-0.3195, $K_2 = 0.6390$, and $K_3 = 0.0435$. From the relationship in (19), we have

$$\begin{bmatrix} K_P \\ K_I \\ K_D \end{bmatrix} = \begin{bmatrix} -1 & -2\rho^2 & 2 \\ \frac{1}{T} & \frac{\rho^2}{T} + \frac{1}{T} & -\frac{1}{T} \\ 0 & \rho^2 T & -T \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0.3099 \\ 0.3243 \\ 0.0048 \end{bmatrix}.$$

Fig. 2 (right) shows the closed-loop poles that lie inside the circle of radius $\rho = 0.275$. The roots are $0.2500 \pm j0.1118$ and $0.2500 \pm j0.0387$.

To illustrate further, we select several sets of stabilizing PID parameters from the set obtained in Example 1 (i.e., $\rho = 1$) and compare the step responses between them. Fig. 3 shows that the maximally deadbeat design produces nearly deadbeat response.

VI. MAXIMUM DELAY TOLERANCE DESIGN

In some control systems an important design parameter is the delay tolerance of the loop, that is the maximum delay that can be inserted into the loop without destabilizing it. In digital control a delay of k sampling instants is represented by z^{-k}. We use this to determine the maximum delay that a control-loop under PID control can be designed to tolerate. This gives the limit of delay tolerance achievable for the given plant under PID control.

Let the plant be $G(z) = (N(z))/(D(z))$. We consider the problem of finding the maximum delay L^* such that the plant can be stabilized by a PID controller. In other words, finding the maximum values of L^* such that the stabilizing PID gain set for the plant

$$z^{-L}G(z) = \frac{N(z)}{z^L D(z)}, \qquad \text{for } L = 0, 1, \ldots, L^* \tag{20}$$

is not empty. Let $\mathcal{S}_i$ be the set of PID gains that stabilizes the plant $z^{-i}G(z)$. Then, it is clear that

$$\cap_{i=0}^{L} \mathcal{S}_i \text{ stabilizes } z^i G(z) \text{ for all } i = 0, 1, \ldots, L. \tag{21}$$

VII. CONCLUDING REMARKS

In this note, we have given a solution to the problem of stabilization of a digital control system using PID controllers. The solution is complete in the sense that a constructive yes or no answer to whether stabilization is possible, is given and in case it is possible the entire set is determined by solving sets of linear inequalies in two variables obtained by gridding over the third variable. This approach is akin to the geometric approach to synthesis and design advocated in [16]. These solution sets open up the possibility of improved and optimal design using PID controllers. The questions of loop shaping, time domain response shaping, and robust designs are important candidates for research.

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Stability Analysis of Swarms

Veysel Gazi and Kevin M. Passino

Abstract—In this note, we specify an "individual-based" continuous-time model for swarm aggregation in n-dimensional space and study its stability properties. We show that the individuals (autonomous agents or biological creatures) will form a cohesive swarm in a finite time. Moreover, we obtain an explicit bound on the swarm size, which depends only on the parameters of the swarm model.

Index Terms—Biological systems, multiagent systems, stability analysis, swarms.

I. INTRODUCTION

For a long time, it has been observed that certain living beings tend to perform swarming behavior. Examples of swarms include flocks of birds, schools of fish, herds of animals, and colonies of bacteria. It is known that such a cooperative behavior has certain advantages such

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as avoiding predators and increasing the chance of finding food but it requires communications and coordinated decision making. Operational principles from such systems can be used in engineering for developing distributed cooperative control, coordination, and learning strategies for autonomous agent systems such as autonomous multi-robot applications, unmanned undersea, land, or air vehicles. There are, however, several key steps to exploit biological principles to develop such highly automated systems. These include modeling, coordination strategy specification, and analysis to show that group dynamics achieve group goals. In this article we develop a simple model describing swarm aggregation and analyze its stability properties. We show that the individuals will form a cohesive swarm in a finite time. Moreover, we obtain a bound on the swarm size, which depends only on the parameters of the swarm model.

