Technical Notes and Correspondence

Stability Analysis of Piecewise Discrete-Time Linear Systems

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***Abstract*—This note presents a stability analysis method for piecewise discrete-time linear systems based on a piecewise smooth Lyapunov function. It is shown that the stability of the system can be established if a piecewise Lyapunov function can be constructed and, moreover, the function can be obtained by solving a set of linear matrix inequalities (LMIs) that is numerically feasible with commercially available software.**

***Index Terms*—Discrete-time systems, linear matrix inequality (LMI), piecewise linear systems, stability.**

I. Introduction

Piecewise linear systems have been a subject of research in the systems and control community for some time, see, for example, [1]–[14]. In fact, the piecewise linear systems arise often in practical control systems when piecewise linear components are encountered. These components include dead-zone, saturation, relays and hysteresis. In addition, many other classes of nonlinear systems can also be approximated by the piecewise linear systems. Thus the piecewise linear systems provide a powerful means of analysis and design for nonlinear control systems.

A number of significant results have been obtained on analysis and controller design of such piecewise continuous time linear systems during the last few years. For example, the authors in [1] studied a basic issue, that is, the well posedness of piecewise linear systems. Necessary and sufficient conditions for bimodal systems to be well-posed have been derived and the extension to the multimodal case has also been discussed. The authors in [2], [3] presented results on stability and optimal performance analysis for piecewise linear systems based on a piecewise continuous Lyapunov function. It has been shown that lower bounds, as well as upper bounds, on the optimal control cost can be obtained by semidefinite programming and the framework of piecewise linear systems can be used to analyze smooth nonlinear systems with arbitrary accuracy. The authors in [4] discussed stability analysis and controller design of piecewise linear systems which may involve multiple equilibrium points based on a common quadratic Lyapunov function and a piecewise quadratic Lyapunov function. It has been shown that stability and performance analysis can be cast as convex optimization problems. A controller design method based on a common quadratic Lyapunov function and linear matrix inequalities (LMIs) has been proposed. Similar work to those in [2]–[4] has also been reported in [5], [6] where the piecewise Lyapunov function might be discontinuous across the region boundaries.

More recently, there appeared a number of result on stability analysis and controller design of piecewise discrete-time systems using global quadratic or piecewise Lyapunov functions in the open literature [7]–[9]. The authors in [7] presented an approach to stabilization of piecewise linear systems based on a global quadratic Lyapuov function. It was shown that the piecewise state feedback control law could be obtained by solving a set of LMIs and a single nonconvex bilinear matrix inequality. The authors in [8] and [9] presented a number of results on stability analysis, controller design, H_∞ analysis and H_∞ controller design for the piecewise linear systems based on a piecewise Lyapunov function. However, only the piecewise linear system without affine terms were discussed in these two papers. As it is known, many piecewise linear systems have nonzero affine terms for those regions which do not contain the origin. Fox example, when a nonlinear system is linearized around a number of operating points in its state space, then the affine terms appear when the operating point is not at the origin. Therefore in this note, we propose a method for stability analysis of the piecewise discrete-time linear systems with affine terms by constructing a novel piecewise Lyapunov function. This function is guaranteed to be decreasing when the state of the system jumps from one region to another. It is shown that the piecewise Lyapunov function can be constructed by solving a set of LMIs. The work presented in this note can be considered as an extension of the work for the piecewise continuous time systems in [2], [3] to their discrete-time counterparts or the extension of the work for the piecewise discrete-time systems without affine terms in [8] and [9] to those with affine terms.

The rest of the note is organized as follows. Section II introduces the piecewise linear system model and a motivating example. Section III presents a method for stability analysis of such systems and some numerical examples. Finally, conclusions are given in Section IV.

