Stability Analysis of Stochastic Networked Control Systems

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Abstract— **In this paper, we study the stability of Networked Control Systems (NCSs) that are subject to time-varying transmission intervals, time-varying transmission delays, packet-dropouts and communication constraints. Communication constraints impose that, per transmission, only one sensor or actuator node can access the network and send its information. Which node is given access to the network at a transmission time is orchestrated by a so-called network protocol. This paper considers NCSs, in which the transmission intervals and transmission delays are described by a random process, having a *continuous* probability density function (PDF). By focussing on linear plants and controllers and periodic and quadratic protocols, we present a modelling framework for NCSs based on stochastic discrete-time switched linear systems. Stability (in the mean-square) of these systems is analysed using convex overapproximations and a finite number of linear matrix inequalities. On a benchmark example of a batch reactor, we illustrated the effectiveness of the developed theory.**

I. INTRODUCTION

Modelling, analysis, and controller design of networked control systems (NCSs) has recently received considerable attention in literature. The main reason for this attention is the advantages that NCSs offer, such as low installation and maintenance costs, reduced system wiring and increased flexibility of the system. A drawback of networking the control system, however, is that it is no longer possible to assume, that delays are constant or perhaps negligible, that sampling occurs equidistantly in time, and that all sensor and actuator signals are available at all times. As a result, a deep understanding of the effects of time-varying delays, time-varying transmission intervals, and constrained communication, (i.e., not all sensor and actuator signals being transmitted at every transmission), on the stability and performance of the control system is needed. Most of the literature studies the effects of only some of the phenomena, while ignoring the others. Clearly, it is important to consider the combined presence of time-varying delays and time-varying transmission intervals, and communication constraints, as in any practical NCS they will be present simultaneously.

Stability of NCSs subject to time-varying transmission intervals and communication constraints has been considered in [1], [2] and time-varying transmission intervals, time-varying delays and communication constraints in [3]–[5]. Given a protocol, such as the well-known Round-Robin (RR) and Try-Once-Discard (TOD) protocol, which orchestrates when a certain communication node is given access to the network, the mentioned papers provide criteria for computing the so-called Maximum Allowable Transmission Interval (MATI) and Maximum Allowable Delay (MAD). Stability is guaranteed as long as the actual transmission intervals and delays are always smaller than the MATI and MAD, respectively.

A common feature of the aforecited references is that conditions for stability are derived, given hard *deterministic* bounds on the various network phenomena. In many situations, however, transmission intervals and delays are modelled as random phenomena that are described by probability distributions. Unfortunately, less results are available that provide conditions for stability when the transmission intervals and delays are random processes. A common approach found in literature, see, e.g., [6]–[9], is to take a finite or countable set of possible transmission intervals and delays and attribute probabilities to each element of the set. In this way, the NCS can be effectively modelled as a Markov jump system [10]. It is however not possible to make any statements about stability when the number of elements in the set are not finite or countable.

In this paper, we focus on linear plants and linear controllers and study the stability (in the mean-square) of NCSs, in the presence time-varying transmission intervals and time-varying delays, which are described by random processes, and communication constraints. Contrary to [6]–[9], we allow for *continuous* probability density functions, which can, possibly, be defined on an unbounded domain, like in [11], [12]. In particular, the techniques we provide are applicable to more general probability distributions, including the exponential probability distribution that was studied in [11] as a special case. Contrary to [12], we can consider both quadratic and periodic protocols, as introduced in [4]. These classes of protocols includes the well-known Try-Once-Discard (TOD) protocol and Round-Robin (RR) protocol as special cases. For reasons of space, however, in this paper we restrict our attention to the analysis for the quadratic protocol. The main difference between between [11], [12] and the work presented in this paper is that [11], [12] use a continuous-time modelling paradigm, while we apply a *discrete-time* modelling framework that leads to a switched linear system model, which is stochastically time-varying. Using a convex overapproximation and newly developed Linear Matrix Inequalities (LMIs), the stability (in the mean-square) of the NCS with the transmission intervals and delays satisfying a continuous probability density function (PDF) can be analysed. We will show the effectiveness of the presented approach on the benchmark example of a batch reactor as also used in [1]–[4], [11].

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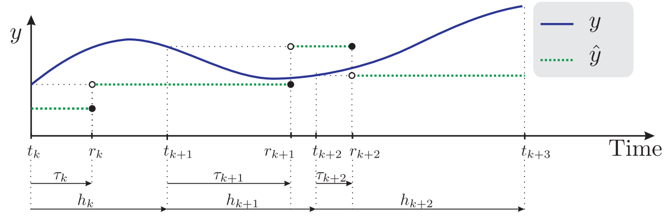

Fig. 1: Illustration of a typical evolution of y and $\hat{y}$.

