Stability analysis of the BDF slowest first multirate methods

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

CASA-Report 07-04
January 2007

Stability analysis of the BDF slowest
first multirate methods

by

A. Verhoeven, E.J.W. ter Maten,
R.M.M. Mattheij, B. Tasić

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

International Journal of Computer Mathematics
Vol. 00, No. 00, Month 200x, 1–26

Stability analysis of the BDF Slowest first multirate methods

A. Verhoeven*† and E.J.W. ter Maten†‡ and R.M.M. Mattheij† and B. Tasić‡

†Technische Universiteit Eindhoven, Den Dolech 2, Eindhoven, 5600 MB, The Netherlands
‡ NXP Semiconductors, High Tech Campus 48, 5656 AE, Eindhoven, The Netherlands

(*v3.2 released October 2006*)

This paper deals with the stability analysis of the BDF Slowest first multirate time-integration methods which are applied to the transient analysis of circuit models. From an asymptotic analysis it appears that these methods are indeed stable if the subsystems are stable and weakly coupled.

Keywords: Backward Difference Formula; Differential-algebraic equations; Multirate time-integration; Stability analysis; Transient analysis

AMS Subject Classifications: 34E13; 65L20; 65L80; 65M55

Index

1. Introduction

Differential-algebraic equations (DAEs) play an important role in many applications, such as electronics, mechatronics, control theory, but also in discretized PDEs. We will consider the following initial value problem

$$\frac{d}{dt}\left[\mathbf{q}(t,\mathbf{x})\right] + \mathbf{j}(t,\mathbf{x}) = \mathbf{0}, \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{1}$$

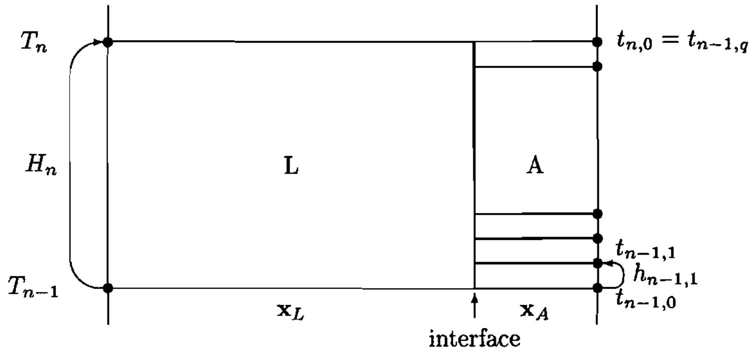
*Corresponding author. Email: averhoev@win.tue.nl

International Journal of Computer Mathematics
ISSN 0020-7160 print/ISSN 1029-0265 online © 200x Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/0020716YYxxxxxxxx

For electrical circuits $\mathbf{x}(t) \in \mathbb{R}^d$ is the time behaviour of the electrical state, while the functions $\mathbf{q}, \mathbf{j} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ represent the charges and currents in the circuit. In general $\mathbf{q}$ and $\mathbf{j}$ can be strongly nonlinear with respect to $\mathbf{x}$ and $\mathbf{q}$ can be not invertible.
In the classical numerical integration methods, the Initial Value Problem (1) is solved by means of implicit integration methods, like the BDF-methods. Each iteration, all equations are discretized by means of the same stepsize.
Often, parts of the model have latency or multirate behaviour. Latency means that parts of the state $\mathbf{x}(t)$ are constant during a certain time interval. Multirate behaviour means that some variables are slowly varying, compared to other variables. In both cases, it would be attractive to integrate these parts with a larger timestep.

1.1. *Partition of the system*

In contradiction to classical integration methods, multirate methods integrate both parts with different stepsizes or even with different schemes. Besides the coarse time-grid $\{T_n, 0 \le n \le N\}$ with stepsizes $H_n = T_n - T_{n-1}$ also a refined time-grid $\{t_{n-1,m}, 1 \le n \le N, 0 \le m \le q\}$ is used with stepsizes $h_{n,m} = t_{n,m} - t_{n,m-1}$ and multirate factor q. The two time-grids are synchronized, which means that $T_n = t_{n,0} = t_{n-1,q}$ for all n.



For a multirate method it is necessary to partition the variables and equations into an active (A) and a latent (L) part. This can be done by the user or automatically. Let $\mathbf{B}_A \in \mathbb{R}^{d_A \times d}$ and $\mathbf{B}_L \in \mathbb{R}^{d_L \times d}$ be selection matrices with $d_A + d_L = d$ such that $\mathbf{B}_A\mathbf{B}_A^T = \mathbf{I}, \mathbf{B}_A\mathbf{B}_L^T = \mathbf{O}$, etc. Then the variables and functions can be split in active (A) and latent (L) parts:

$$\begin{aligned} \mathbf{x} &= \mathbf{B}_A^T\mathbf{x}_A + \mathbf{B}_L^T\mathbf{x}_L, \quad \mathbf{x}_A \in \mathbb{R}^{d_A}, \mathbf{x}_L \in \mathbb{R}^{d_L}, \\ \mathbf{q}(t,\mathbf{x}) &= \mathbf{B}_A^T\mathbf{q}_A(t,\mathbf{B}_A\mathbf{x},\mathbf{B}_L\mathbf{x}) + \mathbf{B}_L^T\mathbf{q}_L(t,\mathbf{B}_A\mathbf{x},\mathbf{B}_L\mathbf{x}), \\ \mathbf{j}(t,\mathbf{x}) &= \mathbf{B}_A^T\mathbf{j}_A(t,\mathbf{B}_A\mathbf{x},\mathbf{B}_L\mathbf{x}) + \mathbf{B}_L^T\mathbf{j}_L(t,\mathbf{B}_A\mathbf{x},\mathbf{B}_L\mathbf{x}). \end{aligned} \tag{2}$$

