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# Stability Analysis of $M$-Dimensional Asynchronous Swarms with a Fixed Communication Topology *

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**Abstract**

Coordinated dynamical swarm behavior occurs when certain types of animals forage for food or try to avoid predators. Analogous behaviors can occur in engineering systems (e.g. in groups of autonomous mobile robots or air vehicles). In this paper, we study a model of an $M$-dimensional ($M \geq 2$) asynchronous swarm with a fixed communication topology, where each member only communicate with fixed neighbors, to provide conditions under which collision-free convergence can be achieved with finite-size swarm members that have proximity sensors and neighbor position sensors that only provide delayed position information. Moreover, we give conditions under which an $M$-dimensional asynchronous mobile swarm with a fixed communication topology following an "edge-leader" can maintain cohesion during movements even in the presence of sensing delays and asynchronism. In addition, the swarm movement flexibility is analyzed. Such stability analysis is of fundamental importance if one wants to understand the coordination mechanisms for groups of autonomous vehicles or robots where inter-member communication channels are less than perfect and collisions must be avoided.

## 1 Introduction

A variety of organisms have the ability to cooperatively forage for food while trying to avoid predators and other risks. For instance, when a school of fish searches for prey, or if it encounters a predator, the fish often make coordinated maneuvers as if the entire group were one organism [1]. Analogous behavior is seen in flocks of birds, herds of wildebeests, swarms of bees, groups of ants, and social bacteria [2, 3, 4, 5]. We call this kind of aggregate motion "swarm behavior." A high-level view of a swarm suggests that the organisms are cooperating to achieve some purposeful behavior and achieve some goal. Naturalists and biologists have studied such swarm behavior for decades. Moreover, computer scientists in the field of "artificial life" have studied how to model and simulate biological swarms to understand how such "social animals" interact, achieve goals, and evolve [6, 7].

Recently, there has been a growing interest in biomimicry of the mechanisms of foraging and swarming for use in engineering applications since the resulting swarm intelligence can be applied in optimization (e.g., in telecommunication systems) [2, 5], robotics [8, 9], traffic patterns in intelligent transportation systems [10, 11, 12], and military applications [13]. For instance, there has been a growing interest in groups (swarms) of flying vehicles [14, 15, 16]. Moreover, it has been proposed that swarms of robots may provide the possibility of enhanced task performance, high reliability (fault tolerance), low unit complexity,

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and decreased cost over traditional systems. Also, it has been argued that a swarm of robots can accomplish some tasks that would be impossible for a single robot to achieve. Particular research includes that of Beni [9] who introduced the concept of cellular robotic systems, and the related study in [17]. The behavior-based control strategy put forward by Brooks [18] is quite well known and it has been applied to collections of simple independent robots, usually for simple tasks. Mataric [19] describes experiments with a homogeneous population of robots acting under different communication constraints. Suzuki [20] considered a number of two-dimensional problems of formation of geometric patterns with distributed anonymous mobile swarm robots, where point-size robots are studied and collisions are allowed. A preliminary study on applying social potential fields to distributed autonomous multi-robot control was presented in [21]. A survey of autonomous search by robots and animals is provided in [22]. Decentralized control of a collective of autonomous robotic vehicles was discussed in [23], where stability of a linear chain of interdependent vehicles spreading out along a line was analyzed. Other approaches and results in this area are summarized in [8, 24].

In this paper, we are interested in mathematical modeling and analysis of stability properties of swarms. Stability is a basic qualitative property of swarms since if it is not present, then it may be impossible for the swarm to achieve any other group objective. Stability analysis of swarms is still an open problem but there have been several areas of relevant progress. In biology, researchers have used "continuum models" for swarm behavior based on non-local interactions, and have studied stability properties [25]. Jin et al. in [26] studied stability of synchronized distributed control of one-dimensional and two-dimensional swarm structures. Interestingly, their model and analysis look similar to the model and proof of stability for the load balancing problem in computer networks [27, 28]. Moreover, swarm "cohesiveness" was characterized as a stability property and a one-dimensional asynchronous swarm model was constructed by putting many identical single finite-size vehicular swarm members together, which have proximity sensors and neighbor position sensors that only provide delayed position information in [29, 30]. For this model, in [29, 30] the authors showed that for a one-dimensional stationary edge-member swarm, total asynchronism leads to asymptotic collision-free convergence and partial asynchronism leads to finite time collision-free convergence even with sensing delays. Furthermore, conditions were given in [30, 31] under which an asynchronous mobile swarm following (pushed by) an "edge-leader" can maintain cohesion during movements even in the presence of sensing delays and asynchronism. In addition, similar results are presented in [32] for stability of a one-dimensional discrete-time asynchronous swarm via an different analysis method from [29]. Recently, in [33] a continuous-time synchronous swarm model has been introduced and conditions for stable cohesion and ultimate swarm member behavior were derived with point-sized agents with no concern for collisions. Next, we would note that there have been several investigations into the stability of inter-vehicle distances in "platoons" in intelligent transportation systems (e.g., in [34, 35] or of the "slinky effect" in [36, 37], and traffic flow in [10, 12]). Finally, we would note that the study of stability properties of aircraft (spacecraft) formations is a relevant and active research area [14, 15].

Swarm stability for the $M \geq 2$ dimensional case will be studied in this paper, where stability is used to characterize the cohesiveness of a swarm. Comparing with our earlier analysis for the one-dimensional case in [29, 30, 31], the $M \geq 2$ dimensional case is more challenging. For example, each member in the one-dimensional swarm has only left and right neighbors, and it can only move to the right or to the left. However, in the $M \geq 2$ dimensional case each member may have many neighbors (it depends on the definition of "neighbor") and each of them has an infinite number of moving directions. Especially for the mobile case, there must exist some constraints for the moving direction of the "leader" besides the constraints on its step size in order to maintain swarm cohesion during movements and simultaneously avoid collisions. All the above significantly complicates the convergence analysis for the $M \geq 2$ dimensional case. In this paper, we will present an $M$-dimensional asynchronous swarm model by putting many single finite-size swarm members together in an $M$-dimensional space, where we assume there exists a fixed communication topology among swarm members and each member only communicates with fixed neighbors via the topology. With the choice of initial conditions of the swarm, a communication topology that specifies that swarm member $i$, $i = 1, 2, \ldots, N$, only communicates with its nearest neighbor $i - 1$ is fixed. We will provide conditions under which an $M$-dimensional swarm will converge to be in a cohesive form even in the presence of sensing delays and asynchronism on the basis of such a swarm model. Furthermore, we consider an $M$-dimensional asynchronous mobile swarm, where member 1 (the "leader") leads and all members communicate only with neighbors according to a communication topology, to present conditions under which it can maintain collision-free cohesion during movements even with sensing delays and asynchronism. Our study uses a discrete time discrete event dynamical system [27] approach and unlike the studies of platoon stability in intelligent transportation systems we avoid detailed characteristics of low level "inner-loop control" and vehicle dynamics in favor of focusing on high level mechanisms underlying qualitative swarm behavior when there are imperfect communications.

# 2 Modeling

First, we will explain the capabilities of a single swarm member and provide a mathematical model for an $M$-dimensional $N$-member asynchronous swarm with a communication topology, where $M \geq 2$ and $N \geq 2$ are fixed. Next, a mathematical model for an $M$-dimensional asynchronous mobile swarm with a communication topology following an "edge-leader" will be given.

## 2.1 Single Swarm Member Model

An $M$-dimensional swarm is a set of $N$ swarm members that move in an $M$-dimensional space. Assume each swarm member has a finite physical size (diameter) $w > 0$. It has a "proximity sensor," which has a sensing range with a radius $\varepsilon > w$ around each member. In the $M = 2$ case, it is a circular-shaped area with a radius $\varepsilon > w$ around each member. Once another swarm member reaches a distance of $\varepsilon$ from it, the sensor *instantaneously* indicates the position of the other member. However, if its neighbors are not in its sensing range, the proximity sensor will return $\infty$ (or, practically, some large number). The proximity sensor is used to help avoid swarm member collisions and ensures that our framework allows for finite-size vehicles, not just points. Each swarm member also has a "neighbor position sensor," which can sense the positions of neighbors around it if they are present. It performs this sensing via communication that is defined below where a communication topology is used. We assume that there is no restriction on how close a neighbor must be for the neighbor position sensor to provide a sensed value of its position. The sensed position information may be subjected to random delays (i.e., each swarm member's knowledge about its neighbors' positions may be outdated). We assume that each swarm member knows its own position with no delay. Notice that we define the position, distance and sensor sensing range of the finite-size swarm member with respect to its center, not its edge.

Swarm members like to be close to each other, but not too close. Suppose $d$ is the desired "comfortable distance" between two adjacent swarm neighbors ("adjacent" will be fully defined below in terms of a communication topology), which is known by every swarm member, and $d > \varepsilon > w$ as shown in Figure 3. Each swarm member senses the inter-swarm member distance via both neighbor position and proximity sensors and makes decisions for movements via some position updating algorithms, which is according to the error between the sensed distance and the comfortable distance $d$. And then, the decisions are input to its "driving device," which provides locomotion for it. Each swarm member will try to move to maintain a comfortable distance to its neighbors. This will tend to make the group move together in a cohesive swarm.

## 2.2 Swarm Model with a Fixed Communication Topology

An $M$-dimensional swarm is formed by putting many of the above single swarm members together on an $M$-dimensional space. An example of an $M = 2$ dimensional $N$-member swarm is shown in Figure 1. Let $x^i(t)$ denote the position vector of swarm member $i$ at time $t$. We have $x^i(t) = [x^i_1(t), x^i_2(t), \ldots, x^i_M(t)]^\top \in R^M$, $i = 1, 2, \ldots, N$, where $x^i_m(t)$, $m = 1, 2, \ldots, M$, is the $m^{th}$ position coordinate of member $i$. We assume that there is a set of times $T = \{0, 1, 2, \ldots\}$ at which one or more swarm members update their positions. Let $T^i \subseteq T, i = 1, 2, \ldots, N$, be a set of times at which the $i^{th}$ member's position $x^i(t), t \in T^i$, is updated. Notice that the elements of $T^i$ should be viewed as the indices of the sequence of physical times at which updates take place, not the real times. These time indices are non-negative integers and can be mapped into physical times. The $T^i$, $i = 1, 2, \ldots, N$, are independent of each other for different $i$. However, they may have intersections (i.e., it could be that $T^i \cap T^j \neq \emptyset$ for $i \neq j$, so two or more swarm members may move simultaneously). Note that our model assumes that swarm member $i$, $i = 2, 3, \ldots, N$, communicates with its neighbor member $i-1$ via its neighbor position sensors within a communication topology (the communication topology will be explained below) to obtain the position information of member $i-1$ (the position information obtained may be subjected to random delays). A variable $\tau^i_{i-1}(t) \in T$, $i = 2, 3, \ldots, N$, is used to denote the time index of the real time at which position information of its communicating neighbor $i-1$ is obtained by member $i$ at $t \in T^i$ and it satisfies $0 \leq \tau^i_{i-1}(t) \leq t$ for $t \in T^i$. Of course, while we model the times at which neighbor position information is obtained as being the same times at which one or more swarm members decide where to move and actually move, it could be that the *real time* at which such neighbor position information is obtained is earlier than the real time where swarm members moved. The difference $t - \tau^i_{i-1}(t)$ between current time $t$ and the time $\tau^i_{i-1}(t)$ can be viewed as a form of communication delay (of course the actual length of the delay depends on what real times correspond to the indices $t$, $\tau^i_{i-1}(t)$). Moreover, it is important to note that we assume that $\tau^i_{i-1}(t) \geq \tau^i_{i-1}(t')$ if $t > t'$ for $t, t' \in T^i$. This ensures that member $i$ uses the most recently obtained communicating neighbor position information. Besides the neighbor position information obtained from its neighbor position sensors, swarm member $i$ also gets some information from its proximity sensors. Assume that if its communicating neighbor $i-1$ is beyond the sensing range of its proximity sensors, it uses the information $x^{i-1}(\tau^i_{i-1}(t))$ from its neighbor position sensors; if its neighbor $i-1$ is inside the sensing range of its proximity sensors, it uses the real-time neighbor position information $x^{i-1}(t)$ provided by its proximity sensors. The information is used for position updating until member $i$ gets more recent information, for example, from its neighbor position sensor. Notice that swarm member $i$ updates its position only at time indices $t \in T^i$ and at all times $t \notin T^i$ its position $x^i(t)$ is left unchanged.

In the below, based on [28] we specify two assumptions that we use to characterize asynchronism for swarms.

**Assumption 1. (Total Asynchronism):** *Assume the sets $T^i$, $i = 1, 2, \ldots, N$, are infinite, and if for each $k$, $t_k \in T^i$ and $t_k \to \infty$ as $k \to \infty$, then $\lim_{k\to\infty} \tau^i_{i-1}(t_k) = \infty$, $i = 2, 3, \ldots, N$.*

This assumption guarantees that each swarm member moves infinitely often and the old position information of neighbors of each swarm member is eventually purged from it. More precisely, given any time $t_1$, there exists a time $t_2 > t_1$ such that $\tau^i_{i-1}(t) \geq t_1$, for $i = 2, 3, \ldots, N$ and $t \geq t_2$. On the other hand, the delays $t - \tau^i_{i-1}(t)$ in obtaining position information of neighbor of member $i$ can become unbounded as $t$ increases. Next, we specify a more restrictive type of asynchronism, but one which is usually easy to implement in practice.

**Assumption 2. (Partial Asynchronism):** *There exists a finite positive integer $B$ (i.e., $B \in Z^+$, where $Z^+$ represents the set of positive integers) such that:*

*(a) For every $i$ and $t \geq 0$, $t \in T$, at least one of the elements of the set $\{t, t+1, \ldots, t+B-1\}$ belongs to $T^i$.*

*(b) There holds $t - B < \tau^i_{i-1}(t) \leq t$ for all $i = 2, 3, \ldots, N$, and all $t \geq 0$ belonging to $T^i$.*

Notice that for the partial asynchronism assumption each member moves at least once within $B$ time indices and the delays $t - \tau^i_{i-1}(t)$ in obtaining position information of neighbors of member $i$ are bounded by $B$, i.e., $0 \leq t - \tau^i_{i-1}(t) < B$.