A. Nomenclature

The following notational conventions will be used. $\text{diag}(A_1, \ldots, A_N)$ denotes a block-diagonal matrix with the entries $A_1, \ldots, A_N$ on the diagonal and $A^\top \in \mathbb{R}^{m \times n}$ denotes the transposed of matrix $A \in \mathbb{R}^{n \times m}$. For a vector $x \in \mathbb{R}^n$, we denote by x^i the i-th component and $\|x\| := \sqrt{x^\top x}$ its Euclidean norm. We denote by $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$ the spectral norm of a matrix A, which is the square-root of the maximum eigenvalue of the matrix $A^\top A$. We sometimes write symmetric matrices of the form $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$, as $\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$. The convex hull and interior of a set $\mathcal{A}$ are denoted by $\text{co}\mathcal{A}$ and $\text{int}\mathcal{A}$, respectively. A probability density function on $\mathbb{R}^n$ is a Lebesgue-integrable function $p : \mathbb{R}^n \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers, that satisfies $\int_{\mathbb{R}^n} p(x)dx = 1$. The expected value of the random variable $x \in \mathbb{R}^n$ is defined as $\mathbb{E}(x) := \int_{\mathbb{R}^n} xp(x)dx$.

II. NCS Model and Problem Statement

In this section, we present the model describing the Networked Control Systems (NCSs), subject to communication constraints, time-varying transmission intervals and delays. Let us consider the linear time-invariant (LTI) continuous-time plant given by

$$\begin{cases} \frac{d}{dt}x^p(t) = A^p x^p(t) + B^p \hat{u}(t) \\ \quad y(t) = C^p x^p(t), \end{cases} \tag{1}$$

where $x^p \in \mathbb{R}^{n_p}$ denotes the state of the plant, $\hat{u} \in \mathbb{R}^{n_u}$ the most recently received control variable, $y \in \mathbb{R}^{n_y}$ the (measured) output of the plant and $t \in \mathbb{R}^+$ the time. The controller, also an LTI system, is assumed to be given by

$$\begin{cases} \frac{d}{dt}x^c(t) = A^c x^c(t) + B^c \hat{y}(t) \\ \quad u(t) = C^c x^c(t) + D^c \hat{y}(t). \end{cases} \tag{2}$$

In this description, $x^c \in \mathbb{R}^{n_c}$ denotes the state of the controller, $\hat{y} \in \mathbb{R}^{n_y}$ the most recently received output of the plant and $u \in \mathbb{R}^{n_u}$ denotes the controller output. At transmission instant $t_k, k \in \mathbb{N}$, (parts of) the outputs of the plant $y(t_k)$ and controller $u(t_k)$ are sampled and are transmitted over the network. We assume that they arrive after a delay τ_k at instant $r_k := t_k + \tau_k$, called the arrival instant, see Fig. 1.

Let us now explain in more detail the functioning of the network and define these 'most recently received' $\hat{y}$ and $\hat{u}$ exactly. The plant is equipped with sensors and actuators that are grouped into N nodes. At each transmission instant t_k, $k \in \mathbb{N}$, one node, denoted by $\sigma_k \in \{1, \ldots, N\}$, gets access to the network and transmits its corresponding values. These transmitted values are received and implemented on the controller and/or the plant at arrival instant r_k. As in [3], a transmission only occurs after the previous transmission has arrived, i.e., $t_{k+1} > r_k \geqslant t_k$, for all $k \in \mathbb{N}$. In other words, we consider the delays to be smaller than the transmission interval. After each transmission and reception, the values in $\hat{y}$ and $\hat{u}$ are updated, while the other values in $\hat{y}$ and $\hat{u}$ remain the same. This leads to the constrained data exchange expressed as

$$\begin{cases} \hat{y}(t) = \Gamma^y_{\sigma_k} y(t_k) + (I - \Gamma^y_{\sigma_k})\hat{y}(t_k) \\ \hat{u}(t) = \Gamma^u_{\sigma_k} u(t_k) + (I - \Gamma^u_{\sigma_k})\hat{u}(t_k) \end{cases} \tag{3}$$

for all $t \in (r_k, r_{k+1}]$, where $\Gamma_{\sigma_k} := \text{diag}(\Gamma^y_{\sigma_k}, \Gamma^u_{\sigma_k})$ is a diagonal matrix, given by

$$\Gamma_i = \text{diag}(\gamma_{i,1}, \ldots, \gamma_{i,n_y+n_u}), \tag{4}$$

when $\sigma_k = i$. In (4), the elements $\gamma_{i,j}$, with $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n_y\}$, are equal to one, if plant output y^j is in node i, elements $\gamma_{i,j+n_y}$, with $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n_u\}$, are equal to one, if controller output u^j is in node i, and are zero elsewhere.

The value of $\sigma_k \in \{1, \ldots, N\}$ in (3) indicates which node is given access to the network at transmission instant t_k, $k \in \mathbb{N}$. Indeed, (3) reflects that the values in $\hat{u}$ and $\hat{y}$ corresponding to node σ_k are updated just after r_k, with the corresponding transmitted values at time t_k, while the others remain the same. A scheduling protocol determines the sequence $(\sigma_0, \sigma_1, \ldots)$ and a particular class of protocols will be made explicit later.

The transmission instants t_k, as well as the arrival instants r_k, $k \in \mathbb{N}$ are not necessarily distributed equidistantly in time. Hence, both the transmission intervals $h_k := t_{k+1} - t_k > 0$, $k \in \mathbb{N}$ and the transmission delays $\tau_k := r_k - t_k \geqslant 0$, $k \in \mathbb{N}$, are varying in time, as is also illustrated in Fig. 1. Furthermore, since $t_{k+1} > r_k$, for all $k \in \mathbb{N}$, we have that $\tau_k < h_k$. We assume that the transmission intervals and transmission delays are described by an Independent and Identically Distributed (IID) random process, characterised by a probability density function (PDF) $p : \mathbb{R}^2 \to \mathbb{R}^+$. The assumptions made above can described by making explicit assumptions on the PDF.