Now equation (1) is equivalent to the following partitioned system

$$\begin{aligned} \tfrac{d}{dt}\left[\mathbf{q}_A(t,\mathbf{x}_A,\mathbf{x}_L)\right] + \mathbf{j}_A(t,\mathbf{x}_A,\mathbf{x}_L) = \mathbf{0}, &\quad \mathbf{x}_A(0) = \mathbf{x}_{A,0}, \\ \tfrac{d}{dt}\left[\mathbf{q}_L(t,\mathbf{x}_A,\mathbf{x}_L)\right] + \mathbf{j}_L(t,\mathbf{x}_A,\mathbf{x}_L) = \mathbf{0}, &\quad \mathbf{x}_L(0) = \mathbf{x}_{L,0}. \end{aligned} \tag{3}$$

Of course it is also possible to extend this partition in a partition of k subsystems, where the k sub-systems have an decreasing activity. Furthermore, each subsystem again can be partitioned in a recursive way.

1.2. *Relaxation*

All multirate methods have the common property that they use waveform relaxation to solve a partitioned system like (3). Each part is integrated on an independent time-grid which depends on its own activity. Usually, the unknowns of the already integrated subsystems are interpolated and used at the new time-grid for the following subsystem.

The slowest part e.g. needs only one large step H, while the faster subsystems are integrated on refined time-grids using smaller microsteps. A basic property of multirate is that not more large steps are done before all faster subsystems are also integrated. For a more general description about waveform relaxation, the reader is referred to [19].

Two very natural types of multirate methods are "Slowest first" and "Fastest first". With the first one the subsystem with the slowest behaviour (or largest time constant) is one step integrated. Afterwards the subsystems with increasingly faster behaviour are integrated. With the "Fastest first" method the subsystem with the fastest behaviour (or smallest time constant) is one step integrated. Afterwards the subsystems with decreasingly faster behaviour are integrated. The last approach has the advantage that it is less hard to predict the slow than the fast interface variables. However, there is a drawback with respect to stepsize control. Because it is possible that the large step-size H has to be reduced, then previous numerical solutions of the active subsystems are required which implies that we need a lot of memory.

1.3. *General types of multirate*

To keep it simple, we will work with a time-independent version of the partitioned system (3) with $\mathbf{y} = \mathbf{x}_A$ the active variable and $\mathbf{z} = \mathbf{x}_L$ the slow variable:

$$\frac{d}{dt}\left[\mathbf{q}_A(\mathbf{y},\mathbf{z})\right] + \mathbf{j}_A(\mathbf{y},\mathbf{z}) = \mathbf{0}, \tag{4}$$

$$\frac{d}{dt}\left[\mathbf{q}_L(\mathbf{y},\mathbf{z})\right] + \mathbf{j}_L(\mathbf{y},\mathbf{z}) = \mathbf{0}. \tag{5}$$

In this section some available multirate methods will be discussed. The multirate methods are independent of the integration method, but are presented for the BDF scheme. The integration order for the slow and fast part are equal to K and k respectively. Furthermore, the coarse and refined time-grids are assumed to be equidistant and synchronized, which means that $t_{n-1,q} = T_n = t_{n,0}$. Multirate schemes have been investigated by more people, which results can be found in [2–5,8,14,18]. We will summarize some common approaches.

Semi-implicit multirate methods only integrate the equations (4) and (5) separately, while the other parts are estimated by means of extrapolation or interpolation. The variable $\mathbf{z}$ really has to be a latent variable, which can be integrated with a large step-size H. This implies that the interpolation of $\mathbf{z}$ is expected to be very accurately. In this case, $\mathbf{z}$ will be rather independent of the prediction errors of the active variables.

The Slow-Fast method (Alg.1) first integrates (5) with one large step-size H, while $\mathbf{y}$ is approximated by means of extrapolation. Afterwards equation (4) is integrated with a small step-size h, while $\mathbf{z}$ is approximated by means of interpolation. Because it is not possible to get an accurate prediction for the fast variable $\mathbf{y}_n$, often just constant extrapolation is used. In this paper we will use interpolation of order $K-1$ of the updated slow interface. For Euler Backward this means constant interpolation of the updated value at the new time-point T_n, with $\mu^0_{n-1,m} = 1$ and $\mu^1_{n-1,m} = 0$. We also will consider Euler Backward with linear interpolation, with $\mu^0_{n-1,m} = \frac{m}{q}$ and $\mu^1_{n-1,m} = \frac{q-m}{q}$.

Because these semi-implicit multirate methods use extrapolation, they could have unstable behavior. To improve the stability, the latent part can be integrated by means of an implicit compound step. The Compound-Fast method (Alg.2) first integrates (4) and (5) together with one large step-size H, which results in $\mathbf{y}_n$ and $\mathbf{z}_n$. Afterwards, only equation (4) is integrated with a small step-size h, while $\mathbf{z}$ is approximated by means of interpolation.