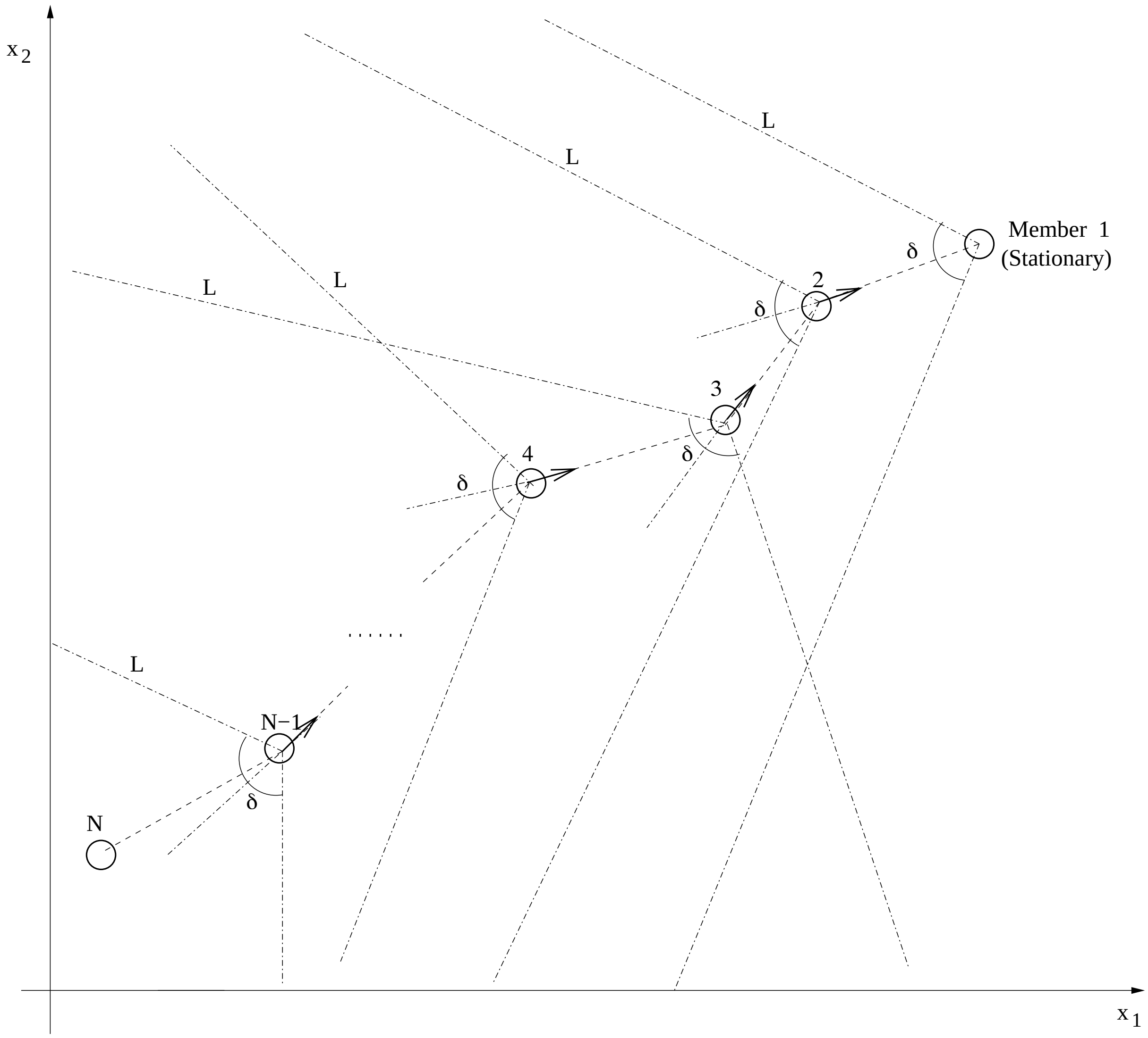


Figure 1: An $M$-dimensional ($M = 2$) $N$-member asynchronous swarm with a fixed communication topology (dashed line along the direction of the arrows), all members moving to be adjacent to the stationary member (member 1).

Next, in order to construct a swarm model we will first explain the communication topology. We will study an asynchronous swarm with certain initial conditions and where a communication topology is fixed. An example of such an $M = 2$ dimensional swarm is shown in Figure 1. Assume that there is a set of $N$ swarm members distributed in an $M$-dimensional space. Assume $|x| = \sqrt{x^\top x}$ and $|x^i(0) - x^j(0)| > d$, for

$i, j = 1, 2, \ldots, N, i \neq j$ initially. Suppose one of the swarm members always remains stationary, which we call member 1. Member 1's nearest neighbor, which we call member 2 (if there is more than one nearest neighbor, choose one randomly), is assumed to be initially located inside a sector area starting from the position of member 1 with a radius $L \gg d$ and a central angle $0 \leq \delta \leq \pi$ as shown in Figure 1 (notice that it is a sector area for $M = 2$ case, and a cone for $M = 3$). Member 1 only communicates with member 2, and member 2 uses the information of member 1 to update its position. Similarly, member 2's nearest neighbor, which we call member 3, is located inside the *overlapping* area of the sector area starting from the position of member 1 and a equal-size sector area starting from the position of member 2, which is also symmetrical about the extension of the connected line between positions of members 1 and 2. Member 2 only communicates with members 1 and 3, and member 3 uses the information of member 2 to update its position. In the same way, assume the nearest neighbor of swarm member $i-1$, $i = 4, 5, \ldots, N$, which we call member $i$, is only located inside the overlapping area of all the sector areas starting from positions of members $1, 2, \ldots, i-1$, respectively. In addition, the sector area starting from the position of member $i-1$ is symmetrical about the extension of the connected line between positions of members $i-2$ and $i-1$. Member $i-1$ only communicates with members $i-2$ and $i$, and member $i$ uses the information of member $i-1$ to update its position. Therefore, there exists a fixed communication topology from member $N$ to member 1 in such a swarm, which is represented by a dashed line along the direction of the arrows in Figure 1.

We may think of the above swarm as a chain of inter-communicating single swarm members. We assume that swarm members will obtain the position information of their nearest neighbor via this communication topology to update their positions at their updating time indices. In particular, swarm member $i$, $i = 2, 3 \ldots, N$, tries to maintain a comfortable distance $d$ to its nearest neighbor member $i-1$ so that it moves only along the connection line of its position and the sensed position of member $i-1$. Clearly with the choice of the above initial conditions, there will be no communication path (the line connecting positions of two communicating neighbors) overlapping in such a fixed communication topology during position updating of swarm members and the communicating neighbor of each member is always its nearest neighbor. In fact, there are underlying reasons that collisions will never happen in the swarm, which will be explained below. Note that if $\delta = 0$, the swarm which satisfies the above initial conditions is a one-dimensional swarm.

Of course, not all possible initial conditions of swarm members can fit in the above constraints. Two examples of $M = 2$ dimensional swarms with illegal initial conditions are shown in Figure 2. Obviously for the swarm in Figure 2(a), it is impossible to build an overlap-free communication topology from member 3 to the stationary member 1. Considering the swarm in Figure 2(b), clearly all members except the first three are not located in the overlapping area requested above so that it is possible to have communication paths overlapping during movements of swarm members although there may exist a communication topology from member 9 to member 1. Hence, the "overlapping" condition is needed due to the asynchronism and delays and our focus on collision avoidance. Next, note that we need the constraint $0 \leq \delta \leq \pi$ above to avoid situations like in Figure 2(b). Why is the upper bound $\pi$? This is due to the fact that member $i$, $i = 2, 3 \ldots, N$, is the nearest neighbor of member $i-1$, and will be discussed more below.

Let $e^i(t) = x^i(t) - x^{i+1}(t),\ i = 1, 2, \ldots, N-1$ denote the inter-neighbor distance vector of members $i+1$ and $i$. Assume the direction of $e^i(t)$ is from the position of member $i+1$ to the position of member $i$ and $|e^i(t)| = \sqrt{(e^i(t))^\top e^i(t)}$, where $|e^i(t)|$ denotes its magnitude. We use "$g$ functions" $g_a(|e^i(t)| - d)$ and $g_f(|e^i(t)| - d)$ (see [29, 30, 31]) to denote two different kinds of attractive and repelling relationships

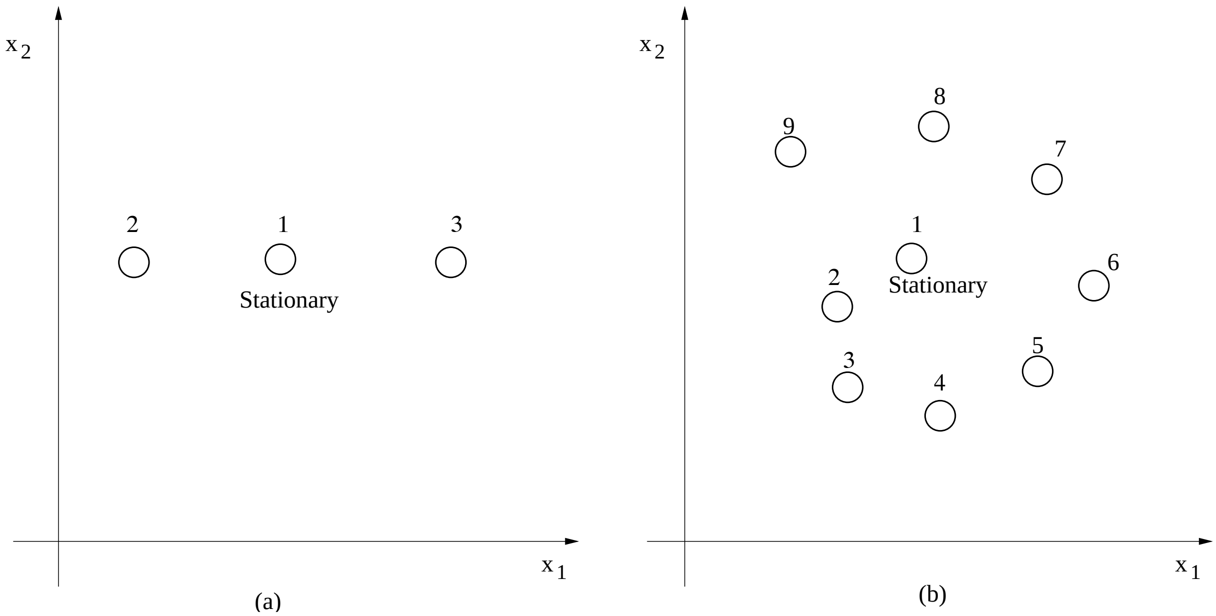


Figure 2: Examples of $M = 2$ dimensional swarms with illegal initial conditions.

between two swarm neighbors, where for a scalar $\beta > 1$, $g_a(|e^i(t)| - d)$ is such that

$$\frac{1}{\beta}(|e^i(t)| - d) < g_a(|e^i(t)| - d) < (|e^i(t)| - d), \text{ if } (|e^i(t)| - d) > 0; \tag{1}$$

$$g_a(|e^i(t)| - d) = (|e^i(t)| - d) = 0, \text{ if } (|e^i(t)| - d) = 0; \tag{2}$$

$$(|e^i(t)| - d) < g_a(|e^i(t)| - d) < \frac{1}{\beta}(|e^i(t)| - d), \text{ if } (|e^i(t)| - d) < 0. \tag{3}$$

and for some scalars $\beta$ and $\eta$, such that $\beta > 1$, and $\eta > 0$, $g_f(|e^i(t)| - d)$ satisfies

$$\frac{1}{\beta}(|e^i(t)| - d) < g_f(|e^i(t)| - d) < (|e^i(t)| - d), \text{ if } (|e^i(t)| - d) > \eta; \tag{4}$$

$$g_f(|e^i(t)| - d) = (|e^i(t)| - d), \text{ if } -\eta \leq (|e^i(t)| - d) \leq \eta; \tag{5}$$

$$(|e^i(t)| - d) < g_f(|e^i(t)| - d) < \frac{1}{\beta}(|e^i(t)| - d), \text{ if } (|e^i(t)| - d) < -\eta. \tag{6}$$

Note that the above $g$ functions (a scalar) only represents the amount of the attractive or repelling force between two communicating neighbors for a given distance vector. The moving direction of swarm members depends on the direction of the distance vector $e^i(t)$.

Next, we will show that in the above $M$-dimensional swarm, collisions will never happen even without proximity sensors. From our assumption, $|e^i(0)| > d$, for $i = 1, 2, \ldots, N-1$, so that at the beginning the proximity sensor of member $i+1$ cannot sense its neighbor $i$. So, swarm members only update their positions according to the chosen $g$ function, which uses the sensed information provided by their neighbor position sensors. Particularly as show in Figure 3 which enlarges a part of Figure 1, member 2 will move towards the stationary member 1 step by step along the connected line between members 1 and 2 via $g$ function at $t \in T^2$ once it gets the unchanged position information of member 1, $x^1(t)$. From the definition of the $g$ function, we have

$$g(|x^1(t) - x^2(t)| - d) \leq |x^1(t) - x^2(t)| - d$$

and so we get

$$|e^1(t)| \geq d, \forall t$$

As we know, member 3's proximity sensors cannot sense member 2 since $|e^2(t)|$ is greater than $d$ at the beginning. Therefore, member 3 updates its position at $t \in T^3$ only according to its sensed position information $x^2(\tau_2^3(t))$ and the update step is equal to $g(|x^2(\tau_2^3(t))-x^3(t)|-d)$. According to our assumptions of asynchronism, we have $\tau_2^3(t) \leq t$. As shown in Figure 3, member 2 already arrives at the point $C$ at $t$, which is $x^2(t)$. However, due to communication delays, the position information of member 2 obtained by member 3 at $t \in T^3$ is $x^2(\tau_2^3(t))$, which is still at the point $B$ (note that if there is no communication delay, i.e., $\tau_2^3(t) = t$, points $B$ and $C$ will overlap). Suppose $\alpha$ is the angle formed by lines of $BC$ and $AB$ in the clockwise direction, and then we have $\pi/2 \leq \alpha \leq 3\pi/2$ since $0 \leq \delta \leq \pi$ and member 3 is only located inside the overlapping area of the two sector areas starting from positions of members 1 and 2 with a central angle $\delta$. Clearly, in the triangle $ABC$, $|AC| \geq |AB|$ ($|AC| = |AB|$ if points $B$ and $C$ overlap), i.e.,

$$|x^2(\tau_2^3(t)) - x^3(t)| \leq |x^2(t) - x^3(t)| \tag{7}$$

According to the definition of the $g$ function and Equation (7), we have

$$g(|x^2(\tau_2^3(t)) - x^3(t)| - d) \leq |x^2(\tau_2^3(t)) - x^3(t)| - d \leq |x^2(t) - x^3(t)| - d \tag{8}$$

From Equation (8), we know the update step of member 3 is always less than or equal to the error between the real distance from member 3 to 2 and $d$ (i.e, $|x^2(t) - x^3(t)| - d$). Hence, the inter-member distance between members 3 and 2 is always greater than or equal to $d$. Clearly, a similar result holds for all other swarm members, so we have

$$|e^i(t)| \geq d, \forall t, i = 1, 2, ..., N-1 \tag{9}$$

Equation (9) implies that all the swarm members' proximity sensors will never sense their nearest neighbor during movements. With the choice of initial conditions there always exists a fixed overlap-free communication topology during movements of swarm members so that each member always communicates with its nearest neighbor and try to maintain a comfortable distance to it. Equation (9) also implies that the distance between every member and its nearest neighbor is larger than or equal to $d$ at any time. Therefore, members will never have collisions in the above swarm even without proximity sensors.