Assumption II.1 For each $k \in \mathbb{N}$, the transmission interval h_k and the transmission delay τ_k are described by an IID random process, characterised by a PDF $p : \mathbb{R}^2 \to \mathbb{R}^+$, with $p(h, \tau) = 0$ for all $(h, \tau) \notin \Theta$, where

$$\Theta = \left\{(h, \tau) \in \mathbb{R}^2 \mid h > 0 \ \wedge \ 0 \leqslant \tau < h\right\}. \tag{5}$$

A. The NCS as a time-varying switched system

To analyse the stability of the NCS described above, we transform it into a discrete-time model. In this framework, we need a discrete-time equivalent of (1) and also of (2) because a continuous-time controller is used. To arrive at this description, let us first define the network-induced error as

$$\begin{cases} e^y(t) := \hat{y}(t) - y(t) \\ e^u(t) := \hat{u}(t) - u(t). \end{cases} \tag{6}$$

The stochastically time-varying discrete-time switched system can now be obtained by describing the evolution of the

$$\bar{x}_{k+1} = \underbrace{\begin{bmatrix} A_{h_k} + E_{h_k}BDC & E_{h_k}BD - E_{h_k-\tau_k}B\Gamma_{\sigma_k} \\ C(I - A_{h_k} - E_{h_k}BDC) & I - D^{-1}\Gamma_{\sigma_k} + C(E_{h_k-\tau_k}B\Gamma_{\sigma_k} - E_{h_k}BD) \end{bmatrix}}_{=:\tilde{A}_{\sigma_k,h_k,\tau_k}} \bar{x}_k \tag{11}$$

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states between t_k and $t_{k+1} = t_k + h_k$. In order to do so, we define $x_k^p := x^p(t_k)$, $u_k := u(t_k)$, $\hat{u}_k := \lim_{t\downarrow r_k} \hat{u}(t)$ and $e_k^u := e^u(t_k)$. Since $\hat{u}$, as in (3), is a left-continuous piecewise constant signal, we can write $\hat{u}_{k-1} = \lim_{t\downarrow r_{k-1}} \hat{u}(t) = \hat{u}(r_k) = \hat{u}(t_k)$. As (3) and (6) yield $\hat{u}_{k-1} = u_k + e_k^u$ and $\hat{u}_{k-1} - \hat{u}_k = \Gamma_{\sigma_k}^u e_k^u$, we can write the exact discretisation of (1) as follows:

$$x_{k+1}^p = e^{A^p h_k} x_k^p + \int_0^{h_k} e^{A^p s} ds B^p (u_k + e_k^u) \\ - \int_0^{h_k-\tau_k} e^{A^p s} ds B^p \Gamma_{\sigma_k}^u e_k^u. \tag{7}$$

A discretised equivalent of (2) is obtained in a similar fashion by defining $x_k^c := x^c(t_k)$, $y_k := y(t_k)$, $e_k^y := e^y(t_k)$, $\hat{y}_k := \lim_{t\downarrow r_k} \hat{y}(t)$, and observing $\hat{y}_{k-1} = \hat{y}(t_k)$, and is given by

$$x_{k+1}^c = e^{A^c h_k} x_k^c + \int_0^{h_k} e^{A^c s} ds B^c (y_k + e_k^y) \\ - \int_0^{h_k-\tau_k} e^{A^c s} ds B^c \Gamma_{\sigma_k}^y e_k^y. \tag{8}$$

We now present two different models each describing a particular NCS. The first covers the situation where both the plant and the controller outputs are transmitted over the network and the second where only the plant outputs y are transmitted over the network and the controller outputs u are sent continuously via an ideal nonnetworked connection. We include this particular case, because it is often used in examples in NCS literature (see, e.g., the benchmark example in [1]–[4], [11]) and it allows us to compare our methodology to the existing ones.

1) The NCS model when both y and u are transmitted: For an NCS having controller (2), the complete NCS model is obtained by combining (3), (6), (7), and (8) and defining

$$\bar{x}_k := \begin{bmatrix} x_k^{p\top} & x_k^{c\top} & e_k^{y\top} & e_k^{u\top} \end{bmatrix}^\top. \tag{9}$$

This results in the discrete-time model (11), as shown on the top of this page, in which $\tilde{A}_{\sigma_k,h_k,\tau_k} \in \mathbb{R}^{n\times n}$, with $n = n_p + n_c + n_y + n_u$, and

$$A_{h_k} := \text{diag}(e^{A^p h_k}, e^{A^c h_k}), \quad B := \begin{bmatrix} 0 & B^p \\ B^c & 0 \end{bmatrix}, \tag{11a}$$

$$C := \text{diag}(C^p, C^c), \qquad D := \begin{bmatrix} I & 0 \\ D^c & I \end{bmatrix}, \tag{11b}$$

$$E_\rho := \text{diag}(\textstyle\int_0^\rho e^{A^p s} ds, \int_0^\rho e^{A^c s} ds), \qquad \rho \in \mathbb{R}. \tag{11c}$$