Biologists have been working on understanding and modeling of swarming behavior for a long time [1]–[4]. The general understanding now is that the swarming behavior is a result of an interplay between a long range attraction and a short range repulsion between the individuals. In [1], Breder suggested a simple model composed of a constant attraction term and a repulsion term which is inversely proportional to the square of the distance between two members, whereas in [2] Warburton and Lazarus studied the affect on cohesion of a family of attraction/repulsion functions. The articles in [3] and [4] provide good background and review of the swarm modeling concepts and literature such as spatial and nonspatial models, individual-based versus continuum models and so on. (See also [5] and references therein for other related work).

Parallel to the mathematical biologists, there are a number of physicists who have done important work on swarming behavior [6]–[11]. The general approach the physicists take is to model each individual as a particle and study the collective behavior due to their interaction. Many of them assume that particles are moving with constant absolute velocity and at each time step assume the average direction of motion of the particles in its neighborhood with some random perturbation. They try to study the affect of the noise on the collective behavior and to validate their models through extensive simulations.

In recent years, engineering applications such as formation control of multirobot teams and autonomous air vehicles have emerged and this has increased the interest of engineers in swarms. For example, in [12], the authors describe formation control strategies for autonomous air vehicles, whereas [13]–[17] describe different approaches for formation control of multiagent (multirobot) teams. In [13], a behavior based formation control of multiple land robots integrated with the other navigational goals of the robots is described, whereas the article in [14] proposes a method that uses only local information. They use the feedback linearization technique for controller design to exponentially stabilize the relative distances of the robots in the formation. Similar results are obtained also in [15] and [16], where the authors use formation constrains and control Lyapunov functions to develop the formation control strategy and prove stability of the formation (i.e., formation maintenance). The results in [17], on the other hand, are based on using virtual leaders and artificial potentials for robot interactions in a group of agents for maintenance of the group geometry. They use the system kinetic energy and the artificial potential energy as a Lyapunov function to prove closed loop stability and employ a dissipative term to achieve asymptotic stability of the formation. Reference [18] describes a systematic framework for studying feasibility of formations for both undirected and directed type formations. In [19], Reif and Wang consider distributed control approach of groups of robots, called *social potential fields* method, which is based on artificial force laws between individual robots and robot groups. The force laws are inverse-power or spring force laws incorporating both attraction and repulsion. Another work on distributed formation control of robots is [20], where the au-

thors consider asynchronous distributed control and geometric pattern formation of multipleanonymous (or identical) robots.

Important work on swarm stability is given by Beni *et al.* in [21] and [22]. In [21], they consider a synchronous distributed control method for discrete one and two dimensional swarm structures and prove stability in the presence of disturbances using Lyapunov methods. On the other hand, [22] is, to best of our knowledge, one of the first stability results for asynchronous methods (with no time delays). There, they consider a *linear* swarm model and prove sufficient conditions for the asynchronous convergence of the swarm to a synchronously achievable configuration.

Swarm stability under *total asynchronism* (i.e., asynchronism with time delays) was first considered in [23] and [24]. In [23] a one dimensional discrete time totally asynchronous swam model is proposed and stability (swarm cohesion) is proved. The authors prove asymptotic convergence under total asynchronism conditions and finite time convergence under *partial asynchronism* conditions (i.e., total asynchronism with a bound on the maximum possible time delay). In [24], on the other hand, the authors consider a mobile swarm model and prove that cohesion will be preserved during motion under certain conditions, expressed as bounds on the maximum possible time delay.

In [25], we obtained similar results to those in [23] for a swarm with a different mathematical model for the intermember interactions and motions using some earlier results developed for parallel and distributed computation in computer networks in [26].

All of these stability investigations have been limited to either one or two-dimensional space. Note that in one dimension, the problem of swarming is very similar to the problem of *platooning* of vehicles in *automated highway systems*, an area that has been studied extensively (see, for example, [27]–[29] and the references therein).

Recent work in [30] is focusing on extending the work in [23] and [24] to the multidimensional case by imposing special constraints on the topology of the "leader" movements and using specific communication topology.

The results in this note were first published in [31]. In [32], we describe a class of attraction/repulsion functions that can be used for swarm aggregations. In [33], on the other hand, we analyzed the stability of swarms moving in a profile of nutrients or toxic substances and showed collective convergence of all the individuals to more favorable regions of the nutrient profile.

Finally, note that it is possible to establish a connection between analysis of swarms and analysis of multibody systems such as groups of planets.

