Stability Analysis of Spherically Symmetric Star in Scalar-Tensor Theories of Gravity

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A stability analysis of a spherically symmetric star in scalar-tensor theories of gravity is given in terms of the frequencies of quasi-normal modes. The scalar-tensor theories have a scalar field which is related to gravitation. There is an arbitrary function, the so-called coupling function, which determines the strength of the coupling between the gravitational scalar field and matter. Instability is induced by the scalar field for some ranges of the value of the first derivative of the coupling function. This instability leads to significant discrepancies with the results of binary-pulsar-timing experiments and hence, by the stability analysis, we can exclude the ranges of the first derivative of the coupling function in which the instability sets in. In this article, the constraint on the first derivative of the coupling function from the stability of relativistic stars is found. Analysis in terms of the quasi-normal mode frequencies accounts for the parameter dependence of the wave form of the scalar gravitational waves emitted from the Oppenheimer-Snyder collapse. The spontaneous scalarization is also discussed.

§1. Introduction

General relativity is not the only viable theory of gravity. Scalar-tensor theories of gravity are among the alternative theories. The scalar-tensor theories contain a gravitational scalar field and an arbitrary function that determines the strength of the coupling between the scalar field and matter. With some choice of the coupling function, the scalar-tensor theories can pass all the present experimental tests of gravitation. From a theoretical point of view, it has been pointed out that the scalar-tensor theories arise naturally as the low-energy limit of string theory.[1], [2] Until now almost all tests of high precision have been performed only in a weak field. It has been pointed out, however, by Damour and Esposito-Farèse[3], [4] that there are considerable deviations of predictions in a strong field given by some viable scalar-tensor theories from those given by general relativity.

In the scalar-tensor theories of gravity, as in the case of general relativity, generation of gravitational waves is a typical phenomenon in a strong field and hence it is expected[5], [6] that the projects (LIGO[7], VIRGO[8], GEO[9] and TAMA[10]) of gravitational wave observations by laser interferometers will test the viable gravitational theories with high accuracy. Unlike general relativity, there is a scalar mode in gravitational waves in the scalar-tensor theories. Since this scalar gravitational wave can be detected by laser interferometers[6] or resonant mass antennas and can be separated from tensor modes, it has crucial importance with respect to the observational tests of the theories of gravity. The scalar gravitational waves from the Oppenheimer-Snyder collapse, i.e., collapse of a spherically symmetric and homogeneous dust ball, have been calculated numerically as reported in Ref. 11), where

behavior of the scalar field within the dust, the observed wave form of the scalar gravitational waves, and its dependence on the value of the first derivative of the coupling function are examined. From the observed wave form of the scalar gravitational waves we can obtain information concerning the first derivative of the coupling function. The dependence of the observed wave form on the first derivative of the coupling function can be explained by stability analysis of the quasi-normal modes since the behavior of the scalar field under the outgoing wave condition before the formation of the horizon is characterized by the quasi-normal modes that satisfy the regularity condition at the center.

The stability analysis is motivated by the gravitational test through pulsar-timing experiments. By measuring the change of the period of pulsation we can obtain information regarding scalar "form factors" of the pulsar, and those form factors are related to the variation of the moment of inertia under the influence of the companion star.[4] It was shown that in some class of the scalar-tensor theories there is a solution with non-perturbative strong-field effects and it has form factors that are significantly different from those in general relativity.[3,4] In order to obtain the correct gravitational theory, it is important to investigate the internal structure of highly relativistic stars and check the consistency with the results of observational tests. The stability analysis reveals whether the deviation of the stable equilibrium solution of the relativistic star from that in general relativity remains small or not. If, with some choice of the coupling function, the stable equilibrium solution suffers from non-perturbative effects by the coupling between the scalar field and matter and has significantly different scalar form factors from those in general relativity, the results of the pulsar-timing experiments will exclude that choice of the coupling function.

The condition for the stability of relativistic stars of a perfect fluid in the scalar-tensor theories for radial and time-symmetric perturbations was examined by Bruckman and Velázquez.[12] They found that the stability depends on the choice of the coupling function.

This paper is organized as follows. In § 2 we review the scalar-tensor theories of gravity and derive the basic equation. In § 3 we introduce the quasi-normal modes of the scalar field and discuss some useful conditions concerning them. In § 4 we introduce stellar models. In § 5 we give numerical results on stability versus instability for these stellar models. In § 6 we summarize the results and discuss their implications. We use the units in which $c = 1$. We follow the MTW[13] sign conventions for the metric tensor, Riemann tensor and Einstein tensor.

§2. Basic equation

2.1. *Scalar-tensor theories of gravity*

We consider a class of scalar-tensor theories of gravity in which gravitation is mediated by a long-range scalar field in addition to space-time curvature. The action in the Einstein frame formulation is given by[14]

$$I = \frac{1}{16\pi G_*} \int \sqrt{-g_*}\,(R_* - 2g_*^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu})\,d^4x + I_m[\Psi_m, A^2(\varphi)g_{*\mu\nu}], \qquad (2\cdot1)$$

where the metric tensor $g_{*\mu\nu}$ in the "unphysical" Einstein frame is related to the metric tensor $g_{\mu\nu}$ in the "physical" Brans-Dicke frame by the conformal transformation as

$$g_{\mu\nu} = A^2(\varphi)g_{*\mu\nu}. \qquad (2\cdot2)$$

Here, G_* is the "bare" gravitational constant and φ is the gravitational scalar field. R_* is the scalar curvature of $g_{*\mu\nu}$, and Ψ_m represents matter fields collectively. The arbitrary function $A(\varphi)$ is related to a coupling function which will be introduced below. The Brans-Dicke frame metric $g_{\mu\nu}$ is universally coupled to matter in the model we consider here, and hence the Einstein equivalence principle is valid.

The field equations are given by

$$G_{*\mu\nu} = 8\pi G_* T_{*\mu\nu} + 2\left(\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g_{*\mu\nu}g_*^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}\right), \qquad (2\cdot3)$$

$$\Box_*\varphi = -4\pi G_*\alpha(\varphi)T_*, \qquad (2\cdot4)$$

and the equations of motion are given by

$$\nabla_{*\nu}T_{*\mu}^{\nu} = \alpha(\varphi)T_*\nabla_{*\mu}\varphi, \qquad (2\cdot5)$$

where

$$T_*^{\mu\nu} \equiv \frac{2}{\sqrt{-g_*}}\frac{\delta I_m[\Psi_m, A^2(\varphi)g_{*\mu\nu}]}{\delta g_{*\mu\nu}} = A^6(\varphi)T^{\mu\nu}, \qquad (2\cdot6)$$

$$T_* \equiv T_{*\mu}{}^{\mu} \equiv T_*^{\mu\nu}g_{*\mu\nu}, \qquad (2\cdot7)$$

$$\alpha(\varphi) \equiv \frac{d\ln A(\varphi)}{d\varphi}. \qquad (2\cdot8)$$

In the above equations, $G_{*\mu\nu}$, $\nabla_{*\nu}$ and $\Box_*$ are the Einstein tensor, covariant derivative and d'Alembertian of $g_{*\mu\nu}$ respectively, and $T^{\mu\nu}$ is the stress-energy tensor of matter in the Brans-Dicke frame.