2) The NCS model when only y is transmitted: In this case we assume that only the outputs of the plant are transmitted over the network and the controller communicates its values continuously and without delay. We therefore have that $u(t) = \hat{u}(t)$, for all $t \in \mathbb{R}^+$, which allows us to combine (1) and (2), yielding

$$\begin{bmatrix} \dot{x}^p(t) \\ \dot{x}^c(t) \end{bmatrix} = \begin{bmatrix} A^p & B^p C^c \\ 0 & A^c \end{bmatrix} \begin{bmatrix} x^p(t) \\ x^c(t) \end{bmatrix} + \begin{bmatrix} B^p D^c \\ B^c \end{bmatrix} \hat{y}(t). \tag{12}$$

Since $\hat{y}$ is still updated according to (3), we can describe the evolution of the states between t_k and $t_{k+1} = t_k + h_k$ in a similar fashion as in (7). In this case, (9) reduces to

$$\bar{x}_k := \begin{bmatrix} x_k^{p\top} & x_k^{c\top} & e_k^{y\top} \end{bmatrix}^\top, \tag{13}$$

resulting in (11), in which

$$A_{h_k} := e^{\left[\begin{smallmatrix} A^p & B^p C^c \\ 0 & A^c \end{smallmatrix}\right] h_k}, \qquad B := \begin{bmatrix} B^p D^c \\ B^c \end{bmatrix}, \tag{14a}$$

$$C := \begin{bmatrix} C^p & 0 \end{bmatrix}, \qquad D := I, \tag{14b}$$

$$E_\rho := \int_0^\rho e^{\left[\begin{smallmatrix} A^p & B^p C^c \\ 0 & A^c \end{smallmatrix}\right] s} ds, \qquad \rho \in \mathbb{R}. \tag{14c}$$

B. The Quadratic Protocol as a Switching Function

Based on the previous modelling steps, the NCS is formulated as a stochastically time-varying discrete-time switched system (11). In this framework, protocols are considered as the switching function determining σ_k. We consider quadratic protocols, as introduced in [4].

A quadratic protocol is a protocol, for which the switching function can be written as

$$\sigma_k = \arg \min_{i=1,\dots,N} \bar{x}_k^\top P_i \bar{x}_k, \tag{15}$$

where P_i, $i \in \{1, \dots, N\}$, are certain given matrices. In fact, the well-known TOD protocol, see, e.g., [1]–[3], sometimes also called Maximum Error First (MEF) protocol, belongs to this class of protocols. In the TOD protocol, the node that has the largest network-induced error, i.e., the largest difference between the latest transmitted values and the current values of the signals corresponding to the node, is granted access to the network. The TOD protocol can be modelled as in (15) by adopting the following structure in the P_i matrices:

$$P_i = \bar{P} - \text{diag}(0, \Gamma_i), \tag{16}$$

in which Γ_i, $i \in \{1, \dots, N\}$, is given by (4) and $\bar{P}$ some arbitrary matrix. Indeed, if we define $\tilde{e}_k^i := \Gamma_i e_k$, where $e_k := [e_k^{y\top}, e_k^{u\top}]^\top$, (15) becomes

$$\sigma_k = \arg\min \left\{ -e_k^\top \Gamma_1 e_k, \dots, -e_k^\top \Gamma_N e_k \right\} \\ = \arg\max \left\{ \|\tilde{e}_k^1\|, \dots, \|\tilde{e}_k^N\| \right\}, \tag{17}$$

which is the TOD protocol. In case two nodes have the same maximal values, one of them can be chosen arbitrarily.

Remark II.2 Although the work presented in this paper considers analysis of NCSs with continuous-time controllers and quadratic protocols only, extensions are possible towards discrete-time controllers and other protocols, such as periodic protocols.

C. Stability of the NCS

The problem studied in this paper is to analyse stability of the stochastically time-varying discrete-time switched linear system (11) with protocol (15), and the transmission intervals and transmission delays by a random process satisfying Assumption II.1. Let us now formally define stability for the NCS.

Definition II.3 System (11) with switching sequences satisfying (15) is said to be *Uniformly Globally Mean-Square Exponentially Stable* (UGMSES) if there exist $c \geqslant 0$ and $0 \leqslant \lambda < 1$, such that for any initial condition $\bar{x}_0 \in \mathbb{R}^n$, and all $k \in \mathbb{N}$, it holds that

$$\mathbb{E}\left(\|\bar{x}_k\|^2\right) \leqslant c\|\bar{x}_0\|^2\lambda^k. \tag{18}$$