II. MODEL OF AN AGGREGATING SWARM

Consider a swarm of M individuals (members) in an n-dimensional Euclidean space. We model the individuals as points and ignore their dimensions. The position of member i of the swarm is described by $x^i \in \mathbb{R}^n$. We assume synchronous motion and no time delays, i.e., all the members move simultaneously and know the exact position of all the other members. The motion dynamics evolve in continuous time. The equation of motion that we consider for individual i is given by

$$\dot{x}^i = \sum_{j=1, j \neq i}^{M} g(x^i - x^j), i = 1, \ldots, M \tag{1}$$

where $g(\cdot)$ represents the function of attraction and repulsion between the members. In other words, the direction and magnitude of motion of each member is determined as a sum of the attraction and repulsion of all the other members on this member. The attraction/repulsion function that we consider is

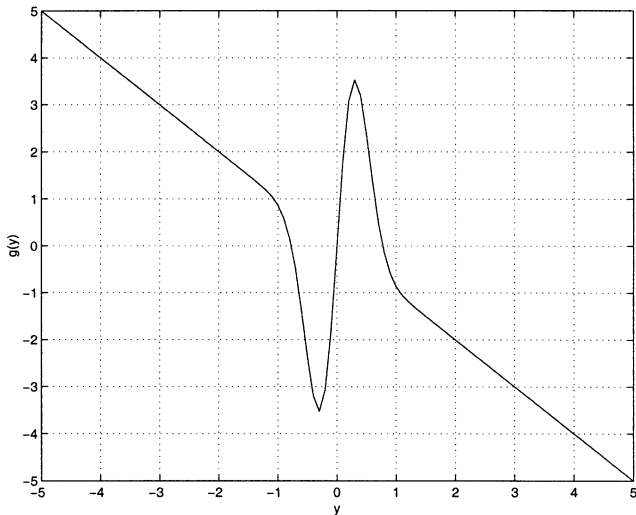$$g(y) = -y \left(a - b \exp \left(-\frac{\|y\|^2}{c} \right) \right) \tag{2}$$

Fig. 1.   Attraction/repulsion function $g(\cdot)$.

where a, b, and c are positive constants such that $b > a$ and $\|y\| = \sqrt{y^\top y}$. For the $y \in \mathbb{R}^1$ case with $a = 1$, $b = 20$, and $c = 0.2$ this function is shown in Fig. 1. In higher dimensions (i.e., $y \in \mathbb{R}^n$), the function is exactly the same as in one-dimensional case, except that it acts on the line connecting the positions of the two members (i.e., the line on which the vector y lies).

Note that the function $g(\cdot)$ constitutes an *artificial social potential* function, similar to the ones in [19] and [2], that governs the interindividual interactions. The parameter a represents the attraction, whereas the term $b \exp(-\|y\|^2/c)$ represents the repulsion. Note that this function is attractive (i.e., a dominates) for large distances and repulsive (i.e., $b \exp(-(\|y\|^2/c))$) dominates) for small distances, which is consistent with interindividual attraction/repulsion in biological swarms. Therefore, it constitutes a crude approximation of biological interactions and also allows us to perform stability analysis. The main drawback with $g(\cdot)$ is that it is not unbounded for infinitesimally small arguments (which may be needed to avoid collisions) and that it has an infinite range (which is inconsistent with biology since no creature has infinite sensing range). However, note that this article is the first step toward stability analysis of swarms and these issues are topic of further research. In fact, we consider those issues in [32], where we describe a class of attraction/repulsion functions that lead to aggregation.

By equating $y(a - b \exp(-(\|y\|^2/c))) = 0$, one can easily find that $g(y)$ switches sign at the set of points defined as $\mathcal{Y} = \{y = 0 \text{ or } \|y\| = \delta = \sqrt{c \ln(b/a)}\}$. The distance δ is the distance at which the attraction and repulsion balance. It is known that there exists such a distance in biological swarms [2], [4].

In this note, *cohesiveness* of the swarm is the main *stability property* that we are concerned with. We define the *size of the swarm* as the radius of the hyperball within which the individuals converge.