Let $e^i(\tau_i^{i+1}(t)) = x^i(\tau_i^{i+1}(t)) - x^{i+1}(t),\ i = 1, 2, \ldots, N-1$ denote the sensed inter-neighbor distance vector of members $i+1$ and $i$. Assume its direction is from the position of member $i+1$ to the sensed position of member $i$ and $|e^i(\tau_i^{i+1}(t))| = \sqrt{(e^i(\tau_i^{i+1}(t)))^\top e^i(\tau_i^{i+1}(t))}$, where $|e^i(\tau_i^{i+1}(t))|$ denotes its magnitude. And so,

$$|e^i(t)| \geq |e^i(\tau_i^{i+1}(t))| \geq d, \forall t, i = 1, 2, ..., N-1 \tag{10}$$

A mathematical model for the above $M$-dimensional swarm is given by

$$\begin{aligned}
x^1(t+1) &= x^1(t), \forall t \in T^1 \\
x^2(t+1) &= x^2(t) + g(|x^1(\tau_1^2(t)) - x^2(t)| - d)\left[\frac{x^1(\tau_1^2(t)) - x^2(t)}{|x^1(\tau_1^2(t)) - x^2(t)|}\right], \forall t \in T^2 \\
\vdots &= \qquad \vdots \\
x^{N-1}(t+1) &= x^{N-1}(t) + g(|x^{N-2}(\tau_{N-2}^{N-1}(t)) - x^{N-1}(t)| - d)\left[\frac{x^{N-2}(\tau_{N-2}^{N-1}(t)) - x^{N-1}(t)}{|x^{N-2}(\tau_{N-2}^{N-1}(t)) - x^{N-1}(t)|}\right], \forall t \in T^{N-1} \\
x^N(t+1) &= x^N(t) + g(|x^{N-1}(\tau_{N-1}^N(t)) - x^N(t)| - d)\left[\frac{x^{N-1}(\tau_{N-1}^N(t)) - x^N(t)}{|x^{N-1}(\tau_{N-1}^N(t)) - x^N(t)|}\right], \forall t \in T^N \\
x^i(t+1) &= x^i(t),\ \forall t \notin T^i,\ i = 1, 2, ..., N
\end{aligned} \tag{11}$$

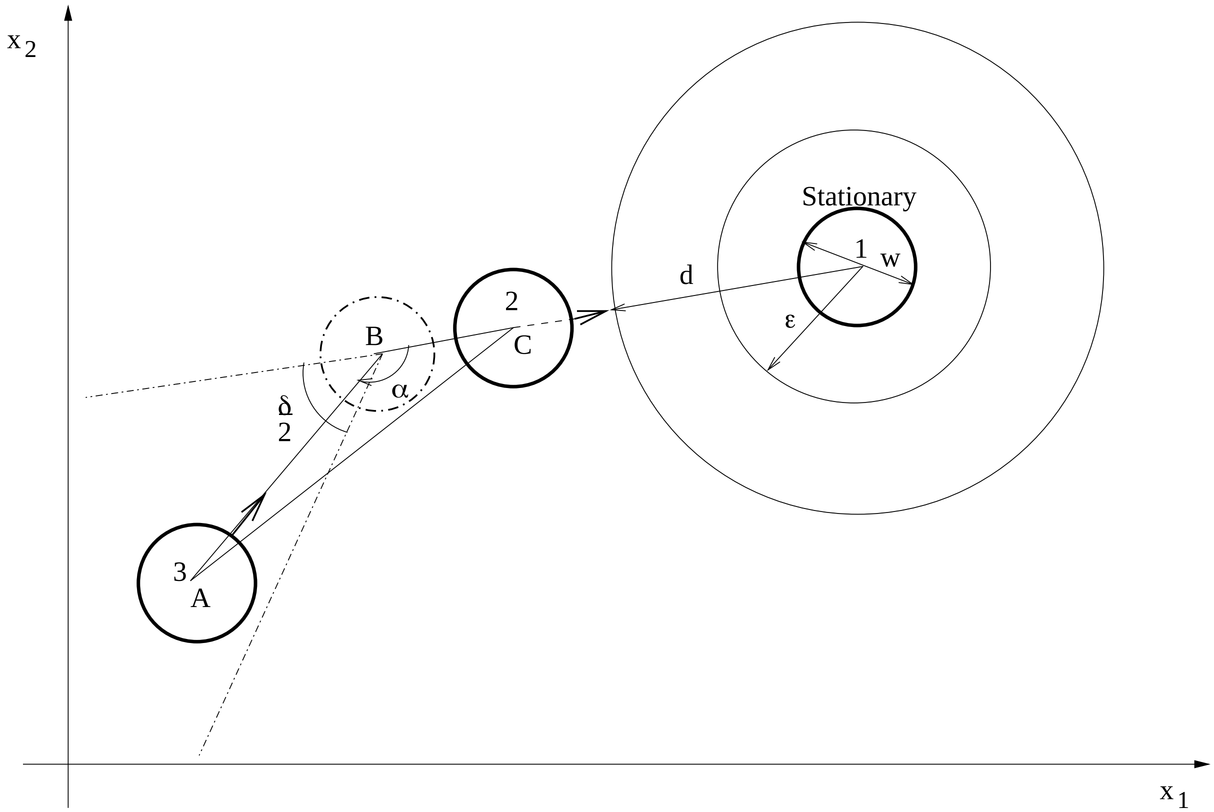


Figure 3: The enlargement of the part about member 1, 2, and 3 in Figure 1

Here, each item in brackets is a unit vector which represents the moving direction of each swarm member which is the direction of its sensed inter-neighbor distance vector, and each $g$ function item in front of the brackets is a scalar, which is the step size of each swarm member. In addition, $x^i(t)$ is the position vector of member $i$. Hence, "+"and "−" above are the addition and subtraction of vectors.

From the above assumptions, we can write the model of Equation (11) into the following form.

$$
\begin{aligned}
e^1(t+1) &= \begin{cases} e^1(t) - g(|e^1(\tau_1^2(t))| - d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right], \forall t \in T^2 \\ e^1(t), \forall t \notin T^2 \end{cases} \\
e^i(t+1) &= \begin{cases} e^i(t) + g(|e^{i-1}(\tau_{i-1}^i(t))| - d)\left[\frac{e^{i-1}(\tau_{i-1}^i(t))}{|e^{i-1}(\tau_{i-1}^i(t))|}\right] - g(|e^i(\tau_i^{i+1}(t))| - d)\left[\frac{e^i(\tau_i^{i+1}(t))}{|e^i(\tau_i^{i+1}(t))|}\right], \\ \qquad \forall t \in T^i \cap T^{i+1},\ i = 2, 3, \ldots, N-1 \\ e^i(t) - g(|e^i(\tau_i^{i+1}(t))| - d)\left[\frac{e^i(\tau_i^{i+1}(t))}{|e^i(\tau_i^{i+1}(t))|}\right], \forall t \in T^{i+1}, t \notin T^i,\ i = 2, 3, \ldots, N-1 \\ e^i(t) + g(|e^{i-1}(\tau_{i-1}^i(t))| - d)\left[\frac{e^{i-1}(\tau_{i-1}^i(t))}{|e^{i-1}(\tau_{i-1}^i(t))|}\right], \forall t \in T^i, t \notin T^{i+1},\ i = 2, 3, \ldots, N-1 \\ e^i(t),\ \forall t \notin T^i \cup T^{i+1},\ i = 2, 3, \ldots, N-1 \end{cases}
\end{aligned} \tag{12}
$$

where $e^i(t)$ is the inter-neighbor distance vector of members $i+1$ and $i$.

## 2.3 Mobile Swarm Model with an Edge-Leader

Assume that in a swarm with a fixed communication topology, a member will consider itself to be an edge-member if it only communicates with one neighbor, and a middle member if it communicates with two neighbors. Therefore, members 1 and $N$ of the swarm in Figure 1 are edge-members since they only communicate with members 2 and $N-1$, respectively. All other members $i$, i=2,3, ... , $N-1$, are middle members since they communicate with both neighbors $i-1$ and $i+1$ assuming a fixed communication topology. Now assume that member 1 (the edge member) moves to some direction with a bounded step as a edge-leader (we will explain the moving direction and step size of the edge-leader below). Member $i+1$ will try to follow $i$, $i = 1, 2, \ldots, N-1$, and at the same time try to maintain a comfortable inter-neighbor distance.

Similar to [30, 31], we assume $[d-\gamma, d+\gamma]$ is a "comfortable distance neighborhood" relative to two communicating neighbors $i$ and $i+1$ (i.e., when $|e^i(t)| \in [d-\gamma, d+\gamma]$, we say that they are in the comfortable distance neighborhood), where $2\gamma$ is the comfortable distance neighborhood size. Assume that $0 < \varepsilon < d-\gamma$ so that we do not consider swarm member $i+1$ to be at a comfortable distance to member $i$ if it is too close to it, where $\varepsilon$ is the sensing range of swarm members' proximity sensors.

Clearly in the two assumptions of asynchronism we specified above, only Assumption 2 (partial asynchronism) will result in cohesiveness for a mobile swarm since the delays in Assumption 1 (total asynchronism), which could be unbounded, will make swarm members lose track of their edge-leader or their neighbors during movements, i.e., the distance between swarm neighbors could become unbounded just because swarm members use arbitrarily old sensed information. Hence, we construct the mobile swarm model based on Assumption 2 (partial asynchronism), which has a finite positive integer $B$ as an "asynchronism measure."

For convenience, assume that $|e^i(0)| = d$, for $i = 1, 2, ..., N-1$ initially, i.e., at the beginning all swarm members are at a comfortable distance to their communicating neighbors. In addition, we assume that the initial positions of swarm members satisfy the constraints we explained for the stationary edge-member case except that all the sector areas starting from the position of member $i$, $i = 2, 3, \ldots, N-1$ are formed by a line starting from the position of member $i$ and the extension of the connected line between positions of members $i-1$ and $i$ with a central angle $\delta/2$ as shown in Figure 4, where $0 \leq \delta \leq \pi$ (note that actually these sector areas are all the left or right half parts of the corresponding sector areas we explained for the stationary edge-member case in Figure 1; we will explain how the $\delta/2$ constraint arises below). Similarly, member $i+1$ only communicates with member $i$ and there exists a fixed communication topology from member $N$ to member 1 represented by a dashed line along the direction of the arrows as shown in Figure 4.

Assume member 1 (the edge-leader) moves only in "legal directions" with a step vector $s(t)$ at time $t \geq 0, t \in T^1$, where $0 < |s(t)| \leq r$ (i.e., the step size is bounded by a finite positive scalar $r$). Here, "legal directions" means those directions to which the distances of member 1 to all other members will monotonically increase along each of its move steps if all other members are stationary and they are defined on the basis of the initial conditions we explained above. In fact, we assume member 1 will calculate the range of legal directions according to the position information of member 2 and 3 before each of its moving steps (note that here we assume there are at least three members in the swarm, and member 1 gets the position information of member 3 via member 2. The two-member case will be discussed in Corollaries 1 and 2). Assume that the moving direction of member 1 forms an angle $\theta$ with the connected line of positions of members 2 and 1 in the clockwise direction, and the connected line between positions of members 1 and 2 forms an angle $\omega$ with the connected line between positions of members 2 and 3 in the clockwise direction. Moreover, assume the connected line between positions of members 2 and 1 forms an angle $\sigma$ with the connected line between positions of members 1 and 3 in the clockwise direction. Then, legal directions are those directions satisfying

$$\pi - \sigma \leq \theta \leq \pi \text{ if } \omega > \pi$$

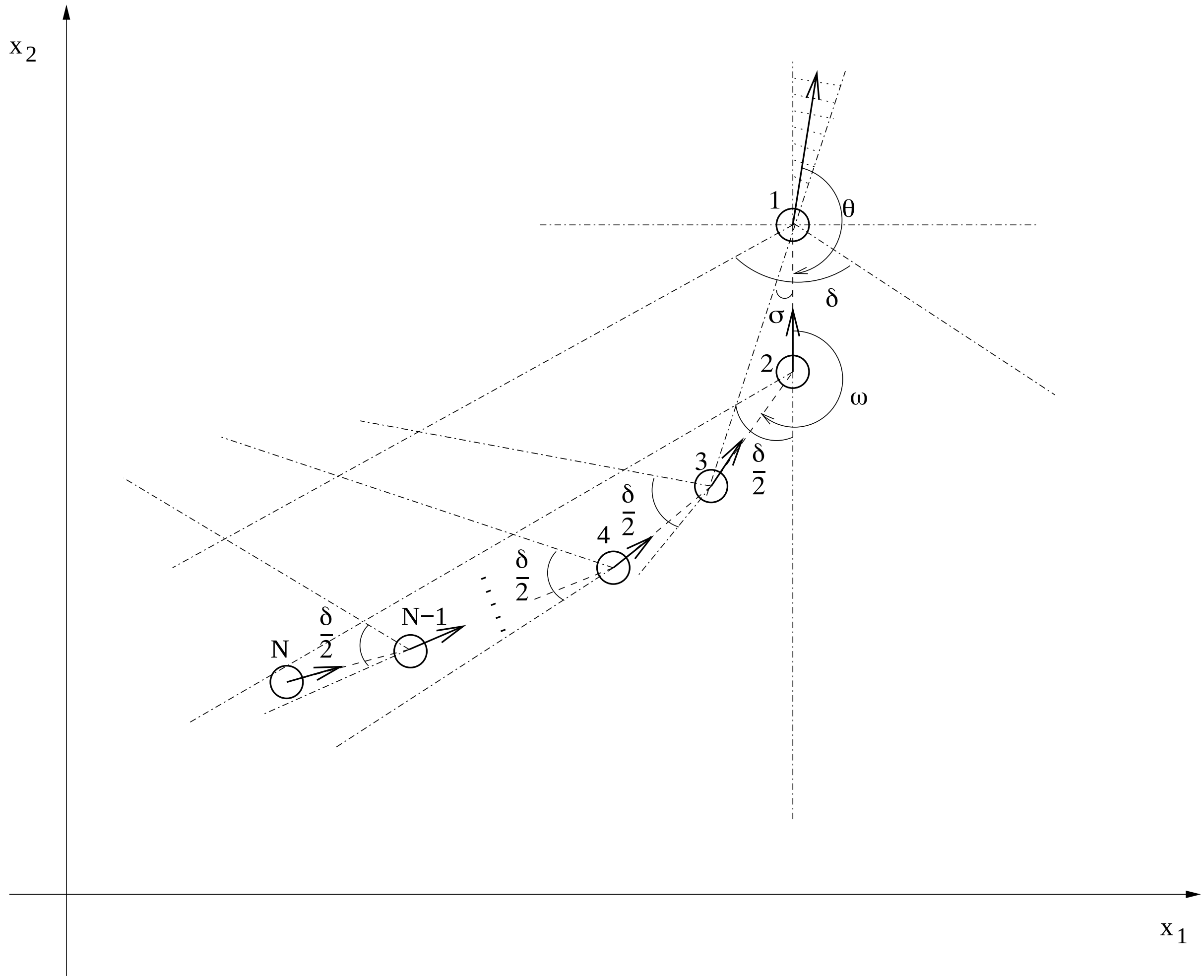


Figure 4: An $M$-dimensional ($M = 2$) $N$-member asynchronous mobile swarm example with a fixed communication topology (dashed line along the direction of the arrows), all members following an edge-leader (member 1) (case (a): the edge-leader's moving direction satisfies $\pi - \sigma \leq \theta \leq \pi$ if $\omega > \pi$).