III. Obtaining a Convex Overapproximation

In the previous section, we obtained an NCS model in the form of a stochastically time-varying discrete-time switched linear system. In the stability conditions developed in the next section, we will employ techniques originally developed for the situation in which the time-varying transmission intervals and delays lie in some bounded set $\bar{\Theta}$, i.e., $(h_k, \tau_k) \in \bar{\Theta}$ for all $k \in \mathbb{N}$, as discussed in [4]. As in [4], $\tilde{A}_{\sigma_k,h_k,\tau_k}$ depends nonlinearly on the uncertain parameters h_k and τ_k. To make the system amenable for analysis, a procedure was proposed to overapproximate $\tilde{A}_{\sigma_k,h_k,\tau_k}$ by a polytopic system with norm-bounded additive uncertainty, i.e.,

$$\bar{x}_{k+1} = \sum_{l=1}^{M} \alpha_k^l \left(\bar{A}_{\sigma_k,l} + \bar{B}_l \Delta_k \bar{C}_{\sigma_k}\right)\bar{x}_k, \tag{19}$$

where $\bar{A}_{\sigma,l} \in \mathbb{R}^{n\times n}$, $\bar{B}_l \in \mathbb{R}^{n\times q}$, $\bar{C}_\sigma \in \mathbb{R}^{q\times n}$, for $\sigma \in \{1,\ldots,N\}$ and $l \in \{1,\ldots,M\}$, with M the number of vertices of the polytope. The vector $\alpha_k = [\alpha_k^1 \ldots \alpha_k^M]^\top \in \mathcal{A}$, $k \in \mathbb{N}$, is time varying with

$$\mathcal{A} = \Big\{\alpha \in \mathbb{R}^M \,\Big|\, \textstyle\sum_{l=1}^{M} \alpha^l = 1 \text{ and } \alpha^l \geqslant 0 \text{ for } l \in \{1,\ldots,M\}\Big\} \tag{20}$$

and $\Delta_k \in \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ is a norm-bounded set of matrices in $\mathbb{R}^{m\times m}$ that describes the additive uncertainty. Equation (19) is an overapproximation of (11), in the sense that for all $\sigma \in \{1,\ldots,N\}$, it holds that

$$\Big\{\tilde{A}_{\sigma,h,\tau} \mid (h,\tau) \in \bar{\Theta}\Big\} \subseteq \Big\{ \textstyle\sum_{l=1}^{M} \alpha^l \left(\bar{A}_{\sigma,l} + \bar{B}_l \Delta \bar{C}_\sigma\right) \,|\, \alpha \in \mathcal{A}, \Delta \in \boldsymbol{\Delta}\Big\}. \tag{21}$$

Contrary to [4], we will not exactly pursue a description satisfying (21), as this would remove all information about the PDF of (h_k, τ_k). As in [4], we partition Θ into triangles $\mathcal{S}_m$, $m \in \{1,\ldots,S\}$, but we make individual overapproximations of $\tilde{A}_{\sigma_k,h_k,\tau_k}$ for each triangle $\mathcal{S}_m$, instead. This allows us to assign a probability $\bar{p}_m = \iint_{\mathcal{S}_m} p(h,\tau)dhd\tau$ to each triangle and adopt this information in the subsequent stability analysis. Roughly speaking, the continuous PDF $p(h,\tau)$ is approximated by a discrete probability distribution that assigns probabilities to (h,τ) in each triangle $\mathcal{S}_m$ in the partitioning of Θ. Since it is typically not possible to achieve a partitioning $\cup_{m=1}^{S}\mathcal{S}_m = \Theta$, (as we use a finite number of bounded triangles $\mathcal{S}_m$, $m \in \{1,\ldots,S\}$, and Θ can be an unbounded set), we will propose a method to deal with the 'remainder', i.e., with $\Theta^c := \Theta\backslash(\cup_{m=1}^{S}\mathcal{S}_m)$, and select it to be small in the sense that $\int_{\Theta^c} p(h,\tau)dhd\tau < \varepsilon$ for some suitably chosen $\varepsilon \geqslant 0$.

The proposed overapproximation is such that for each $\mathcal{S}_m$, $m \in \{1,\ldots,S\}$, and for all $\sigma \in \{1,\ldots,N\}$, it holds that

$$\Big\{\tilde{A}_{\sigma,h,\tau} \mid (h,\tau) \in \mathcal{S}_m\Big\} \subseteq \Big\{ \textstyle\sum_{l=1}^{3} \alpha^l \bar{A}_{\sigma,m,l} + \bar{B}_m \Delta \bar{C}_\sigma \,|\, \alpha \in \mathcal{A}, \Delta \in \boldsymbol{\Delta}\Big\}, \tag{22}$$

where $\bar{A}_{\sigma,m,l} \in \mathbb{R}^{n\times n}$, $\bar{B}_m \in \mathbb{R}^{n\times q}$ and the procedure to obtain this convex overapproximation is given below.

Procedure III.1

- Select triangles $\mathcal{S}_m \subseteq \Theta$, $m \in \{1,\ldots,S\}$, satisfying

$$\mathcal{S}_m = \text{co}\{(\tilde{h}_{m,1}, \tilde{\tau}_{m,1}), (\tilde{h}_{m,2}, \tilde{\tau}_{m,2}), (\tilde{h}_{m,3}, \tilde{\tau}_{m,3})\} \tag{23}$$

  where $(\tilde{h}_{m,l}, \tilde{\tau}_{m,l})$, $l \in \{1,2,3\}$ denote the vertices of the triangle $\mathcal{S}_m$. Moreover, for all $m, p \in \{1,\ldots,S\}$ and $p \neq m$, $\text{int}\mathcal{S}_p \cap \text{int}\mathcal{S}_m = \emptyset$, $\text{int}\mathcal{S}_m \neq \emptyset$, and $\int_{\Theta\backslash(\cup_{m=1}^{S}\mathcal{S}_m)} p(h,\tau)dhd\tau < \varepsilon$, for some small $\varepsilon \geqslant 0$.
- Define

$$\bar{A}_{\sigma,m,l} := \tilde{A}_{\sigma,\tilde{h}_{m,l},\tilde{\tau}_{m,l}}. \tag{24}$$

