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STABILITY ANALYSIS OF SYMMETRICAL ROTOR-BEARING SYSTEMS WITH INTERNAL DAMPING USING FINITE ELEMENT METHOD

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ABSTRACT

This paper deals with the stability analysis of self-excited bending vibrations of linear symmetrical rotor-bearing systems caused by internal damping using the finite element method. The rotor system consists of uniform circular Rayleigh shafts with internal viscous damping, symmetrical rigid disks, and discrete undamped isotropic bearings. By combining the sensitivity method and the matrix representation of the rotor dynamic equations in complex form to assess stability, it is proved theoretically that the stability threshold speed and the corresponding whirling speed coincide with the first forward critical speed regardless of the magnitude of the internal damping.

INTRODUCTION

It is well known that the stability of rotors is influenced by the internal damping. The early works of Kimball (1924) and Newkirk (1924) showed that internal damping destabilizes the whirling motion if the rotational speed of the rotor exceeds the first critical speed. The first analytical stability analysis of rotors was done by Smith (1933), who studied the motion of massless shafts carrying one or more rigid disks, mounted in isotropic bearings with both "stationary and rotary" damping present. It was found that *"when rotary damping but no stationary damping is present, the motion is unstable above the lowest critical speed of positive precession if the bearings are symmetrical, ..."*

The stability problems of rotors with both internal and external damping have been discussed by several authors (Dimentberg, 1961; Ehrich, 1964; Tondl, 1965 and others). In most of the works by the investigators listed above, however, the gyroscopic effects are neglected.

Of the many researchers studying the stability problems of rotors using finite elements, Zorzi and Nelson (1977) carried out first the numerical stability analysis of such rotor systems including the effects of rotatory inertia, gyroscopic moments, and both internal viscous and hysteretic damping. By using the numerical examples for a uniform circular shaft with *viscous damping coefficient*, supported at its ends by two indentical undamped isotropic bearings, they have found that the first and second forward precessional modes become unstable at the first and second critical speeds, respectively.

The purpose of this paper is to show that the numerical results above are also valid for a more general rotor system (with viscous internal damping, isotropic undamped bearings) using the finite element method. To this end, the system equations of the rotor are written in complex form using a note by Nelson (1985). Then, by applying the sensitivity method for the associated eigenvalue problem, it is proved theoretically that *the stability threshold speed of the rotor always coincides with the first forward critical speed regardless of the magnitude of the internal viscous damping coefficient.*

EQUATIONS OF MOTION IN COMPLEX FORM

In this section the equations of motion for a rigid disk, finite shaft element with internal viscous damping, undamped isotropic bearing, and the complete rotor system are solely written in complex form by making use of a note by Nelson (1985) and the paper by Zorzi and Nelson (1977). Note that the equation of motion for the shaft element presented in complex form by Nelson (1985) does not contain internal damping, whereas in the paper by Zorzi and Nelson (1977), the effects of both viscous and hysteretic internal damping are included into the finite element model.

Consider a symmetric rotor system as shown in Fig 1. The

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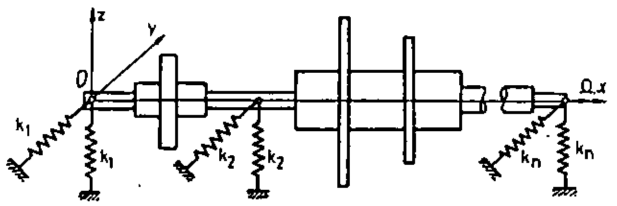