(the shaded region in Figure 4) and

$$\pi \leq \theta \leq \pi + \sigma \text{ if } \omega \leq \pi$$

by symmetry, which are two different cases for the $M = 2$ dimensional swarm (for $M = 3$ case, the shaded region becomes a one-fourth cone). Note that if members 3 and 2 are located in the same line with member 1, i.e., $\omega = \pi$, the unique legal moving direction for member 1 is $\theta = \pi$. Moreover, the $\delta/2$ constraint of initial conditions (compared to $\delta$ constraint in the stationary case) guarantees that the legal directions we defined always hold during movements of the swarm, i.e., with the choice of initial conditions member 1 will always move far away from all other members as long as it moves in legal directions. Clearly with the constraint of legal directions and the choice of the above initial conditions, member 1 cannot make sharp turns or move towards member 2 during the moving process. It only moves far away from all other members so that collisions can be avoided and an overlap-free communication topology always exists in the swarm, which is a prerequisite for the mobile swarm to keep the cohesiveness. Obviously you could define other strategies that would allow for sharper turns but this will come at the expense of the leader needing more position information from its followers.

From the above, member 1 must have the position information of members 2 and 3 to decide its moving direction. At the beginning, members 3 and 2 are stationary with a comfortable distance to members 2 and 1, respectively. Therefore, member 1 starts its first move at $t \in T^1, t \geq 0$ after it obtains the position information of members 2 and 3, and all other members also start moving one by one at their updating time sets. However, member 1 cannot immediately use its neighbor position information to move further at the next $t \in T^1, t \geq 0$ since its information about member 2 obtained via its neighbor position sensor $x^2(\tau_2^1(t))$ may include random delays (at this time its proximity sensor doesn't work since the inter-neighbor distance is equal to or larger than $d$). In the same way, its information of member 3 passed by member 2 may include random delays, too. Therefore, we assume that member 1 has to use, we call, a "wait $4B-1$ steps strategy" under Assumption 2 (partial asynchronism) to get the information necessary for continuing its moving. Simply speaking, the "wait $4B-1$ steps strategy" is that member 1 has to wait $4B-1$ time indices to make another move step after its previous move, i.e., if member 1 moves at $t^1 \in T^1$, its next moving step will be at the first time index which satisfies $t \geq t^1 + 4B - 1$, $t \in T^1$. The underlying idea is that we assume at $t \in T^1$, member 1 moves with a step vector $s(t)$ only when it has enough information to decide its legal directions; otherwise, it remains stationary (waits). According to Assumption 2 (partial asynchronism), the maximum possible neighbor position delay is $B-1$. On the basis of this, we can prove that the leader can be guaranteed to get the direction information of the position information of members 2 and 3 within the $4B-1$ time steps, which will be explained in detail in a proof below.

Furthermore, we assume that member 2 follows member 1 in order to be in a comfortable distance to member 1. It updates its position at $t \in T^2$ towards member 1 along the connected line of its current position and its obtained position of member 1. Similarly, all other swarm members $i$, $i = 3, 4, \ldots, N$ move to follow their moving communicating neighbors $i-1$ along the connected line of their positions and their obtained position of $i-1$ and try to be at the comfortable distance to them. We think of the swarm as maintaining the cohesiveness if all the swarm members are in the comfortable distance neighborhood to their communicating neighbors during movements. Note that the leader's moving step size bound $r$ and the asynchronism measure $B$ can be used as a measure of how fast a cohesive asynchronous swarm moves.

Next, we will show that in the above $M$-dimensional mobile swarm, collisions will never happen even without proximity sensors. From our assumption, $|e^i(0)| = d$, for $i = 1, 2, \ldots, N-1$, so that at the beginning the proximity sensor of member $i+1$ cannot sense its neighbor $i$. So, swarm members only update their positions according to the $g$ function, which uses the information provided by their neighbor position sensors. Moreover, member 1 always moves far away from all other members. Similar to the analysis for the stationary edge-member case in the last section, we can prove that Equation (10) also holds for the above mobile case according to the definition of the $g$ function and assumptions of asynchronism. This implies there will be no collisions in the above mobile swarm even without proximity sensors. Thus, we can write a model in the below for the above $M$-dimensional mobile swarm. For the edge-leader (member 1), we have

$$\begin{aligned} x^1(t+1) &= \begin{cases} x^1(t) + s(t), & \text{if } t - t_p(t) \geq 4B-1, t \in T^1; \\ x^1(t), & \text{otherwise.} \end{cases} \\ t_p(t+1) &= \begin{cases} t, & \text{if } t - t_p(t) \geq 4B-1, t \in T^1; \\ t_p(t), & \text{otherwise.} \end{cases} \end{aligned} \tag{13}$$

where $B \in Z^+$ is the asynchronism measure in Assumption 2, and $t_p(t)$ denotes the last time index that member 1 moved to a new position at time $t \in T$. Furthermore, we assume that at $t = 0$ member 1 receives members 2 and 3's initial position information and $t_p(0) = -(4B-1)$ so that member 1 will start to move at the first time index $t \in T^1$, $t \geq 0$.

For all other swarm members, we have

$$
\begin{aligned}
x^2(t+1) &= x^2(t) + g(|x^1(\tau_1^2(t)) - x^2(t)| - d)\left[\frac{x^1(\tau_1^2(t)) - x^2(t)}{|x^1(\tau_1^2(t)) - x^2(t)|}\right], \forall t \in T^2 \\
\vdots &= \qquad \vdots \\
x^{N-1}(t+1) &= x^{N-1}(t) + g(|x^{N-2}(\tau_{N-2}^{N-1}(t)) - x^{N-1}(t)| - d)\left[\frac{x^{N-2}(\tau_{N-2}^{N-1}(t)) - x^{N-1}(t)}{|x^{N-2}(\tau_{N-2}^{N-1}(t)) - x^{N-1}(t)|}\right], \forall t \in T^{N-1} \\
x^N(t+1) &= x^N(t) + g(|x^{N-1}(\tau_{N-1}^{N}(t)) - x^N(t)| - d)\left[\frac{x^{N-1}(\tau_{N-1}^{N}(t)) - x^N(t)}{|x^{N-1}(\tau_{N-1}^{N}(t)) - x^N(t)|}\right], \forall t \in T^N \\
x^i(t+1) &= x^i(t),\ \forall t \notin T^i,\ i = 2, 3, \ldots, N \qquad (14)
\end{aligned}
$$

which is similar to the model of the stationary edge-member case in Equation (11). Similar to the stationary edge-member case, we can write the model of Equations (13) and (14) into the following form:

$$
\begin{aligned}
e^1(t+1) &= \begin{cases}
e^1(t) + s(t) - g(|e^1(\tau_1^2(t))| - d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right], & \text{if } t \in T^2, t - t_p(t) \geq 4B - 1, t \in T^1; \\
e^1(t) + s(t), & \text{if } t - t_p(t) \geq 4B - 1, t \in T^1, t \notin T^2; \\
e^1(t) - g(|e^1(\tau_1^2(t))| - d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right], & \text{if } t \in T^2, t \notin T^1 \text{ or } t \in T^2 \cup T^1, t - t_p(t) < 4B - 1; \\
e^1(t), & \text{if } t \notin T^2 \cup T^1 \text{ or } t \notin T^2, t \in T^1, t - t_p(t) < 4B - 1.
\end{cases} \\
t_p(t+1) &= \begin{cases}
t, & \text{if } t - t_p(t) \geq 4B - 1, t \in T^1; \\
t_p(t), & \text{otherwise.}
\end{cases} \\
e^i(t+1) &= \begin{cases}
e^i(t) + g(|e^{i-1}(\tau_{i-1}^i(t))| - d)\left[\frac{e^{i-1}(\tau_{i-1}^i(t))}{|e^{i-1}(\tau_{i-1}^i(t))|}\right] - g(|e^i(\tau_i^{i+1}(t))| - d)\left[\frac{e^i(\tau_i^{i+1}(t))}{|e^i(\tau_i^{i+1}(t))|}\right], \\
\qquad \text{if } t \in T^i \cap T^{i+1},\ i = 2, 3, \ldots, N-1; \\
e^i(t) - g(|e^i(\tau_i^{i+1}(t))| - d)\left[\frac{e^i(\tau_i^{i+1}(t))}{|e^i(\tau_i^{i+1}(t))|}\right], & \text{if } t \in T^{i+1}, t \notin T^i,\ i = 2, 3, \ldots, N-1; \\
e^i(t) + g(|e^{i-1}(\tau_{i-1}^i(t))| - d)\left[\frac{e^{i-1}(\tau_{i-1}^i(t))}{|e^{i-1}(\tau_{i-1}^i(t))|}\right], & \text{if } t \in T^i, t \notin T^{i+1},\ i = 2, 3, \ldots, N-1; \\
e^i(t), & \text{if } t \notin T^i \cup T^{i+1},\ i = 2, 3, \ldots, N-1
\end{cases} \qquad (15)
\end{aligned}
$$

where $e^i(t)$ is the inter-member distance vector between members $i+1$ and $i$ and we have the same assumption for $t_p(t)$ as the above.

# 3 Convergence Analysis of $M$-Dimensional Swarms with a Fixed Communication Topology

In this section, we will study stability properties of $M$-dimensional asynchronous swarms with a fixed communication topology on the basis of mathematical models we built earlier and provide conditions under which the swarm will obtain and keep the cohesiveness even in the presence of sensing delays and asynchronism. First we will consider a stationary edge-member asynchronous $N$-member swarm, and then we will investigate an $N$-member asynchronous mobile swarm following an edge-leader.

## 3.1 Convergence of a Stationary Edge-Member $M$-Dimensional Swarm

Here, we will provide conditions under which the swarm in Figure 1 will converge to be adjacent to a stationary edge-member. We begin with the two-member case, then consider the general $N$-member case where the proofs will depend on the $N = 2$ case.

#### 3.1.1 Convergence for a Two-Member Swarm

Suppose there is an $M$-dimensional two-member swarm, which has member $i$ and $i+1$, where member $i$ always remains stationary and member $i+1$ communicates with member $i$ and tries to move to maintain a comfortable distance $d$ to it.

**Lemma 1.** *For an $N = 2$ $M$-dimensional totally asynchronous swarm modeled by*

$$\begin{aligned} e^i(t+1) &= e^i(t) - g(|e^i(\tau_i^{i+1}(t))| - d)\left[\frac{e^i(\tau_i^{i+1}(t))}{|e^i(\tau_i^{i+1}(t))|}\right], \forall t \in T^{i+1} \\ e^i(t+1) &= e^i(t), \ \forall t \notin T^{i+1}, \end{aligned} \tag{16}$$

*where member i remains stationary, $|e^i(0)| > d$, and $g = g_a$, it is the case that for any $\gamma$, $0 < \gamma \le |e^i(0)| - d$, there exists a time $t'$ such that $|e^i(t')| \in [d, d+\gamma]$ and also $\lim_{t\to\infty} |e^i(t)| = d$.*

**Proof.** Define a Lyapunov-like function,

$$V_i(t) = |e^i(t)| - d, \forall t \in T^{i+1}, \tag{17}$$

that measures how close swarm member $i+1$ is to the comfortable distance from member $i$. Notice that

$$\begin{aligned} V_i(t+1) &= |e^i(t+1)| - d \\ &= |e^i(t) - g(|e^i(\tau_i^{i+1}(t))| - d)\left[\frac{e^i(\tau_i^{i+1}(t))}{|e^i(\tau_i^{i+1}(t))|}\right]| - d, \forall t \in T^{i+1} \end{aligned}$$

Since member $i$ remains stationary, we have

$$e^i(\tau_i^{i+1}(t)) = e^i(t), \forall t \in T^{i+1}$$

Therefore,

$$\begin{aligned} V_i(t+1) &= |e^i(t) - g(|e^i(t)| - d)\left[\frac{e^i(t)}{|e^i(t)|}\right]| - d \\ &= |e^i(t)| - g(|e^i(t)| - d) - d, \forall t \in T^{i+1} \end{aligned} \tag{18}$$

As we know, initially $|e^i(t)| > d$, and by Equations (17) and (18),

$$\Delta V_i = V_i(t+1) - V_i(t) = -g_a(|e^i(t)| - d) < 0 \tag{19}$$

Moreover, if at some $t \in T^{i+1}$, $|e^i(t)| > d$, from Equations (1) and (16), we have $|e^i(t+1)| > d$. So Equation (19) always holds in this case.

