

# Stability Analysis of Networked Control Systems: A Sum of Squares Approach

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**Abstract**—This paper presents a sum of squares (SOS) approach to the stability analysis of networked control systems (NCSs) incorporating time-varying delays and time-varying transmission intervals. We will provide mathematical models that describe these NCSs and transform them into suitable hybrid systems formulations. Based on these hybrid systems formulations we construct Lyapunov functions using SOS techniques that can be solved using LMI-based computations. This leads to several advantages: (i) we can deal with nonlinear polynomial controllers and systems, (ii) we can allow for non-zero lower bounds on the delays and transmission intervals in contrast with various existing approaches, (iii) we allow more flexibility in the Lyapunov functions thereby possibly obtaining improved bounds for the delays and transmission intervals than existing results, and finally (iv) it provides an automated method to address stability analysis problems in NCS.

## I. INTRODUCTION

Stability of networked control systems (NCSs) received considerable attention in recent years and several approaches are currently available for tackling this challenging problem.

The first line of research that can be distinguished is the *discrete-time modeling* approach, see e.g. [5]–[7], [10], [11], [14], [25], [26], which applies to linear plants and linear controllers and is based on exact discretization of the NCS between two transmission times. After a polytopic overapproximation step, robust stability analysis methods are used to obtain LMI-based conditions for stability of the NCS.

The *sampled-data approach* uses continuous-time models that describe the NCS dynamics in the continuous-time domain (so without exploiting any form of discretization) and perform stability analysis based on these sampled-data NCS models directly, see e.g. [8], [9], [28], [29]. The models are in the form of delay-differential equations (DDEs) and Lyapunov-Krasovskii-functionals are used to assess stability based on LMIs. An alternative approach, recently proposed in [18], [19], is based on impulsive DDEs that explicitly take into account the piecewise constant nature of the control signal, thereby reducing conservatism with respect to the work based on DDEs. Constructive LMI-based stability conditions in the latter line of work apply for linear plants and linear controllers and non-zero lower bounds on sampling intervals and delays.

A third line of research is formed by the *continuous-time modeling* (or emulation) approach, which is inspired by the work in [27], and extended in [2], [3], [13], [20], [21]. To describe the NCS, this research line exploits hybrid modeling formalisms as advocated in [12]. The stability of the resulting hybrid system model is based on Lyapunov functions constructed by combining separate Lyapunov functions for the network-free closed-loop system (which has to

be designed to satisfy certain stability properties) and the network protocol (or, alternatively, adopting directly small gain arguments). The available stability conditions all apply for the case of zero lower bounds for the transmission intervals and delays.

In this paper we propose an alternative computational method for stability analysis of NCSs, which from a modeling point of view is closest to the continuous-time modeling approach as just discussed, although it includes also the models based on impulsive DDEs [18], [19], see Remark 1 below. In particular, we will consider here NCSs that exhibit varying transmission intervals and varying delays, while dropouts can be included by prolongations of the transmission intervals. These models will be converted into a hybrid systems formulations as in [12]. Assuming piecewise polynomial plant dynamics (including piecewise affine systems) and a piecewise polynomial controller Lyapunov functions can be constructed using sum of squares (SOS) tools [15], [23], [24]. As a result, this will lead to LMI-based tests for stability given bounds on the delays and transmission intervals. With respect to the existing methods, this approach has various advantages:

- 1) we can deal with nonlinear (piecewise) polynomial controllers and systems, while the constructive conditions in the discrete-time and sampled-data approach only can handle linear plants and controllers;
- 2) we can easily incorporate non-zero lower bounds on the transmission interval and delays, as opposed to the sampled-data approach and emulation approaches;
- 3) we allow more flexibility in the Lyapunov functions thereby obtaining less conservative results;
- 4) we obtain an automated method to address stability analysis problems in NCS;
- 5) we do not have to discretize and perform any polytopic overapproximations as in the discrete-time approach.

Due to these advantages, the SOS-based stability analysis for NCS appears to be a valid alternative in various situations.

## II. NCS DESCRIPTION

In this section, we describe a NCS model that includes time-varying delays and time-varying sampling intervals. In addition, dropouts might be included by modeling them as prolongations of transmission intervals. For the sake of brevity, we will not consider communication constraints and network protocols, which is also possible based on the general NCS model discussed in [13] which extends earlier work [20], inspired by [27]. In the extended version [1] of this paper, this general setup and the usage of SOS

techniques for the stability analysis of NCSs including these communication constraints is discussed.