Fig.1. Rotor system in isotropic bearings

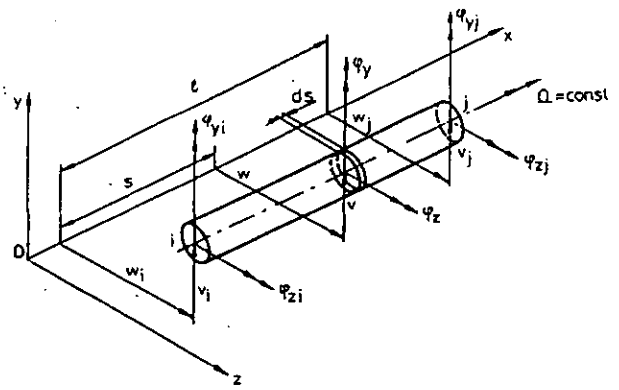

Fig.2. Typical shaft element

rotor system consists of symmetrical rigid disks with negligible thicknesses, uniform circular Rayleigh shafts with viscous internal damping, and n isotropic undamped bearings with stiffnesses k_i $(i=1,2,...,n)$. The rotor is balanced and rotates at a constant speed Ω. The reference system $Oxyz$ is fixed in space. The external damping, axial load and the gravity are neglected.

The degrees of freedom of any node i of the rotor system are illustrated using a typical finite shaft element of length ℓ as shown in Fig.2. Node i has four degrees of freedom: two translations (v_i, w_i) in the (y,z) directions, and two rotations $(\varphi_{yi}, \varphi_{zi})$ about the (y,z) axes, respectively. Fig.2 also shows the two translations (v,w) and rotations (φ_y, φ_z) of any cross section of the finite shaft element located at a distance s from the left node i.

The complex displacement vector $\mathbf{p}_i$ at node i of the shaft element is defined by complex coordinates as

$$\mathbf{p}_i = \begin{bmatrix} r_i \\ \varphi_i \end{bmatrix} = \begin{bmatrix} v_i & +iw_i \\ \varphi_{yi} & +i\varphi_{zi} \end{bmatrix}. \qquad (1)$$

For later use, we also write the complex deflection $r = v+iw$ and the complex rotation $\varphi = \varphi_y + i\varphi_z$ of any cross section of the shaft element as

$$r = \mathbf{h}^T\mathbf{p}^e, \qquad \varphi = i\mathbf{h}'^T\,\mathbf{p}^e \qquad (2\text{-}3)$$

where $(\mathbf{p}^e)^T = [\mathbf{p}_i^T, \mathbf{p}_j^T]$ is the (4x1) complex nodal displacement vector of the shaft element, the prime denotes differentiation with respect to s, and

$$\mathbf{h} = [\psi_1\ -i\psi_2\ \psi_3\ -i\psi_4]^T. \qquad (4)$$

Here ψ_1, ψ_2, ψ_3 and ψ_4 are the translational shape functions defined by Zorzi and Nelson (1977).

Rigid Disk. The equation of motion for a rigid disk in complex form is given by

$$(\mathbf{M}_t^d + \mathbf{M}_r^d)\ddot{\mathbf{p}}^d - \Omega\,\mathbf{G}^d\dot{\mathbf{p}}^d = \mathbf{F}^d, \qquad (5)$$

where $\mathbf{p}^d$ is the displacement vector, corresponding to four degrees of freedom $(v^d, w^d, \varphi_y^d, \varphi_z^d)$ of the node at which the disk is attached. The definition of the translational and rotational mass matrices $(\mathbf{M}_t^d, \mathbf{M}_r^d)$ and the gyroscopic matrix $\mathbf{G}^d$ are given in the note by Nelson (1985). Here we also write the translational and rotational kinetic energies (T_t^d, T_r^d) for the disk in terms of $\mathbf{p}^d$ and its conjugate $\bar{\mathbf{p}}^d$:

$$T_t^d = \frac{1}{2}m^d\left[(\dot{v}^d)^2 + (\dot{w}^d)^2\right] = \frac{1}{2}(\dot{\bar{\mathbf{p}}}^d)^T\mathbf{M}_t^d\dot{\mathbf{p}}^d, \qquad (6)$$

$$T_r^d = \frac{1}{2}I_D\left[(\dot{\varphi}_y^d)^2 + (\dot{\varphi}_z^d)^2\right] = \frac{1}{2}(\dot{\bar{\mathbf{p}}}^d)^T\mathbf{M}_r^d\dot{\mathbf{p}}^d, \qquad (7)$$

where m^d and I_D are the mass and diametral mass moment of inertia of the disk, respectively.