Define the *center* of the swarm members as $\bar{x} = (1/M) \sum_{i=1}^{M} x^i$. Note that because of the symmetry of $g(\cdot)$ the center $\bar{x}$ is stationary for all t. In other words, since $g(\cdot)$ is symmetric with respect to the origin, member i moves toward every other member j exactly the same amount as j moves toward i. We express this more formally in the following lemma.

Lemma 1: The center $\bar{x}$ of the swarm described by the model in (1) with an attraction/repulsion function $g(\cdot)$ as given in (2) is stationary for all t.

Proof: Let $g_1(x^i - x^j) = a - b \exp(-(\|x^i - x^j\|^2/c))$ and note that

$$\dot{\bar{x}} = -\frac{1}{M} \sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M} (x^i - x^j) g_1(x^i - x^j) = 0. \quad \blacksquare$$

Basically, this lemma says that, on average, the swarm described by (1) with an attraction/repulsion function as given in (2) is not drifting. Note, however, that although it states that the center of the swarm is stationary, it does not say anything about the relative motions of the members with respect to it. It may be the case that the members diverge from the center while it stays stationary. Intuitively, however, we would expect the members to move toward the center for the given swarm model. In several of the results and discussions to follow we either implicitly or explicitly will use the fact that $\bar{x}$ is stationary.

III. ANALYSIS OF SWARM COHESION

Our first result is about a swarm member which does not have any neighbors in its repulsion range. We call such a member a *free agent*.

Definition 1: A swarm member i is called a *free agent* at time t if

$$\left\| x^i(t) - x^j(t) \right\| > \delta, \forall j \in S, j \neq i$$

where $S = \{1, \ldots, M\}$ is the set of members of the swarm.

Note that since the distance from all the other members to a free agent is greater than δ, there will not be any repulsion force and the total force on this member will be a combined effect of all the attraction imposed by all the other members. We will show that this force is pointing toward the center $\bar{x}$ of the swarm and, therefore, the member is moving toward it. Before stating this result more rigorously, we define $e^i = x^i - \bar{x}$, for each individual $i = 1, \ldots, M$.

Lemma 2: Assume that a member i of the swarm described by the model in (1) with an attraction/repulsion function $g(\cdot)$ as given in (2) is a free agent at time t and that its distance to the center $\bar{x}$ of the swarm is greater then δ, i.e.,

$$\left\| e^i(t) \right\| = \left\| x^i(t) - \bar{x} \right\| > \delta.$$

Then, at time t, its motion is in a direction of decrease of $\|e^i(t)\|$ (i.e., toward the center $\bar{x}$).

Proof: From the definition of the center $\bar{x}$ of the swarm, we have $\sum_{j=1}^{M} x^j = M\bar{x}$. Subtracting from both sides Mx^i we obtain

$$\sum_{j=1}^{M} (x^i - x^j) = M(x^i - \bar{x}) = Me^i. \quad (3)$$

Then, the motion of member i can be represented as

$$\dot{x}^i = -\sum_{j=1, j \neq i}^{M} (x^i - x^j) \left[a + b \exp\left(-\frac{\|x^i - x^j\|^2}{c} \right) \right]$$
$$= -aMe^i + b \sum_{j=1, j \neq i}^{M} \exp\left(-\frac{\|x^i - x^j\|^2}{c} \right)(x^i - x^j)$$

where on the first line we used the definition of function $g(\cdot)$ in (2) and added $a(x^i - x^i) = 0$, and substituted the value of $\sum_{j=1}^{M}(x^i - x^j)$ from (3) on the second.

Choosing the Lyapunov function candidate for member i as $V_i = (1/2)e^{i\top}e^i$ and taking its derivative, we can show that $\dot{V}_i$ is bounded by

$$\dot{V}_i \leq -aM\|e^i\|^2 + \sum_{j=1, j \neq i}^{M}$$
$$b \exp\left(-\frac{\|x^i - x^j\|^2}{c} \right)\|x^i - x^j\|\|e^i\|. \quad (4)$$

Since member i is a free agent at time t (note that we dropped the time index t throughout the proof for convenience), we have $\|x^i - $

$x^j\| > \delta, \forall j \neq i$ and note that for that range the function $\exp(-(\|x^i - x^j\|^2/c))\|x^i - x^j\|$ is a decreasing function of the distance with the maximum occurring at $\|x^i - x^j\| = \delta$. Using these facts, we have

$$\dot{V}_i \leq -a\|e^i\|^2 - (M-1)\left[a\|e^i\| - b\delta\exp\left(-\frac{\delta^2}{c}\right)\right]\|e^i\|. \quad (5)$$