The function $\alpha(\varphi)$ is a coupling function between the scalar field and trace of the stress-energy tensor of matter, as seen in Eq. (2·4). If $\alpha(\varphi)$ is constant, the theory reduces to the Brans-Dicke theory. [15] If $\alpha(\varphi) = 0$, the theory reduces to general relativity. Hereafter we consider an asymptotically flat space-time and we assume $\varphi \to \varphi_0$ at spatial infinity. We use units in which $G_* = 1$, and we fix the freedom of the constant conformal transformation by requiring $A(\varphi_0) = 1$.

Here we briefly summarize the present constraints on the coupling function $\alpha(\varphi)$ through the results of solar-system test experiments. The solar system is a laboratory in a weak gravitational field, and hence the parametrized-post-Newtonian (PPN) formalism [16] works well. In the scalar-tensor theories introduced above, the so-called Eddington parameters among the PPN parameters are expressed by the coupling function $\alpha(\varphi)$ as [14]

$$1 - \gamma_{\text{Edd}} = \frac{2\alpha_0^2}{1 + \alpha_0^2}, \qquad (2\cdot9)$$

$$\beta_{\text{Edd}} - 1 = \frac{\beta_0 \alpha_0^2}{2(1 + \alpha_0^2)^2}, \tag{2.10}$$

where

$$\alpha_0 \equiv \alpha(\varphi_0), \tag{2.11}$$

$$\beta_0 \equiv \frac{d\alpha}{d\varphi}(\varphi_0), \tag{2.12}$$

and the values of the other PPN parameters that enter in the first post-Newtonian approximation are identical to those for general relativity. [16] The results of the experiments of the light deflection constrain γ_{Edd} as [17]

$$\gamma_{\text{Edd}} = 0.9996 \pm 0.0017, \tag{2.13}$$

and this constraint reduces to

$$\alpha_0^2 < 0.001. \tag{2.14}$$

The results of the lunar laser-ranging experiments constrain β_{Edd} as [18]

$$\beta_{\text{Edd}} = 0.9998 \pm 0.0006, \tag{2.15}$$

and this constant reduces to

$$\beta_0 \alpha_0^2 = -0.0004 \pm 0.0012. \tag{2.16}$$

2.2. Perturbation equations

From this point we restrict our attention to the scalar-tensor theories in which

$$\alpha_0 = 0. \tag{2.17}$$

This choice is completely consistent with the experimental constraints (2.14) and (2.16). The scalar-tensor theories in which $\alpha_0 = 0$ pass all weak-field tests of solar-system experiments that have been carried out until now, for an arbitrary value of β_0. Further by this assumption we can seek the pure effect of the first derivative β_0 of the coupling function $\alpha(\varphi)$ on the stability of a star.

For $\alpha_0 = 0$, the field equations (2.3) and (2.4) allow the solution

$$g_{*\mu\nu} = g_{\mu\nu}^{(E)}, \tag{2.18}$$

$$T_{*\mu\nu} = T_{\mu\nu}^{(E)}, \tag{2.19}$$

$$\varphi = \varphi_0, \tag{2.20}$$

where the set of $g_{\mu\nu}^{(E)}$ and $T_{\mu\nu}^{(E)}$ is a solution of the Einstein equation, i.e.,

$$G_{\mu\nu}^{(E)} = 8\pi T_{\mu\nu}^{(E)}, \tag{2.21}$$

and therefore

$$\nabla_\nu^{(E)} T_\mu^{(E)\nu} = 0. \tag{2.22}$$

Here, $\nabla_\nu^{(E)}$ and $G_{\mu\nu}^{(E)}$ are the covariant derivative and Einstein tensor of $g_{\mu\nu}^{(E)}$, respectively. In this solution the Einstein frame expression agrees with the Brans-Dicke frame expression:

$$g_{\mu\nu} = g_{*\mu\nu},$$
$$T_{\mu\nu} = T_{*\mu\nu}.$$

If we consider linear perturbations of this solution, the coupled differential equations for the perturbations decouple to an equation for the scalar field perturbation and equations for the metric and matter perturbations. This is because, for $\alpha_0 = 0$, the perturbation of the conformal factor $A^2(\varphi)$ defined in Eq. (2·2) by the scalar field perturbation from the constant value φ_0 is a second-order infinitesimal. For the same reason, the perturbations $\delta g_{*\mu\nu}$ and $\delta T_{*\mu\nu}$ of $g_{*\mu\nu}$ and $T_{*\mu\nu}$ in the Einstein frame are the same as $\delta g_{\mu\nu}$ and $\delta T_{\mu\nu}$ of $g_{\mu\nu}$ and $T_{\mu\nu}$ in the Brans-Dicke frame up to linear order:

$$g_{*\mu\nu} = g_{\mu\nu}^{(E)} + \delta g_{\mu\nu} + \text{higher order terms}, \tag{2·23}$$

$$T_{*\mu\nu} = T_{\mu\nu}^{(E)} + \delta T_{\mu\nu} + \text{higher order terms}, \tag{2·24}$$

$$\varphi = \varphi_0 + \delta\varphi. \tag{2·25}$$

From Eqs. (2·3)~(2·5), we obtain the field equations for the linear perturbations as

$$\delta G_{\mu\nu} = 8\pi\delta T_{\mu\nu}, \tag{2·26}$$

$$\Box^{(E)}\delta\varphi = -4\pi\beta_0 T^{(E)}\delta\varphi, \tag{2·27}$$

and the equations of motion as

$$\delta(\nabla_\nu T_\mu^\nu) = 0, \tag{2·28}$$

where $\Box^{(E)}$ is the d'Alembertian of $g_{\mu\nu}^{(E)}$. Equations (2·26) for the metric and matter perturbations are identical to those for general relativity. Since we are now interested in instability induced by the gravitational scalar field we do not consider these equations here. Thus it is sufficient for our purposes to consider only Eq. (2·27) for the scalar field perturbation. In what follows we omit the symbol (E) to simplify the notation.

We assume that the unperturbed solution is static and spherically symmetric. Then, without loss of generality, the metric is written in the simple form

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Psi(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2·29}$$

In this space-time, Eq. (2·27) becomes

$$-e^{-2\Phi}\frac{\partial^2\delta\varphi}{\partial t^2} + \frac{e^{-\Phi-\Psi}}{r^2}\frac{\partial}{\partial r}\left(e^{\Phi-\Psi}r^2\frac{\partial\delta\varphi}{\partial r}\right)$$
$$+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\delta\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\delta\varphi}{\partial\phi^2}\right] + 4\pi\beta_0 T\delta\varphi = 0. \tag{2·30}$$

We concentrate on perturbations with sinusoidal time dependence, and define $\psi_{\omega lm}(r)$ as

$$\delta\varphi = e^{i\omega t} \sum_{l,m} \frac{\psi_{\omega lm}(r)}{r} Y_m^l(\theta, \phi), \tag{2.31}$$

where we have followed the sign convention of Ref. 19). Hereafter we abbreviate $\psi_{\omega lm}(r)$ as $\psi(r)$. Then from Eq. (2.30) we obtain a Schrödinger-like equation as

$$\frac{d^2\psi}{dr_*^2} + [\omega^2 - V(r_*)]\psi = 0, \tag{2.32}$$

where we have defined the "generalized tortoise coordinate" r_* as

$$dr_* \equiv e^{\Psi - \Phi} dr. \tag{2.33}$$

The effective potential, $V(r_*)$, is defined as

$$V(r_*) \equiv \frac{\Phi' - \Psi'}{r} e^{2(\Phi - \Psi)} + \frac{l(l+1)}{r^2} e^{2\Phi} - 4\pi\beta_0 T e^{2\Phi}, \tag{2.34}$$

where the prime denotes d/dr and β_0 has been defined in Eq. (2.12). On the right-hand side of Eq. (2.34), the first term is a "curvature potential", the second is a "centrifugal potential", and the third comes from the fact that the coupling function $\alpha(\varphi)$ depends on the scalar field φ, and hence $\alpha(\varphi)$ is not constant when the perturbation of the scalar field φ is present.