#### A. Description of the NCS

Consider the continuous-time plant

$$\dot{x}_p = f_p(x_p, \hat{u}), \quad y = g_p(x_p) \quad (1)$$

in which  $x_p \in \mathbb{R}^{n_p}$  denotes the state of the plant,  $\hat{u} \in \mathbb{R}^{n_u}$  denotes the control values being implemented at the plant and  $y \in \mathbb{R}^{n_y}$  is the output of the plant. The plant is controlled over a communication network by a controller, given by

$$\dot{x}_c = f_c(x_c, \hat{y}), \quad u = g_c(x_c, \hat{y}), \quad (2)$$

where the variable  $x_c \in \mathbb{R}^{n_c}$  is the state of the controller,  $\hat{y} \in \mathbb{R}^{n_y}$  contains the most recent output measurements of the plant that are available at the controller and  $u \in \mathbb{R}^{n_u}$  denotes the controller output. The presence of a communication network causes  $u \neq \hat{u}$  and  $y \neq \hat{y}$ , as will be explained next. In particular, the considered NCS setup assumes that the sensor acts in a time-driven fashion and that both the controller and the actuator act in an event-driven fashion (i.e. responding instantaneously to newly arrived data). The controller, sensors, and actuators are connected through a shared network subject to varying transmission intervals and varying delays:

1) *Varying Transmission Intervals*: At the transmission instants,  $t_k \in \mathbb{R}_{\geq 0}$ ,  $k \in \mathbb{N}$ , the plant outputs and control values are sampled and sent over the network. The transmission instants  $t_k$  satisfy  $t_k = \sum_{i=0}^{k-1} h_i \quad \forall k \in \mathbb{N}$ , which are non-equidistantly spaced in time due to the time-varying transmission intervals  $h_k := t_{k+1} - t_k > 0$ , with  $h_k \in [h_{\min}, h_{\max}]$  for all  $k \in \mathbb{N}$ , for some  $0 \leq h_{\min} \leq h_{\max}$ . We assume that the transmission instants  $t_0, t_1, t_2, \dots$  satisfies  $t_{k+1} > t_k$ , for all  $k \in \mathbb{N}$  and  $\lim_{k \rightarrow \infty} t_k = \infty$ .

2) *Varying Delays*: The transmitted input and output values are received after a delay  $\tau_k \in \mathbb{R}_{\geq 0}$ , with  $\tau_k \in [\tau_{\min}, \tau_{\max}]$ , for all  $k \in \mathbb{N}$  where  $0 \leq \tau_{\min} \leq \tau_{\max}$ . To describe the admissible range of transmission intervals and delays, the following standing assumption is adopted

**Assumption 1** *The transmission intervals satisfy  $0 \leq h_{\min} \leq h_k \leq h_{\max}$  and  $h_k > 0$  for all  $k \in \mathbb{N}$  such that  $\lim_{k \rightarrow \infty} t_k = \infty$ , and the delays satisfy  $0 \leq \tau_{\min} \leq \tau_k \leq \min\{\tau_{\max}, h_k\}$ ,  $k \in \mathbb{N}$*

The latter condition implies that each transmitted packet arrives before the next sample is taken. Hence, without loss of generality we can assume that  $\tau_{\max} \leq h_{\max}$ .

The networked-induced errors, defined as  $e_y(t) = \hat{y}(t) - y(t)$  and  $e_u(t) = \hat{u}(t) - u(t)$ , describe the difference between what is the most recent information that is available at the controller/plant and the current value of the plant/controller output, respectively. In between the updates of the values of  $\hat{y}$  and  $\hat{u}$ , the network is assumed to operate in a zero-order-hold (ZOH) fashion. At times  $t_k + \tau_k$ ,  $k \in \mathbb{N}$ , the updates satisfy

$$\hat{y}((t_k + \tau_k)^+) = y(t_k) \quad (3a)$$

$$\hat{u}((t_k + \tau_k)^+) = u(t_k) \quad (3b)$$

at  $t_k + \tau_k$ . Based on (3) we can derive how the network-induced error behaves at the update times  $t_k + \tau_k$  as

$$e((t_k + \tau_k)^+) = e(t_k + \tau_k) - e(t_k). \quad (4)$$

See [13] for more details on (4) and the NCS setup.

The problem that we aim to solve in this paper is to determine stability of the NCS given the bounds  $h_{\min}, h_{\max}, \tau_{\min}$  and  $\tau_{\max}$  as in Assumption 1, or determine these bounds such that stability is guaranteed.