For the second term to be negative semidefinite, we need $\|e^i\| \geq (b\delta/a)\exp(-(\delta^2/c))$. Note, however, that $(b/a)\exp(-(\delta^2/c)) = 1$, which implies that we need $\|e^i\| \geq \delta$, which, on the other hand, holds by our hypothesis. Therefore, we have

$$\dot{V}_i \leq -a\|e^i\|^2 = -2aV_i$$

which proves the assertion. ∎

Remark: From the attraction/repulsion function $g(\cdot)$ in (2) one can see that one term in $g(\cdot)$ always gives attraction and the other repulsion and the resultant force is their sum. This leads to similar terms in the derivative of the Lyapunov function in (4). If an individual is away from all the other individuals, the second term in the Lyapunov function is negligibly small compared to the first term and it moves toward the center. If it is close to the other individuals (i.e., in their repulsion range), then the second term becomes significant. ∎

Note that Lemma 2 does not imply that x^i will converge to $\bar{x}$ for all i. Intuitively, once a member gets to the vicinity of another member, then the repulsive force will be in effect and the conditions of Lemma 2 will not be satisfied anymore. However, it is important because it gives us an idea of the tendency of the individuals to move toward the center of the swarm. Therefore, it is normal to expect that the members will (potentially) aggregate and form a cluster around $\bar{x}$. To prove this, we need to analyze the motion of the members which are not necessarily free agents and that is done in the next result.

Theorem 1: Consider the swarm described by the model in (1) with an attraction/repulsion function $g(\cdot)$ as given in (2). As time progresses, all the members of the swarm will converge to a hyperball

$$B_\varepsilon(\bar{x}) = \{x : \|x - \bar{x}\| \leq \varepsilon\}$$

where

$$\varepsilon = \frac{b}{a}\sqrt{\frac{c}{2}}\exp\left(-\frac{1}{2}\right).$$

Moreover, the convergence will occur in finite time bounded by

$$\bar{t} = \max_{i \in S}\left\{-\frac{1}{2a}\ln\left(\frac{\varepsilon^2}{2V_i(0)}\right)\right\}.$$

Proof: Choose any swarm member i. Let $V_i = (1/2)e^{i\top}e^i$ be the corresponding Lyapunov function for which we have (see the proof of Lemma 2)

$$\dot{V}_i \leq -aM\|e^i\|^2 + \sum_{j=1,j\neq i}^{M} b\exp\left(-\frac{\|x^i - x^j\|^2}{c}\right)\|x^i - x^j\|\|e^i\|. \quad (6)$$

Now, note that each of the functions $\exp(-(\|x^i - x^j\|^2/c))\|x^i - x^j\|$ is a bounded function whose maximum occurs at $\|x^i - x^j\| = \sqrt{c/2}$ and is given by $\sqrt{c/2}\exp(-(1/2))$. Substituting this in the above equation we obtain that $\dot{V}_i < 0$ as long as

$$\|e^i\| > \frac{b(M-1)}{aM}\sqrt{\frac{c}{2}}\exp\left(-\frac{1}{2}\right).$$

Define $\varepsilon = (b/a)\sqrt{c/2}\exp(-(1/2))$ and note that $\varepsilon > (b(M-1)/aM)\sqrt{c/2}\exp(-(1/2))$. This implies that as $t \to \infty$, e^i converges within the ball around $\bar{x}$ defined by $(b(M-1)/aM)\sqrt{c/2}\exp(-(1/2))$. Since $\varepsilon > (b(M-1)/aM)\sqrt{c/2}\exp(-(1/2))$ we have $e^i \to B_\varepsilon$.