Note that $\mathbf{y}_n$ is twice computed by the Compound-Fast method. Another possibility is the Generalized Compound-Fast method, which computes $\mathbf{y}_{n-1,\alpha q}$ and $\mathbf{z}_n$ simultaneously with $\alpha q \in \mathbb{N}$. For a more detailed description of this method the reader is referred to [17]. This family of Generalized Compound-Fast methods contains the Compound-Fast method itself ($\alpha = 1$) and the Mixed Compound-Fast method ($\alpha = \frac{1}{q}$), which computes $\mathbf{y}_{n-1,1}$ and $\mathbf{z}_n$ simultaneously. This Mixed Compound-Fast approach is also used by the MROW method [1,15]. The first active solution $\mathbf{y}_{n-1,1}$ is already computed by means of the compound step. Note

ALGORITHM 1.1 The BDF Slow-Fast multirate method

Solve for $\mathbf{z}_n$:

$$\rho_0\mathbf{q}_L(\hat{\mathbf{y}}_n,\mathbf{z}_n)+\ldots+\rho_K\mathbf{q}_L(\mathbf{y}_{n-K},\mathbf{z}_{n-K})+H\mathbf{j}_L(\hat{\mathbf{y}}_n,\mathbf{z}_n)=\mathbf{0} \tag{6}$$

$$\hat{\mathbf{y}}_n-\mathbf{y}_{n-1}=\mathbf{0} \tag{7}$$

Solve for $\mathbf{y}_{n-1,m}$ $(m=1,\ldots,q)$:

$$\bar{\rho}_0\mathbf{q}_A(\mathbf{y}_{n-1,m},\hat{\mathbf{z}}_{n-1,m})+\ldots+\bar{\rho}_k\mathbf{q}_A(\mathbf{y}_{n-1,m-k},\hat{\mathbf{z}}_{n-1,m-k})+h\mathbf{j}_A(\mathbf{y}_{n-1,m},\hat{\mathbf{z}}_{n-1,m})=\mathbf{0} \tag{8}$$

$$\hat{\mathbf{z}}_{n-1,m}-(\mu^0_{n-1,m}\mathbf{z}_n+\ldots+\mu^K_{n-1,m}\mathbf{z}_{n-K})=\mathbf{0} \tag{9}$$

ALGORITHM 1.2 The BDF Compound-Fast multirate method

Solve for $\mathbf{z}_n$ and $\mathbf{y}_n$:

$$\rho_0\mathbf{q}_A(\mathbf{y}_n,\mathbf{z}_n)+\ldots+\rho_K\mathbf{q}_A(\mathbf{y}_{n-K},\mathbf{z}_{n-K})+H\mathbf{j}_A(\mathbf{y}_n,\mathbf{z}_n)=\mathbf{0} \tag{10}$$

$$\rho_0\mathbf{q}_L(\mathbf{y}_n,\mathbf{z}_n)+\ldots+\rho_K\mathbf{q}_L(\mathbf{y}_{n-K},\mathbf{z}_{n-K})+H\mathbf{j}_L(\mathbf{y}_n,\mathbf{z}_n)=\mathbf{0} \tag{11}$$

Solve for $\mathbf{y}_{n-1,m}$ $(m=1,\ldots,q)$:

$$\bar{\rho}_0\mathbf{q}_A(\mathbf{y}_{n-1,m},\hat{\mathbf{z}}_{n-1,m})+\ldots+\bar{\rho}_k\mathbf{q}_A(\mathbf{y}_{n-1,m-k},\hat{\mathbf{z}}_{n-1,m-k})+h\mathbf{j}_A(\mathbf{y}_{n-1,m},\hat{\mathbf{z}}_{n-1,m})=\mathbf{0} \tag{12}$$

$$\hat{\mathbf{z}}_{n-1,m}-(\mu^0_{n-1,m}\mathbf{z}_n+\ldots+\mu^K_{n-1,m}\mathbf{z}_{n-K})=\mathbf{0} \tag{13}$$

that $\mathbf{y}_{n-1,1}$ is equal to the solution of the integration of the fastest part for $m=0$. The Compound-Fast method has the benefit that it is more stable and is easier to implement, while the Mixed Compound-Fast method results in better scaled nonlinear equations which are easier to solver with the Newton method.

Although the used integration methods for the sub-circuits can be A-stable, this will not be the case for the multirate version [14]. For multirate methods, the results also depend on the extrapolated or interpolated results of the other part. Thus the stability will always strongly depend on the used partition and on the coupling. In particular, the extrapolation may cause unstable behavior. Therefore, it is expected that the methods with an implicit compound step are more stable methods, because they do not use explicit extrapolation.

Besides the previous methods, in [14] also implicit multirate methods are proposed. Now, the compound step and refinement are written as one huge system of algebraic equations which is simultaneously solved. This means that no interpolation or extrapolation is necessary. Compared to the other multirate methods, it needs much more computational time but also has better stability properties.

1.4. *Dynamical properties of the active part*

For a proper implementation of the previous multirate schemes, we assume that the solvability is preserved for the active part. Furthermore, it is also very useful if the active part of a stable DAE is also stable and has the same index as the original DAE.

Consider the linear time-invariant system

$$\Sigma: \quad \mathbf{C}\dot{\mathbf{y}}+\mathbf{G}\mathbf{y}=\mathbf{u}. \tag{14}$$

It is well-known that

$$\begin{aligned} \text{the system (14) is solvable } &\Leftrightarrow \sigma(\Sigma) \text{ is a finite set,} \\ \text{the system (14) is stable } &\Leftrightarrow \forall_{\lambda \in \sigma(\Sigma)} \mathrm{Re}[\lambda] < 0, \end{aligned}$$

where $\sigma(\Sigma) = \{\lambda \in \mathbb{C} : \det(\lambda \mathbf{C} + \mathbf{G}) = 0\}$. For a general partition these properties are not preserved for the active part of a DAE. For example, consider the linear 2-dimensional problem $\Sigma : \mathbf{C}\dot{\mathbf{x}} + \mathbf{G}\mathbf{x} = \mathbf{s}$, where

$$\mathbf{C} = \mathbf{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This DAE is solvable because $\det(\lambda \mathbf{C} + \mathbf{G}) = -(\lambda + 1)^2$ which is only equal to zero for $\lambda = -1$, so $\sigma(\Sigma) = \{-1\}$ is a finite set. If we take the partition with

$$\mathbf{B}_A = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \mathbf{B}_L = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

we get for the refinement the unsolvable problem

$$0\dot{\mathbf{y}} + 0\mathbf{y} = \mathbf{s}_1.$$

Notice that the active part of an ODE is always solvable, because then $\mathbf{C} = \mathbf{I}$ is an invertible matrix. However, the stability is not automatically preserved for both ODEs and DAEs. If we take

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} -1 & -2 \\ 2 & 2 \end{pmatrix},$$

we have a stable ODE with eigenvalues $-\frac{1}{2} \pm \frac{1}{2}\sqrt{7}i$, but for the refinement we get the following unstable differential equation

$$\dot{\mathbf{y}} = \mathbf{y} + \mathbf{s}_1.$$

Finally it can be shown that also the index of the active part is not always preserved.