#### B. Hybrid System Formulation

To facilitate the stability analysis, the NCS model is transformed into a hybrid system [12], [13] of the form

$$\dot{\xi} = F(\xi), \quad \xi \in C, \quad (5a)$$

$$\xi^+ = G(\xi), \quad \xi \in D, \quad (5b)$$

where  $C$  and  $D$  are subsets of  $\mathbb{R}^{n_\xi}$ ,  $F: C \rightarrow \mathbb{R}^{n_\xi}$  and  $G: D \rightarrow \mathbb{R}^{n_\xi}$  are mappings and  $\xi^+$  denotes the value of the state directly after the reset. We denote the hybrid system (5) for shortness sometimes by its data  $(C, D, F, G)$ .

To transform the NCS setup (1)-(2) and (3) into (5), the auxiliary variables  $s \in \mathbb{R}^{n_e}$ ,  $\tau \in \mathbb{R}_{\geq 0}$  and  $\ell \in \{0, 1\}$  are introduced to reformulate the model in terms of so-called flow equations (5a) and reset equations (5b). The variable  $s$  is an auxiliary variable containing the memory storing the value  $e(t_k)$  at  $t_k$  for the update of  $e$  at the update instant  $t_k + \tau_k$  as in (4),  $\tau$  is a timer to constrain both the transmission interval as well as the transmission delay and  $\ell$  is a Boolean keeping track whether the next event is a transmission event or an update event. To be precise, when  $\ell = 0$  the next event will be related to transmission (at times  $t_k$ ,  $k \in \mathbb{N}$ ) and when  $\ell = 1$  the next event will be an update (at times  $t_k + \tau_k$ ,  $k \in \mathbb{N}$ ).

The state of our hybrid system  $\Sigma_{NCS}$  is chosen as  $\xi = (x, e, s, \tau, \ell) \in \mathbb{R}^{n_\xi}$ , where  $x = (x_p, x_c)$ . The continuous flow map  $F$  can now be defined as

$$F(\xi) := (f(x, e), g(x, e), 0, 1, 0), \quad (6)$$

where  $f, g$  are appropriately defined functions depending on  $f_p, g_p, f_c$  and  $g_c$ . See [20] for the explicit expressions of  $f$  and  $g$ . Flow according to  $\dot{\xi} = F(\xi)$  occurs when the state  $\xi$  lies in the flow set

$$C := \{\xi \in \mathbb{R}^{n_\xi} \mid (\ell = 0 \wedge \tau \in [0, h_{\max}]) \vee \vee (\ell = 1 \wedge \tau \in [0, \tau_{\max}])\}, \quad (7)$$

where  $\wedge$  denotes the logical ‘and’ operator and  $\vee$  denotes the logical (non-exclusive) ‘or’ operator. The jump map  $G$  inducing resets

$$(x^+, e^+, s^+, \tau^+, \ell^+) = G(x, e, s, \tau, \ell),$$

is obtained by combining the “transmission reset relations,” active at the transmission instants  $\{t_k\}_{k \in \mathbb{N}}$ , and the “update reset relations,” active at the update instants  $\{t_k + \tau_k\}_{k \in \mathbb{N}}$ . Using (4), the jump map  $G$  is defined at the transmission resets (when  $\ell = 0$ ) as

$$G(x, e, s, \tau, 0) = (x, e, e, 0, 1) \quad (8)$$

and the update resets (when  $\ell = 1$ ) as

$$G(x, e, s, \tau, 1) = (x, s - e, 0, \tau, 0). \quad (9)$$

The jump map  $G$  is allowed to reset the system when the state is in the jump set

$$D := \{\xi \in \mathbb{R}^{n_\xi} \mid (\ell = 0 \wedge \tau \in [h_{\min}, h_{\max}]) \vee (\ell = 1 \wedge \tau \in [\tau_{\min}, \tau_{\max}])\}. \quad (10)$$

Finally, we define the equilibrium set of the hybrid system  $\mathcal{A} = \{\xi \in \mathbb{R}^{n_\xi} \mid x = 0 \wedge e = s = 0\}$  for which we would like to prove stability. Hence, the informal stability problem phrased at the end of Section II-A translates now to the question of determining global asymptotic stability (GAS) of the set  $\mathcal{A}$  for  $\Sigma_{NCS} := (C, D, F, G)$  (see [12] for exact definitions of global asymptotic stability of sets). For the remainder of the paper, we will define  $\chi := (x, e, s) \in \mathbb{R}^{n_\chi}$ .

**Remark 1** *The sampled-data system as considered in [17], which lumped the sensor-controller and controller-actuator delays into one delay, was modeled as an impulsive delay-differential equation and focused on linear dynamics with system matrix  $A$ , input matrix  $B$  and state feedback controllers of the form  $u = -Kx_p$ . This model can also be expressed in this hybrid framework by omitting  $e_u$  and  $x_c$  and taking  $y = x_p = x$ ,  $f(x, e) = (A - BK)x - BKe$  and  $g(x, e) = (-A + BK)x + BKe$ .*

### III. STABILITY ANALYSIS

In this section, we will show how the set  $\mathcal{A}$  of the hybrid NCS model  $\Sigma_{NCS}$  can be shown to be GAS by exploiting SOS techniques. We will first state some fundamental hybrid system stability results relevant to our purposes and then present the corresponding SOS theorems, which will be exploited to set up SOS stability conditions for the presented NCS model.