- To bound the approximation error, first construct the matrix $\bar{\Lambda}$, that, depending on the NCS model defined in Section II-A, is given by

$$\bar{\Lambda} = \begin{cases} \text{diag}(A^p, A^c), \text{ if (11) is as in Section II-A.1,} \\ \begin{bmatrix} A^p & B^pC^c \\ 0 & A^c \end{bmatrix}, \text{ if (11) is as in Section II-A.2.} \end{cases} \tag{25}$$

  Write the matrix $\bar{\Lambda}$ in its real Jordan form [13], i.e. $\bar{\Lambda} := T\Lambda T^{-1}$, where T is an invertible matrix and

$$\Lambda = \text{diag}(\Lambda_1,\ldots,\Lambda_L) \tag{26}$$

  with $\Lambda_i \in \mathbb{R}^{n_i\times n_i}$, $i \in \{1,\ldots,L\}$, the i-th real Jordan block of $\bar{\Lambda}$.
- Compute for each real Jordan block Λ_i, $i \in \{1,\ldots,L\}$ the worst case approximation error, i.e.

$$\delta_{i,m}^{A} = \sup_{\substack{\sum_{l=1}^{3}\alpha^l = 1,\\ \alpha^l \geqslant 0}} \Big\| e^{\Lambda_i \sum_{l=1}^{3}\alpha^l \tilde{h}_{m,l}} - \sum_{l=1}^{3} \alpha^l e^{\Lambda_i \tilde{h}_{m,l}} \Big\|, \tag{27a}$$

$$\delta_{i,m}^{E_h} = \sup_{\substack{\sum_{l=1}^{3}\alpha^l = 1,\\ \alpha^l \geqslant 0}} \Big\| \sum_{l=1}^{3} \alpha^l \int_{\tilde{h}_{m,l}}^{\sum_{l=1}^{3}\alpha^l \tilde{h}_{m,l}} e^{\Lambda_i s}ds \Big\|, \tag{27b}$$

$$\delta_{i,m}^{E_{h-\tau}} = \sup_{\substack{\sum_{l=1}^{3}\alpha^l = 1,\\ \alpha^l \geqslant 0}} \Big\| \sum_{l=1}^{3} \alpha^l \int_{\tilde{h}_{m,l}-\tilde{\tau}_{m,l}}^{\sum_{l=1}^{3}\alpha^l (\tilde{h}_{m,l}-\tilde{\tau}_{m,l})} e^{\Lambda_i s}ds \Big\|. \tag{27c}$$

  For a detailed explanation of the origin of the approximation error bounds, the reader is referred to [4].

- Finally, define

$$\bar{C}_\sigma := \begin{bmatrix} T^{-1} & 0 \\ T^{-1}BDC & T^{-1}BD \\ 0 & -T^{-1}B\Gamma_\sigma \end{bmatrix} \tag{28}$$

and

$$\bar{B}_m := \begin{bmatrix} T & T & T \\ -CT & -CT & -CT \end{bmatrix} \cdot U_m, \tag{29}$$

in which

$$U_m = \mathrm{diag}(\delta^A_{1,m} I_1, \dots, \delta^A_{L,m} I_L, \delta^{E_h}_{1,m} I_1, \dots, \delta^{E_h}_{L,m} I_L, \delta^{E_{h-\tau}}_{1,m} I_1, \dots, \delta^{E_{h-\tau}}_{L,m} I_L), \tag{30}$$

with I_i the identity matrix of size n_i, complying with the i-th real Jordan Block. The additive uncertainty set $\mathbf{\Delta} \subseteq \mathbb{R}^{3(n_p+n_c)\times 3(n_p+n_c)}$ is now given by

$$\mathbf{\Delta} = \big\{ \mathrm{diag}(\Delta^1, \dots, \Delta^{3L}) \mid \Delta^{i+jL} \in \mathbb{R}^{n_i \times n_i}, \|\Delta^{i+jL}\| \leqslant 1, i \in \{1, \dots, L\}, j \in \{0, 1, 2\} \big\}. \tag{31}$$

Remark III.2 In the special case that there exist h_{nom} or τ_{nom} such that $p(h,\tau) = 0$, either for all $h \neq h_{\text{nom}}$ or for all $\tau \neq \tau_{\text{nom}}$, i.e., the transmission interval or delay is constant, Procedure III.1 has to be modified slightly. This is because we proposed to form triangles $\mathcal{S}_m \subseteq \Theta \subset \mathbb{R}^2$, $m \in \{1, \dots, S\}$, having the property that $\mathrm{int}\mathcal{S}_m \neq \emptyset$, which is not useful in this case. In this case, we propose to form line-segments $\mathcal{S}_m$, $m \in \{1, \dots, S\}$ instead, such that for each $\mathcal{S}_m$, $m \in \{1, \dots, S\}$, it holds that

$$\mathcal{S}_m = \mathrm{co}\{(\tilde{h}_{m,1}, \tilde{\tau}_{m,1}), (\tilde{h}_{m,2}, \tilde{\tau}_{m,2})\}, \tag{32}$$

where $(\tilde{h}_{m,l}, \tilde{\tau}_{m,l})$, $l \in \{1, 2\}$, now denote the vertices of the line segment $\mathcal{S}_m$. All other properties of $\mathcal{S}_m$, $m \in \{1, \dots, S\}$ still hold and the remainder of the procedure can be applied *mutatis mutandis*.