II. Piecewise Linear System Model

Consider autonomous piecewise discrete-time linear systems of the form

$$\begin{aligned} x(t+1) =& A_l x(t) + a_l, \quad \text{for } x \in S_l \\ l =& 1, 2, \ldots, m \end{aligned} \tag{2.1}$$

where $\{S_l\}_{l\in L} \subseteq \mathcal{R}^n$ denotes a partition of the state space into a number of closed polyhedral subspaces, L is the index set of subspaces, $x(t) \in \mathcal{R}^n$ the system state variables, (A_l, a_l) the lth local model of the system and a_l the offset term. For the definition of state trajectory and solution to the piecewise linear system (2.1) please refer to [1]–[3] for details. Here we assume that given any initial condition $x(0) = x_0$, the difference (2.1) has a solution for all $t > 0$. We also assume that when the state of the system transits from the region S_l to S_j at the time t, the dynamics of the system is governed by the dynamics of the local model of S_l at that time. For future use, we also define a set Ω that represents all possible transitions from one region to another, that is

$$\Omega := \{l, j | x(t) \in S_l, x(t+1) \in S_j, j \neq l\}.$$

Remark 2.1: It is noted that the system models defined in (2.1) are in fact *affine* systems instead of linear systems. They include an additional affine or offset term. In this note, the notation of linear systems has been *abused* to represent the affine systems.

Define $L_0 \subseteq L$ as the set of indexes for subspaces that contain the origin and $L_1 = L \backslash L_0 \subseteq L$ the set of indexes for the subspaces that do not contain the origin. It is assumed that $a_l = 0$ for all $l \in L_0$.

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For convenient notation, we introduce

$$\bar{A}_l = \begin{bmatrix} A_l & a_l \\ 0 & 1 \end{bmatrix} \quad \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}. \tag{2.2}$$

Then using this notation, the system model (2.1) can be expressed as

$$\bar{x}(t+1) = \bar{A}_l \bar{x}(t),\ x(t) \in S_l \tag{2.3}$$

It is well known that for the aforementioned piecewise discrete-time system with $a_l = 0, l \in L$, it is possible to show its stability using a globally quadratic Lyapunov function $V(x) = x^T P x$ if a common positive–definite matrix P can be found. A computational approach to finding such a matrix P can be cast as a set of LMIs as in the following lemma.

Lemma 2.1: If there exists a symmetric positive–definite matrix P such that

$$A_l^T P A_l - P < 0,\ l \in L \tag{2.4}$$

then the state trajectory of the system (2.1) with $a_l = 0, l \in L$ tends to the origin exponentially.

It is noted that the condition (2.4) is a set of LMIs in matrix P that can be easily solved by commercially available software such as Matlab. It is also noticed that the condition (2.4) is a sufficient condition and many stable systems may not satisfy such a condition. In order to verify that no common matrix P to (2.4) exists, it is useful to consider the following dual problem [2]: if there exist positive–definite matrices R_l, $l \in L$ such that

$$\sum_{l \in L} \left(A_l^T R_l A_l - R_l \right) > 0 \tag{2.5}$$

then the inequalities in (2.4) do not admit a common positive–definite solution P.

As argued in [2] for the continuous-time case, we present the following example to demonstrate that the Lemma 1 is conservative in the sense that a piecewise discrete-time linear system is stable, but it does not satisfy (2.4).

Example 2.1: Consider a piecewise discrete-time linear system

$$x(t+1) = \begin{cases} A_1 x(t) & if\ x_1 \le 0 \\ A_2 x(t) & if\ x_1 > 0 \end{cases}$$

where

$$A_1 = \begin{bmatrix} 1 & 0.5 & 0 \\ -0.3 & 0.8 & 0 \\ 0 & 0 & 0.4 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0.4 & 0.01 \\ -0.1 & 0.8 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}.$$

By solving the dual problem stated in (2.5), one can easily verify that there exists no positive–definite matrix solution P to (2.4). However, the simulation results indicate that the system is stable.

As an alternative to a globally quadratic Lyapunov function, recently, piecewise Lyapunov functions have been developed for stability test for the piecewise continuous linear systems [2]. Motivated by the work in [2], we will construct a piecewise Lyapunov function which is guaranteed to be decreasing when the state of the system stays in a region, or transits from one region to another in the next section. The similar idea has also been independently developed in [8] and [9].

