_2)\sin\left(\pm\frac{q\pi}{2}\right) \\ &= -\sin\omega\tau. \end{aligned}$$

From above two equations, one has

$$\begin{aligned} &\Big(|\omega|^{2q}\cos q\pi + |\omega|^q(k_1+k+k_2)\cos\frac{q\pi}{2} \\ &+k_1(k+k_2)\Big)^2 + \Big(|\omega|^{2q}\sin(\pm q\pi) \\ &+|\omega|^q(k_1+k+k_2) \\ &\times\sin\left(\pm\frac{q\pi}{2}\right)\Big)^2 = 1, \end{aligned}$$

i.e,

$$\begin{aligned} &|\omega|^{4q} + 2(k_1+k+k_2)^2\cos\frac{q\pi}{2}\cdot|\omega|^{3q} \\ &+\left((k_1+k+k_2)^2 + 2k_1(k+k_2)\cos q\pi\right)|\omega|^{2q} \\ &+2k_1(k+k_2)(k_1+k+k_2)\cos\frac{q\pi}{2}\cdot|\omega|^q \\ &+k_1^2(k+k_2)^2 = 0. \end{aligned} \tag{13}$$

Obviously, when $0 < q < 1$ and the following condition holds

$$k_1 \geq 0, \qquad k_2 \geq 0, \tag{14}$$

then (13) has no real solutions, meaning that (12) has no purely imaginary roots under assumption (14).

If we choose

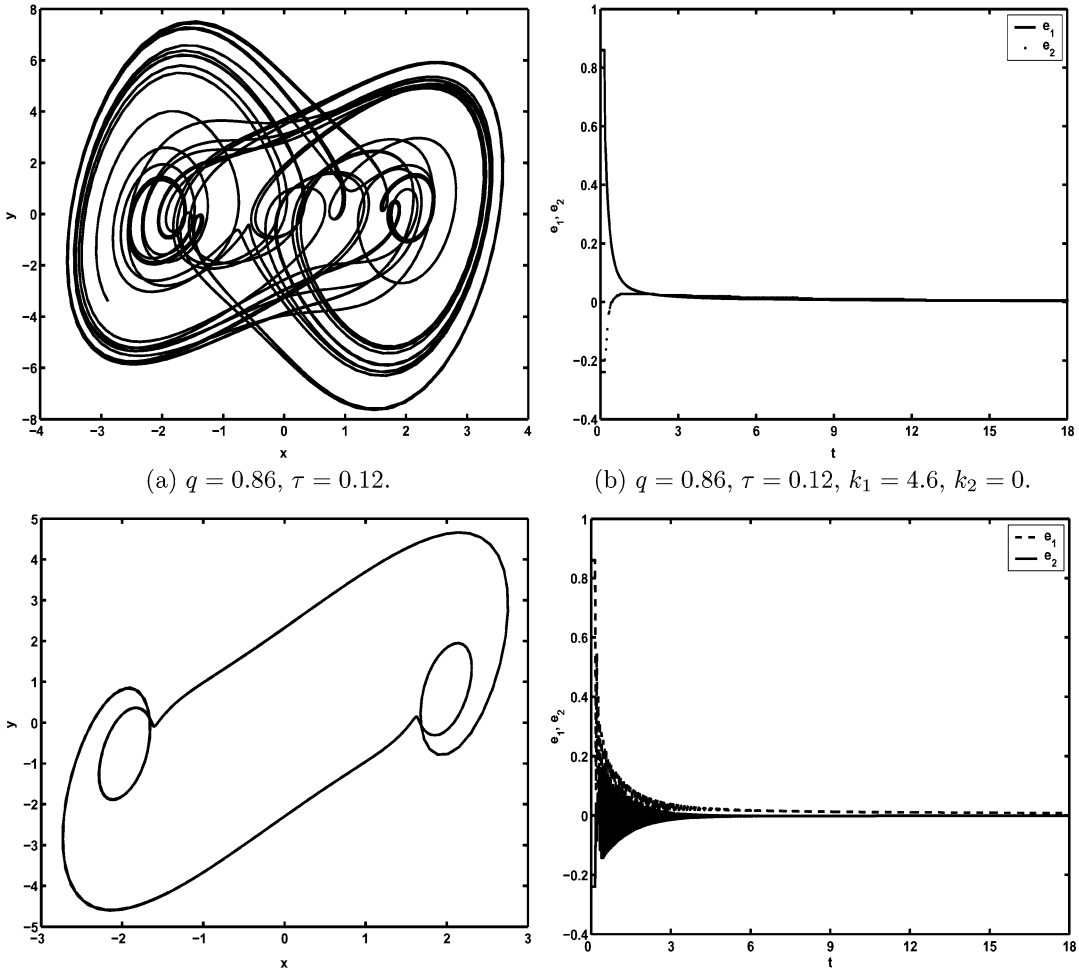
$$k_1 > 0, \quad k_2 > \frac{1}{k_1} - k, \tag{15}$$

then two eigenvalues of coefficient matrix A of (11) are negative.

So, when k_1 and k_2 satisfy both (14) and (15), system (11) satisfy the conditions of Corollary 3. Therefore, we have the following theorem.