§3. Stability analysis

3.1. *The quasi-normal modes*

For any complex number ω, a general solution of Eq. (2.32) consists of a linear combination of two independent solutions as

$$\psi(r_*) = C^+(\omega)\psi^+(r_*) + C^-(\omega)\psi^-(r_*), \tag{3.1}$$

where ψ^+ and ψ^- are purely ingoing and outgoing waves. That is,

$$\psi^+ \longrightarrow e^{i\omega r_*}, \tag{3.2}$$

$$\psi^- \longrightarrow e^{-i\omega r_*}, \tag{3.3}$$

in the limit $r_* \to \infty$. The coefficients $C^+(\omega)$ and $C^-(\omega)$ are determined by

$$C^+(\omega) = \frac{W(\psi, \psi^-)}{W(\psi^+, \psi^-)}, \tag{3.4}$$

$$C^-(\omega) = \frac{W(\psi, \psi^+)}{W(\psi^-, \psi^+)}, \tag{3.5}$$

where $W(f, g)$ is the Wronskian defined as

$$W(f, g) \equiv f\frac{dg}{dr_*} - \frac{df}{dr_*}g. \tag{3.6}$$

For a general complex number ω there may be no solution that satisfies the regularity condition ($\psi = 0$) at the origin and outgoing wave boundary condition at infinity ($C^+(\omega) = 0$) simultaneously. For some special, discrete values of the frequency ω there exists a purely outgoing wave solution that satisfies the regularity condition. We have special interest in this solution as an eigenmode of the scalar field. We call the solution quasi-normal mode and this special complex number ω quasi-normal mode frequency. The physical meaning of the quasi-normal mode of the scalar field is that when the scalar field is perturbed by some effect other than the incidence of scalar waves, the scalar field oscillates with radiation of scalar waves, and its characteristic oscillation modes are described by the quasi-normal modes.

3.2. *Stability of the quasi-normal modes*

With the sign convention in Eq. (2·31), if the imaginary part of the quasi-normal mode frequency ω is positive, the quasi-normal oscillation is damped, and its damping rate is given by this imaginary part. If the imaginary part of ω is negative, however, the quasi-normal mode is unstable, and the amplitude of the perturbation grows exponentially in the growth rate given by the absolute value of the imaginary part. The unstable quasi-normal mode develops from a regular initial set of data, because the amplitude of the perturbation decreases exponentially to zero in the limit $r_* \to \infty$. Whether the quasi-normal mode is stable or not is thus determined by whether its complex frequency is in the upper half plane or in the lower half plane.

As for the stability of the quasi-normal modes of Eq. (2·32), the following conditions, similar to those for the quasi-normal modes of spherical nonrotating stars in general relativity,[20] hold: (i) An unstable quasi-normal mode of any l has a purely imaginary frequency, i.e., an unstable mode grows exponentially in time without oscillation. (ii) If the effective potential V is non-negative everywhere, there is no unstable quasi-normal mode. (iii) Consider a sequence of effective potentials smoothly parametrized by a real variable λ. Then there are discrete quasi-normal mode frequencies, and each quasi-normal mode frequency, denoted by $\omega_n(\lambda)$, moves smoothly along a trajectory in the complex plane as λ varies smoothly. (iv) A quasi-normal mode frequency is bounded. (v) Consider a sequence of effective potentials smoothly parametrized by a real variable λ. If there is a critical value $\lambda_n^{\mathrm{crit}}$ for any quasi-normal mode frequency ω_n such that $\mathrm{Im}[\omega_n(\lambda)] > 0$ for $\lambda > \lambda_n^{\mathrm{crit}}$ and $\mathrm{Im}[\omega_n(\lambda)] < 0$ for $\lambda < \lambda_n^{\mathrm{crit}}$, then $\omega_n(\lambda_n^{\mathrm{crit}}) = 0$. In other words, when the stability of a quasi-normal mode changes as λ varies smoothly, its frequency is zero.

Condition (i) is the direct result of the fact that the finiteness of the norm of an unstable quasi-normal mode and hermitianity of the derivative operator

$$-\frac{d^2}{dr_*^2} + V, \qquad (3·7)$$

guarantee that the eigenvalue ω^2 is real for the unstable quasi-normal mode. Condition (ii) is equivalent to the absence of bound states in the non-negative potential in quantum mechanics. The proof of condition (iii) is almost the same as that for the quasi-normal mode frequencies of gravitational waves in general relativity.[21]

The sketch of the proof is as follows. First we take $u \equiv r^{-l-1}\psi$ in place of ψ (cf. Appendix A). The regularity condition requires $du/dr = 0$ at the origin, and we fix the normalization by requiring $u = 1$ at the origin. The solutions u, and therefore ψ depend smoothly on λ because each term in the differential equation (2·32) depends smoothly on λ. So $C^+(\omega; \lambda)$, which is defined by $C^+(\omega)$ for λ, is a smooth function of λ. The solutions u and therefore ψ are also analytic functions of ω because the analyticity of each term for ω in the differential equation (2·32) guarantees the Cauchy-Riemann relations for its solutions. Therefore $C^+(\omega; \lambda)$ is an analytic function of ω. From the property of an analytic function, any zero of $C^+(\omega; \lambda)$ for any given λ, denoted by $\omega_n(\lambda)$, is isolated, and the number of the zeroes is, at most, countably infinite. If we were to assume the discontinuity of each quasi-normal mode frequency $\omega_n(\lambda)$ with respect to λ, a contradiction would result with the smoothness of $C^+(\omega; \lambda)$ with respect to λ through the maximum principle. Therefore we conclude that each quasi-normal mode frequency $\omega_n(\lambda)$ varies smoothly with respect to λ. Condition (iv) can be easily shown by observing that in the limit $|\omega| \to \infty$, the solution regular at the origin is not a purely outgoing wave. Condition (v) is the combined result of conditions (i), (iii) and (iv).

We define a critical value β_0^{crit} as a value of β_0 such that for $\beta_0 > (<)\beta_0^{\text{crit}}$ there is no unstable mode and for $\beta_0 < (>)\beta_0^{\text{crit}}$ there is an unstable mode. The corollary of condition (ii) is that if $T = T^\mu_\mu < 0$ in the stellar interior, which is considered to be satisfied in most of the physical situations, then there is a critical value β_0^{crit}, and for any β_0 larger than this value there is no unstable quasi-normal mode. The corollary of condition (iii) is that each quasi-normal mode frequency moves smoothly in the complex plane as we change β_0 smoothly. The corollary of condition (v) is that there is a zero-frequency quasi-normal mode at $\beta_0 = \beta_0^{\text{crit}}$.