III. Piecewise Quadratic Stability

As argued in [2], it is not necessary to require a globally valid common positive–definite matrix P for a piecewise linear system since the dynamics described by A_l is only valid within the cell S_l. Consequently, a piecewise Lyapunov function might be sufficient to guarantee the stability of the piecewise linear system. For example, if a piecewise Lyapunov function defined by

$$V(x) = x^T P_l x, \qquad x \in S_l \tag{3.1}$$

with

$$A_l^T P_l A_l - P_l < 0, \qquad l \in L \tag{3.2}$$

is also decreasing when the state transits across boundaries, then the system (2.1) with $a_l = 0, l \in L$, can be guaranteed to be stable in the sense of Lyapunov.

As shown in [2], in order to reduce the conservatism of (3.2), the so-called S-procedure [15] can be used. It is noted that the following matrices $\bar{E}$'s can be constructed for each cell since they are polyhedra such that:

$$\bar{E}_l \bar{x} \ge 0 \tag{3.3}$$

where $\bar{E}_l = [\, E_l \quad e_l \,]$ with $e_l = 0$ for $l \in L_0$. It should be noted that the above vector inequality means that each entry of the vector is nonnegative. Then, when $a_l = 0, l \in L$, and (3.2) can be replaced by

$$A_l^T P_l A_l - P_l + E_l^T U_l E_l < 0, \qquad l \in L \tag{3.4}$$

where U_l is a matrix with nonnegative entries.

Then we are ready to present the following main result of this note.

Theorem 3.1: Consider the piecewise linear system (2.1). If there exist symmetric matrices $P_l, l \in L_0$, $\bar{P}_l, l \in L_1$, U_l, W_l and Q_{lj} such that U_l, W_l and Q_{lj} have nonnegative entries and the following LMIs are satisfied:

$$0 < P_l - E_l^T U_l E_l, \quad l \in L_0 \tag{3.5}$$

$$A_l^T P_l A_l - P_l + E_l^T W_l E_l < 0, \quad l \in L_0 \tag{3.6}$$

$$0 < \bar{P}_l - \bar{E}_l^T U_l \bar{E}_l, \quad l \in L_1 \tag{3.7}$$

$$\bar{A}_l^T \bar{P}_l \bar{A}_l - \bar{P}_l + \bar{E}_l^T W_l \bar{E}_l < 0, \quad l \in L_1 \tag{3.8}$$

$$A_l^T P_j A_l - P_l + E_l^T Q_{lj} E_l < 0, \quad l, j \in \Omega \cap L_0 \tag{3.9}$$

$$\bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l + \bar{E}_l^T Q_{lj} \bar{E}_l < 0, \quad l, j \in \Omega \cap L_1 \tag{3.10}$$

$$\bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l + \bar{E}_l^T Q_{lj} \bar{E}_l < 0, \quad l, j \in \Omega, l \in L_1, j \in L_0 \tag{3.11}$$

$$\bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l + \bar{E}_l^T Q_{lj} \bar{E}_l < 0, \quad l, j \in \Omega, l \in L_0, j \in L_1 \tag{3.12}$$

where we define $\bar{P}_j = [\, I_{n\times n} \quad 0_{n\times 1} \,]^T P_j [\, I_{n\times n} \quad 0_{n\times 1} \,]$ for $j \in L_0$ in (3.11) and $\bar{P}_l = [\, I_{n\times n} \quad 0_{n\times 1} \,]^T P_l [\, I_{n\times n} \quad 0_{n\times 1} \,]$, for $l \in L_0$ in (3.12), then the origin of the piecewise linear system is exponentially stable, that is, $x(t)$ tends to the origin exponentially for every trajectory in the state space.

Proof: Consider the following Lyapunov function candidate $V(t)$:

$$V(t) = \begin{cases} x^T P_l x, & x \in S_l, l \in L_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{P}_l \begin{bmatrix} x \\ 1 \end{bmatrix}, & x \in S_l, l \in L_1 \end{cases} \tag{3.13}$$

It is obvious from (3.13) that there exists a constant $\gamma > 0$ such that for $x \in S_l, l \in L_0$

$$V(t) \le \gamma \|x\|^2 \tag{3.14}$$

and for $x \in S_l, l \in L_1$

$$V(t) \le \gamma \left(\|x\|^2 + 1 \right) \le \gamma \left(\|x\|^2 + \frac{\|x\|^2}{c} \right) = \frac{\gamma(1+c)}{c} \|x\|^2 \tag{3.15}$$

where $c := \min_{x \in S_l, l \in L_1} \|x\|^2 > 0$ since $x \neq 0$ for $x \in S_l$, $l \in L_1$.