1.5. *Overview of this paper*

This paper investigates the stability of the Slow-Fast and the Compound-Fast multirate versions of the multistep BDF scheme. For a stability analysis of the Generalized Compound-Fast version of the Euler Backward scheme one may consult [17]. Although also other implicit methods can be used, like Runge Kutta methods, we use BDF integration methods because they use less function evaluations and they are very well suited for interpolation. For linear multistep methods the solution can always be represented by a piecewise polynomial, which can be used to interpolate the latent interface variables without accuracy loss.

Although BDF methods are A-stable for order $p \in \{1, 2\}$ and $A(\alpha)$-stable for $p \in \{3, \ldots, 6\}$, their multirate version will have different stability properties. One of the major problems of multirate schemes is the lack of general theoretical results that guarantees stability [4,8]. The efficiency gain of the multirate schemes could be destroyed by the instability, blowing up the global errors.

This problem has already been investigated in other papers like [3,4,7,8,12–14]. There one considers the stability of the multirate schemes applied to a two-dimensional real linear test equation. In [4] an overview is given of this previous work. It appears possible to derive stability conditions for the elements of the corresponding companion matrix. It is also shown how for a fixed multirate factor q and stiffness graphs of stability regions can be constructed. It appears that the Euler Backward multirate algorithm which uses constant extrapolation of the updated slow part is more stable than for linear interpolation

Nevertheless, the derived stability conditions do not have a direct relation with the original test equation. Like for the stability analysis for ODEs we want conditions for the test equation. This paper uses an asymptotic analysis to simplify the stability conditions for the companion matrix by taking the limit $H \to 0$ or $q \to \infty$. It appears possible to express the stability conditions directly in terms of the elements of the matrix $\mathbf{A}$ in the test equation. Furthermore, the available stability conditions are rather algebraic and do not explain anything; in particular for multistep methods, because the companion matrix is more complex here. We also derive simplified stability conditions for the BDF multirate algorithm of higher order. It is even allowed that the integration orders for the slow and fast parts are different.
This important topic will be analyzed in this paper. The paper is organized as follows. Section 2 gives a general introduction to the stability analysis of multirate schemes for ODEs and DAEs. Section 3 contains a stability analysis for the onestep version (Euler Backward) applied to a 2-dimensional linear ODE. Then section 4 investigates the stability of the numerical scheme for the BDF methods of higher order on the same test equation. Finally, section 5 closes this paper with some concluding remarks.

2. Stability analysis of multirate schemes

Multirate methods have less good stability properties than ordinary integration methods. Therefore this section explains how the stability of those multirate methods can be analyzed for the Slow-Fast and Compound-Fast methods.

2.1. *Stability of multirate scheme*

Consider the partitioned nonlinear DAE in (4,5) with property $\mathbf{j}_A(\mathbf{0},\mathbf{0}) = \mathbf{j}_L(\mathbf{0},\mathbf{0}) = \mathbf{0}$. We assume that the origin is a stable stationary solution, which implies that for all initial conditions $\mathbf{y}(t) \to \mathbf{0}$ and $\mathbf{z}(t) \to \mathbf{0}$ if $t \to \infty$. The stability of multirate schemes will only be analyzed for DAEs with these properties. Furthermore the analysis is done for equidistant time-grids.

Definition 2.1 Let $\mathbf{y}_n$ and $\mathbf{z}_n$ be the numerical approximations of the multirate scheme at the time-point $T_n = nH$ on the coarse equidistant time-grid for a solvable DAE. The scheme is called (conditionally) stable if for all initial conditions $\mathbf{y}_n \to \mathbf{0}$ and $z_n \to \mathbf{0}$ if $n \to \infty$. The multirate scheme is A-stable (or unconditionally stable) if it is stable for all solvable DAEs with $\mathbf{y}(t), \mathbf{z}(t) \to \mathbf{0}$ for $t \to \infty$ and for all $H, q > 0$.

Such criterion would require a stability analysis of a nonlinear multi-dimensional recurrence relation, which is very complex. In [12] it has been shown that all semi-implicit and implicit Euler Backward multirate methods are stable if the system (4,5) is monotonically max-norm stable and satisfies an additional stability condition. Another possible approach is to consider the Prothero-Robinson equation [1,11], which is used to analyze the stability on a given trajectory $\mathbf{y} = \tilde{\mathbf{y}}(t), \mathbf{z} = \tilde{\mathbf{z}}(t)$:

$$\begin{aligned} \tfrac{d}{dt}\left[\mathbf{q}_A(\mathbf{y}-\tilde{\mathbf{y}},\mathbf{z}-\tilde{\mathbf{z}})\right] + \mathbf{j}_A(\mathbf{y}-\tilde{\mathbf{y}},\mathbf{z}-\tilde{\mathbf{z}}) = \mathbf{j}_A(\mathbf{0},\mathbf{0}) = \mathbf{0}, \\ \tfrac{d}{dt}\left[\mathbf{q}_L(\mathbf{y}-\tilde{\mathbf{y}},\mathbf{z}-\tilde{\mathbf{z}})\right] + \mathbf{j}_L(\mathbf{y}-\tilde{\mathbf{y}},\mathbf{z}-\tilde{\mathbf{z}}) = \mathbf{j}_L(\mathbf{0},\mathbf{0}) = \mathbf{0}. \end{aligned} \tag{15}$$