§4. Models

To obtain further understanding of the instability induced by the scalar field we consider stellar models that describe relativistic stars. First we take as a stellar model the equilibrium solution of an incompressible fluid (see Box 23.2 of MTW [13]). In this case the matter is a perfect fluid:

$$T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu). \tag{4·1}$$

The trace of the stress-energy tensor is given by

$$T = T^\mu_\mu = -\rho + 3P. \tag{4·2}$$

The energy density and pressure are given by

$$\rho(r) = \begin{cases} \rho_0 & (0 \leq r < R) \\ 0 & (R \leq r) \end{cases}, \tag{4·3}$$

$$P(r) = \begin{cases} \rho_0 \left[\dfrac{\left(1 - \dfrac{2Mr^2}{R^3}\right)^{1/2} - \left(1 - \dfrac{2M}{R}\right)^{1/2}}{3\left(1 - \dfrac{2M}{R}\right)^{1/2} - \left(1 - \dfrac{2Mr^2}{R^3}\right)^{1/2}} \right] & (0 \le r < R) \\ 0 & (R \le r) \end{cases}, \quad (4\cdot4)$$

where M and R are the gravitational mass and radius of the star,

$$M = \frac{4}{3}\pi\rho_0 R^3. \qquad (4\cdot5)$$

The metric is given in the form of Eq. (2·29), where

$$e^{2\Phi(r)} = \begin{cases} \dfrac{1}{4}\left[3\left(1 - \dfrac{2M}{R}\right)^{1/2} - \left(1 - \dfrac{2Mr^2}{R^3}\right)^{1/2}\right]^2 & (0 \le r < R) \\ 1 - \dfrac{2M}{r} & (R \le r) \end{cases}, \quad (4\cdot6)$$

$$e^{2\Psi(r)} = \left[1 - \frac{2m(r)}{r}\right]^{-1}, \qquad (4\cdot7)$$

$$m(r) = \begin{cases} \frac{4}{3}\pi\rho_0 r^3 & (0 \le r < R) \\ M & (R \le r) \end{cases}. \qquad (4\cdot8)$$

There exists a smallest radius of a star of homogeneous density. From this, we have the condition

$$\frac{R}{M} > \frac{9}{4}. \qquad (4\cdot9)$$

We can easily see that the negativity of T in the stellar interior holds if and only if

$$\frac{R}{M} > \frac{18}{5}, \qquad (4\cdot10)$$

and the dominant energy condition holds if and only if

$$\frac{R}{M} > \frac{8}{3}. \qquad (4\cdot11)$$

Next we introduce a polytropic stellar model. The metric for a static and spherically symmetric space-time is written in the form of Eq. (2·29). The stress-energy tensor of a perfect fluid is given by Eq. (4·1). We integrate the Tolman-Oppenheimer-Volkoff equation

$$\frac{dP}{dr} = -(P + \rho)\frac{m + 4\pi r^3 P}{r[r - 2m]}, \qquad (4\cdot12)$$

where $m(r)$ is defined by Eq. (4·7) and determined by

$$m(r) = 4\pi \int_0^r \rho(r')r'^2 dr'. \qquad (4\cdot13)$$

The function $\Phi(r)$ is determined by integrating the differential equation

$$\frac{d\Phi}{dr} = -\frac{1}{P + \rho}\frac{dP}{dr}, \qquad (4\cdot14)$$

and by the requirement of matching to the Schwarzschild space-time at the stellar surface. We consider the following polytropic equation of state:

$$\rho = nm_b + \frac{Kn_0m_b}{\Gamma - 1}\left(\frac{n}{n_0}\right)^{\Gamma},\qquad(4\cdot15)$$

$$P = Kn_0m_b\left(\frac{n}{n_0}\right)^{\Gamma},\qquad(4\cdot16)$$

$$m_b = 1.66 \times 10^{-24}\text{g},\qquad(4\cdot17)$$

$$n_0 = 0.1\text{fm}^{-3}.\qquad(4\cdot18)$$

We then take the parameters $\Gamma = 2.34$ and $K = 0.0195$ [4] to fit the more realistic equation of state for high-density nuclear matter. [22]

The effective potentials of the stellar model of an incompressible fluid are shown

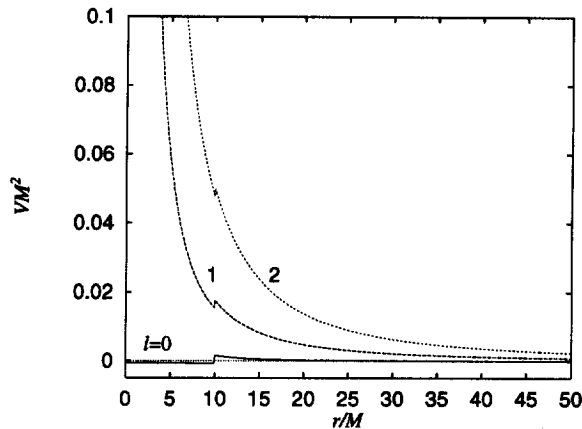
Fig. 1. l-dependence on the effective potential. The ordinate is the effective potential and the abscissa is the areal coordinate. The stellar radius R is fixed to $10M$. The theoretical parameter β_0 is fixed to 0.
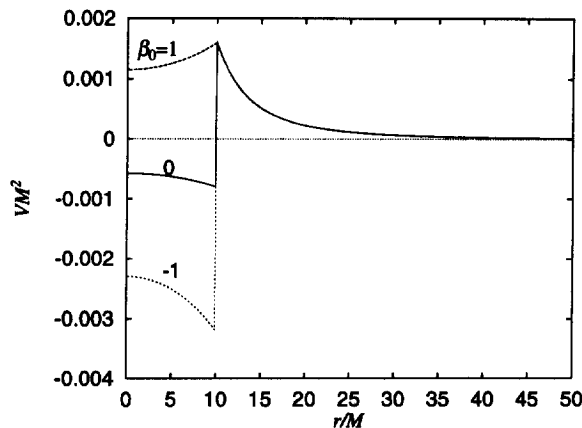

Fig. 2. β_0-dependence on the effective potential. The ordinate and abscissa are the same as in Fig. 1. The stellar radius R is fixed to $10M$. The angular momentum l is fixed to 0.
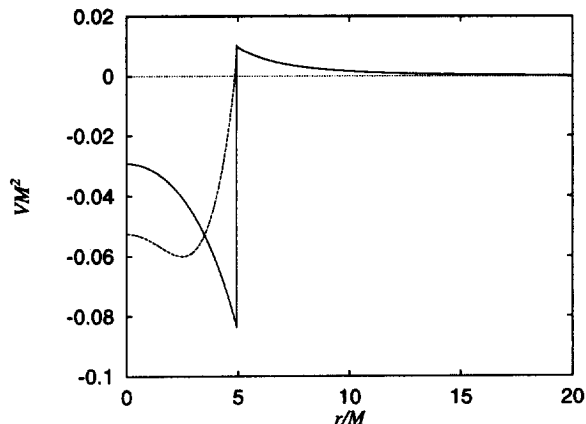
Fig. 3. Comparison between the effective potentials of an $l = 0$ mode for $\beta_0 = -5$ of the incompressible fluid model and polytropic model. The ordinate and abscissa are the same as in Fig. 1. The solid and dashed lines denote the incompressible fluid model and polytropic model respectively. In the polytropic model, the total central density is $1.16 \times 10^{15} \mathrm{g/cm^3}$, and the stellar mass is $1.74 M_\odot$. The stellar radius is $4.94 M$ in both models. The details of the polytropic stellar model are given in § 4 and Appendix B.