Finite Shaft Element. The equation of motion for the finite rotating shaft element with internal viscous damping in complex form is given by

$$(\mathbf{M}_t^e + \mathbf{M}_r^e)\ddot{\mathbf{p}}^e + (\eta\mathbf{K}_b^e - \Omega\mathbf{G}^e)\dot{\mathbf{p}}^e + (\mathbf{K}_b^e + \eta\Omega\mathbf{K}_c^e)\mathbf{p}^e = \mathbf{F}^e. \quad (8)$$

Here η is the internal viscous damping coefficient, $\mathbf{K}_c^e = -i\mathbf{K}_b^e$ the complex circulation matrix. The mass matrices $(\mathbf{M}_t^e, \mathbf{M}_r^e)$ and the bending stiffness matrix $\mathbf{K}_b^e$ are also defined in the paper by Nelson (1985).

The translational and rotational kinetic energies (T_t^e, T_r^e), and the bending strain energy U^e of the shaft element can be expressed through the use of equations (2-4):

$$T_t^e = \frac{1}{2}\int_0^\ell \mu(\dot{v}^2 + \dot{w}^2)ds = \frac{1}{2}(\dot{\bar{p}}^e)^T \mathbf{M}_t^e \dot{\mathbf{p}}^e, \quad (9)$$

$$T_r^e = \frac{1}{2}\int_0^\ell I_d(\dot{\varphi}_y^2 + \dot{\varphi}_z^2)ds = \frac{1}{2}(\dot{\bar{p}}^e)^T \mathbf{M}_r^e \dot{\mathbf{p}}^e, \quad (10)$$

$$U^e = \frac{1}{2}\int_0^\ell EI[(v'')^2 + (w'')^2]ds = \frac{1}{2}(\bar{\mathbf{p}}^e)^T \mathbf{K}_b^e \mathbf{p}^e, \quad (11)$$

where μ is the mass per unit length, I_d the diametral mass moment of inertia of the shaft per unit length, EI the bending rigidity of the shaft element,

$$\mathbf{M}_t^e = \int_0^\ell \mu\bar{\mathbf{h}}\mathbf{h}^T ds, \;\; \mathbf{M}_r^e = \int_0^\ell I_d\bar{\mathbf{h}}'(\mathbf{h}')^T ds, \;\; \mathbf{K}_b^e = \int_0^\ell EI\bar{\mathbf{h}}''(\mathbf{h}'')^T ds. \quad (12\text{-}14)$$

Isotropic Bearing. The linear equation of an undamped isotropic bearing model can be written in complex form:

$$\mathbf{K}^b\mathbf{p}^b = \mathbf{F}^b, \quad (15)$$

where

$$\mathbf{K}^b = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}^b \quad (16)$$

is the bearing stiffness matrix, $\mathbf{p}^b$ is the complex displacement vector at the bearing location (node) and $\mathbf{F}^b$ is the complex bearing external force vector. Clearly, the strain energy U^b of the bearing can be expressed as

$$U^b = \frac{1}{2}(\bar{\mathbf{p}}^b)^T \mathbf{K}^b \mathbf{p}^b \quad (17)$$

System Equations. The equations of motion of the complete rotor system can be obtained by assembling all component equations of form equations (5), (8) and (15). The resulting equation is of the form:

$$\mathbf{M}\ddot{\mathbf{p}} + (\eta\mathbf{K}_b - \Omega\mathbf{G})\dot{\mathbf{p}} + (\mathbf{K}_s - i\eta\Omega\mathbf{K}_b)\mathbf{p} = 0, \quad (18)$$

where

$$\mathbf{p}^T = [\mathbf{p}_1^T \mathbf{p}_2^T \ldots \mathbf{p}_N^T] \quad (19)$$

is the (2Nx1) complex nodal displacement vector of the rotor system (N equals the number of nodes). The stiffness matrix $\mathbf{K}_s$ can be written as

$$\mathbf{K}_s = \mathbf{K}_b + \mathbf{K}^B \quad (20)$$

where $\mathbf{K}^B$ is a diagonal matrix the nonzero elements of which are the stiffnesses of the isotropic bearings. The $\mathbf{M}, \mathbf{K}_s$ and $\mathbf{G}$ matrices are the mass, bending stiffness, and gyroscopic matrices of the system obtained by assembling the element matrices.