Otherwise it is only possible to prove local stability for the linearized system around the origin. Then we get the following multi-dimensional linear time-invariant DAE or ODE

$$\underbrace{\begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix}}_{\mathbf{C}} \begin{pmatrix} \dot{\mathbf{y}} \\ \dot{\mathbf{z}} \end{pmatrix} + \underbrace{\begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix}}_{\mathbf{G}} \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{16}$$

$$\begin{pmatrix} \dot{\mathbf{y}} \\ \dot{\mathbf{z}} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix}. \tag{17}$$

In [14] it is proved that Euler Backward multirate methods are stable for (17) if the matrix

$$\begin{pmatrix} \mu(\mathbf{A}_{11}) & \|\mathbf{A}_{12}\| \\ \|\mathbf{A}_{21}\| & \mu(\mathbf{A}_{22}) \end{pmatrix}$$

is stable, where μ is a logarithmic norm, that is

$$\mu(\mathbf{A}) = \lim_{h\to 0^+} \frac{\|\mathbf{I}+h\mathbf{A}\|-1}{h} = \lim_{h\to 0^+} \frac{\log(\|e^{h\mathbf{A}}\|)}{h}.$$

This is the case if $\mu(\mathbf{A}_{11}) < 0, \mu(\mathbf{A}_{22}) < 0$ and $\|\mathbf{A}_{12}\|\|\mathbf{A}_{21}\| < \mu(\mathbf{A}_{11})\mu(\mathbf{A}_{22})$. In qualitative terms this means that each subsystem is stable and the couplings between the subsystems are weak.

2.2. *Two-dimensional test equations*

For ordinary integration methods stability can be studied by looking at the scalar test equation $\dot{x} = \lambda x$ with $\lambda \in \mathbb{C}$ [10]. For multirate methods for DAEs with two time-steps h and H, the following two-dimensional test equation could be studied, where y and z are the active and latent variable respectively

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}}_{\mathbf{C}} \begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} + \underbrace{\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}}_{\mathbf{G}} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{18}$$

For ordinary differential equations the following (real) linear test equation is usually studied [8, 14]

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} y \\ z \end{pmatrix} \tag{19}$$

From now on we will only consider the stability of multirate schemes for (19). Let y_n and z_n be the numerical approximations at the time-point $T_n = nH$ on the coarse time-grid. For Euler Backward multirate schemes the numerical solutions y_n and z_n satisfy the following two-dimensional recurrence relation

$$\begin{pmatrix} z_n \\ y_n \end{pmatrix} = \underbrace{\begin{pmatrix} \sigma & \rho \\ \tau & \nu \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} z_{n-1} \\ y_{n-1} \end{pmatrix}. \tag{20}$$

Note that the order of y_n and z_n in (20) is different from the order of y, z in (19). The multirate method is stable if y_n and z_n tend to zero for $n \to \infty$, which is the case if $\rho(\mathbf{M}) < 1$. For $q = 1$, the stability behaviour of the multirate methods is independent of the used coordinate system. However, for $q > 1$ this is only the case if the linear system is decoupled. Otherwise the stability does not only depend on the eigenvalues but also on the eigenvectors of the matrix $\mathbf{A}$.

The dynamics of multistep methods can not be described by (20). Assume that the compound step uses a BDF method of order K, while the refinement is done with a BDF method of order k. We introduce the following vectors

$$\mathbf{z}_n := \begin{pmatrix} z_n \\ \vdots \\ z_{n-K+1} \end{pmatrix} \in \mathbb{R}^K, \quad \mathbf{y}_n := \begin{pmatrix} y_n \\ \vdots \\ y_{n-k+1} \end{pmatrix} \in \mathbb{R}^k. \tag{21}$$

Then the dynamics of a multirate linear multistep method obey the following multi-dimensional recurrence relation

$$\begin{pmatrix} \mathbf{z}_n \\ \mathbf{y}_n \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{S} & \mathbf{R} \\ \mathbf{T} & \mathbf{N} \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} \mathbf{z}_{n-1} \\ \mathbf{y}_{n-1} \end{pmatrix}, \tag{22}$$

where $\mathbf{M} \in \mathbb{R}^{(K+k)\times(K+k)}$ is the companion matrix. The multistep multirate method is stable if $\mathbf{y}_n$ and $\mathbf{z}_n$ tend to zero for $n \to \infty$. Again this is the case if $\rho(\mathbf{M}) < 1$.
Thus in both cases the stability of multirate schemes can be determined from $\rho(\mathbf{M})$ where $\mathbf{M} \in \mathbb{R}^{(K+k)\times(K+k)}$. The schemes applied to (19) are A-stable if $\rho(\mathbf{M}) < 1$ for all $H, q > 0$ and stable matrices $\mathbf{A}$ [14]. Because of simplicity, we start with the stability of the first order Euler Backward multirate method. Afterwards we also consider BDF multirate methods of higher order.

3. Stability analysis of Euler Backward multirate algorithm

This section deals with the stability analysis of the Slow-Fast and Compound-Fast versions of the Euler Backward multirate algorithm. First we will show that the dynamics really can be described by the recurrence relation in (20). Both constant and linear interpolation of the latent part will be included. Afterwards we will state a theorem which gives us stability conditions for the matrix $\mathbf{A}$. Because these conditions are rather complex to interpret, we state two other theorems which are based on an asymptotic analysis for $H \to 0$ or $q \to \infty$.