in Figs. 1~3. Figure 1 shows the l-dependence of the shape of the effective potential of the incompressible fluid model for $R = 10M$. The parameter β_0 is fixed to 0. The centrifugal potential raises significantly the total effective potential for $l \geq 1$ modes. Therefore, from condition (i), an $l = 0$ (spherically symmetric) mode is more likely to be unstable than any $l \geq 1$ mode, and therefore we concentrate our attention to the $l = 0$ mode. Figure 2 shows the β_0-dependence of the shape of the effective potential of the incompressible fluid model for $l = 0$ and $R = 10M$. The term which comes from the coupling between the scalar field and matter depends on β_0. As β_0 becomes smaller, the quasi-normal modes become more likely to be unstable because the potential well becomes deeper. In particular, from condition (i) and Fig. 2, no unstable quasi-normal mode exists in the theory in which $\beta_0 = 1$ for $R = 10M$ in this model. Figure 3 shows the effective potential of the stellar model of an incompressible fluid and that of the polytropic stellar model of an $l = 0$ mode for $\beta_0 = -5$. In Fig. 3 we fix the central total energy density to be $1.16 \times 10^{15} \mathrm{g/cm^3}$ in the polytropic stellar model and obtain the stellar radius 12.7km, mass $1.74 M_\odot$ and radius-to-mass ratio 4.94. We plot, for comparison, the effective potential of the model of an incompressible fluid whose radius-to-mass ratio is also fixed to 4.94. The details of the numerical calculation of the polytropic stellar model are described in Appendix B.

§5. Numerical results

To search for β_0^{crit} we integrate the differential equation (2·32) for $\omega = 0$ from the origin under the regularity condition. The regularity condition is given by the Taylor expansion of the solution around the origin. The expansion of the regular solution

T. Harada

around the origin is described in Appendix A. After we integrate Eq. (2·32) to the stellar surface we calculate the Wronskian with an exterior solution that is regular at $r = \infty$. As shown in Appendix C, we know that the exterior solution for $\omega = 0$ that is regular at $r = \infty$ is given by the hypergeometric function. Then we examine whether or not the Wronskian vanishes. However, the existence of a zero frequency quasi-normal mode does not necessarily imply the onset of instability because condition (v) does not deny the existence of a λ_n^{false} for ω_n such that $\omega_n(\lambda_n^{\text{false}}) = 0$ but $\text{Im}[\omega_n(\lambda)] > 0$ for $\lambda > \lambda_n^{\text{false}}$ and also for $\lambda < \lambda_n^{\text{false}}$. In order to check the onset of instability we relax the assumption $\omega = 0$. We consider a pure imaginary frequency, i.e.,
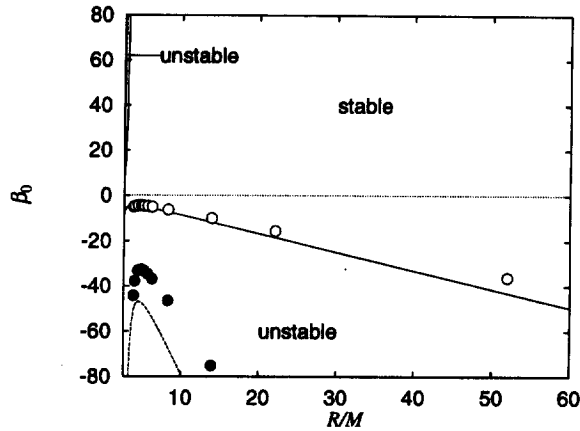
$$\omega = -ic, \quad (c > 0) \tag{5·1}$$

Fig. 4. Stability versus instability. The ordinate is the theoretical parameter β_0 and the abscissa is the stellar radius. The solid lines denote the lines $\beta_0 = \beta_0^{\text{crit}}$, and the dashed lines denote the onset of instability of another quasi-normal mode in the incompressible fluid model. The open circles denote the points $\beta_0 = \beta_0^{\text{crit}}$, and the filled circles the onset of instability of another quasi-normal mode in the polytropic model.
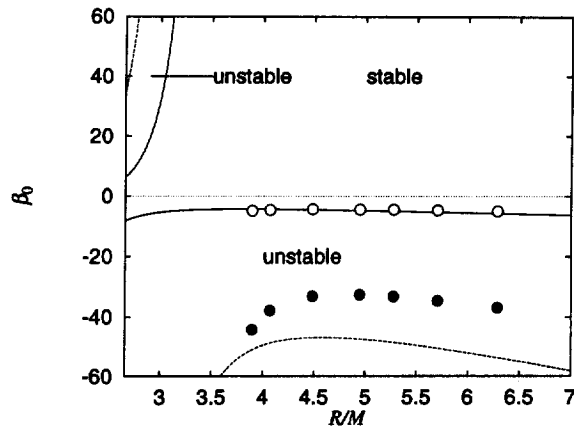

Fig. 5. Magnification of the region of smaller stellar radius in Fig. 4.

and under the regularity condition we integrate Eq. (2·32) from the origin to the surface and thereafter from the surface to the asymptotic region $|\omega|r \gg 1$ and $r \gg M$. Then we calculate the Wronskian with the asymptotic expansion of ψ^- and the coefficient $C^+(\omega)$. The details of the asymptotic expansion are described in Appendix D. When β_0 is changed to a slightly smaller (larger) value than the value for the emergence of a zero-frequency mode, we can confirm that an unstable quasi-normal mode emerges by observing that $C^+(\omega = -ic)$ crosses zero as c is increased from zero.

Table I. The critical values for the polytropic stellar models. ρ_c, M and R are the total energy density at the center, gravtitational mass and radius, respectively. Only the equilibrium solutions that satisfy the necessary condition for stability $dM/d\rho_c > 0$ are in the Table.

$\rho_c(\text{g/cm}^3)$	$M/M_\odot$	$R(\text{km})$	β_0^{crit}
2.26E+15	1.95	11.2	-4.90
1.92E+15	1.93	11.6	-4.59
1.45E+15	1.85	12.2	-4.35
1.16E+15	1.74	12.7	-4.39
1.03E+15	1.67	12.9	-4.48
8.96E+14	1.57	13.2	-4.67
7.70E+14	1.45	13.4	-4.97
5.32E+14	1.12	13.7	-6.21
3.10E+14	0.670	13.7	-9.94
2.04E+14	0.410	13.3	-15.5
1.00E+14	0.161	12.3	-36.2

The results of numerical calculations are shown in Figs. 4~6. Figures 4 and 5 are diagrams of stability versus instability in the β_0 vs R/M plane. The solid and dashed lines denote the emergence of the first and second unstable modes for the incompressible fluid model. Here we have restricted our attention to the stellar model satisfying $R > (8/3)M$ so that the dominant energy condition may be satisfied. If we consider more realistic models, there is the necessary condition, $dM/d\rho_c > 0$, for stability, where ρ_c is the total density at the center. From this condition, the ratio R/M cannot be smaller than some critical value that depends on the equation of state, for example, $3.18 - 4.63$ for models in Ref. 23). If the radius is smaller than this critical radius, the stellar model is already unstable even if $\beta_0 = 0$, i.e., even if there is no coupling between the scalar field and matter. The unstable region above the line $\beta_0 = 0$, as seen in Figs. 4 and 5, is due to the violation of negativity of $T = -\rho + 3P$ for the stellar radius $R < (18/5)M$. As seen in Fig. 4, for $R \gg M$ the curve of $\beta_0 = \beta_0^{\text{crit}}$ can be fitted by the straight line