Positive Definite Matrices. Since kinetic energy, by definition, cannot be negative, the last kinetic energy relations represented by equations (6-7) and (9-10) are called positive quadratic (Hermitian) forms, and the mass matrices $\mathbf{M}_t^d, \mathbf{M}_r^d, \mathbf{M}_t^e$ and $\mathbf{M}_r^e$ are called positive definite matrices. Similarly, from the strain energy expressions, equations (11) and (17), it can be seen that $\mathbf{K}_b^e$ and $\mathbf{K}^b$ are also positive definite matrices. Evidently, the system matrices $\mathbf{M}, \mathbf{K}_b, \mathbf{K}^B$ and $\mathbf{K}_s$ are positive definite Hermitian matrices. Thus the following relations hold:

$$\bar{\mathbf{p}}^T\mathbf{M}\mathbf{p} > 0, \;\; \bar{\mathbf{p}}^T\mathbf{K}_b\mathbf{p} > 0, \;\; \bar{\mathbf{p}}^T\mathbf{K}_s\mathbf{p} > 0, \quad (\mathbf{p} \neq 0). \quad (21)$$

Note that the system gyroscopic matrix $\mathbf{G}$ is not Hermitian, however, by using the definitions of the gyroscopic matrices $(\mathbf{G}^d, \mathbf{G}^e)$ listed in the Appendix of Nelson's paper (1985) it can be expressed as a product of the imaginary unit and a corresponding positive definite Hermitian matrix $\mathbf{M}_g$:

$$\mathbf{G} = i\mathbf{M}_g, \quad (22)$$

$$\bar{\mathbf{p}}^T\mathbf{M}_g\mathbf{p} > 0 \qquad (\mathbf{p} \neq 0). \quad (23)$$

STABILITY ANALYSIS

On seeking a solution to equation (18) of the form

$$\mathbf{p} = \mathbf{P}\,e^{\lambda t}, \quad (24)$$

we obtain the eigenvalue problem:

$$[\lambda^2\mathbf{M} + \lambda(\eta\mathbf{K}_b - \Omega\mathbf{G}) + \mathbf{K}_s - i\eta\Omega\mathbf{K}_b]\mathbf{P} = 0 \quad (25)$$

with 4N eigenvalues λ_j and corresponding eigenvectors $\mathbf{P}_j (j = 1,2,\ldots,4N)$. The eigenvalues λ are of the form:

$$\lambda = \alpha + i\omega, \quad (26)$$

where α is the damping coefficient, ω the damped natural frequency or whirl speed.

Instability occurs if one of the eigenvalues has a positive real part. Thus, the problem of determining the limit of stability of the rotor is reduced to finding the shaft speed Ω_s (*stability threshold speed*) at which the greatest real part of all eigenvalues λ_j equals zero. The corresponding imaginary part ω_w is the whirling speed.

To find the possible limit Ω, we substitute the eigenvalue of the form

$$\lambda = i\omega \quad (27)$$

into equation (25), which after premultiplying it by the complex conjugate eigenvector $\bar{\mathbf{P}}^T$ leads to the following complex scalar equation:

$$(-\omega^2 m + \omega\Omega g + k_s) + i\eta k_b(\omega - \Omega) = 0. \quad (28)$$

By using equation (22) and inequalities (21), (23) it can be seen that the scalars m, k_s, k_b and g in equation (28) are in all positive real quantities defined by

$$\bar{\mathbf{P}}^T\mathbf{M}\mathbf{P} = m > 0, \;\; \bar{\mathbf{P}}^T\mathbf{K}_s\mathbf{P} = k_s > 0, \;\; \bar{\mathbf{P}}^T\mathbf{K}_b\mathbf{P} = k_b > 0,$$

$$\bar{\mathbf{P}}^T\mathbf{G}\mathbf{P} = ig, \quad (g>0). \quad (29\text{-}32)$$

From equation (28) we get

$$\omega = \Omega \quad (\Omega > 0), \quad (33)$$

and

$$m - g = \frac{k_s}{\omega^2} > 0. \quad (34)$$

By substituting equations (27) and (33) into equation (25), we obtain

$$(-\Omega^2\mathbf{M} - i\Omega^2\mathbf{G} + \mathbf{K}_s)\mathbf{P} = 0, \quad (35)$$

which is identical with the eigenvalue problem for the forward (undamped) bending critical speeds of the rotor system. Thus, the possible limit of stability of the rotor coincides with one of its forward critical speeds.