3.1. *Derivation of the recurrence relation*

LEMMA 3.1 *Consider the Slow-Fast and the Compound-Fast versions of the Euler Backward multirate scheme. Then $\{z_n\}$ and $\{y_n\}$ are solutions of the following recurrence relation*

$$\begin{pmatrix} z_n \\ y_n \end{pmatrix} = \underbrace{\begin{pmatrix} \sigma & \rho \\ \tau & \nu \end{pmatrix}}_{M} \begin{pmatrix} z_{n-1} \\ y_{n-1} \end{pmatrix}, \tag{23}$$

where for the Slow-Fast version,

$$\rho = \tfrac{a_{21}H}{1-a_{22}H}, \sigma = \tfrac{1}{1-a_{22}H} \tag{24}$$

and, for the Compound-Fast version,

$$\rho = \tfrac{a_{21}H}{1-(a_{11}+a_{22})H+(a_{11}a_{22}-a_{12}a_{21})H^2}, \sigma = \tfrac{1-a_{11}H}{1-(a_{11}+a_{22})H+(a_{11}a_{22}-a_{12}a_{21})H^2}. \tag{25}$$

For both versions, if constant interpolation is used,

$$\nu = \gamma^q + \textstyle\sum_{l=0}^{q-1} \gamma^l \delta\rho, \tau = \sum_{l=0}^{q-1} \gamma^l \delta\sigma \tag{26}$$

and for linear interpolation

$$\nu = \gamma^q + \textstyle\sum_{l=0}^{q-1} \gamma^l \rho\delta(1 - \frac{l}{q}), \tau = \sum_{l=0}^{q-1} \gamma^l \delta(\frac{l}{q}(1-\sigma) + \sigma), \tag{27}$$

where

$$\gamma = \tfrac{1}{1-a_{11}h}, \delta = \tfrac{a_{12}h}{1-a_{11}h}. \tag{28}$$

Proof

Analysis of the compound step In both the Slow-Fast and the Compound-Fast methods the latent variable is integrated first. Using constant extrapolation of y_{n-1} for the Slow-Fast method we obtain the relation

$$\frac{z_n - z_{n-1}}{H} = a_{21} y_{n-1} + a_{22} z_n. \tag{29}$$

From (29) it indeed follows that

$$z_n = \rho y_{n-1} + \sigma z_{n-1}, \tag{30}$$

where ρ, σ are given in (24). For the Compound-Fast method, we get a recurrence relation for $\{y_n\}$ and $\{z_n\}$:

$$\begin{cases} \frac{y_n - y_{n-1}}{H} = a_{11} y_n + a_{12} z_n, \\ \frac{z_n - z_{n-1}}{H} = a_{21} y_n + a_{22} z_n. \end{cases} \tag{31}$$

The solution satisfies again (30) with different values for ρ and σ in (25).

Analysis of the refinement step For both methods $z_{n-1,j}$ is estimated for $j \in \{1, \ldots, q-1\}$ employing z_{n-1} and z_n as follows:

$$\begin{aligned} \text{Constant interpolation: } & \hat{z}_{n-1,j} = z_n, \\ \text{Linear interpolation: } & \hat{z}_{n-1,j} = z_{n-1} + \tfrac{j}{q}(z_n - z_{n-1}) = \tfrac{q-j}{q} z_{n-1} + \tfrac{j}{q} z_n. \end{aligned} \tag{32}$$

Finally, the active part is integrated on the time interval $[T_{n-1}, T_n]$ with q steps h:

$$\frac{y_{n-1,j} - y_{n-1,j-1}}{h} = a_{11} y_{n-1,j} + a_{12} \hat{z}_{n-1,j}. \tag{33}$$

The recurrence relation (33) is equivalent to

$$y_{n-1,j} = \frac{\frac{1}{h}}{\frac{1}{h} - a_{11}} y_{n-1,j-1} + \frac{a_{12}}{\frac{1}{h} - a_{11}} \hat{z}_{n-1,j} = \gamma y_{n-1,j-1} + \delta \hat{z}_{n-1,j}, \tag{34}$$

where $\gamma = \frac{1}{1 - a_{11} h}$ and $\delta = \frac{a_{12} h}{1 - a_{11} h}$. If constant interpolation is used we have for $j \in \{1, \ldots, q\}$

$$\begin{aligned} y_{n-1,j} &= \gamma y_{n-1,j-1} + \delta z_n \\ &= \gamma^j y_{n-1,0} + \textstyle\sum_{k=0}^{j-1} \gamma^{j-1-k} \delta z_n. \end{aligned} \tag{35}$$

If linear interpolation is used we have for $j \in \{1, \ldots, q\}$

$$\begin{aligned} y_{n-1,j} &= \gamma y_{n-1,j-1} + \delta(1 - \tfrac{j}{q}) z_{n-1} + \delta \tfrac{j}{q} z_n \\ &= \gamma^j y_{n-1,0} + \textstyle\sum_{k=0}^{j-1} \gamma^{j-1-k} \left(\delta(1 - \tfrac{k+1}{q}) z_{n-1} + \delta \tfrac{k+1}{q} z_n \right). \end{aligned} \tag{36}$$