#### A. Stability of Hybrid Systems

First we will use the following definition to specify a Lyapunov function candidate  $V(\xi) : \text{dom } V \rightarrow \mathbb{R}$ , with  $\text{dom } V \subseteq \mathbb{R}^{n_\xi}$ , for a hybrid system as in (5). We will use the concept of a sublevel set of  $V(\xi)$  on a subset  $\Xi$  of  $\text{dom } V$ , which is a set of the form  $\{\xi \in \Xi \mid V(\xi) \leq c\}$  for some  $c \in \mathbb{R}$ .

**Definition 1** [12] *Consider a hybrid system  $\Sigma = (C, D, F, G)$  and a compact set  $\mathcal{A} \subseteq \mathbb{R}^{n_\xi}$ . The function  $V : \text{dom } V \rightarrow \mathbb{R}$  is a Lyapunov function candidate for  $(\Sigma, \mathcal{A})$  if*

- (i.)  $V$  is continuous and nonnegative on  $(C \cup D) \setminus \mathcal{A} \subset \text{dom } V$ ,
- (ii.)  $V$  is continuously differentiable on an open set  $\mathcal{O}$  satisfying  $C \setminus \mathcal{A} \subset \mathcal{O} \subset \text{dom } V$ ,
- (iii.)

$$\lim_{x \rightarrow \mathcal{A}, x \in \text{dom } V \cap (C \cup D)} V(x) = 0.$$

- (iv.) the sublevel sets of  $V$  on  $\text{dom } V \cap (C \cup D)$  are compact

To prove GAS of the set  $\mathcal{A}$ , we will make use of the following theorem.

**Theorem 1** *Consider a hybrid system  $\Sigma = (C, F, D, G)$  and a compact set  $\mathcal{A} \subset \mathbb{R}^{n_\xi}$  satisfying  $G(D \cap \mathcal{A}) \subset \mathcal{A}$ . If every solution of  $\Sigma$  exists for all times  $t \in [0, \infty)$  and there exists a Lyapunov function candidate  $V$  for  $(\Sigma, \mathcal{A})$  that satisfies Definition 1 and*

$$\langle \nabla V(\xi), F(\xi) \rangle < 0 \text{ for all } \xi \in C \setminus \mathcal{A} \quad (11)$$

$$V(G(\xi)) - V(\xi) \leq 0 \text{ for all } \xi \in D \setminus \mathcal{A}, \quad (12)$$

then the set  $\mathcal{A}$  is GAS.

#### B. Stability using SOS techniques

Constructing suitable Lyapunov functions to prove stability is known to be a hard problem, certainly in the nonlinear and hybrid context. Here, we provide a computational approach to this problem based on polynomial Lyapunov functions and sum of squares techniques (SOS) [4], [15], [22]–[24]. The main idea is that a polynomial  $p(x)$  that can be written as a sum of squares, i.e. there exist polynomials  $p_1(x), p_2(x), \dots, p_m(x)$  such that  $p(x) = \sum_{i=1}^m p_i^2(x)$  for all  $x$ , is clearly nonnegative for all  $x$ . As such, inequalities, as in (11) and (12), can be guaranteed if their left-hand sides can be expressed as sums of squares (where S-procedure like relaxations can be used to incorporate the regional information  $\xi \in C \setminus \mathcal{A}$  in (11) and  $\xi \in D \setminus \mathcal{A}$  in (12)). The appeal of SOS is that the solution can be computed using convex semidefinite programming techniques. Indeed  $p(x) = \sum_{i=1}^m p_i^2(x)$  can be checked by finding a positive semidefinite matrix  $Q$ , and a vector of monomials  $Z(x)$  such that  $p(x) = Z^\top(x)QZ(x)$ , see e.g. [24].

In the context of stability of hybrid systems (5), when  $F$  and  $G$  are piecewise polynomial functions (which in the case of the NCS models presented earlier, is true when  $f_c, g_c, f_p, g_p$  are piecewise polynomial) on their domains  $C$  and  $D$ , the Lyapunov stability conditions in Theorem 1 can be transformed into a set of polynomial inequalities. To formalize this idea, we provide the following two definitions, where we use the notation  $\mathbb{R}[x_1, \dots, x_n]$  to denote the set of polynomials in  $n$  variables  $x_1, \dots, x_n$  with real coefficients.