Since member i was an arbitrary member, the result holds for all the members. To prove the finite-time convergence, note that for $\|e^i\| \geq \varepsilon$, we have

$$\dot{V}_i \leq -a\|e^i\|^2 = -2aV_i.$$

Therefore, the solution of V_i satisfies

$$V_i(t) \leq V_i(0)e^{-2at}$$

for which it can be shown that crosses the $\|e^i\| = \varepsilon$ boundary in a time bounded by

$$t_i \leq -\frac{1}{2a}\ln\left(\frac{\varepsilon^2}{2V_i(0)}\right)$$

and this proves the theorem. ∎

This result is important not only because it proves the cohesiveness of the swarm, but also it provides an explicit bound on the size of the swarm. Note that the bound ε makes intuitive sense. To see this note that increasing parameter a (i.e., increasing attraction) decreases the size of the bound ε. In contrast, increasing parameter b (i.e., increasing repulsion magnitude) or parameter c (increasing repulsion range) increases ε and these are intuitively expected results. For the $g(\cdot)$ function given in Fig. 1 with parameters $a = 1$, $b = 20$, and $c = 0.2$, we have $\varepsilon \approx 3.8$.

Remark: Note that the bound on the swarm size $(b(M-1)/(aM))\sqrt{c/2}\exp(-(1/2))$ depends on M. Therefore, for swarms with a small number of members the bound will differ significantly for different values of M. However, in biological swarms the number of the members M can be very large and as $M \to \infty$ we have $(b(M-1)/(aM))\sqrt{c/2}\exp(-(1/2)) \to \varepsilon$. In other words, ε is the maximum possible bound on the swarm size independent of the number of the individuals in the swarm. ∎

Remark: In view of the aforementioned remark, for large values of M the size of the cohesive swarm is relatively independent of the number of the members (individuals). In other words, it is almost constant independent of the number of the members. This implies that as the number of the members increases the density of the swarm will also increase. This is inconsistent with some biological examples and is due to the particular attraction/repulsion function $g(\cdot)$ that we chose. ∎

Remark: Note that even the bound $(b(M-1)/(aM))\sqrt{c/2}\exp((-1/2))$ is very conservative, because in the aforementioned proof, we used $(x^i - x^j)^\top e^i \leq \|x^i - x^j\|\|e^i\|$ and also assumed that the functions $\exp(-(\|x^i - x^j\|^2)/c)\|x^i - x^j\|$ are at their peak values for all i and j and these both are *never* the case. Therefore, the actual size of the swarm is, in general, much smaller than ε. ∎

Note also that even though the results here were developed for the attraction repulsion fungtion $g(\cdot)$ in (2), they can be extended and generalized to a *class* of attraction repulsion functions as was done in [32].

IV. ANALYSIS OF SWARM MEMBER BEHAVIOR IN A COHESIVE SWARM

Theorem 1 shows only the region where the swarm members will converge and provides a bound on the size of the swarm. It does not, however, say anything about whether the swarm members will stop their motion or will start an oscillatory motion within the region and this issue needs to be investigated further. To this end, we first define the state x of the system as the vector of the positions of the swarm members $x = [x^{1\top}, \ldots, x^{M\top}]^\top$. Let the invariant set of equilibrium points be

$$\Omega_e = \{x : \dot{x} = 0\}.$$

We will prove that as $t \to \infty$ the state $x(t)$ converges to Ω_e, i.e., the configuration of the swarm members converges to a constant arrangement.

Theorem 2: Consider the swarm described by the model in (1) with an attraction/repulsion function $g(\cdot)$ as given in (2). As $t \to \infty$ we have $x(t) \to \Omega_e$.

Proof: We choose the Lyapunov function

$$J(x) = \frac{1}{2} \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \left[a \| x^i - x^j \|^2 + bc \exp\left(-\frac{\|x^i - x^j\|^2}{c}\right) \right]$$

which is an artificial potential function. Then, one can show that the gradient of $J(x)$ with respect to each x^i is given by $\nabla_{x^i} J(x) = -\dot{x}^i$. Now, taking the time derivative of the Lyapunov function along the motion of the system we obtain

$$\dot{J}(x) = [\nabla_x J(x)]^\top \dot{x} = \sum_{i=1}^{M} [\nabla_{x^i} J(x)]^\top \dot{x}^i$$
$$= \sum_{i=1}^{M} \left[-\dot{x}^i \right]^\top \dot{x}^i = -\sum_{i=1}^{M} \left\| \dot{x}^i \right\|^2 \leq 0$$

for all t. Then, using the LaSalle's Invariance Principle we conclude that as $t \to \infty$ the state x converges to the largest invariant subset of the set defined as

$$\Omega = \left\{ x : \dot{J}(x) = 0 \right\} = \{ x : \dot{x} = 0 \} = \Omega_e.$$