Now we shall prove that the rotor is unstable at all speeds above the first forward critical speeds Ω_{f1}, that is the stability threshold speed coincides with Ω_{f1}. To this end, we apply the eigenvalue sensitivity analysis. Let us consider the rotor speed Ω as a system parameter, and differentiate equation (25) with respect to Ω:

$$[2\lambda\frac{\partial\lambda}{\partial\Omega}\mathbf{M} + \frac{\partial\lambda}{\partial\Omega}(\eta\mathbf{K}_b - \Omega\mathbf{G}) - \lambda\mathbf{G} - i\eta\mathbf{K}_b]\mathbf{P}$$
$$+[\lambda^2\mathbf{M} + (\eta\mathbf{K}_b - \Omega\mathbf{G}) + \mathbf{K}_s - i\eta\Omega\mathbf{K}_b]\frac{\partial\mathbf{P}}{\partial\Omega} = 0. \quad (36)$$

The quantity $\partial\lambda/\partial\Omega$ in the above equation is referred to as an eigenvalue sensitivity coefficient (Rajan et. al., 1986), which can be written, with the aid of equation (26), in the form:

$$\frac{\partial\lambda}{\partial\Omega} = \frac{\partial\alpha}{\partial\Omega} + i\frac{\partial\omega}{\partial\Omega}. \quad (37)$$

To calculate the real part $\partial\alpha/\partial\Omega$ from equation (36) at any value Ω of the forward critical speeds. we substitute again equation (27) into equation (36), and premultiply it by $\bar{\mathbf{P}}^T$. We then obtain the following expression for the damping sensitivity coefficient $\partial\alpha/\partial\Omega$:

$$\frac{\partial\alpha}{\partial\Omega} = \frac{2\eta\Omega k_b(m-g)}{(\eta k_b)^2 + \Omega^2(2m-g)^2}. \quad (38)$$

Here we have applied the premultiplication of equation (25) by the term $\partial\bar{\mathbf{P}}^T/\partial\Omega$ and also the properties of the system matrices $\mathbf{M}, \mathbf{K}_s$ and $\mathbf{G}$ as presented below:

$$\bar{\mathbf{M}}^T = \mathbf{M}, \quad \bar{\mathbf{K}}_s^T = \mathbf{K}_s, \quad \bar{\mathbf{G}}^T = -\mathbf{G}. \quad (39)$$

By using equations (31) and (34), it is clear from equation (38) that the damping sensitivity coefficient $\partial\alpha/\partial\Omega$ is positive at each forward critical speed. Consequently, there is a sign-change in the real part of the corresponding eigenvalue from negative (stable) to positive (unstable). If we assume that the rotor has n_f forward critical speeds, then any k th forward whirl mode of the rotor $(k \le n_f)$ becomes unstable at the k th forward critical speed Ω_{fk} regardless of the magnitude of the internal damping, and remains unstable for higher speeds.

As can be seen, the stability threshold speed Ω_t of the rotor system coincides with Ω_{f1}.

In addition, as follows from equation (33), the whirling speed ω_w of the rotor is also equal to Ω_{f1} regardless of the magnitude of internal damping. Furthermore, from equations (24) and (35) we see that the whirl mode induced at the stability threshold always occurs in the first forward whirl mode of the undamped rotor.

It is noteworthy that all backward whirl modes are stable for any rotational speed.

CONCLUDING REMARKS

In this paper a finite element stability analysis of general symmetric rotor systems supported by undamped isotropic bearings with internal viscous has been presented using complex coordinates and the sensitivity analysis. It is proved theoretically that the stability threshold speed and the whirling speed coincide with the first forward critical speed regardless of the magnitude of the internal damping.

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