**Definition 2** *A set  $\mathcal{D}$  is called a basic semialgebraic set if it can be described as*

$$\mathcal{D} = \{x \in \mathbb{R}^n \mid e_i(x) \geq 0, i = 1, \dots, M_e \text{ and } f_j(x) = 0, j = 1, \dots, M_f\}$$

for certain polynomials  $e_i(x) \in \mathbb{R}[x_1, \dots, x_n]$ ,  $i = 1, \dots, M_e$  and  $f_j(x) \in \mathbb{R}[x_1, \dots, x_n]$ ,  $j = 1, \dots, M_f$ .

**Definition 3** A function  $p : \Omega \rightarrow \mathbb{R}$  with  $\Omega \subseteq \mathbb{R}^n$  is called *piecewise polynomial* if there are  $M$  basic semialgebraic sets  $\Omega_1, \dots, \Omega_M$  such that

- (i)  $\Omega = \bigcup_{i=1}^M \Omega_i$
- (ii)  $\forall x \in \mathbb{R}^n$  there exists an  $i \in \{1, \dots, M\}$   
such that  $p(x) = p_i(x)$  when  $x \in \Omega_i$

To apply SOS techniques to the hybrid model (5),  $F : C \rightarrow \mathbb{R}^{n_\epsilon}$  and  $G : D \rightarrow \mathbb{R}^{n_\epsilon}$  need to be piecewise polynomial as in Definition 3. The sets  $C$  and  $D$  can then be expressed as  $C = \bigcup_{i=1}^I C_i$  and  $D = \bigcup_{m=1}^M D_m$  with  $C_i, i = 1, \dots, I$  and  $D_m, m = 1, \dots, M$  basic semialgebraic sets, meaning that

$$C_i = \{\xi \in \mathbb{R}^{n_\epsilon} \mid c_{i,j}(\xi) \geq 0, \text{ for } j = 1, \dots, m_C^i, \quad \bar{c}_{i,l}(\xi) = 0, \text{ for } l = 1, \dots, n_C^i\}, \quad (13)$$

$$D_m = \{\xi \in \mathbb{R}^{n_\epsilon} \mid d_{m,j}(\xi) \geq 0, \text{ for } j = 1, \dots, m_D^m, \quad \bar{d}_{m,l}(\xi) = 0, \text{ for } l = 1, \dots, n_D^m\} \quad (14)$$

where  $c_{i,j}(\xi), \bar{c}_{i,l}(\xi), d_{m,j}(\xi)$  and  $\bar{d}_{m,l}(\xi) \in \mathbb{R}[\xi]$  are polynomials. Hence, the hybrid system (5) can now be written in the form

$$\dot{\xi} = F_i(\xi), \quad \xi \in C_i, \quad i = 1, \dots, I \quad (15a)$$

$$\xi^+ = G_m(\xi), \quad \xi \in D_m, \quad m = 1, \dots, M. \quad (15b)$$

We will use the above notation to expand Theorem 1 in the spirit of [23] by applying a technique similar to the S-procedure, called the positivstellensatz [15], [24], in order to encode the information that the inequalities (11) and (12) only have to be satisfied on the sets  $C \setminus \mathcal{A}$  and  $D \setminus \mathcal{A}$ .

**Theorem 2** Given a hybrid system  $\Sigma = (C, F, D, G)$  as in (15) with the sets  $C = \bigcup_i C_i$  and  $D = \bigcup_m D_m$  where  $C_i$  is of the form (13) and  $D_m$  is of the form (14) and  $F_i$  and  $G_i$  polynomial functions for all  $i = 1, \dots, I$  and  $m = 1, \dots, M$ . Furthermore, consider a compact set  $\mathcal{A} \subset \mathbb{R}^{n_\epsilon}$  satisfying  $G(D \cap \mathcal{A}) \subset \mathcal{A}$ . If every solution of  $\Sigma$  exists for all times  $t \in [0, \infty)$  and there exist (i.) a function  $V(\xi)$  for  $(\Sigma, \mathcal{A})$  that satisfies Definition 1, (ii.) polynomials  $\bar{r}_{i,l}(\xi)$  and  $\bar{s}_{m,l}(\xi) \in \mathbb{R}[\xi]$  and (iii.) SOS polynomials  $r_{i,j}(\xi)$  and  $s_{m,j}(\xi) \in \mathbb{R}[\xi]$  such that

$$\langle \nabla V(\xi), F_i(\xi) \rangle + \sum_{j=1}^{m_C^i} r_{i,j}(\xi) c_{i,j}(\xi) + \sum_{l=1}^{n_C^i} \bar{r}_{i,l}(\xi) \bar{c}_{i,l}(\xi) < 0 \quad \forall \xi \notin \mathcal{A}, \quad i = 1, \dots, I, \quad (16)$$

$$V(G_m(\xi)) - V(\xi) + \sum_{j=1}^{m_D^m} s_{m,j}(\xi) d_{m,j}(\xi) + \sum_{l=1}^{n_D^m} \bar{s}_{m,l}(\xi) \bar{d}_{m,l}(\xi) \leq 0 \quad \forall \xi \notin \mathcal{A}, \quad m = 1, \dots, M, \quad (17)$$

then the set  $\mathcal{A}$  is GAS.