From Equations (1) and (19), we know that if $|e^i(t)|$ is beyond the $\gamma-$range of d (i.e., $|e^i(t)| > d + \gamma$, where $0 < \gamma \le |e^i(0)| - d$), we get

$$\frac{\gamma}{\beta} < \frac{1}{\beta}(|e^i(t)| - d) < g_a(|e^i(t)| - d)$$

So member $i+1$ will move toward member $i$ with a moving step at least larger than $\frac{\gamma}{\beta}$. Hence, member $i+1$ needs at most

$$\frac{\beta}{\gamma}(|e^i(0)| - d - \gamma)$$

update time steps, and at least one update time step, to make their inter-member distance inside $[d, d+\gamma]$, where $|e^i(0)|$ is the initial inter-member distance. So there exists a time $t'$ such that $|e^i(t')| \in [d, d+\gamma]$.

Moreover, from Equations (19), we know that for $t \in T^{i+1}$, $V_i(t)$ will asymptotically tend to zero (i.e., $\lim_{t\to\infty} |e^i(t)| = d$). **Q.E.D.**

**Lemma 2.** *For an $N = 2$ $M$-dimensional partially asynchronous swarm modeled by Equation (16) but with $g = g_f$, where member $i$ remains stationary, $|e^i(0)| > d$, the inter-member distance of members $i+1$ and $i$, $|e^i(t)|$ will converge to $d$ in some finite time, that is bounded by $B[\frac{\beta}{\eta}(|e^i(0)| - d - \eta) + 2]$.*

**Proof.** According to Assumption 2, we know that at most after time $B$ from the beginning, member $i+1$ will sense member $i$'s position. Then we get the results from the proof of Lemma 1 after replacing $g_a$ with $g_f$ and choosing $\gamma = \eta$.

For the case $0 < (|e^i(t)| - d) \leq \eta$, $|e^i(t)|$ will converge to $d$ in *one* time step according to Equation (5). Similar to the proof of Lemma 1 we can prove that if $|e^i(t)| > d + \eta$ (here $\eta = \gamma$), member $i+1$ will move towards member $i$ with a step at least larger than $\frac{\eta}{\beta}$. Hence, after at most $\frac{\beta}{\eta}(|e^i(0)| - d - \eta)$ update time steps, and at least one update time step, $|e^i(t)|$ will converge to be inside $[d, d+\eta]$ . From Assumption 2, we know that for a partially asynchronous swarm, the maximum update time interval is $B$. Also, according to Equation (5), member $i+1$ will reach a comfortable distance to member $i$ in the next update time step. So the total time, including delay time and moving time, needed to achieve convergence is bounded by

$$B + B[\frac{\beta}{\eta}(|e^i(0)| - d - \eta)] + B = B[\frac{\beta}{\eta}(|e^i(0)| - d - \eta) + 2] \tag{20}$$

**Q.E.D.**

### 3.1.2 Convergence for an $N$-Member $M$-Dimensional Swarm

Here, we will show that all members in an $N$-member swarm with a fixed communication topology form members $N$ to 1 will converge to be at the comfortable distance $d$ from their communicating neighbors on the basis of the above analysis of a two-member swarm.

**Theorem 1.** **(Partial Asynchronism, Finite Time Convergence):** *For an $N$-member $M$-dimensional swarm modeled by Equation (12) with $g = g_f$, $N \geq 2$, Assumption 2 (partial asynchronism) holds, and $|e^i(0)| > d$, all the inter-member distances of communicating neighbors $|e^i(t)|$, $i = 1, 2, \ldots, N-1$, will converge to the comfortable distance $d$ in some finite time, that is bounded by*

$$B[\frac{\beta}{\eta}(\sum_{i=1}^{N-1}(|e^i(0)| - d) - \eta) + 2]$$

*where $|e^i(0)|$ are the initial inter-neighbor distances.*

**Proof.** We will use a mathematical induction method, where our induction hypothesis will be that $|e^i(t)|$, $i = 1, 2, \ldots, k$, converge to the comfortable distance $d$ in some finite time and from this we will show that $|e^{k+1}(t)|$ will converge to $d$ after some finite time.

First, for $k = 1$, member 2 moves towards the stationary member 1 to be in a comfortable distance $d$ to it, and we have

$$e^1(t+1) = \begin{cases} e^1(t) - g(|e^1(\tau_1^2(t))| - d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right], \forall t \in T^2 \\ e^1(t), \ \forall t \notin T^2 \end{cases} \tag{21}$$

According to Lemma 2, $|e^1(t)|$ will converge to $d$ in some finite time.

Next we must show that given the induction hypothesis, the inter-neighbor distance of members $k+1$ and $k+2$, $|e^{k+1}(t)|$ in the $N$-member $M$-Dimensional swarm will converge to $d$ after some finite time.

According to our induction hypothesis we know that there exists a finite time $t^*$ such that $|e^i(t)| = d$, $i = 1, 2, \dots, k$, which means the first $k+1$ members of the $N$-member swarm remains stationary since they already stay in a comfortable distance to their reference neighbors.

Now considering the updating of $|e^{k+1}(t)|$, from Equation (12), we have

$$e^{k+1}(t+1) = \begin{cases} e^{k+1}(t) + g(|e^k(\tau_k^{k+1}(t))| - d)\left[\frac{e^k(\tau_k^{k+1}(t))}{|e^k(\tau_k^{k+1}(t))|}\right] - g(|e^{k+1}(\tau_{k+1}^{k+2}(t))| - d)\left[\frac{e^{k+1}(\tau_{k+1}^{k+2}(t))}{|e^{k+1}(\tau_{k+1}^{k+2}(t))|}\right], \\ \qquad \forall t \in T^{k+1} \cap T^{k+2} \\ e^{k+1}(t) - g(|e^{k+1}(\tau_{k+1}^{k+2}(t))| - d)\left[\frac{e^{k+1}(\tau_{k+1}^{k+2}(t))}{|e^{k+1}(\tau_{k+1}^{k+2}(t))|}\right], \forall t \in T^{k+2}, t \notin T^{k+1} \\ e^{k+1}(t) + g(|e^k(\tau_k^{k+1}(t))| - d)\left[\frac{e^k(\tau_k^{k+1}(t))}{|e^k(\tau_k^{k+1}(t))|}\right], \forall t \in T^{k+1}, t \notin T^{k+2} \\ e^{k+1}(t), \ \forall t \notin T^{k+1} \cup T^{k+2} \end{cases} \tag{22}$$

After $t \geq t^*$, we have $|e^k(t)| = d$ so that after $t \geq t^* + B$, we have

$$|e^k(\tau_k^{k+1}(t))| = |e^k(t)| = d$$

according to Assumption 2. From Equation (5), we have

$$g(|e^k(\tau_k^{k+1}(t))| - d) = 0, \ t \geq t^* + B$$

And so, we can write Equation (22) into the following form:

$$e^{k+1}(t+1) = \begin{cases} e^{k+1}(t) - g(|e^{k+1}(\tau_{k+1}^{k+2}(t))| - d)\left[\frac{e^{k+1}(\tau_{k+1}^{k+2}(t))}{|e^{k+1}(\tau_{k+1}^{k+2}(t))|}\right], \forall t \in T^{k+2}, t \geq t^* + B \\ e^{k+1}(t), \ \forall t \notin T^{k+2}, t \geq t^* + B \end{cases} \tag{23}$$

Therefore, after $t \geq t^* + B$, member $k+2$ moves towards member $k+1$, which already remains stationary, to be in a comfortable distance $d$ to it. Clearly, from Lemma 2, $|e^{k+1}(t)|$ will converge to $d$ after some finite time. This ends the induction step.

Next, we will try to bound the amount of converging time for the $N$-member $M$-dimensional swarm. In Lemma 2 we deduce that for a two-member swarm the time needed to achieve convergence is bounded by

$$B[\frac{\beta}{\eta}(|e^i(0)| - d - \eta) + 2]$$

For the $N$-member swarm, we already know that

$$|e^i(t)| \geq d, \forall t, i = 1, 2, \dots, N-1$$

so that swarm members never hinder their neighbors' movements. As we know, swarm members move to their reference neighbors with a step at least larger than $\frac{\eta}{\beta}$ when their inter-neighbor distance is beyond $\eta$-range of the comfortable distance due to the definition of $g_f$. Considering the worst case, all $N$ members except the stationary one move to the same direction on a line. As we know, under Assumption 2 (partial asynchronism) all the swarm members will converge to be in a comfortable distance to their reference neighbors one by one in this case. Therefore, we can use the total time in the worst case taken by member $N$ to reach its final position to bound the total converging time of the swarm, which is

$$B[\frac{\beta}{\eta}(\sum_{i=1}^{N-1}(|e^i(0)| - d) - \eta) + 2]$$

**Q.E.D.**

**Remark 1:** Notice that for an $N$-member $M$-dimensional totally asynchronous swarm modeled by Equation (12) with $g = g_a$, $N \geq 2$, Assumption 1 (total asynchronism) holds, and $|e^i(0)| > d$, similarly we can use Lemma 1 to prove that all the inter-member distances of communicating neighbors $|e^i(t)|$, $i = 1, 2, \ldots, N-1$, will asymptotically converge to the comfortable distance $d$.

**Remark 2:** Notice that the one-dimensional asynchronous swarm results in [29, 30] can be seen as a special case of our results if we assume $|e^i(0)| > d$ initially.

## 3.2 Convergence of an $M$-Dimensional Mobile Swarm Following an Edge-Leader

Next, we will study cohesiveness of an $M$-dimensional mobile swarm. First, we will study the case of using the $g_f$ function, and then what happens if a different $g$ function is used that does not require a swarm member to move to be adjacent to its neighbor in one step if it gets very close to it.

### 3.2.1 Convergence for an $N$-Member Asynchronous Mobile Swarm

First, we choose $g_f$ as the $g$ function in Equation (15) and assume $\gamma = \eta$ ($\eta$ is used in the definition of $g_f$). We will show that all members in an $N$-member mobile swarm will be in a comfortable distance neighborhood from their communicating neighbors during movements if there are constraints on the leader's moving direction, moving frequency, and the partial asynchronism measure, and constraints on the leader's moving step bound, the number of swarm members, and the comfortable distance neighborhood size.

**Theorem 2.** *For an $N$-member $M$-dimensional asynchronous mobile swarm with a fixed communication topology modeled by Equation (15), where $g$ is $g_f$, $N > 2$, Assumption 2 (partial asynchronism) holds, $|e^i(0)| = d$, $i = 1, 2, ..., N-1$, and the edge-leader (member 1) only moves in legal directions defined above via the "wait $4B-1$ steps strategy," if*

$$0 < r \leq \frac{2\gamma}{N} \tag{24}$$

*for a given $\gamma$, all the swarm members will be in the comfortable distance neighborhood $[d, d+\gamma]$ of their communicating neighbors during movements, where $r$ is the upper bound of the edge-leader's moving step size $|s(t)|$, $\gamma$ (choose $\gamma = \eta$) is the comfortable distance neighborhood size, and $B \in Z^+$ is the partial asynchronism measure.*

**Proof.** For such an $N$-member mobile swarm, each swarm member follows its preceding communicating neighbor except the edge-leader. We know from Equation (10) that there are no collisions between members. This decouples the problem so that we can consider each pair of neighboring swarm members individually.

First, we consider the relationship between members 1, 2 and 3 to explain why member 1 can get the position information of members 2 and 3 with the "wait $4B-1$ steps strategy" even in the "worst" case. Here, the worst case means that members 1 and 2 and member 2 and 3 have the maximum delay in obtaining each other's position information; member 1 updates its position to the direction of $\theta = \pi$ (which is the one-dimensional case) with a maximum possible step size $r$ at the earliest time satisfying $t \in T^1$, $t - t_p(t) = 4B-1$, where $T^1 = T$; members 2 and 3 only update their positions at one element of the time set $\{t, t+1, ..., t+B-1\}$ for $t \in T$ so that they move as slow as possible.

Consider the worst case. In the first time set $\{0, 1, \ldots, B-1\}$, member 1 starts its first move step at $t = 0$ according to Equation (13) since we assume it gets members 2 and 3's initial position information at $t = 0$ and $t_p(0) = -(4B-1)$. At time $t = 1$, member 1 arrives its new position and remains stationary at least until at $t = 4B-1$ from the "wait $4B-1$ steps strategy." So, in the worst case we have

$$|e^1(1)| = d + r$$

From Assumption 2, members 1 and 2's delay in knowing about the position information of each other can be as large as $B-1$. Member 2 remains stationary at its updating time index of the first time set since it still thinks member 1 is still in the initial position due to the delay. However, in the second time set $\{B, B+1, \ldots, 2B-1\}$ it at least receives the new position information of member 1 at $t = B$ since the maximum delay is $B-1$. So, in the worst case it moves towards the new position of member 1 via the $g_f$ function at $t = 2B-1$ in the second time set. From Equation (15), we have

$$e^1(t+1)) = e^1(t) - g(|e^1(\tau_1^2(t))| - d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right] \tag{25}$$

where $e^1(\tau_1^2(2B-1)) = e^1(B) = d + r$. From Equation (24) and the fact that $N > 2$, we have

$$|e^1(B)| - d = r \leq (N-1)r \leq \gamma \tag{26}$$

Since we choose $\gamma = \eta$, from Equation (26), and the definition of $g_f$, we then have

$$g_f(|e^1(B)| - d) = |e^1(B)| - d = r$$

Therefore,

$$|e^1(2B)| = d$$

And then, member 2 remains stationary at its new position since it is already in a comfortable distance to member 1.

Similarly, in the third time set $\{2B, 2B+1, \ldots, 3B-1\}$, member 3 at least knows member 2's new position at $t = 3B-1$. In the worst case, assume member 3 just updates its position at $t = 3B-2$ in the time set. Then it has to wait to update its position again at the next time set $\{3B, 3B+1, \ldots, 4B-1\}$, i.e., it moves towards member 2 at $t = 4B-2$. And so, at $t = 4B-1$ members 2 and 3 are all at a new position which is in a comfortable distance to their communicating neighbors even in the worst case. Clearly, their new position information can be calculated by member 1, which is updated from their previous position information with only one moving step (via the $g_f$ function). Therefore, at $t = 4B-1$, member 1 has enough information to decide its legal moving directions and is ready to have another moving step. This explains that member 1 can get the position information of members 2 and 3 with the "wait $4B-1$ steps strategy" even in the worst case. In the future time set, all three members will repeat the above process. Moreover, we can conclude from the above that the maximum possible value of inter-neighbor distance $|e^1(t)|$ and $|e^2(t)|$ is $d + r$, which is in the range of comfortable distance neighborhood $[d, d+\gamma]$ from Equation (24).