Theorem 2. *If k_1 and k_2 satisfy both (14) and (15), then the zero solution of system (11) is Lyapunov globally asymptotically stable. Therefore, synchronization between systems (9) and (10) can be achieved.*

Numerical simulations are given in Fig. 1. Here, we find a chaotic attractor for $q = 0.86$, $\tau = 0.12$ and a limit cycle for $q = 0.66$, $\tau = 0.12$, see Fig. 1(a) and (c), respectively. These limit sets can be synchronized by the drive-response configuration (9) and (10). For the



(a) $q = 0.86,\ \tau = 0.12.$ (b) $q = 0.86,\ \tau = 0.12,\ k_1 = 4.6,\ k_2 = 0.$

(c) $q = 0.66,\ \tau = 0.12.$ (d) $q = 0.66,\ \tau = 0.12,\ k_1 = 6,\ k_2 = 6.$

Fig. 1 The chaotic attractor and limit cycle of delayed fractional Duffing oscillator and their synchronization between systems (9) and (10), where $k = 0.25$, $B = 7.5$. (a) Chaotic attractor, where $q = 0.86$, $\tau = 0.12$. (b) Synchronization error of chaotic attractor, where $q = 0.86$, $\tau = 0.12$, $k_1 = 4.6$, $k_2 = 0$. (c) Limit cycle, where $q = 0.66$, $\tau = 0.12$. (d) Synchronization error of limit cycle, where $q = 0.66$, $\tau = 0.12$, $k_1 = 6$, $k_2 = 6$

chaotic attractor, the control parameters are chosen as $k_1 = 4.6, k_2 = 0$, and for the limit cycle, the control parameters are chosen as $k_1 = 6, k_2 = 6$, see Fig. 1(b) and (d). Of course, there are some other choices for the control parameters values as long as they satisfy Theorem 2. In all simulations, we apply the Adams–Bashforth–Moulton scheme, for references see [21, 22].

5 Conclusions

In the present paper, we study the stability of fractional systems with multiple time delays. The characteristic equation for such systems is first defined. Based on this introduced characteristic equation, several interesting stability criterions are derived. Using these obtained results, we successfully determine a sufficient stability condition for a delayed fractional differential equation. We also apply our results to the synchronization of limit sets between fractional Duffing systems with retard time. These two examples are in line with the theoretical analysis.

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References

1. Hale, J.K., Verdugn Lunel, S.M., Introduction to Functional Differential Equations. Springer-Verlag, New York (1991)
2. Deng, W.H., Wu, Y.J., Li, C.P.: Stability analysis of differential equations with time-dependent delay. Int. J. Bif. Chaos. **16**(2), 465–472 (2006)
3. Li, C.P., Sun, W.G., Kurths, J.: Synchronization of complex dynamical networks with time delays. Physica A **361**, 24–34 (2006)
4. Li, C.P., Sun, W.G., Xu, D.: Synchronization of complex dynamical networks with nonlinear inner-coupling functions and time delays. Prog. Theor. Phys. **114**(4), 749–761 (2005)
5. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley-Interscience, New York (1993)
6. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
7. Butzer, P.L., Westphal, U.: An Introduction to Fractional Calculus. World Scientific, Singapore (2000)
8. Agrawal, O.P., Tenreiro Machado, J.A., Sabatier, J.: Introduction. Nonlinear Dyn. **38**(1–2), 1–2 (2004)
9. Kempfle, S., Schäfer, I., Beyer, H.: Fractional calculus: theory and applications. Nonlinear Dyn. **29**(1–4), 99–127 (2002)
10. Matignon, D.: Stability result on fractional differential equations with applications to control processing. In: Proceedings of IMACS-SMC, pp. 963–968, Lille, France (1996)
11. Chen, Y., Moore, K.L.: Analytical stability bound for a class of delayed fractional-order dynamic systems. Nonlinear Dyn. **29**, 191–200 (2002)
12. Muth, E.J.: Transform Methods with Applications to Engineering and Operations Research. Prentice-Hall, New Jersey (1977)
13. Franklin, P.: Functions of Complex Variables. Prentice-Hall, New Jersey (1958)
14. Chen, Y., Moore, K.L.: Analytical stability bound for delayed second-order systems with repeating poles using Lambert function ω. Automatica **38**, 891–895 (2002)
15. Deng, W.H., Li, C.P.: Synchronization of chaotic fractional Chen system. J. Phys. Soc. Jpn. **74**, 1645–1648 (2005)
16. Deng, W.H., Li, C.P.: Chaos synchronization of the fractional Lü system. Physica A **353**, 61–72 (2005)
17. Lu, J.G.: Nonlinear observer design to synchronize fractional-order chaotic system via a scaler transmitted signal. Physica A **359**, 107–118 (2005)
18. Lu, J.G.: Synchronization of a class of fractional-order chaotic systems via a scalar transmitted signal. Chaos Solitons Fractals **27**(2), 519–525 (2006)
19. Zhou, T.S., Li, C.P.: Synchronization in fractional-order differential systems. Physica D **212**, 111–125 (2005)
20. Li, C.P., Deng, W.H., Xu, D.: Chaos synchronization of the Chua system with a fractional order. Physica A **360**, 171–185 (2006)
21. Diethelm, K., Ford, N.J., Freed, A.D.: A predictor–corrector approach for the numerical solution of fractional differential equations. Nonlinear Dyn. **29**(1–4), 3–22 (2002)
22. Li, C.P., Peng, G.J.: Chaos in Chen's system with a fractional order. Chaos Solitons Fractals **22**, 443–450 (2004)