Inserting (30) into (35) and (36) for $j = q$ results in

$$\begin{aligned} y_n = y_{n-1,q} &= \gamma^q y_{n-1,0} + \textstyle\sum_{k=0}^{q-1} \gamma^{q-1-k} \delta(\rho y_{n-1,0} + \sigma z_{n-1}) \\ &= \nu y_{n-1,0} + \tau z_{n-1} = \nu y_{n-1} + \tau z_{n-1}, \end{aligned} \tag{37}$$

where ν, τ are given in (26) and

$$\begin{aligned} y_n = y_{n-1,q} &= \gamma^q y_{n-1,0} + \left(\sum_{k=0}^{q-1} \gamma^{q-1-k} \delta (1 - \tfrac{k+1}{q})\right) z_{n-1} + \\ &\quad \left(\sum_{k=0}^{q-1} \gamma^{q-1-k} \delta \tfrac{k+1}{q}\right) (\rho y_{n-1,0} + \sigma z_{n-1}) \\ &= \nu y_{n-1,0} + \tau z_{n-1} = \nu y_{n-1} + \tau z_{n-1}, \end{aligned} \tag{38}$$

where ν, τ are given in (27). From (30), (37) and (38) it indeed follows that $\{y_n\}, \{z_n\}$ satisfy the recurrence relation in (23). □

3.2. *Stability conditions*

Before we state the stability conditions for (23) we need the following Lemma.

LEMMA 3.2 *Let $\phi(\lambda) = \det(M - \lambda I) = \lambda^2 - tr(M)\lambda + \det(M)$ be the characteristic polynomial of M, where $M \in \mathbb{R}^{2\times 2}$. Using the Routh-Hurwitz criterion one can easily show that [4, 6]*

$$\rho(M) < 1 \Leftrightarrow \begin{cases} \phi(-1) = 1 + tr(M) + \det(M) > 0, \\ \phi(0) = \det(M) < 1, \\ \phi(1) = 1 - tr(M) + \det(M) > 0. \end{cases} \tag{39}$$

Proof We are looking for conditions for the coefficients of $\phi(\lambda)$ such that

$$\phi(\lambda) = 0 \Rightarrow |\lambda| < 1. \tag{40}$$

Let $S = \{\lambda : |\lambda| < 1\} \subset \mathbb{C}$. Let $w : S \to \mathbb{C}^-$ be the transformation $w = \frac{\lambda - 1}{\lambda + 1}$. This maps the boundary of S into the imaginary axis $Re(w) = 0$, and the interior of S into the half-plane $Re(w) < 0$. Indeed for $\lambda = e^{i\theta} \in \partial S$ we obtain

$$w = \frac{e^{i\theta} - 1}{e^{i\theta} + 1} = \frac{e^{i\theta} - 1}{e^{i\theta} + 1} = \frac{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}{e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}} = \frac{2i\sin(\frac{\theta}{2})}{2\cos(\frac{\theta}{2})} = \tan(\frac{\theta}{2})i.$$

Furthermore $w(0) = -1 \in \mathbb{C}^-$. The inverse transformation $\lambda : \mathbb{C}^- \to S$ satisfies $\lambda = \frac{1+w}{1-w}$. Now we can rewrite formula (40) as

$$\psi(w) = (1 - w)^2 \phi(\frac{1 + w}{1 - w}) = 0 \Rightarrow Re(w) < 0. \tag{41}$$

Because

$$\psi(w) = (1 - w)^2 \phi(\frac{1 + w}{1 - w}) = (1 + \mathrm{tr(M)} + \det(\mathrm{M}))w^2 + 2(1 - \det(\mathrm{M}))w + (1 - \mathrm{tr(M)} + \det(\mathrm{M})),$$

we obtain

$$\psi(w) = \underbrace{(1 + \mathrm{tr(M)} + \det(\mathrm{M}))}_{\psi_0} w^2 + \underbrace{2(1 - \det(\mathrm{M}))}_{\psi_1} w + \underbrace{(1 - \mathrm{tr(M)} + \det(\mathrm{M}))}_{\psi_2} = 0 \Rightarrow Re(w) < 0. \tag{42}$$

Because of the Routh-Hurwitz criterion [6] this is the case if and only if the coefficients $\psi_0 = \phi(-1), \psi_1 = 1 - \phi(0)$, $\psi_2 = \phi(1)$ are positive. This is indeed the case if and only if the conditions in (39) are fulfilled. □

THEOREM 3.3 *Consider the recurrence relation in (23) which describes the dynamical behaviour of the Slow-Fast and Compound-Fast versions of the Euler Backward multirate scheme for the stable test equation*

(19). Then the schemes using constant interpolation are stable for all H, q if

$$\begin{array}{rl} (1+\sigma)(1+\gamma^q)+\rho\delta\sum_{l=0}^{q-1}\gamma^l & > 0, \\ 1-\sigma\gamma^q & > 0, \\ (1-\sigma)(1-\gamma^q)-\rho\delta\sum_{l=0}^{q-1}\gamma^l & > 0, \end{array} \quad (43)$$

and the schemes using linear interpolation are stable for all H, q if

$$\begin{array}{rl} (1+\sigma)(1+\gamma^q)-\rho\delta\sum_{l=0}^{q-1}\gamma^l(\frac{2l}{q}-1) & > 0, \\ \frac{\rho\delta}{q}\sum_{l=0}^{q-1}\gamma^l l-\sigma\gamma^q+1 & > 0, \\ (1-\sigma)(1-\gamma^q)-\rho\delta\sum_{l=0}^{q-1}\gamma^l & > 0. \end{array} \quad (44)$$

Proof The methods are stable if $\rho(\mathbf{M}) < 1$ for all $H, q > 0$ and stable matrices $\mathbf{A}$. Because $\mathbf{M} \in \mathbb{R}^{2\times 2}$ Lemma 3.2 gives us

$$\rho(\mathbf{M}) < 1 \Leftrightarrow \begin{cases} \phi(-1) = 1 + \mathrm{tr}(\mathbf{M}) + \det(\mathbf{M}) > 0, \\ \phi(0) = \det(\mathbf{M}) < 1, \\ \phi(1) = 1 - \mathrm{tr}(\mathbf{M}) + \det(\mathbf{M}) > 0. \end{cases} \quad (45)$$