$$\beta_0^{\text{crit}} = -\frac{\pi^2}{12}\frac{R}{M}, \tag{5·2}$$

and this line is derived by the emergence of the unstable mode in the square well potential ignoring the space-time curvature and matter pressure, and assuming a homogeneous density distribution. The open and filled circles denote the emergence of the first and second unstable quasi-normal modes for the polytropic stellar models that satisfy the necessary condition for stability, $dM/d\rho_c > 0$. They are also shown in Table I. Figures 4 and 5 suggest that the critical value β_0^{crit} for the emergence of the first unstable mode has little dependence on the equation of state, as a function of the ratio R/M. Figure 6 displays the emergence of an unstable quasi-normal mode for the incompressible fluid model of $R = 10M$. Since the eigenvalue of the first unstable mode is approximately given by the depth of the effective potential V and the third term dominates other terms in Eq. (2·34), if β_0 is considerably smaller
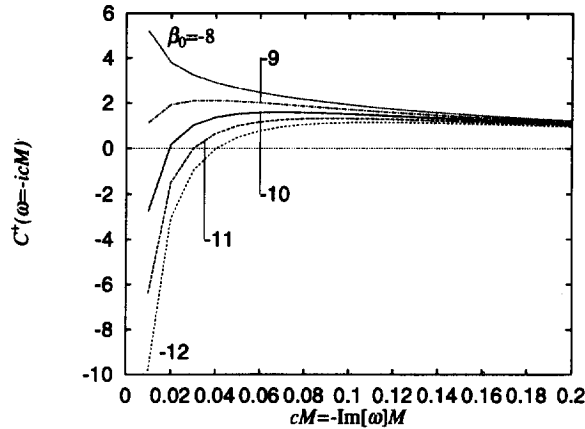
Fig. 6. Emergence of an unstable quasi-normal mode. The ordinate is $C^+ (\omega = -ic)$ and the abscissa is $c = -\text{Im}[\omega]$.

than β_0^{crit}, the growth time τ for the first unstable mode is approximately given by

$$\tau \sim \sqrt{\frac{1}{|\beta_0 T|}} \sim \frac{\tau_{\text{ff}}}{\sqrt{|\beta_0|}}, \tag{5.3}$$

where τ_{ff} denotes the free-fall time of the star.

§6. Summary and discussion

In the scalar-tensor theories of gravity in which $\alpha_0 \equiv \alpha(\varphi_0) = 0$, the field equations allow a solution that consists of a constant scalar field and a solution of the Einstein equation for metric tensor and matter. The field equations for linear perturbations of this solution are decoupled to an equation for the scalar field perturbation and equations for the metric and matter perturbations. The equations for the metric and matter perturbations are identical to those in general relativity.

The stability analysis of the equation for the scalar field perturbation is given in terms of the quasi-normal mode frequencies. From general arguments on the quasi-normal modes of the scalar field in the static and spherically symmetric space-time, the following results turn out to be true: An unstable mode grows monotonically. The presence of a non-negative effective potential implies the absence of unstable quasi-normal modes. When a quasi-normal mode changes its stability, its frequency is zero.

From the shape of the effective potential, we find the following tendencies with regard to stability: As l becomes smaller, the quasi-normal modes of the scalar field become more likely to be unstable . Stability of the quasi-normal modes also depends on the first derivative β_0 of the coupling function $\alpha(\varphi)$ between the scalar field and matter. As β_0 becomes smaller, the quasi-normal modes of the scalar field become more likely to be unstable if $T = -\rho + 3P < 0$ in the stellar interior.

In order to seek for the critical value β_0^{crit} for the onset of instability of a quasi-normal mode of the scalar field, we have examined whether or not a zero-

frequency exists, because the frequency of the marginally stable quasi-normal mode that changes its stability must be zero. Then we have confirmed the onset of instability by observing the emergence of an unstable mode.

Using the incompressible fluid model and polytropic model of a star we found the critical value β_0^{crit} numerically as a function of R/M. If $R/M \simeq 4$ for neutron stars and if these stellar models approximate such neutron stars well, our numerical results suggest that a constraint on the parameter β_0 would be $\beta_0 \gtrsim -5$ from the stability of the neutron stars. In the theories in which $\alpha_0 = 0$ and $\beta_0 \lesssim -5$ (this value depends slightly on the equation of state for high-density nuclear matter), the coupling between the scalar field and matter may change the stability of a stellar solution that consists of a solution for the Einstein equation and a constant scalar field. Even if a stable equilibrium solution exists for $\beta_0 \lesssim -5$, such a solution must contain non-perturbative effects due to the coupling between the scalar field and matter and it will have significantly different form factors for the pulsar-timing experiments from those in general relativity.[4] Even for non-relativistic stars, such as white dwarfs, main-sequence stars, planets and so on, there is a critical value $\beta_0^{\mathrm{crit}} \sim -R/M$ for their stability.

Damour and Esposito-Farése discovered that, for a coupling function of the form

$$\alpha(\varphi) = \beta\varphi, \tag{6.1}$$

when $\beta \lesssim -4$, non-perturbative effects develop for massive neutron stars. That is, the star has a nontrivial finite "scalar charge", and the scalar field displays nontrivial configuration even when the asymptotic value φ_0 is extremely small or zero.[3],[4] The solution with the non-perturbative effects is significantly different from that in general relativity. These results raise a paradoxical problem in the limit $\varphi_0 \to 0$. If $\varphi_0 = 0$, both the field equations and boundary condition possess symmetry under the reflection transformation $\varphi - \varphi_0 \to -(\varphi - \varphi_0)$, and hence they have the trivial solution $\varphi(x) = \varphi_0 = 0$, and the matter and metric tensor satisfy the Einstein equation in general relativity. It seems that this fact implies a discontinuity in the sequence of the solutions against φ_0. Recently it has been discussed that the emergence of the non-perturbative effects on the equilibrium solutions of neutron stars is interpreted as a "spontaneous scalarization" analogous to a spontaneous magnetization of the ferromagnets.[4] The critical value $\beta_0^{\mathrm{crit}} \sim -5$ for the onset of instability of a quasi-normal mode for relativistic stars obtained here by our analysis is approximately equal to the critical value in which an equilibrium solution with non-perturbative effects for relativistic stars emerges for $\alpha_0 = 0$.[3],[4] In the context of spontaneous scalarization, we have shown that if $\beta_0 > \beta_0^{\mathrm{crit}}$, the equilibrium solution with symmetry for the reflection transformation $\varphi - \varphi_0 \to -(\varphi - \varphi_0)$ is stable, but if $\beta_0 < \beta_0^{\mathrm{crit}}$ the solution with the symmetry is no longer stable. This fact strongly suggests that a spontaneous symmetry breaking occurs at $\beta_0 = \beta_0^{\mathrm{crit}}$. It is still important to examine the stability of the equilibrium solution with non-perturbative effects.

It was shown in Ref. 11) that in the Oppenheimer-Snyder collapse the scalar field within the dust decays with oscillation for non-negative values of β_0 (see Fig. 14 of Ref. 11)) and grows without oscillation for a large negative value of β_0 (see

Fig. 15 of Ref. 11)). This leads to the β_0-dependence of the observed wave form of the scalar gravitational waves, as seen in Figs. 10~12 of Ref. 11). The results of the stability analysis given above account for the behavior of the scalar field in terms of the quasi-normal mode frequencies. For $\beta_0 > \beta_0^{\rm crit}$ no unstable quasi-normal mode exists, and only the damped oscillation modes are allowed, but for $\beta_0 < \beta_0^{\rm crit}$ unstable modes exist and grow exponentially without oscillation. Those purely outgoing wave modes characterize the time development of the scalar field within the dust and therefore the observed wave form of the scalar gravitational waves. For $\beta_0 > \beta_0^{\rm crit}$, the quasi-normal mode oscillations appear in the observed wave form before the last quasi-normal mode ringing of the formed black hole, as seen in Fig. 11 of Ref. 11). For $\beta_0 < \beta_0^{\rm crit}$, the growing quasi-normal mode appears in the wave form before the last quasi-normal mode ringing of the black hole, as seen in Fig. 12 of Ref. 11).