Next, we try to find the maximum possible inter-neighbor distance between members 4 and 3 and members 5 and 4. For this purpose, a special case (that is different from the worst case for members 1, 2 and 3 above) is considered as follows. As we know, in the above worst case, at $t = 4B-1$ member 3 reaches its new position, and member 1 starts its second moving step. Then, we have

$$|e^1(4B-1)| = |e^2(4B-1)| = d$$

Assume member 4 also has the maximum delay about the position information of member 3, and so member 4 still remains stationary at its initial position. We have

$$|e^3(4B-1)| = d + r$$

Now, different from above, we assume that since $t = 4B - 1$, members 2 and 3 get the position information of their communicating neighbors without any delay, and they update their position synchronously in order to maintain a comfortable distance $d$ at $t = 4B - 1$. Therefore, at $t = 4B$, we get

$$|e^1(4B)| = |e^2(4B)| = d$$

and

$$|e^3(4B)| = d + 2r$$

Note that here we consider the one-dimensional case since we try to find the maximum possible inter-neighbor distance. Then, members 1, 2, and 3 remain stationary until $t = 8B - 2$ from the "wait $4B - 1$ steps strategy." Due to the delay, member 4 knows member 3's new position at $t = 5B - 1$ and adjusts its distance to member 3 to be comfortable via the $g_f$ function at least at $t = 6B - 1$. Then, the inter-neighbor distance of members 4 and 3, $|e^3(t)|$ will bounce between $d$ and $d + r$ in the future time sets even in the above special case. Similar to the above analysis, we have the same conclusion for the inter-neighbor distance between members 5 and 4 as that for members 4 and 3. Therefore, the maximum possible value of inter-neighbor distance $|e^3(t)|$ and $|e^4(t)|$ is $d + 2r$, which is also in the range of comfortable distance neighborhood $[d, d + \gamma]$ from Equation (24). In the same way, we can find that the maximum possible inter-neighbor distance between members $N - 1$ and $N$ is $d + (N - 1)r/2$ if $N$ is an odd number, and is $d + (Nr)/2$ if $N$ is an even number, which is the largest of all possible inter-neighbor distances in the time set $T$. Hence, we conclude that the inter-neighbor distance bound for $N$ members is $d + (Nr)/2$.

From Equation (24), we have

$$d + (Nr)/2 \leq d + \gamma$$

and from Equation (10), we then have

$$d \leq |e^i(t)| \leq d + (Nr)/2 \leq d + \gamma, \text{ for } i = 1, 2, ..., N - 1$$

which means all members will always be in the comfortable distance neighborhood $[d, d + \gamma]$ with their neighbors. So all members can keep the distance from their communicating neighbors in the range of comfortable distance neighborhood even in the worst case. **Q.E.D.**

**Remark 3:** Note that in Theorem 2, Equation (24) provides bound on how far the leader can move in one step for a given $\gamma$, and the "wait $4B - 1$ steps strategy" provides bound on how frequent the leader can move for a given $B$. They work together to provide how fast a $N$-member swarm can move while still maintaining the type of cohesiveness characterized by $\gamma$.

**Remark 4:** From Theorem 1, we can see that if member 1 (the edge-leader) stops moving (i.e., $s(t) = 0$, for $t \in T^1$, $t - t_p(t) = 4B - 1$), all the inter-neighbor distances will converge to be the comfortable distance $d$.

In the above, we study the convergence property of mobile swarms with at least three members. Now we consider the $N = 2$ case, which is even simpler. In the two-member case, legal directions of the leader are defined as $\pi/2 \leq \theta \leq 3\pi/2$. Moreover, member 1 only needs the information about the connected line of its position and the position of member 2 to decide its moving direction before further moving so that it use the "wait $2B-1$ steps strategy," which is the same as the "wait $4B-1$ steps strategy" except the length of the waiting time. Similarly, we have the following corollary.

**Corollary 1.** *For a two-member $M$-dimensional asynchronous mobile swarm modeled by*

$$e^1(t+1) = \begin{cases} e^1(t)+s(t)-g(|e^1(\tau_1^2(t))|-d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right], & \text{if } t\in T^2, t-t_p(t)\geq 2B-1, t\in T^1; \\ e^1(t)+s(t), \quad \text{if } t-t_p(t)\geq 2B-1, t\in T^1, t\notin T^2; \\ e^1(t)-g(|e^1(\tau_1^2(t))|-d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right], & \text{if } t\in T^2, t\notin T^1 \text{ or } t\in T^2\cup T^1, t-t_p(t)<2B-1; \\ e^1(t), \quad \text{if } t\notin T^2\cup T^1 \text{ or } t\notin T^2, t\in T^1, t-t_p(t)<2B-1. \end{cases}$$

$$t_p(t+1) = \begin{cases} t, & \text{if } t-t_p(t)\geq 2B-1, t\in T^1; \\ t_p(t), & \text{otherwise.} \end{cases} \tag{27}$$

*where $g$ is $g_f$, Assumption 2 (partial asynchronism) holds, $|e^1(0)|=d$, $t_p(0)=-(2B-1)$, and the edge-leader (member 1) only moves to legal directions via the "wait $2B-1$ steps strategy," if*

$$0<r\leq\gamma \tag{28}$$

*for a given $\gamma$, the two members will be in the comfortable distance neighborhood $[d, d+\gamma]$ during movements, where $r$ is the upper bound of the edge-leader's moving step size $|s(t)|$, $\gamma$ (choose $\gamma=\eta$) is the comfortable distance neighborhood size, and $B\in Z^+$ is the partial asynchronism measure.*

**Proof.** Similar to the proof of Theorem 2, it is easy to find the maximum possible inter-neighbor distance of members 1 and 2 is equal to $d+r$ by analyzing the worst case. From Equation (28), the two members always keep their distance in the range of comfortable distance neighborhood. **Q.E.D.**

### 3.2.2 Analysis of Movement Flexibility

In Theorem 2, we provide conditions under which an $N$-member mobile swarm can keep cohesion during movements and avoid collisions as long as the leader always moves in legal directions which has a $\sigma$-range, where $\sigma$ is the angle formed by the two connected lines between positions of members 2 and 1 and between positions of members 1 and 3. Clearly, the movement flexibility depends on how large $\sigma$ is since $\sigma$ is the maximum possible turning angle that the swarm can make in one step. Therefore, we will analyze the change of $\sigma$ during movements of the swarm.

**Theorem 3.** *For an $N$-member $M$-dimensional asynchronous cohesive mobile swarm satisfying the conditions of Theorem 2, if initially $\sigma>0$ and the swarm keeps moving, $\sigma$ monotonically decreases and*

$$\sigma\rightarrow 0 \text{ as } t\rightarrow\infty$$

*and if initially $\sigma=0$,*

$$\sigma=0, \ \forall t$$

*during movements of the swarm, where $\sigma$ is the angle formed by the two connected lines between positions of members 2 and 1 and between positions of members 1 and 3.*

**Proof.** For an $N$-member $M$-dimensional asynchronous cohesive mobile swarm satisfying the conditions of Theorem 2, member 1 has to obtain the position information of members 2 and 3 to calculate $\sigma$ before it moves. It can decide its legal moving directions with $\sigma$. As shown in Figure 5, assume at time $t$,

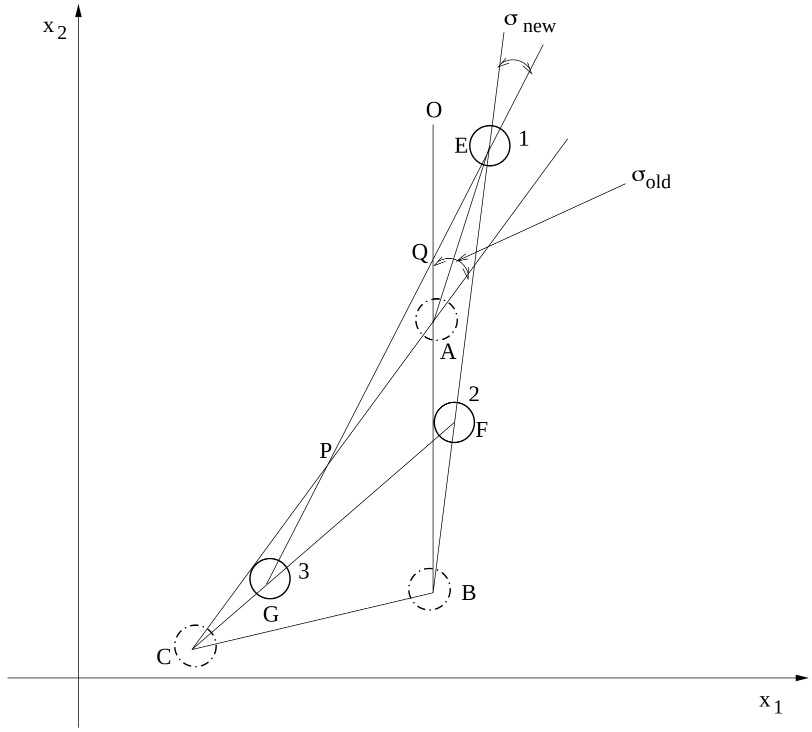


Figure 5: The change of $\sigma$ during movements of the swarm.

$t \in T^1$, $t - t_p(t) = 4B - 1$, and members 1, 2, and 3 stay at positions $A$, $B$, and $C$, respectively (dashed circles). Assume the current $\sigma$ is equal to $\sigma_{old} > 0$. Member 1 moves to a new position in a legal direction. At $t + 1$, member 1 arrives the position $E$ and waits until its next moving step from the "wait $4B - 1$ steps strategy." Then, member 2 will update its position at $t = t + 2B - 1$ in the worst case (see the proof of Theorem 2) to maintain a comfortable distance with member 1, and assume it reaches the position $F$. Similarly, member 3 moves towards the new position of member 2 and reaches the position $G$. Therefore, at $t = t + 4B - 1$, members 2 and 3 are both at a new position. Member 1 can calculate a new $\sigma$ (which we call $\sigma_{new}$) from these new positions to decide its next moving direction. Next, we will consider the relationship between $\sigma_{old}$ and $\sigma_{new}$ in Figure 5.

Clearly, in the triangle $APQ$, we have

$$\sigma_{old} = \angle APQ + \angle AQP \tag{29}$$

Similarly, in the triangle $BEQ$,

$$\angle EQO = \angle BEQ + \angle EBQ \tag{30}$$

and we also have

$$\sigma_{new} = \angle BEQ \tag{31}$$

$$\angle AQP = \angle EQO \tag{32}$$

From Equations (29), (30), (31) and (32), we get

$$\sigma_{old} = \angle EQO + \angle AQP = \angle BEQ + \angle EBQ + \angle AQP = \sigma_{new} + \angle EBQ + \angle AQP$$

And so,

$$\Delta\sigma = \sigma_{old} - \sigma_{new} = \angle EBQ + \angle AQP \tag{33}$$

where $\angle AQP$ is the angle formed by the two connected lines between the new positions and the previous positions of members 1 and 2 ($EF$ and $AB$), and $\angle EBQ$ is the angle formed by the two connected lines between the new positions and the previous positions of members 1 and 3 ($EG$ and $AC$).

Obviously, we have

$$\angle AQP \geq 0 \tag{34}$$

and

$$\angle AQP = 0$$

if and only if member 1 moves along the extension of line $AB$ (the one-dimensional case, where $EF$ and $AB$ are overlapped). Moreover, we have

$$\angle EBQ > 0 \text{ if } \sigma_{old} > 0 \tag{35}$$

and

$$\angle EBQ = 0 \text{ if and only if } \sigma_{old} = 0 \tag{36}$$

Then, from Equations (33), (34) and (35), we have

$$\Delta\sigma > 0 \text{ if } \sigma_{old} > 0 \tag{37}$$

Therefore, $\sigma$ monotonically decreases and

$$\sigma \to 0 \text{ as } t \to \infty$$

if initially $\sigma > 0$ and the swarm keeps moving.

If initially $\sigma = 0$ (the one-dimensional case, where $AC$ and $AB$ are overlapped and member 1 only moves along the extension of line $AB$), from Equations (33), (34), and (36), we get

$$\Delta\sigma = 0 \tag{38}$$

And so,

$$\sigma = 0, \ \forall t$$

during movements of the swarm. **Q.E.D.**

**Remark 5:** Notice that from Theorem 3, an $N$-member $M \geq 2$ dimensional asynchronous cohesive mobile swarm satisfying the conditions of Theorem 2 will gradually become a one-dimensional swarm as $\sigma \to 0$ and its length is bounded by $(N-1)(d+\gamma)$.

**Remark 6:** Notice that after an $N$-member $M \geq 2$ dimensional asynchronous cohesive mobile swarm becomes a one-dimensional swarm, i.e., all members move along the same line, legal directions of the leader could be set to $\pi/2 \leq \theta \leq 3\pi/2$ so that the swarm could later make turns in ways that avoid collisions.

### 3.2.3 Alternative Convergence Conditions

Now we consider the case of using another $g$ function in Equation (15). Assume that for a scalar $c > 0$, $g_c(|e^i(t)| - d)$ is such that

$$|e^i(t)| - d - c \leq g_c(|e^i(t)| - d) < |e^i(t)| - d, \text{ if } |e^i(t)| - d > c \quad (39)$$

$$g_c(|e^i(t)| - d) = 0, \text{ if } -c \leq |e^i(t)| - d \leq c \quad (40)$$

$$|e^i(t)| - d < g_c(|e^i(t)| - d) \leq |e^i(t)| - d + c, \text{ if } |e^i(t)| - d < -c \quad (41)$$

As shown in Figure 6, these relationships are similar to those for the $g_f$ function. However, the $g_c$ function has two different bounds $|e^i(t)| - d - c$ and $|e^i(t)| - d + c$ in Equations (39) and (41), which guarantee the following members are in the $c-$neighborhood of desired comfortable distance of their leading neighbors after each update step so that the following members can keep up with the movements of their leading neighbors. Moreover, the $g_c$ function is equal to 0 when the inter-neighbor distance is already in the $c$-neighborhood of the comfortable distance. Note that here we assume $0 < c < \eta$ since we choose $\gamma = \eta$ before.