Combining (3.14) and (3.15) leads to that there exists a constant $\beta > 0$ such that

$$V(t) \leq \beta \|x\|^2.$$

Moreover, (3.5) and (3.7) imply, respectively, that there exists a constant $\alpha > 0$ such that

$$\begin{aligned}\alpha\|x\|^2 \leq & x^T \left(P_l - E_l^T U_l E_l\right) x \leq x^T P_l x \\ \alpha\|x\|^2 \leq & \alpha\|\bar{x}\|^2 \leq \bar{x}^T \left(\bar{P}_l - \bar{E}_l^T U_l \bar{E}_l\right) \bar{x} \leq \bar{x}^T \bar{P}_l \bar{x}\end{aligned}$$

for $x \in S_l$. That is

$$\alpha\|x\|^2 \leq V(t). \tag{3.16}$$

Thus, from (3.15) and (3.16), we have

$$\alpha\|x\|^2 \leq V(t) \leq \beta\|x\|^2. \tag{3.17}$$

Then we consider the difference of the Lyapunov function candidate. Along trajectories of the system, there exist six possible cases. It should be noted that we have made the following assumption. That is, the dynamics of the system is governed by the dynamics of the local model of S_l when the state of the system transits from the region S_l to S_j at the time t. This assumption is useful for the proofs of Case 3)–Case 6).

Case 1) $x \in S_l$, $l \in L_0$. It follows from (3.6) that there exists a constant $\rho > 0$ such that

$$A_l^T P_l A_l - P_l + E_l^T W_l E_l + \rho I < 0.$$

Then we have

$$\begin{aligned}\Delta V(t) =& V(t) - V(t-1) \\ =& x(t-1)^T \left[A_l^T P_l A_l - P_l\right] x(t-1) \\ \leq& x(t-1)^T \left(-\rho I - E_l^T W_l E_l\right) x(t-1) \\ \leq& -\rho\|x(t-1)\|^2.\end{aligned}$$

Case 2) $x \in S_l$, $l \in L_1$. Similar to the Case 1), one can easily show by using (3.8) that

$$\Delta V(t) \leq -\rho\,\|\bar{x}(t-1)\|^2 \leq -\rho\|x(t-1)\|^2.$$

Case 3) $x(t-1) \in S_l$, $x(t) \in S_j l, j \in \Omega \cap L_0$. It follows from (3.9) that there exists a constant $\rho > 0$ such that

$$A_l^T P_j A_l - P_l + E_l^T Q_{lj} E_l + \rho I < 0.$$

Then we have

$$\begin{aligned}\Delta V(t) =& V(t) - V(t-1) \\ =& x(t-1)^T \left[A_l^T P_j A_l - P_l\right] x(t-1) \\ \leq& x(t-1)^T \left(-\rho I - E_l^T Q_{lj} E_l\right) x(t-1) \\ \leq& -\rho\|x(t-1)\|^2.\end{aligned}$$

Case 4) $x(t-1) \in S_l$, $x(t) \in S_j$, $l, j \in \Omega \cap L_1$. Similar to the Case 3), one can easily show by using (3.10) that

$$\Delta V(t) \leq -\rho\,\|\bar{x}(t-1)\|^2 \leq -\rho\|x(t-1)\|^2.$$

Case 5) $x(t-1) \in S_l$, $x(t) \in S_j$, $l, j \in \Omega$, $l \in L_1$ and $j \in L_0$. It follows from (3.11) that there exists a constant $\rho > 0$ such that

$$\bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l + \bar{E}_l^T Q_{lj} \bar{E}_l + \rho I < 0.$$