Although we have assumed implicitly that the unperturbed solution has a regular origin, we can extend the results obtained in § 3 to a solution that has an event horizon. In this case we have only to replace the regularity condition at the origin $r = 0$ with the ingoing wave condition at the horizon, $r_* \to -\infty$. If we consider a Schwarzschild or Reissner-Nordtström black hole, coupling between the scalar fileld and matter does not exist because $T = 0$ for the vacuum and electromagnetic field. Therefore the effective potential (2·34) is identical to that for a non-gravitational scalar field in general relativity. There are, however, some exotic black hole solutions in which $T \neq 0$, and the stability of such black holes may be altered due to the coupling between the gravitational scalar field and matter.

Although we have assumed $\alpha_0 = 0$, the stability analysis given here applies to the perturbation equations truncated up to zeroth order in α_0 expansions around general relativity, [11] if $\alpha_0 \neq 0$. When we need to consider the effect of the nonzero value of α_0 seriously, we must treat the coupled equations for the perturbations of the scalar field, metric tensor and matter fields. The assumption that α_0 is extremely small seems to be reasonable if the first derivative β_0 of the coupling function $\alpha(\varphi)$ is positive or if the scalar field has a small mass m_φ (10^6km $\lesssim \hbar/m_\varphi \lesssim H_0^{-1}$) and the location φ_m of the minimum of the potential of φ , $V(\varphi) = (1/8\pi)m_\varphi^2(\varphi - \varphi_m)^2$, coincides with a zero of the coupling function $\alpha(\varphi)$, because in these cases the cosmological attraction mechanism is effective and α_0 is attracted toward zero. [4], [24]

Appendix A
—— *Expansion of Regular Solution around the Origin* ——

To suppress numerical errors in the stellar interior we integrate not ψ but u defined as

$$\psi \equiv r^{l+1}u. \tag{A.1}$$

Then the differential equation for u is

$$u'' + \left[(\Phi' - \Psi') + \frac{2(l+1)}{r} \right] u'$$
$$+ \left[\omega^2 e^{2\Psi - 2\Phi} + \frac{\Phi' - \Psi'}{r}l - \frac{l(l+1)}{r^2}(e^{2\Psi} - 1) + 4\pi\beta_0 T e^{2\Psi} \right] u = 0. \tag{A.2}$$

In the exterior vacuum region the differential equation for ψ is

$$\psi'' + \left(1 - \frac{2M}{r}\right)^{-1}\frac{2M}{r^2}\psi'$$
$$+ \left(1 - \frac{2M}{r}\right)^{-2}\left[\omega^2 - \left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right)\left(1 - \frac{2M}{r}\right) \right]\psi = 0. \tag{A.3}$$

In order to obtain the expansion of the solution u around the origin, we expand T, Φ and Ψ around the origin as

$$T = T_0 + T_2 r^2 + T_4 r^4 + \cdots, \tag{A.4}$$

$$\Phi = \Phi_0 + \Phi_2 r^2 + \Phi_4 r^4 + \cdots, \tag{A.5}$$

$$\Psi = \Psi_0 + \Psi_2 r^2 + \Psi_4 r^4 + \cdots. \tag{A.6}$$

If we take the incompressible fluid model, the coefficients are given by

$$T_0 = -2\frac{3\gamma - 2}{3\gamma - 1}\rho_0, \tag{A.7}$$

$$T_2 = -8\pi\frac{\gamma}{(3\gamma - 1)^2}\rho_0^2, \tag{A.8}$$

$$\Phi_0 = \ln\frac{3\gamma - 1}{2}, \tag{A.9}$$

$$\Phi_2 = \frac{4\pi}{3(3\gamma - 1)}\rho_0, \tag{A.10}$$

$$\Phi_4 = \frac{8}{9}\pi^2\frac{3\gamma - 2}{(3\gamma - 1)^2}\rho_0^2, \tag{A.11}$$

$$\Psi_0 = 0, \tag{A.12}$$

$$\Psi_2 = \frac{4}{3}\pi\rho_0, \tag{A.13}$$

$$\Psi_4 = \frac{16}{9}\pi^2\rho_0^2, \tag{A.14}$$

where we have defined

$$\gamma \equiv \sqrt{1 - \frac{2M}{R}}, \tag{A.15}$$

and $1/3 < \gamma < 1$, as seen from Eq. (4.9).

Using these expansions, the regular solution u is expanded around the origin as

$$u = \sum_{\nu=0}^{\infty} a_\nu r^\nu, \tag{A.16}$$

where the coefficients $\{a_\nu\}$ are determined using the coefficients in Eqs. (A·4)~(A·6) as

$$a_1 = a_3 = \cdots = 0, \tag{A·17}$$

$$a_2 = -\frac{1}{2(2l+3)}\left[\omega^2 e^{-2\Phi_0} + 2l\Phi_2 - 2l(l+2)\Psi_2 + 4\pi\beta_0 T_0\right]a_0, \tag{A·18}$$

$$\begin{aligned}
a_4 &= -\frac{1}{4(2l+5)}\left\{\left[2\omega^2 e^{-2\Phi_0}(\Psi_2 - \Phi_2) + 4l(\Phi_4 - \Psi_4)\right.\right.\\
&\quad \left.-2l(l+1)(\Psi_4 + \Psi_2^2) + 4\pi\beta_0(T_2 + 2T_0\Psi_2)\right]a_0\\
&\quad \left.+ \left[\omega^2 e^{-2\Phi_0} + 2(l+2)\Phi_2 - 2(l^2+2l+2)\Psi_2 + 4\pi\beta_0 T_0\right]a_2\right\} \tag{A·19}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8(2l+3)(2l+5)}\left\{\left[\omega^2 e^{-2\Phi_0} + (4+2l)\Phi_2 - 2(l^2+2l+2)\Psi_2\right.\right.\\
&\quad \left.+4\pi\beta_0 T_0\right]\left[\omega^2 e^{-2\Phi_0} + 2l\Phi_2 - 2l(l+2)\Psi_2 + 4\pi\beta_0 T_0\right]\\
&\quad -2(2l+3)\left[2\omega^2 e^{-2\Phi_0}(\Psi_2 - \Phi_2) + 4l(\Phi_4 - \Psi_4)\right.\\
&\quad \left.\left.-2l(l+1)(\Psi_4 + \Psi_2^2) + 4\pi\beta_0(T_2 + 2T_0\Psi_2)\right]\right\}a_0, \tag{A·20}
\end{aligned}$$

$$a_6 = \cdots,$$

and so on.