IV. Stability of NCS with Stochastic Uncertainty

In section II, we discussed the NCS model and in Section III, we proposed a way to overapproximate it by a switched polytopic system with a norm-bounded uncertainty. A specific feature of this overapproximation is that an individual overapproximation is made for each triangle $\mathcal{S}_m \subseteq \Theta$, $m \in \{1, \dots, S\}$ which enables us to preserve the characteristics of the PDF. In this section we will use this overapproximation to develop conditions to verify stability of the NCS model (11) with transmission intervals and delays (h_k, τ_k), characterised by an IID random process satisfying Assumption II.1.

Stability of the class of quadratic protocols given by (15), of which the TOD protocol is a special case, can be analysed using the ideas in [14], in which only switched linear systems without any form of uncertainty are considered. To analyse the stability of (11) having this switching function, we introduce the non-quadratic Lyapunov function

$$V(\bar{x}_k) = \min_{i=1,\dots,N} \bar{x}_k^\top P_i \bar{x}_k, \tag{33}$$

Furthermore, we introduce the sets

$$\mathcal{M} := \Big\{ \Pi \in \mathbb{R}^{N\times N} \mid \sum_{j=1}^{N} \pi_{ji} = 1 \text{ for } i \in \{1, \dots, N\} \text{ and } \pi_{ji} \geqslant 0 \text{ for } i, j \in \{1, \dots, N\} \Big\} \tag{34}$$

and

$$\mathcal{R} = \{ \mathrm{diag}(r_1 I_1, \dots, r_L I_L, r_{L+1} I_1, \dots, r_{3L} I_L) \in \mathbb{R}^{3(n_p+n_c)\times 3(n_p+n_c)} \mid r_i > 0 \}, \tag{35}$$

where I_i is an identity matrix of size n_i. The main result of this section is presented next.

Theorem IV.1 *Suppose there exist triangles S_m, $m \in \{1, \dots, S\}$ and a convex overapproximation as in* (19) *satisfying* (22)*, for all $\sigma \in \{1, \dots, N\}$, a matrix $\Pi = \{\pi_{ji}\} \in \mathcal{M}$, a positive scalar μ, a positive definite matrices P_i satisfying $P_i \prec \mu I$, matrices $U_{i,m}$, and matrices $R_{i,m,l} \in \mathcal{R}$, $i \in \{1, \dots, N\}$, $m \in \{1, \dots, S\}$, and $l \in \{1, 2, 3\}$, satisfying*

$$\begin{bmatrix} U_{i,m} & 0 & \bar{p}_m \bar{A}^\top_{i,m,l} \sum_{j=1}^N \pi_{ji} P_j & C_i^\top R_{i,m,l} \\ \star & R_{i,m,l} & \bar{p}_m \bar{B}_m^\top \sum_{j=1}^N \pi_{ji} P_j & 0 \\ \star & \star & \bar{p}_m \sum_{j=1}^N \pi_{ji} P_j & 0 \\ \star & \star & \star & R_{i,m,l} \end{bmatrix} \succ 0 \tag{36}$$

for all $i \in \{1, \dots, N\}$, $m \in \{1, \dots, S\}$, $l \in \{1, 2, 3\}$, in which $\bar{p}_m := \iint_{\mathcal{S}_m} p(h,\tau) dh d\tau$, and satisfying

$$P_i - \sum_{m=1}^{S} U_{i,m} - \mu \iint_{\Theta \setminus (\cup_{m=1}^S \mathcal{S}_m)} \|\tilde{A}_{i,h,\tau}\|^2 p(h,\tau) dh d\tau I \succ 0, \tag{37}$$

for all $i \in \{1, \dots, N\}$. Then, the switching law (15) *renders the system* (11) *UGMSES.*

We can now briefly comment on the conditions presented in Theorem IV.1: Firstly, the stability of (11) is guaranteed for h and τ satisfying a continuous PDF, because the PDF is also 'overapproximated' by assigning $\bar{p}_m := \iint_{\mathcal{S}_m} p(h,\tau) dh d\tau$ to each triangle $\mathcal{S}_m$, $m \in \{1, \dots, S\}$. To be more precise, the probability $\bar{p}_m$ is the probability that the pair $(h,\tau) \in \mathcal{S}_m$. Secondly, in case the triangles can be chosen such that $\Theta \setminus (\cup_{m=1}^S \mathcal{S}_m) = \emptyset$, the conditions in (37) simplify as the integral in the left-hand side of (37) vanishes. This is possible, if there exists some $\tilde{h} > 0$, such that $p(h,\tau) = 0$ for all $h > \tilde{h}$. In other cases, condition (37) can be satisfied by finding an upper bound on the integral in (37). Since $\|\tilde{A}_{i,h,\tau}\|^2$ can bounded by $\|\tilde{A}_{i,h,\tau}\|^2 \leqslant c e^{\lambda h}$, for some constant $c > 0$ and a constant λ that depends on the eigenvalues of $\bar{\Lambda}^\top + \bar{\Lambda}^\top$, with $\bar{\Lambda}$ as in (25). The satisfaction of (37) requires the existence of the integral in its left-hand side, which is satisfied when the PDF $p(h,\tau)$ decays exponentially faster than the bound $c e^{\lambda} h$, when the transmission intervals approach infinity. Hence, $p(h,\tau) \leqslant \tilde{c} e^{-\tilde{\lambda} h}$ for some $\tilde{\lambda} > \lambda$ guarantees finiteness of the integral in (37). The fact that the PDF decays exponentially fast also allows us to bound the expected value of the evolution of (1) and (2) in between two subsequent transmissions, i.e., the so-called intersample