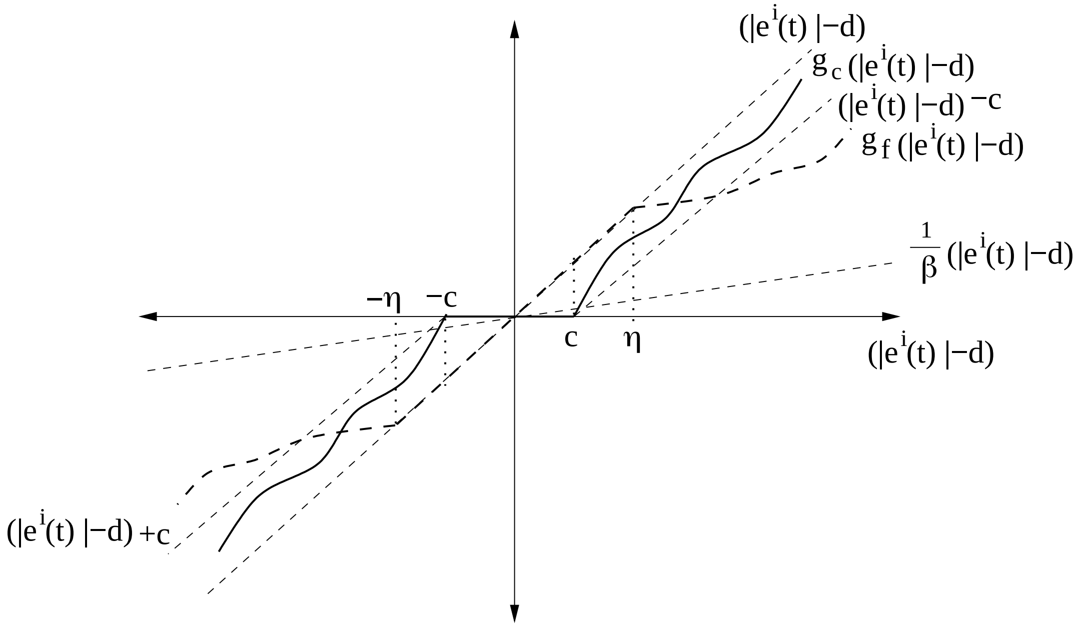


Figure 6: The function $g_c(e^i(t) - d)$ (solid) and $g_f(e^i(t) - d)$ (dashed).

Similarly, we will show that with the $g_c$ function, all members can also be in a comfortable distance neighborhood from their communicating neighbors during movements under some constraints on the leader's moving direction, moving frequency, and the partial asynchronism measure, and constraints on the leader's moving step bound, the number of swarm members, the comfortable distance neighborhood size, and the parameters of the $g_c$. Note that with the $g_c$ function, member 1 has to use the " wait $5B - 2$ steps strategy" instead of the " wait $4B - 1$ steps strategy" since member 1 cannot calculate the position information of member 2 and 3 (it only knows they are inside the $c$-neighborhood of the comfortable distance to their communicating neighbors). It has to wait $B - 1$ time indices more to receive the position information in the case of the maximum communication delay. In this case, we can modify the model in Equation (15) as follows:

$$
\begin{aligned}
e^1(t+1) &= \begin{cases} e^1(t) + s(t) - g(|e^1(\tau_1^2(t))| - d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right], & \text{if } t \in T^2, t - t_p(t) \geq 5B-2, t \in T^1; \\ e^1(t) + s(t), & \text{if } t - t_p(t) \geq 5B-2, t \in T^1, t \notin T^2; \\ e^1(t) - g(|e^1(\tau_1^2(t))| - d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right], & \text{if } t \in T^2, t \notin T^1 \text{ or } t \in T^2 \cup T^1, t - t_p(t) < 5B-2; \\ e^1(t), & \text{if } t \notin T^2 \cup T^1 \text{ or } t \notin T^2, t \in T^1, t - t_p(t) < 5B-2. \end{cases} \\
t_p(t+1) &= \begin{cases} t, & \text{if } t - t_p(t) \geq 5B-2, t \in T^1; \\ t_p(t), & \text{otherwise.} \end{cases} \\
e^i(t+1) &= \begin{cases} e^i(t) + g(|e^{i-1}(\tau_{i-1}^i(t))| - d)\left[\frac{e^{i-1}(\tau_{i-1}^i(t))}{|e^{i-1}(\tau_{i-1}^i(t))|}\right] - g(|e^i(\tau_i^{i+1}(t))| - d)\left[\frac{e^i(\tau_i^{i+1}(t))}{|e^i(\tau_i^{i+1}(t))|}\right], \\ \qquad \text{if } t \in T^i \cap T^{i+1},\ i = 2, 3, \ldots, N-1; \\ e^i(t) - g(|e^i(\tau_i^{i+1}(t))| - d)\left[\frac{e^i(\tau_i^{i+1}(t))}{|e^i(\tau_i^{i+1}(t))|}\right], & \text{if } t \in T^{i+1}, t \notin T^i,\ i = 2, 3, \ldots, N-1; \\ e^i(t) + g(|e^{i-1}(\tau_{i-1}^i(t))| - d)\left[\frac{e^{i-1}(\tau_{i-1}^i(t))}{|e^{i-1}(\tau_{i-1}^i(t))|}\right], & \text{if } t \in T^i, t \notin T^{i+1},\ i = 2, 3, \ldots, N-1; \\ e^i(t), & \text{if } t \notin T^i \cup T^{i+1},\ i = 2, 3, \ldots, N-1. \end{cases}
\end{aligned} \tag{42}
$$

**Theorem 4.** *For an $N$-member $M$-dimensional asynchronous mobile swarm modeled by Equation (42), where $g$ is $g_c$, $N > 2$, Assumption 2 (partial asynchronism) holds, $|e^i(0)| = d$, $i = 1, 2, ..., N-1$, and the edge-leader (member 1) only moves in legal directions defined above via the "wait $5B-2$ steps strategy," if*

$$0 < r \leq \frac{2(\gamma - c)}{N} \tag{43}$$

*for a given $\gamma$, all the swarm members will be in the comfortable distance neighborhood $[d, d+\gamma]$ of their communicating neighbors during the moving process, where $r$ is the upper bound of the edge-leader's moving step size $|s(t)|$, $\gamma$ is the comfortable distance neighborhood size, $B \in Z^+$ is the partial asynchronism measure, and $c$ $(0 < c < \gamma)$ is the parameter of $g_c$ function.*

**Proof.** Similar to the proof of Theorem 2, we can find that the maximum possible inter-neighbor distance between members $N-1$ and $N$ is $d+(N-1)r/2+c$ if $N$ is an odd number, and is $d+(Nr)/2+c$ if $N$ is an even number, which is the largest of all possible inter-neighbor distances in the time set $T$. Hence, we conclude that the inter-neighbor distance bound for $N$ members is $d + (Nr)/2 + c$.

From Equation (43), we have

$$d + (Nr)/2 + c \leq d + \gamma$$

and from Equation (10), we then have

$$d \leq |e^i(t)| \leq d + (Nr)/2 + c \leq d + \gamma, \text{ for } i = 1, 2, ..., N-1$$

which means all members will always be in the comfortable distance neighborhood $[d, d+\gamma]$ with their neighbors. So all members can keep the distance to their communicating neighbors in the range of comfortable distance neighborhood even in the worst case. **Q.E.D.**

Now we consider the $N = 2$ case with the $g_c$ function. Similarly, member 1 only needs the information about the position of member 2 to decide its moving direction before further moving. Therefore, it uses the "wait $3B-1$ steps strategy." So, we have the following corollary.

**Corollary 2.** *For a two-member M-dimensional asynchronous mobile swarm modeled by*

$$
\begin{aligned}
e^1(t+1) &= \begin{cases} e^1(t) + s(t) - g(|e^1(\tau_1^2(t))| - d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right], & \text{if } t \in T^2, t - t_p(t) \geq 3B-1, t \in T^1; \\ e^1(t) + s(t), & \text{if } t - t_p(t) \geq 3B-1, t \in T^1, t \notin T^2; \\ e^1(t) - g(|e^1(\tau_1^2(t))| - d)\left[\frac{e^1(\tau_1^2(t))}{|e^1(\tau_1^2(t))|}\right], & \text{if } t \in T^2, t \notin T^1 \text{ or } t \in T^2 \cup T^1, t - t_p(t) < 3B-1; \\ e^1(t), & \text{if } t \notin T^2 \cup T^1 \text{ or } t \notin T^2, t \in T^1, t - t_p(t) < 3B-1. \end{cases} \\
t_p(t+1) &= \begin{cases} t, & \text{if } t - t_p(t) \geq 3B-1, t \in T^1; \\ t_p(t), & \text{otherwise.} \end{cases}
\end{aligned} \tag{44}
$$

*where $g$ is $g_c$, Assumption 2 (partial asynchronism) holds, $|e^1(0)| = d$, $t_p(0) = -(3B-1)$, and the edge-leader (member 1) only moves to legal directions via the "wait $3B-1$ steps strategy," if*

$$0 < r \leq \gamma - c \tag{45}$$

*for a given $\gamma$, the two members will be in the comfortable distance neighborhood $[d, d+\gamma]$ during movements, where $r$ is the upper bound of the edge-leader's moving step size $|s(t)|$, $\gamma$ is the comfortable distance neighborhood size, $B \in Z^+$ is the partial asynchronism measure, and $c$ ($0 < c < \gamma$) is the parameter of $g_c$ function.*

**Proof.** Similar to the proof of Corollary 1, it is easy to find the maximum possible inter-neighbor distance of members 1 and 2 is equal to $d + r + b$ by analyzing the worst case. From Equation (45), the two members always keep their distance in the range of comfortable distance neighborhood. **Q.E.D.**

**Remark 7:** Notice that similar to Theorem 3, we can prove that $\sigma$ monotonically goes to zero for an $N$-member $M \geq 2$ dimensional asynchronous cohesive mobile swarm satisfying the conditions of Theorem 4.

**Remark 8:** Notice that we can write equivalent theorems of Theorems 2 and 4 as follows:

- With the same conditions in Theorem 2, assume that the edge-leader (member 1) only moves to legal directions defined above via the "wait $2(N-1)B-1$ steps strategy." Then, if

  $$0 < r \leq \gamma \tag{46}$$

  for a given $\gamma$, all members of the $N$-member $M \geq 2$ dimensional asynchronous mobile swarm will be in the comfortable distance neighborhood $[d, d+\gamma]$ of their communicating neighbors during movements.

- With the same conditions in Theorem 4, assume that the edge-leader (member 1) only moves to legal directions defined above via the "wait $2(N-1)B-1$ steps strategy." Then, if

  $$0 < r \leq \gamma - c \tag{47}$$

  for a given $\gamma$, all members of the $N$-member $M \geq 2$ dimensional asynchronous mobile swarm will be in the comfortable distance neighborhood $[d, d+\gamma]$ of their communicating neighbors during movements.

Proofs of the above theorems are similar to those of Theorems 2 and 4. Comparing the above theorems with Theorems 2 and 4, we can see that in order to keep cohesiveness of the $N$-member mobile swarm in the presence of delays and asynchronism, if the leader uses a strategy of waiting more time steps at one position, it can move with a bigger step size in future updating time indices. On the other hand, if it uses a strategy of waiting less time steps, it has to move with a smaller step size.

# 4 Simulation Studies

Here, we will provide simulation examples to illustrate convergence properties of $M$-dimensional asynchronous swarms. First, we will simulate a three-dimensional swarm converging to be adjacent to a stationary member under the partial asynchronism assumption in some finite time, which is summarized in Theorem 1. And then, a simulation example of a three-dimensional cohesive asynchronous mobile swarm, which satisfies all the conditions in Theorem 2, will be given.

In the simulation, let $T = \{0, 1, 2, ...\}$ represent the indices of the sequence of real times. For convenience, we assume it corresponds to the real time set $\{0, 0.1, 0.2, 0.3, 0.4, ...\}$ on a uniform grid of size 0.1 sec at which one or more swarm members update their positions. And we randomly select the time index set $T^i \subseteq T, i = 1, 2, ..., N$, at which the $i^{th}$ member's position $x^i(t), t \in T^i$, is updated. The $T^i$, $i = 1, 2, ..., N$, are independent of each other for different $i$. However, they may have intersections so that two or more swarm members may move simultaneously. Moreover, in order to satisfy the partial asynchronism assumption, we assume $B = 4$ and add constraints to the updating time index set $T^i$ to guarantee each member updates at least once in the $B$ time index interval and the index delays in obtaining neighbor positions are bounded by $B$.

## 4.1 Stationary Edge Member Asynchronous Swarms Simulation

Assume we have a three-dimensional 10-member asynchronous swarm and initially (i.e, $t = 0$ sec), ten members from member 1 to member 10 with a physical size $w = 6$ are in order located at the positions of $(70, 70, 70)$, $(60, 68, 65)$, $(55, 60, 50)$, $(53, 50, 45)$, $(30, 50, 40)$, $(20, 36, 30)$, $(18, 20, 25)$, $(5, 20, 10)$, $(-8, 10, 0)$, $(-8, 0, -10)$ on a $(x_1, x_2, x_3)$ space respectively at $t = 0$, as shown in Figure 7. Note that their initial positions satisfy all the constraints required in Theorem 1. Assume the comfortable distance $d = 10$, and the sensing range of proximity sensors $\varepsilon = 4$. All members will update their positions in their updating time sets $T^i$ except member 1 remains stationary. The communication topology from member 10 to member 1 is fixed according to their initial conditions. Assume the partial asynchronism assumption holds for this swarm with $B = 4$ and we choose a $g_f$ function with a $\eta = 1$ satisfying Equations (1), (2), and (3) to define the attractive and repelling relationship. In particular, $g_f(|e^i(t)| - d) = |e^i(t)| - d$ if $||e^i(t)| - d| \leq \eta$, and $g_f(|e^i(t)| - d) = 0.4(|e^i(t)| - d)$ if $||e^i(t)| - d| > \eta$. Here, we choose $\beta = 10000$ since $|g_f(e^i(t) - d)| = 0.4|(|e^i(t)| - d)| > \frac{1}{\beta}|(|e^i(t)| - d)|$ is required if $||e^i(t)| - d| > \eta$, where $\beta > 1$ and also we want to allow a very small movement at any step. With all the above conditions, we get the finite-time convergence according to Theorem 1.