Then we have

$$\begin{aligned}\Delta V(t) =& V(t) - V(t-1) \\ =& x(t)^T P_j x(t) - \bar{x}(t-1)^T \bar{P}_l \bar{x}(t-1) \\ =& \left(A_l x(t-1) + a_l\right)^T P_j \left(A_l x(t-1) + a_l\right) \\ & - \bar{x}(t-1)^T \bar{P}_l \bar{x}(t-1) \\ =& \bar{x}(t-1)^T \bar{A}_l [I \quad 0]^T P_j [I \quad 0] \bar{A}_l \bar{x}(t-1) \\ & - \bar{x}(t-1)^T \bar{P}_l \bar{x}(t-1) \\ =& \bar{x}(t-1)^T \left[\bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l\right] \bar{x}(t-1) \\ \leq& \bar{x}(t-1)^T \left(-\rho I - \bar{E}_l^T Q_{lj} \bar{E}_l\right) \bar{x}(t-1) \\ \leq& \bar{x}(t-1)^T (-\rho I) \bar{x}(t-1) \\ \leq& -\rho\|x(t-1)\|^2\end{aligned}$$

where $\bar{P}_j = [I_{n\times n} \quad 0_{n\times 1}]^T P_j [I_{n\times n} \quad 0_{n\times 1}]$.

Case 6) $x(t-1) \in S_l$, $x(t) \in S_j$, $l, j \in \Omega$, $l \in L_0$ and $j \in L_1$. It follows from (3.12) that there exists a constant $\rho > 0$ such that

$$\bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l + \bar{E}_l^T Q_{lj} \bar{E}_l + \rho I < 0.$$

Then we have

$$\begin{aligned}\Delta V(t) =& V(t) - V(t-1) \\ =& \bar{x}(t)^T \bar{P}_j \bar{x}(t) - x(t-1)^T P_l x(t-1) \\ =& \begin{bmatrix} A_l x(t-1) \\ 1 \end{bmatrix}^T \bar{P}_j \begin{bmatrix} A_l x(t-1) \\ 1 \end{bmatrix} \\ & - x(t-1)^T P_l x(t-1) \\ =& \bar{x}(t-1)^T \begin{bmatrix} A_l & 0 \\ 0 & 1 \end{bmatrix}^T \bar{P}_j \begin{bmatrix} A_l & 0 \\ 0 & 1 \end{bmatrix} \bar{x}(t-1) \\ & - x(t-1)^T P_l x(t-1) \\ =& \bar{x}(t-1)^T \bar{A}_l^T \bar{P}_j \bar{A}_l \bar{x}(t-1) - \bar{x}(t-1)^T \bar{P}_l \bar{x}(t-1) \\ =& \bar{x}(t-1)^T \left[\bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l\right] \bar{x}(t-1) \\ \leq& \bar{x}(t-1)^T \left(-\rho I - \bar{E}_l^T Q_{lj} \bar{E}_l\right) \bar{x}(t-1) \\ \leq& \bar{x}(t-1)^T (-\rho I) \bar{x}(t-1) \\ \leq& -\rho\|x(t-1)\|^2\end{aligned}$$

where $\bar{P}_l = [I_{n\times n} \quad 0_{n\times 1}]^T P_l [I_{n\times n} \quad 0_{n\times 1}]$.

Summarizing the previous cases leads to

$$\Delta V(t) \leq -\rho\|x\|^2. \tag{3.18}$$

Therefore, the desired result follows directly from (3.17) and (3.18), based on the standard Lyapunov theory. $\nabla\nabla$

The aformentioned conditions are LMIs in the variables P_l, $\bar{P}_l$, U_l, W_l, and Q_{lj}. A solution to those inequalities ensures $V(t)$ defined in (3.13) to be a Lyapunov function for the system. The LMI in (3.5) or (3.7) for each region guarantees that the function is positive and the LMI in (3.6) or (3.8) guarantees that the function decreases along all system trajectories in each region. The LMIs (3.9)–(3.12) guarantee that the function is decreasing when the state transits from one region to another. The terms involving E_l, $\bar{E}_l$, U_l, W_l and Q_{lj} are related to the S-procedure to reduce the conservatism of those inequalities.