**Proof** See [1]. ■

**Remark 2** The SOS relaxation technique as in Theorem 2 can also be applied to encode that the function  $V(\xi)$  only has to be nonnegative on  $(C \cup D) \setminus \mathcal{A}$  into polynomial inequalities (as required in Definition 1) in a similar way.

SOS conditions only guarantee non-negativity of polynomials (i.e. non-strict inequalities) but, of course, proving asymptotic stability requires the Lyapunov derivative (16) being negative definite (satisfying a strict inequality). Thus, we need a way to verify that a given polynomial function is negative or positive definite by checking SOS (positive semidefinite) conditions. We will use the following proposition from [22] to check for positive definiteness of a given polynomial.

**Proposition 1** Given a polynomial  $p(\xi) \in \mathbb{R}[\xi]$  of degree  $2d$ , let  $W(\xi) = \sum_{i=1}^{n_\epsilon} \sum_{j=1}^d \epsilon_{i,j} \xi_i^{2j}$  be such that

$$\sum_{j=1}^d \epsilon_{i,j} > \gamma \quad \text{for all } i = 1, \dots, n \quad (18)$$

with  $\gamma$  a positive number, and  $\epsilon_{i,j} \geq 0$  for all  $i$  and  $j$ . Then the condition

$$p(\xi) - W(\xi) \geq 0 \quad (p(\xi) - W(\xi) \text{ is SOS}) \quad (19)$$

guarantees the positive definiteness of  $p(\xi)$ , i.e.  $p(\xi) > 0$  for all  $\xi \neq 0$ .

Proposition 1 and Theorem 2 form the basis to build the SOS programs that can prove stability of our NCS model (5) with (6)-(10).

### C. Stability of Hybrid NCS models via SOS techniques

In this section we will specify how to set up and verify GAS of the set  $\mathcal{A} = \{\xi \in \mathbb{R}^{n_\epsilon} \mid \chi = 0\}$  of the hybrid NCS models using SOS techniques. The essential steps are the formulation of the hybrid model (5) with  $F : C \rightarrow \mathbb{R}^{n_\epsilon}$  and  $G : D \rightarrow \mathbb{R}^{n_\epsilon}$  being piecewise polynomial as in Definition 3, and applying Theorem 2 and Proposition 1 to derive a suitable SOS program.

Given the definitions of  $C$  and  $D$  for  $\Sigma_{NCS}$ , it is necessary to partition  $C$  and  $D$  by the discrete state  $\ell \in \{0, 1\}$  in the following way

$$C_0 = \{\xi \in \mathbb{R}^{n_\epsilon} \mid \ell = 0, \tau \geq 0, h_{max} - \tau \geq 0\}, \quad (20a)$$

$$C_1 = \{\xi \in \mathbb{R}^{n_\epsilon} \mid \ell = 1, \tau \geq 0, \tau_{max} - \tau \geq 0\}, \quad (20b)$$

with corresponding polynomial flow map

$$F_0(\xi) = F_1(\xi) = F(\chi, \tau, \ell) = (f(x, e), g(x, e), 0, 1, 0) \quad (21)$$

and

$$D_0 = \{\xi \in \mathbb{R}^{n_\epsilon} \mid \ell = 0, \tau - h_{min} \geq 0, h_{max} - \tau \geq 0\}, \quad (22a)$$

$$D_1 = \{\xi \in \mathbb{R}^{n_\epsilon} \mid \ell = 1, \tau - \tau_{min} \geq 0, \tau_{max} - \tau \geq 0\}, \quad (22b)$$