The results of the simulation are given by providing eight plots of swarm member positions from $t = 0$ sec to $t = 14.9$ sec as shown in Figure 7. In the $t = 3.6$ sec plot, each member moves towards its communicating neighbor due to its attractive relationship. In the $t = 5.5$ sec plot, the first 6 members already converged to be in a comfortable distance to their communicating neighbors. In the last two plots, all members already remain stationary at positions adjacent to the position of member 1. We provide all inter-member distances of communicating neighbors during the convergence process in Figure 8. Clearly, all inter-neighbor distances are larger than or equal to 10 (i.e., there are no collisions) and converged to the comfortable distance 10 after $t \geq 9$ sec. It is interesting to note that the inter-member distances do *not* asymptotically decrease at each step; sometimes the inter-member distances could increase, then later decrease (this actually complicated the theoretical analysis in the last section).

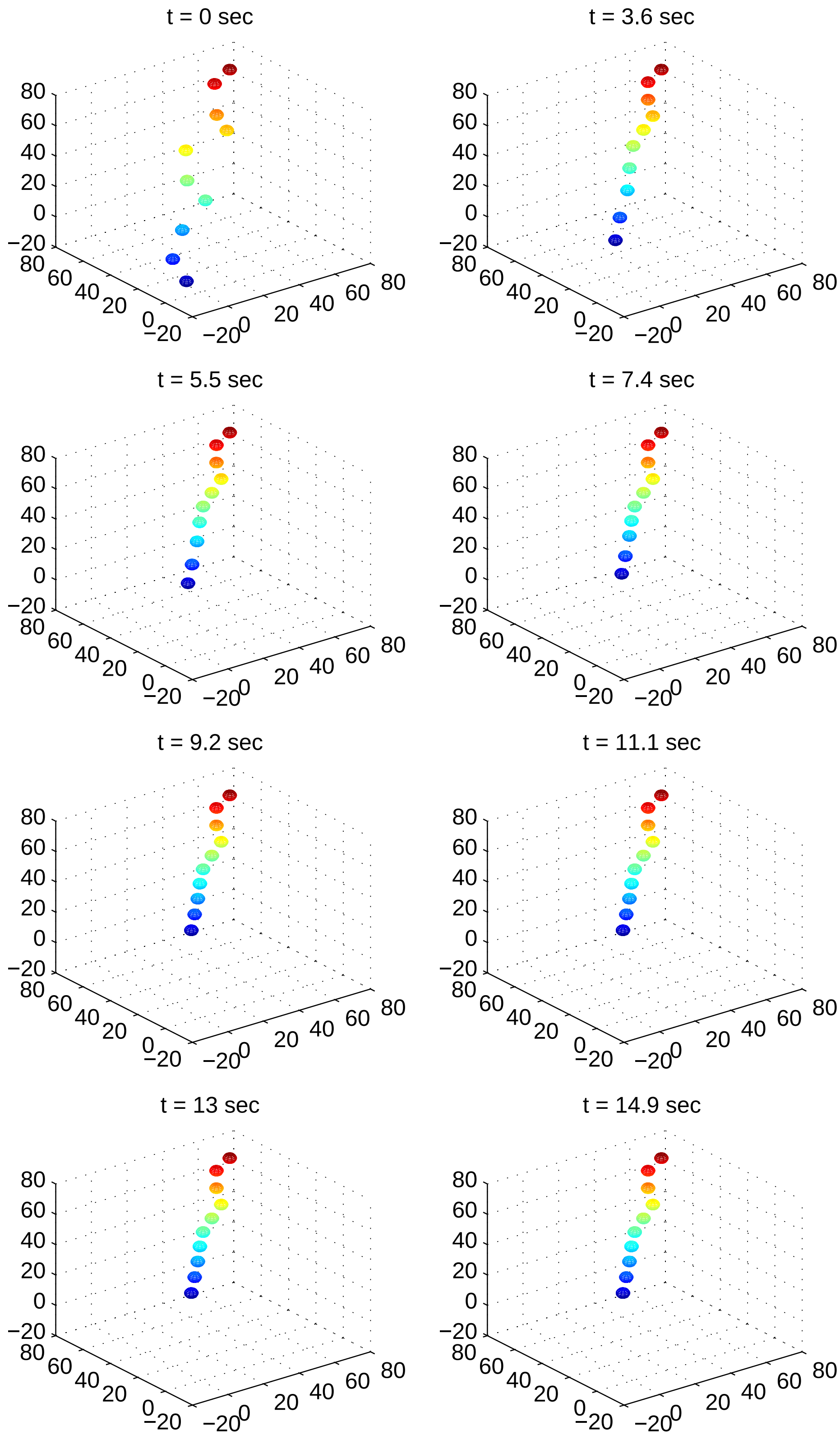


Figure 7: Simulation of $M = 3$ dimensional asynchronous 10-member asynchronous swarm with a fixed communication topology converging behavior.

## 4.2 Asynchronous Mobile Swarm with an Edge-Leader Simulation

Assume we have a three-dimensional 10-member asynchronous mobile swarm and at the beginning ten members from member 1 to member 10 with a physical size $w = 6$ are in order located at the posi-

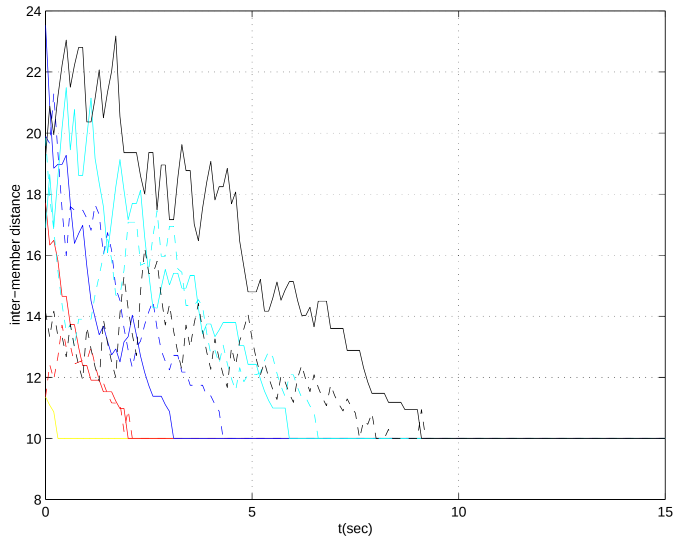


Figure 8: Inter-member distances of communicating neighbors in Figure 7 during the convergence process.

tions of $(71, 70, 70)$, $(68.4, 61.4, 65.7)$, $(62.8, 56.4, 59.1)$, $(56.3, 51.5, 53.2)$, $(49.7, 46.5, 47.6)$, $(42.8, 41.8, 42)$, $(35.8, 37.3, 36.6)$, $(28.7, 32.8, 31.1)$, $(21.4, 28.4, 25.9)$, $(14, 24.1, 20.8)$ on a $(x_1, x_2, x_3)$ space respectively at $t = 0$, as shown in Figure 9. Note that their initial positions satisfy all the constraints required in Theorem 2. Assume the comfortable distance $d = 10$, and the comfortable distance neighborhood size $\gamma = 5$. Member 1 uses the "wait $4B - 1$ steps strategy" to move only in legal directions defined by $\sigma$. We use the same $g_f$ function as above. According to Theorem 2, the edge-leader's moving step is bounded by $r$, where $0 < r \leq \frac{2\gamma}{N} = 1$, in order for the asynchronous mobile swarm to maintain cohesiveness, i.e., all mobile swarm members are at a comfortable distance neighborhood $[10, 15]$ from their neighbors while the swarm moves. Hence, we choose $r = 1$.

The results of the simulation are given by providing eight plots of swarm member positions from $t = 0$ sec to $t = 99.9$ sec as shown in Figure 9. We found that all mobile swarm members maintain a distance inside the comfortable neighborhood range $[10, 15]$ from their neighbors in all time indices. Clearly, there are no collisions during movements and the mobile swarm maintains cohesion. In addition, we show the change of $\sigma$ during movements of the swarm in Figure 10. Obviously, $\sigma$ monotonically decreases to zero as time increases, which verifies the conclusion of Theorem 3. Also, note that at $t = 99.9$ sec in Figure 9, the first five members already move in the same dimension. If we extend the simulation time, all other members will gradually move on the same dimension as $\sigma$ goes to zero so that the three-dimensional 10-member asynchronous swarm becomes a one-dimensional swarm.

## 5 Conclusions

We constructed a mathematical model for an $M$-dimensional asynchronous swarm with a fixed communication topology by putting $N$ identical single swarm members together. We proved that all the inter-member distances of communicating neighbors in an $M$-dimensional asynchronous swarm will converge to the comfortable distance so that it can obtain cohesion even in the presence of delays and asynchronism. Moreover, an $M$-dimensional asynchronous mobile swarm following an edge-leader with a fixed communication topology is modeled and different conditions under which it can maintain cohesion during movements are provided. In addition, the swarm movement flexibility is analyzed. Simulation studies are given to illustrate swarm convergence properties. Note that our analysis, which allows for finite-size swarm

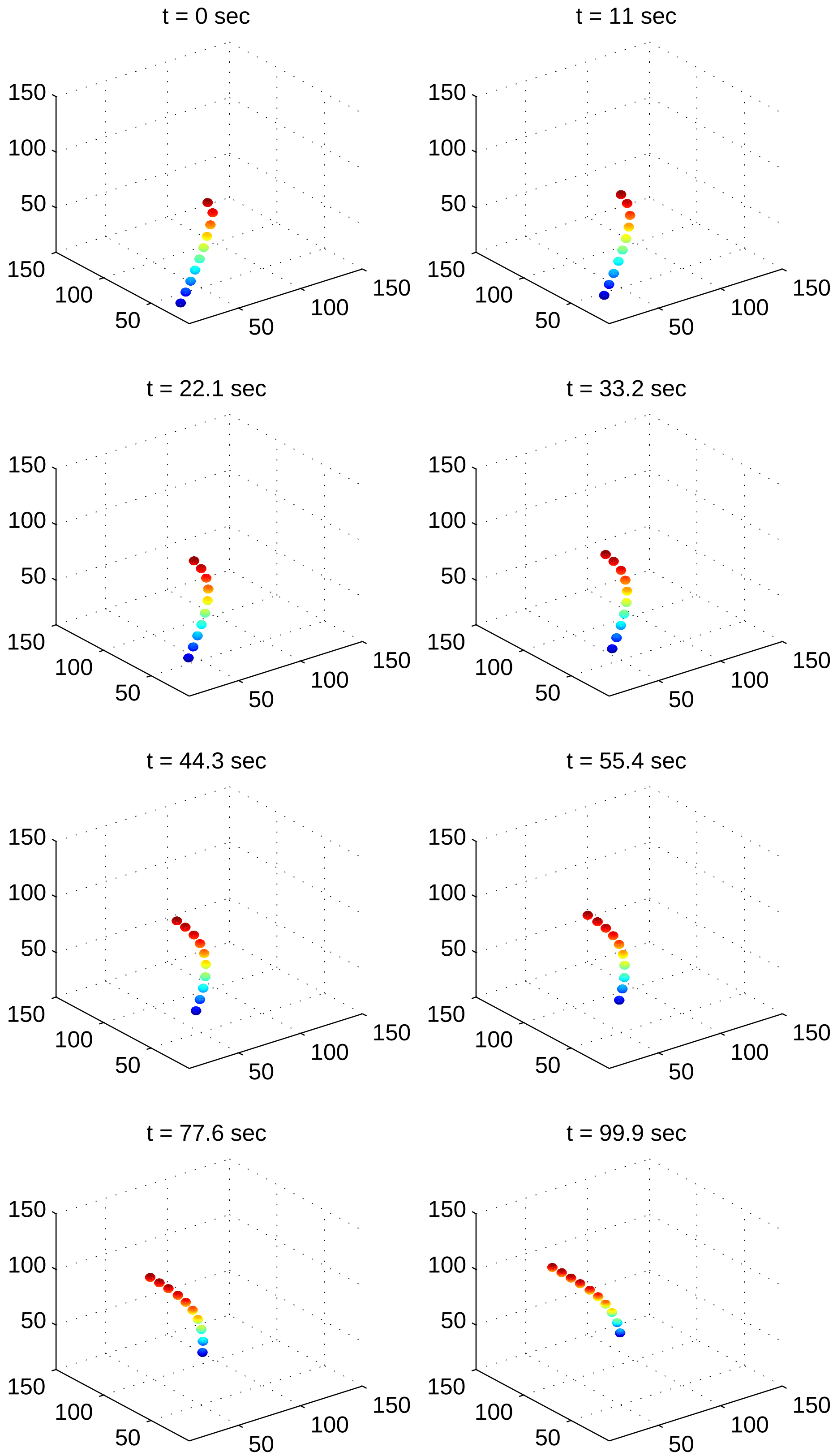


Figure 9: Simulation of $M = 3$ dimensional asynchronous 10-member asynchronous mobile swarm with a fixed communication topology following an edge-leader.

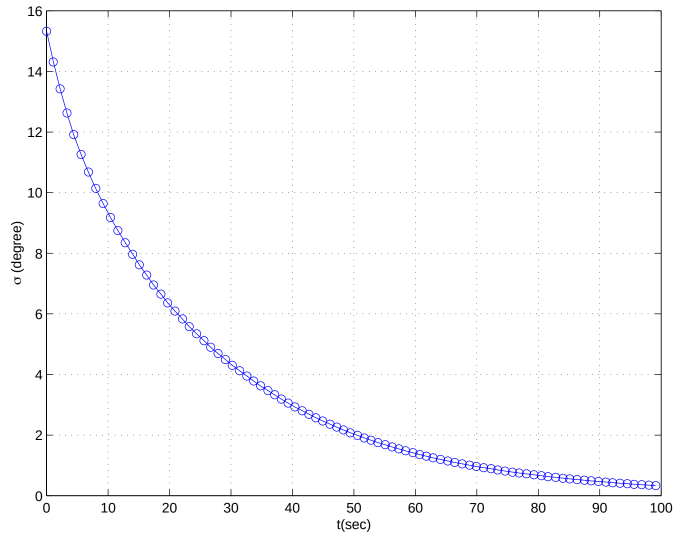


Figure 10: The change of $\sigma$ during movements of the swarm in Figure 9.

members and ensures collision-free swarming, significantly complicates the analysis compared to the case where point-size vehicles are studied and collisions are allowed (e.g., as in [20, 33]) and in some cases clearly cannot allow for as strong of stability results. That is, as you would intuitively expect, asynchronism and delays adversely affect swarm cohesion.

Swarm stability for the case where a communication topology is dynamically generated or updated in a distributed fashion according to the positions of swarm members will be studied in the future.

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