Appendix B
—— *Numerical Calculation of the Polytropic Stellar Model* ——

We make the variables dimensionless:

$$\tilde{n} \equiv \frac{n}{n_0} = \frac{n}{0.1\text{fm}^{-3}}, \tag{B·1}$$

$$\tilde{\rho} \equiv \frac{\rho}{n_0 m_b} = \frac{\rho}{1.66 \times 10^{14}\text{gcm}^{-3}}, \tag{B·2}$$

$$\tilde{P} \equiv \frac{P}{n_0 m_b} = \frac{P}{1.49 \times 10^{35}\text{gcm}^{-1}s^{-2}}, \tag{B·3}$$

$$\tilde{r} \equiv \frac{r}{(n_0 m_b)^{-1/2}} = \frac{r}{9.02 \times 10^6\text{cm}}, \tag{B·4}$$

$$\tilde{m} \equiv \frac{m}{(n_0 m_b)^{-1/2}} = \frac{m}{1.22 \times 10^{35}\text{g}}. \tag{B·5}$$

Hereafter we omit tildes for simplicity. Then we obtain the following ordinary differential equations for n, m and Φ from Eqs. (4·12)~(4·14):

$$\frac{dn}{dr} = -\frac{n(P+\rho)}{\Gamma P}\frac{m + 4\pi r^3 P}{r(r-2m)}, \tag{B·6}$$

$$\frac{dm}{dr} = 4\pi\rho r^2, \tag{B·7}$$

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P}{r(r-2m)}, \tag{B·8}$$

where

$$\rho = n + \frac{P}{\Gamma - 1}, \tag{B.9}$$

$$P = Kn^{\Gamma}. \tag{B.10}$$

We integrate Eqs. (B·6)~(B·8) from $r = 0$. The initial values of n, m and Φ are

$$n = n_c, \tag{B.11}$$

$$m = 0, \tag{B.12}$$

$$\Phi = 0. \tag{B.13}$$

In order to match the internal solution to the exterior Schwarzschild space-time at the stellar surface, we modify the calculated value Φ_{calc} to the true value of Φ_{true} by

$$\Phi_{\text{true}}(r) = \Phi_{\text{calc}}(r) - \Phi(R)_{\text{calc}} + \frac{1}{2}\ln\left(1 - \frac{2M}{R}\right). \tag{B.14}$$

To guarantee regularity at the origin, we use the following expansions of the solution around the origin:

$$n = n_c + n_2 r^2 + n_4 r^4 + \cdots, \tag{B.15}$$

$$m = m_3 r^3 + m_5 r^5 + \cdots, \tag{B.16}$$

$$\Phi = \Phi_2 r^2 + \Phi_4 r^4 + \cdots, \tag{B.17}$$

$$\rho = \rho_c + \rho_2 r^2 + \cdots, \tag{B.18}$$

$$P = P_c + P_2 r^2 + P_4 r^4 + \cdots. \tag{B.19}$$

Then the coefficients are given by the following relations.

$$P_c = Kn_c^{\Gamma}, \tag{B.20}$$

$$\rho_c = n_c + \frac{P_c}{\Gamma - 1}, \tag{B.21}$$

$$m_3 = \frac{4}{3}\pi\rho_c, \tag{B.22}$$

$$n_2 = -n_c \frac{P_c + \rho_c}{2\Gamma P_c}(m_3 + 4\pi P_c), \tag{B.23}$$

$$P_2 = \Gamma P_c \frac{n_2}{n_c}, \tag{B.24}$$

$$\rho_2 = n_2 + \frac{P_2}{\Gamma - 1}, \tag{B.25}$$

$$m_5 = \frac{4}{5}\pi\rho_2, \tag{B.26}$$

$$\Phi_2 = -\frac{P_2}{P_c + \rho_c}, \tag{B.27}$$

$$n_4 = -\frac{1}{4\Gamma P_c}\left[\left\{n_2(P_c + \rho_c) + n_c\left(\rho_2 - \frac{\rho_c}{P_c}P_2\right)\right\}(m_3 + 4\pi P_c)\right.$$

$$\left. + n_c(P_c + \rho_c)(m_5 + 4\pi P_2) + 2n_c(P_c + \rho_c)(m_3 + 4\pi P_c)m_3\right], \quad \text{(B·28)}$$

$$P_4 = \frac{1}{2}\Gamma(\Gamma - 1)P_c\left(\frac{n_2}{n_c}\right)^2 + \Gamma P_c\left(\frac{n_4}{n_c}\right), \quad \text{(B·29)}$$

$$\Phi_4 = -\frac{P_4}{P_c + \rho_c} + \frac{P_2(P_2 + \rho_2)}{2(P_c + \rho_c)^2}. \quad \text{(B·30)}$$

Appendix C
—— *Static Solution in the Exterior Schwarzschild Space-Time* ——

If we assume the scalar field ψ does not depend on t, Eq. (2·32) reduces to

$$\frac{d^2\psi}{dr^2} + \left(1 - \frac{2M}{r}\right)^{-1}\frac{2M}{r^2}\frac{d\psi}{dr} - \left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right)\left(1 - \frac{2M}{r}\right)^{-1}\psi = 0. \quad \text{(C·1)}$$

Here we define

$$\zeta \equiv \frac{2M}{r}, \quad \text{(C·2)}$$

and

$$w \equiv \frac{\psi}{\zeta^l}. \quad \text{(C·3)}$$

Then Eq. (C·1) transforms into

$$\zeta(1-\zeta)\frac{d^2w}{d\zeta^2} + [(2+2l) - (2l+3)\zeta]\frac{dw}{d\zeta} - (l+1)^2 w = 0. \quad \text{(C·4)}$$

This is the hypergeometric equation. Regularity at $r = \infty$ requires

$$\psi \propto \frac{1}{r^l}, \quad \text{(C·5)}$$

in the limit $r \to \infty$. When we require this condition, the unique solution can be written as

$$\psi = q\left(\frac{2M}{r}\right)^l F\left(l+1, l+1, 2l+2; \frac{2M}{r}\right), \quad \text{(C·6)}$$

where F is the hypergeometric function and q is an arbitrary constant.

Appendix D
—— *Asymptotic Expansions* ——

In Eq. (A·3), if $\omega \neq 0$, $r = \infty$ is an irregular singular point. A general solution is expressed as a linear combination of the purely ingoing and outgoing waves, as seen in Eq. (3·1). The asymptotic expansions for ψ^+ and ψ^- in the asymptotic region $|\omega|r \gg 1$ and $r \gg M$ are

$$\psi^+(r_*) = e^{i\omega r_*}\sum_{\nu=0}^{\infty} b_\nu^+ r^{-\nu}, \quad \text{(D·1)}$$

$$\psi^-(r_*) = e^{-i\omega r_*} \sum_{\nu=0}^{\infty} b_\nu^- r^{-\nu}, \tag{D·2}$$

where the coefficients $\{b_\nu^\pm\}$ are given by the following recursive relations:

$$b_1^\pm = \pm i \frac{l(l+1)}{2\omega} b_0^\pm, \tag{D·3}$$

$$b_2^\pm = \frac{1}{8\omega^2}[-(l-1)l(l+1)(l+2) \pm i4\omega M]b_0^\pm, \tag{D·4}$$

$$b_\nu^\pm = \mp \frac{i}{2\omega\nu}\left[\{(\nu-1)\nu - l(l+1) \pm i4\omega M(\nu-1)\}b_{\nu-1}^\pm \right.$$
$$-2M\{(\nu-1)(2\nu-3) - l(l+1)\}b_{\nu-2}^\pm$$
$$\left. +4M^2(\nu-2)^2 b_{\nu-3}^\pm\right] \qquad \text{for } \nu \geq 3. \tag{D·5}$$

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