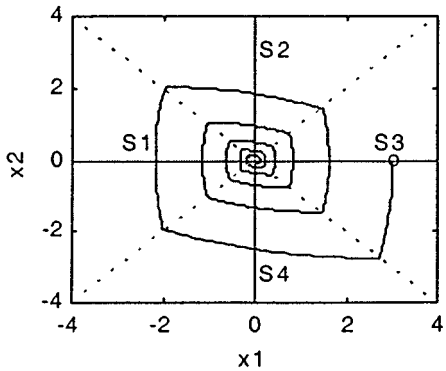

Fig. 1. Trajectory of Example 3.2 from initial condition $x(0) = [3 \quad 0]^T$.

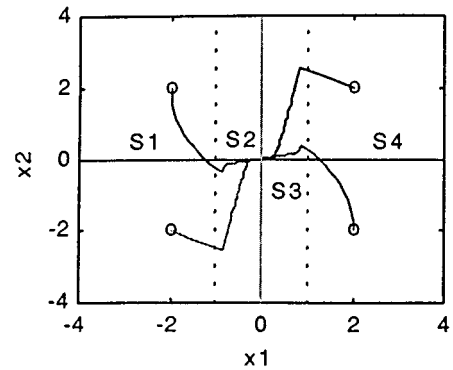

Fig. 2. Trajectories of Example 3.3 from four initial conditions.

Remark 3.1: The set Ω can be determined by the reachability analysis [10]. If it is possible for the transitions happen between all regions, then $\Omega = L^2$, which is defined as a set of $\{l, j | l, j \in L,\ j \neq l\}$.

In the case of $a_l \equiv 0$ for all $l \in L$, we then have the following corollary.

Corollary 3.1: Consider the piecewise linear system (2.1) with $a_l \equiv 0$ for all $l \in L$. If there exist symmetric matrices P_l, $l \in L$, symmetric matrices U_l, W_l and Q_{lj} such that U_l, W_l and Q_{lj} have nonnegative entries and the following LMIs are satisfied:

$$0 < P_l - E_l^T U_l E_l, \quad l \in L, \tag{3.19}$$

$$A_l^T P_l A_l - P_l + E_l^T W_l E_l < 0, \qquad l \in L \tag{3.20}$$

$$A_l^T P_j A_l - P_l + E_l^T Q_{lj} E_l < 0, \qquad l, j \in \Omega \cap L \tag{3.21}$$

then the origin of the piecewise linear system is exponentially stable, that is, $x(t)$ tends to the origin exponentially for every trajectory in the state space.

Example 3.1: Reconsider the piecewise discrete-time linear system in Example 2.1. Using Theorem 3.1 or Corollary 3.1, and noticing the region characterizing matrices

$$E_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we can find the solutions to the LMIs

$$P_1 = \begin{bmatrix} 91.67 & 33.51 & 0.33 \\ 33.51 & 170.20 & 0.80 \\ 0.33 & 0.80 & 43.19 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 88.45 & 47.14 & 1.01 \\ 47.14 & 169.92 & 1.94 \\ 1.01 & 1.94 & 46.16 \end{bmatrix}$$

and thus verify that the system is exponentially stable.

Example 3.2: Consider a piecewise discrete-time linear system

$$x(t+1) = A_l x(t)$$

with region partition shown in Fig. 1. The system matrices are given by

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.01 \\ -0.05 & 0.99 \end{bmatrix} \quad A_2 = A_4 = \begin{bmatrix} 1 & 0.05 \\ -0.01 & 0.99 \end{bmatrix}.$$

The trajectory of a simulation result with initial condition $x(0) = [3 \quad 0]^T$ in Fig. 1 indicates that the system is stable though there does not exist a common positive–definite matrix P for the system. The matrices characterizing the regions are given by

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},\ E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

By solving the LMIs in Theorem 3.1 or Corollary 3.1, we find the following matrices:

$$P_1 = P_3 = \begin{bmatrix} 0.9344 & 0.0809 \\ 0.0809 & 0.2793 \end{bmatrix}$$
$$P_2 = P_4 = \begin{bmatrix} 0.2614 & 0.1257 \\ 0.1257 & 0.9674 \end{bmatrix}$$

and thus one can verify that the system is exponentially stable.