with corresponding polynomial jump map

$$G_0(\xi) = G_0(\chi, \tau, \ell) = (x, e, e, 0, 1), \quad (23a)$$

$$G_1(\xi) = G_1(\chi, \tau, \ell) = (x, s - e, 0, \tau, 0). \quad (23b)$$

Note that  $C = C_0 \cup C_1$ , with  $C_i, i = 0, 1$  basic semialgebraic sets, satisfying (13) and  $D = D_0 \cup D_1$ , with  $D_m, m = 0, 1$  semialgebraic sets, satisfying (14). In addition, the mappings  $G_0(\xi), G_1(\xi)$  and  $F_0(\xi) = F_1(\xi) = F(\xi)$  are polynomial functions, provided that  $f(x, e)$  and  $g(x, e)$  are. This shows that  $F : C \rightarrow \mathbb{R}^{n_\xi}$  and  $G : D \rightarrow \mathbb{R}^{n_\xi}$  are piecewise polynomial, under the standing assumption that  $f(x, e)$  and  $g(x, e)$  are polynomial. Using the above expressions for  $C_i, i = 0, 1$  and  $D_m, m = 0, 1$ , the polynomials  $c_{i,j}$  and  $d_{m,j}$  are defined as shown in Table I.

$c_{i,j}(\xi)$	$d_{m,j}(\xi)$
$c_{0,1} = \tau$	$d_{0,1} = \tau - h_{min}$
$c_{0,2} = h_{max} - \tau$	$d_{0,2} = h_{max} - \tau$
$c_{1,1} = \tau$	$d_{1,1} = \tau - \tau_{min}$
$c_{1,2} = \tau_{max} - \tau$	$d_{1,2} = \tau_{max} - \tau$

TABLE I: SOS relaxations for NCS

We did not include the equality constraints (e.g.  $\ell = 0$  for  $C_0$  or  $\ell = 1$  for  $C_1$ ) as we will encode them through the use of multiple Lyapunov functions explicitly depending on  $\ell$ . The Lyapunov function candidate we propose to use is of the form<sup>1</sup>

$$V(\xi) = V_\ell(\chi, \tau) = \varphi_\ell(\tau) \tilde{W}_\ell(\chi). \quad (24)$$

We specify that the function  $\varphi_\ell(\tau)$  is a polynomial with odd degree and  $\tilde{W}_\ell(\chi)$  is a polynomial with an even degree. This choice of Lyapunov function is inspired by [2], [13]. Combining Proposition 1 and Theorem 2 leads to the polynomial constraints as shown in Table II, where the inequalities will be implemented through SOS conditions. The notation

Constraint Set	
1	$\sum_{j=1}^d \epsilon_{\ell,i,j} \geq \gamma, \quad \epsilon_{\ell,i,j} \geq 0$
2	$V_\ell(\chi, \tau) - \sum_{j=1}^{m_C} q_{\ell,j}(\chi, \tau) c_{\ell,j}(\tau) \geq 0$
3	$-\langle \nabla V_\ell(\chi, \tau), F(\xi) \rangle - W_\ell(\chi) - \sum_{j=1}^2 r_{\ell,j}(\chi, \tau) c_{\ell,j}(\tau) \geq 0$
4a	$V_0(\chi, \tau) - V_1(\bar{G}_0(\chi, \tau)) - \sum_{j=1}^2 s_{0,j}(\chi, \tau) d_{0,j}(\tau) \geq 0$
4b	$V_1(\chi, \tau) - V_0(\bar{G}_1(\chi, \tau)) - \sum_{j=1}^2 s_{1,j}(\chi, \tau) d_{1,j}(\tau) \geq 0$
5	$q_{\ell,j}(\chi, \tau) \geq 0, r_{\ell,j}(\chi, \tau) \geq 0, s_{\ell,j}(\chi, \tau) \geq 0$

TABLE II: SOS program for a NCS

$\bar{G}_i(\chi, \tau), i = 0, 1$  denotes the jump map  $G_i(\xi), i = 0, 1$  restricted to the elements corresponding to  $\chi$  and  $\tau$ , i.e.  $\bar{G}_0(\chi, \tau) = (x, e, e, 0)$  and  $\bar{G}_1(\chi, \tau) = (x, s - e, 0, \tau)$ . The constraints must hold for all  $\ell \in \{0, 1\}$  and  $i \in \{1, 2, \dots, n_\chi\}$ . The function  $W_\ell(\chi), \ell = 0, 1$  is defined as

$$W_\ell(\chi) = \sum_{i=1}^{n_\chi} \sum_{j=1}^d \epsilon_{\ell,i,j} \chi_i^{2j} \quad (25)$$

<sup>1</sup>Note that the multiple Lyapunov function  $V(\xi) = V_\ell(\chi, \tau)$  can be written as one single polynomial Lyapunov function  $V(\xi) = \ell V_1(\chi, \tau) + (1 - \ell) V_0(\chi, \tau)$ .

as in Proposition 1. This function only needs to depend on  $\chi = (x, e, s)$  to guarantee (16) of Theorem 2 because  $\mathcal{A} = \{\xi \in \mathbb{R}^{n_\xi} \mid \chi = 0\}$ . Note that Constraint 3 is derived from combining (19) and (16).