Since each point in Ω_e is an equilibrium, Ω_e is an invariant set and this proves the result. ∎

Remark: The proof of the aforementioned theorem shows the distributed aspect of the swarming behavior. In fact, it shows that the swarm members are performing *distributed optimization* (function minimization) of a common function (the Lyapunov or cost function) using a *distributed gradient method*. In other words, each member computes its part of the gradient of the global function at its position (i.e., computes the gradient with respect to its motion variables) and moves along the negative direction of that gradient. The global function in this case is a function of the distances between the members. ∎

Remark: Another view on the distributed nature of the approach can be as follows. Define

$$J_i(x) = \frac{1}{2} \sum_{j=1, j \neq i}^{M} \left[a \left\| x^i - x^j \right\|^2 + bc \exp\left(\frac{\|x^i - x^j\|^2}{c}\right) \right].$$

Then, note that

$$\dot{x}^i = -\nabla_{x^i} J_i(x) = -\nabla_{x^i} J(x).$$

This can be interpreted as each member i performing an optimization of its local cost function $J_i(x)$, which results in minimizing of the combined cost function

$$J(x) = \frac{1}{2} \sum_{i=1}^{M} J_i(x)$$

to obtain the overall behavior of the swarm. ∎

Remark: Note that in any of the above analysis we did not use the dimension of the state space n. Therefore, the results obtained hold for any dimension n. ∎

Remark: The results here are global. This is a consequence of the definition of the attraction/repulsion function $g(\cdot)$ in (2) over the entire domain. ∎

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Delay-Dependent/Delay-Independent Stability of Linear Systems With Multiple Time-Varying Delays

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Abstract—Delay-dependent/delay-independent uniform asymptotic stability and uniform stability criteria for linear systems with multiple time-varying delays are established respectively in this note. The results are derived based on a new-type stability theorem for retarded dynamical systems and a new analysis technique for estimating the derivative of a Lyapunov function along the solution of a system at certain specific instants. Four remarks together with an illustrative example are given to compare the obtained results with and to show their superiority to those in the literature.

Index Terms—Linear systems, Lyapunov methods, stability, time delay.

I. INTRODUCTION

Recently, new-type stability theorems for retarded dynamical systems have been established by Xu [1]. A new analysis technique has also been proposed in [1] for estimating the derivative of a Lyapunov function along the solution of a system at certain specific instants. In this note, the delay-dependent/delay-independent stability for the following linear system with multiple time-varying delays:

$$
\begin{cases}
\dot{x}(t) = A_0 x(t) + \displaystyle\sum_{k \in J_1} A_k x\left(t - \tau_k(t)\right) \\
\qquad + \displaystyle\sum_{k \in J_2} A_k x\left(t - \tau_k(t)\right) \\
\qquad t \geq t_0 \in R; \\
x_{t_0}(\theta) = x(t_0 + \theta) = \phi(\theta) \quad t_0 + \theta = E_{t_0} \\
E_{t_0} = \displaystyle\bigcup_{k=1}^{m} \left\{ t - \tau_k(t) \,|\, t - \tau_k(t) \leq t_0, \right. \\
\qquad \left. t \geq t_0 \right\} \cup \{ t_0 \}
\end{cases}
\tag{1}
$$

is studied based on the new-type stability theorem for retarded dynamical systems presented in the Appendix [1], where $\phi \in C_n$, $x \in R^n$, $A_k \in R^{n \times n}$ for $k = 0, 1, \ldots, m$ are constant matrices, $\tau_k(t) \leq$

$\tau_{kM} \leq \tau$ for $k \in J_1$ and $\tau_k(t) \leq \tau < \infty$ for $k \in J_2$ with known constant $\tau_{kM} > 0$ and unknown constant $\tau < \infty$ are the time-varying delays, J_1 and J_2 denote two sets satisfying $J_1 \cup J_2 = \{1, 2, \ldots, m\}$, $m_1 = c(J_1)$ and $m_2 = c(J_2)$ denote the cardinalities of the sets J_1 and J_2, respectively. After some preliminaries in Section II, the delay-dependent/delay-independent uniform asymptotic stability and uniform stability criteria for (1) are derived, respectively, in Section III based on Xu's stability theorem and technique in the Appendix [1]. Four remarks together with one illustrative example are given to compare the obtained results with and to show their superiority to those in the literature. The conclusion is given in Section IV.