Example 3.3: Consider a piecewise discrete-time linear system with affine terms

$$x(t+1) = A_l x(t) + a_l$$

with region partition shown in Fig. 2. The system matrices are given by

$$A_1 = \begin{bmatrix} 0.9 & -0.1 \\ 0.1 & 1 \end{bmatrix} \quad a_1 = \begin{bmatrix} 0 \\ -0.02 \end{bmatrix}$$
$$A_2 = A_3 = \begin{bmatrix} 1 & -0.02 \\ 0.02 & 0.9 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} 0.9 & -0.1 \\ 0.1 & 1 \end{bmatrix}, \quad a_4 = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}.$$

Similar to the previous examples, the simulation results in Fig. 2 indicate that the system is stable though there exists no globally quadratic Lyapunov function. The matrices characterizing the regions are given by

$$\bar{E}_1 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$
$$E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \bar{E}_4 = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Using the Theorem 3.1 with $\Omega = \{1,2;2,3;3,2;4,3\}$, we can find the following solutions to those LMIs:

$$P_1 = 10^6 \begin{bmatrix} 4.00 & 0.64 & -0.67 \\ 0.64 & 3.47 & 0.57 \\ -0.67 & 0.57 & 2.75 \end{bmatrix}$$
$$P_2 = P_3 = 10^6 \begin{bmatrix} 1.83 & -0.40 \\ -0.40 & 1.38 \end{bmatrix},$$
$$P_4 = 10^6 \begin{bmatrix} 4.18 & 0.62 & 0.71 \\ 0.62 & 3.49 & -0.57 \\ 0.71 & -0.57 & 2.63 \end{bmatrix}$$

and, thus, one can verify that the system is exponentially stable.

IV. Conclusion

In this note, a new method is developed to test stability of piecewise discrete-time linear systems based on a piecewise Lyapunov function. It is shown that the stability can be determined by solving a set of LMIs. The approach can be extended to performance analysis of such systems as in [2] and [3] for their continuous counterparts.

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A Note on the Relation Between Weak Derivatives and Perturbation Realization

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***Abstract*—This note studies the relationship between two important approaches in perturbation analysis (PA)—perturbation realization (PR) and weak derivatives (WDs). Specifically, we study the relation between PR and WDs for estimating the gradient of stationary performance measures of a finite state-space Markov chain. Will show that the WDs expression for the gradient of a stationary performance measure can be interpreted as the expected PR factor where the expectation is carried out with respect to a distribution that is given through the weak derivative of the transition kernel of the Markov chain. Moreover, we present unbiased gradient estimators.**

***Index Terms*—Markov chains, perturbation analysis (PA), weak derivatives (WDs).**

I. Introduction

Today, *perturbation analysis* (PA) is the most widely accepted gradient estimation technique; see [5]–[7] for details. In this note, we work in particular with the interpretation of PA via *perturbation realization* (PR) *factors*, see [1]. The aim of our analysis is to establish a connection between PR and the concept of *weak derivatives* (WDs), see [8]. Whereas PA is a sample-path based approach, WDs are a measure theoretic approach to gradient estimation.

WDs translate the analysis of the gradient into a particular splitting of the sample path into two subpaths and observing these subpaths until they couple, that is, until the perturbation dies out. The basic principle for PA with PR is as follows. A small change in parameters induces a sequence of changes (either small perturbations in timing, or big jumps in states) in a sample path; the effect of such a change on a performance in a long term can be measured by the PR factors, which can be estimated on a single sample path. Thus, the performance gradient can be obtained by the expectation (in some sense depending on the problem) of the realization factor.

In this note, we study the gradient of stationary performance measures of (discrete time) finite state-space Markov chains via WDs and PR. Our analysis will show that the WDs expression for the gradient of a stationary performance measure of finite state Markov chain can be interpreted as the expected PR factor where the expectation is carried out with respect to a distribution that is given through the weak derivative of the transition probability matrix of the Markov chain.

The note is organized as follows. Section II provides a short introduction to PR and WDs. In Section III, we illustrate the relation between the PA via PR and the weak derivative estimator for the stationary performance of a finite state-space Markov chain. In Section IV, we show the application of realization factors to the weak derivative of the transition matrix. In Section V, we deduce unbiased estimators from the

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