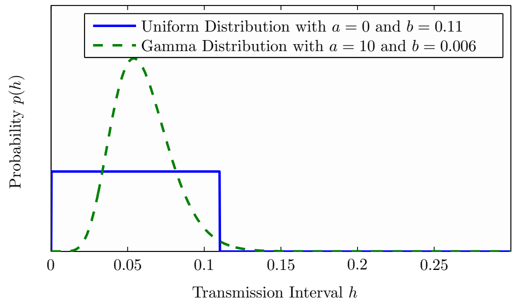

Fig. 2: Illustration of the considered PDFs.

behaviour. As a consequence, UGMSES of the discrete-time NCS model (11) with switching function (15) also implies mean-square exponential stability of the underlying continuous-time NCS given by (1), (2), (3) and (6), with protocol (15).

V. Illustrative Example

In this section, we illustrate the presented theory using a well-known benchmark example in the NCS literature [1]–[4], [11], consisting of a model of a batch reactor. The details of the linearised model of the batch reactor model used in this example and the continuous-time controller can be found in the aforementioned references.

We will analyse the NCS as was done in [1]–[4], [11], where it is assumed that the controller is directly connected to the actuator, i.e., only the two outputs are transmitted via the network. Furthermore, we consider the TOD protocol and assume for simplicity that delays are absent, i.e., $p(h_k, \tau_k) = 0$ for all $\tau_k \neq 0$, $k \in \mathbb{N}$. In this example, we consider two different PDFs, namely a uniform distribution

$$p(h,\tau) = \begin{cases} \frac{1}{b-a} & \text{for } a \leqslant h \leqslant b \text{ and } \tau = 0 \\ 0 & \text{elsewhere} \end{cases} \tag{38}$$

with $a = 10^{-5}$ and $b = 0.11$ and the Gamma distribution

$$p(h,\tau) = \begin{cases} \frac{1}{d^c\Gamma(c)} h^{c-1} e^{-\frac{h}{d}} & \text{for } h > 0 \text{ and } \tau = 0 \\ 0 & \text{elsewhere} \end{cases} \tag{39}$$

with $c = 10$ and $d = 0.006$, in which $\Gamma(c)$ denotes the Gamma function, [15]. The resulting PDFs are shown in Fig. 2.

In order to assess the bounds on the allowable transmission intervals, we first define our NCS model as in Section II-A.2. This model appropriately describes the situation as discussed in this example, where only the plant outputs y are transmitted over the network and the controller outputs u are sent continuously via a nonnetworked connection. Then, we derive the uncertain polytopic system (19), using Procedure III.1. For the uniform distribution, we construct 80 line segments, as discussed in Remark III.2, $\mathcal{S}_m = [(\frac{0.11}{79}(m-1), 0), (\frac{0.11}{79}m, 0)]$, for $m \in \{1, \ldots, 80\}$. For the Gamma distribution, we construct 40 line segments $\mathcal{S}_m = [(\frac{0.3}{39}(m-1), 0), (\frac{0.3}{39}m, 0)]$, $m \in \{1, \ldots, 30\}$. We now check the matrix inequalities of Theorem IV.1, using the structure of the P_i-matrices as in (16). Using this procedure we obtain a feasible solution of LMIs of Theorem IV.1, on the basis of which we conclude that the TOD protocol stabilises the NCS when the transmission intervals are given by an IID random process satisfying the aforementioned PDFs. In [4], we obtained a hard *deterministic* maximum allowable transmission interval of 0.066, which includes all PDFs for which holds that $p(h, \tau) = 0$ for all $h > 0.066$ and all $\tau \neq 0$, and we can therefore conclude that incorporating probabilistic information on the distribution of the transmission intervals can prove stability for situations not covered by earlier results in the literature.

VI. Conclusions

In this paper, we studied the stability of Networked Control Systems (NCSs) that are subject to communication constraints, time-varying transmission intervals and time-varying delays. We analysed the stability of the NCS when the transmission intervals and transmission delays are described by a random process, having a *continuous* probability density function, and the communication sequence is determined by a quadratic protocol. This analysis was based on a stochastically time-varying discrete-time switched linear system of the NCS. We provided conditions for stability (in the mean-square) using a convex overapproximation and a finite number of linear matrix inequalities. On a benchmark example, we illustrated the effectiveness of the developed theory. Although the work presented in this paper considers analysis of NCSs with continuous-time controllers and quadratic protocols only, extensions are possible towards discrete-time controllers and other protocols, such as periodic protocols.

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