Notation: $R = (-\infty, \infty)$; $R_+ = [0, \infty)$; R^n denotes the real vector space of dimension n; $R^{n \times n}$ denotes the real matrix space of dimension $n \times n$; C_n denotes the Banach space of continuous functions mapping $[-\tau, 0]$ into R^n, where $\tau > 0$; A^T denotes the transpose of A; $A > 0$ (or <0) denotes a positive–definite (or negative–definite) matrix; $A \leq B$ means that $A - B \leq 0$ is negative semidefinite; $y_t(\theta) \in R^n$ denotes $y(t + \theta) \in R^n$ for $t \in R$ and $\theta \in R$ so that $y(t) = y_t(0)$; $\| \cdot \|$ denotes the Euclidean norm in R^n; $\|\phi\|_r = \sup_{-\tau \leq \theta \leq 0} \|\phi(\theta)\|$ with $\phi(\theta) \in R^n$ for given $\phi \in C_n$; and $|z|$ denotes the absolute value of $z \in R$; and finally, I_n is the $n \times n$ identity matrix.

II. PRELIMINARIES

The following preliminaries are needed in the next section.

Lemma 1: Let $P > 0 \in R^{n \times n}$ and $D \in R^{n \times n}$ be constant matrices, $x, y \in R^n$ and $\rho > 0$. Then

$$
2 x^T P D y \leq \frac{1}{\rho} x^T P D P^{-1} D^T P x + \rho y^T P y
\tag{2}
$$

and the equality holds if and only if $D^T P x = \rho P y$.

Proof: Note that $(D^T P x - \rho P y)^T P^{-1} (D^T P x - \rho P y) \geq 0$ for any $\rho > 0$ and any $x, y \in R^n$ and $(D^T P x - \rho P y)^T P^{-1} (D^T P x - \rho P y) = 0$ if and only if $D^T P x - \rho P y$. We obtain the result immediately. Q.E.D.

Lemma 2: Let $P > 0 \in R^{n \times n}$ and $D \in R^{n \times n}$ be constant matrices, and let $X(K) = \{x \in R^n | x^T P x = K\}$ with a $K > 0$. Then, for any given $\widetilde{x} \in X(K)$ satisfying $\widetilde{x}^T P D P^{-1} D^T P \widetilde{x} > 0$, there exists a positive $\rho_M > 0$ such that

$$
\max_{y^T P y = K} \{2 \widetilde{x}^T P D y\} \leq \frac{1}{\rho_M} \widetilde{x}^T P D P^{-1} D^T P \widetilde{x} + \rho_M \widetilde{x}^T P \widetilde{x}
\tag{3}
$$

and

$$
\frac{1}{\rho_M} \widetilde{x}^T P D P^{-1} D^T P \widetilde{x} + \rho_M \widetilde{x}^T P \widetilde{x} \\
\leq \frac{1}{\rho} \widetilde{x}^T P D P^{-1} D^T P \widetilde{x} + \rho \widetilde{x}^T P \widetilde{x}
\tag{4}
$$

for any $\rho > 0$.

Proof: Note that $\widetilde{x}^T P D P^{-1} D^T P \widetilde{x} > 0$ and $\widetilde{x} \in X(K)$ so that there must exist a positive such that $\rho_M > 0$ such that $\rho_M^2 K = \widetilde{x}^T P D P^{-1} D^T P \widetilde{x}$ holds. Then, (3) is an immediate result of Lemma 1. Furthermore, from the right-hand side of (4), let

$$
f(\rho) = \frac{\rho_M^2}{\rho} + \rho.
\tag{5}
$$

Then, note that $(df(\rho))/(d\rho) = -(\rho_M^2/\rho^2) + 1$ and $(d^2 f(\rho))/(d\rho^2) = (2\rho_M^2/\rho^3) > 0$ for any $\rho > 0$. Therefore, we have

$$
2 \rho_M \leq f(\rho) = \frac{\rho_M^2}{\rho} + \rho
\tag{6}
$$