Feasibility of this SOS setup proves stability of a NCS with varying delays and varying sampling intervals that satisfy Assumption 1.

#### IV. COMPARATIVE EXAMPLES

We will illustrate our SOS approach on two different NCS examples.

##### A. Example 1 - Sampled Data

A ‘classic’ and well studied system (see [16] and the reference therein), is given by  $\dot{x}_p(t) = u(t)$ ,  $u(t_k) = -x_p(t_k)$ . For constant sampling interval and no delays, the system can be guaranteed to be stable for sampling times up to 2 seconds. In [16], stability of the system for variable sampling intervals is guaranteed for sampling intervals  $h_k \in [0, 1.99]$ ,  $k \in \mathbb{N}$  in a delay-free situation, which corresponds to  $h_{min} = 0$  and a  $h_{max}$  of 1.99. This does not include much conservatism, as can be concluded from the constant sampling interval result. The results obtained in [16], when delays are present, are given in Figure 1.

Two SOS programs (SOSPs) are constructed using the setup in Table II. Both programs use a quadratic  $\tilde{W}_\ell(x, e, s)$  function, however, the first program uses a linear function  $\varphi(\tau)$  and the second program uses a third order function for  $\varphi(\tau)$ . Already with  $\varphi(\tau)$  being a polynomial of third order, the results of [16] are almost replicated, as shown in Figure 1, whereas taking  $\varphi(\tau)$  to be linear results in more conservative results. The flexibility of our SOS approach allows to gradually increase the order of  $\varphi(\tau)$  and reduce conservatism in the results, as Fig. 1 shows.

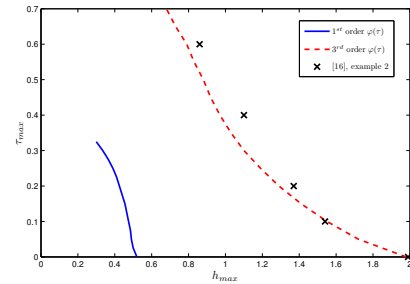


Fig. 1: Tradeoff curves for sampled data NCS.

##### B. Example 2 - Polynomial Sampled-Data

In this example we will show that our method can find Lyapunov functions for a plant with polynomial dynamics. The system we consider is given by  $\dot{x}_p(t) = -x_p^3(t) + x_p^2(t)u(t)$ , which is stabilized by a stabilizing state feedback  $u(t) = -x_p(t)$  when a network is not present.

The constraints from Table II are implemented in a SOS program. We specify the order of  $\tilde{W}_\ell(x, e, s)$  to be six and the function  $\varphi(\tau)$  to be linear, which results in a seventh order  $V(\xi)$ . Tradeoff curves are calculated and shown in Figure 2, showing that indeed, we can analyze a NCS with a polynomial plant and controller in a systematic manner.

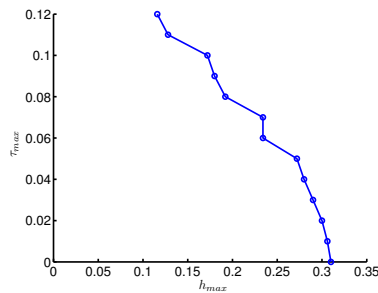


Fig. 2: Tradeoff curves for system with polynomial dynamics.

## V. CONCLUSIONS

In this paper we have presented a sum of squares (SOS) approach for the stability analysis of NCSs that display varying delays and varying sampling intervals. The NCS was modeled as a hybrid system which allowed for general continuous-time polynomial plant and controller dynamics. In order to use SOS techniques, the flow and jump map of the hybrid system were transformed into piecewise polynomial functions. This transformation was explicitly shown for the cases consisting of a sampled-data system without communication constraints. As expected, increasing of the order of the polynomial Lyapunov function leads to improved bounds on the delays and transmission intervals (at the cost of more computational complexity). Next to a reduction in conservatism, our method offers various other advantages with respect to existing approaches, such as dealing with non-zero lower bounds on varying delays and transmission intervals, dealing with nonlinear (polynomial) plants and controllers, not requiring an overapproximation of the NCS (as needed in the discrete-time approach) and finally, the SOS-based approach offers an automated method to tackle the stability problem for NCS including varying delays and transmission intervals. Interestingly, the consideration of communication constraints and network protocols is also possible in the presented framework using the general NCS models in [13], see the extended version [1] of this paper for details. Actually it is shown in [1], for the NCS benchmark example of the batch reactor, that this SOS-based approach provides improved bounds for the delays and transmission intervals compared to the recent results in [13].

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