Using the properties $\mathrm{tr}(\mathbf{M}) = \sigma + \nu$ and $\det(\mathbf{M}) = \sigma\nu - \rho\tau$, we get the following stability conditions for the elements of $\mathbf{M}$

$$\begin{array}{rl} 1+\sigma+\nu+\sigma\nu-\rho\tau & > 0, \\ 1-\sigma\nu+\rho\tau & > 0, \\ 1-\sigma-\nu+\sigma\nu-\rho\tau & > 0. \end{array} \quad (46)$$

After substituting the expressions for ν and τ in (26) and (27) we obtain the three stability conditions in (43) and (44) respectively. □

Notice that we can write the stability conditions in (46) in the following form

$$\begin{cases} (1+\sigma)(1+\nu) > \rho\tau, \\ \sigma\nu - 1 < \rho\tau, \\ (1-\sigma)(1-\nu) > \rho\tau. \end{cases}$$

3.3. *Asymptotic stability conditions*

Because the stability conditions (43) and (44) are rather complex, we will derive more compact stability conditions by means of an asymptotic analysis. First we will prove that the studied multirate schemes are always conditionally stable. Second we also will give sufficient conditions for $q \to \infty$ such that the methods are stable for all H.

Conditional stability THEOREM 3.4 *Both the Slow-Fast and Compound-Fast versions of the Euler Backward multirate schemes using constant or linear interpolation applied to the stable test equation (19) are always conditionally stable.*

Proof The multirate methods are conditionally stable if the stability conditions in (43) or (44) are valid for $H \to 0$. Therefore we will derive asymptotic approximations of these conditions. It easily follows that $\sigma = 1 + a_{22}H + O(H^2)$, $\gamma = 1 + \frac{a_{11}}{q}H + O(H^2)$, $\gamma^q = 1 + a_{11}H + O(H^2)$ and $\rho\delta = \frac{a_{12}a_{21}}{q}H^2 + O(H^3)$. The higher order terms of these numbers are not independent of q. Using these approximations, we can derive that $(1+\sigma)(1+\gamma^q) = 4 + O(H)$, $1 - \sigma\gamma^q = -(a_{11} + a_{22})H + O(H^2)$, $(1-\sigma)(1-\gamma^q) = a_{11}a_{22}H^2 + O(H^3)$,

$\rho\delta\sum_{l=0}^{q-1}\gamma^l(\frac{2l}{q}-1) = O(H^2)$ and $\rho\delta\sum_{l=0}^{q-1}\gamma^l = a_{12}a_{21}H^2 + O(H^3)$. In this way we obtain

$$\begin{array}{ll} (1+\sigma)(1+\gamma^q) + \rho\delta\sum_{l=0}^{q-1}\gamma^l = 4 + O(H), \\ \quad 1-\sigma\gamma^q \qquad\qquad = -(a_{11}+a_{22})H + O(H^2), \\ (1-\sigma)(1-\gamma^q) - \rho\delta\sum_{l=0}^{q-1}\gamma^l = (a_{11}a_{22}-a_{12}a_{21})H^2 + O(H^3). \end{array} \tag{47}$$

and

$$\begin{array}{ll} (1+\sigma)(1+\gamma^q) - \rho\delta\sum_{l=0}^{q-1}\gamma^l(\frac{2l}{q}-1) & = 4 + O(H), \\ \frac{\rho\delta}{q}\sum_{l=0}^{q-1}\gamma^l l - \sigma\gamma^q + 1 & = -(a_{11}+a_{22})H + O(H^2), \\ (1-\sigma)(1-\gamma^q) - \rho\delta\sum_{l=0}^{q-1}\gamma^l & = (a_{11}a_{22}-a_{12}a_{21})H^2 + O(H^3). \end{array} \tag{48}$$

After inserting these asymptotic expressions into (43) and (44), we obtain the following asymptotic stability conditions for (23), which coincide with the ones for (19)

$$\begin{array}{rl} \mathrm{tr}(\mathbf{A}) = & a_{11}+a_{22} \quad < 0, \\ \det(\mathbf{A}) = & a_{11}a_{22} - a_{12}a_{21} > 0. \end{array} \tag{49}$$

Thus indeed the Slow-Fast and Compound-Fast multirate methods using constant or linear interpolation are stable for $H \to 0$ (conditionally stable) because $\mathbf{A}$ is a stable matrix. □

Unconditional stability for $q \to \infty$ Now we will prove a theorem which gives sufficient stability conditions such that both methods are conditionally stable for $q \to \infty$. In the proof we need the following Lemma which is given below without proof.

LEMMA 3.5 *Consider the following rational function $P : \mathbb{R}^+ \to \mathbb{R}$ with*

$$\forall_{H>0} P(H) = \frac{A - BH}{A - CH - DH^2}$$

and $A, B, C, D \in \mathbb{R}$. If $A > 0, C < 0, D < 0, |B| < |C|$, this rational function P satisfies

$$\forall_{H>0} |P(H)| < 1.$$

THEOREM 3.6 *Consider the Euler Backward Slow-Fast and Compound-Fast multirate schemes using constant or linear interpolation applied to the stable test equation (19). If*

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ |a_{12}a_{21}| < |a_{11}a_{22}|, \end{cases} \tag{50}$$

the Slow-Fast version is unconditionally stable for $q \to \infty$. If

$$\begin{cases} a_{11} < 0, \\ -a_{11}a_{22} - 2a_{11}^2 < a_{12}a_{21} < a_{11}a_{22}, \end{cases} \tag{51}$$

the Compound-Fast version is unconditionally stable for $q \to \infty$.

Notice that the Compound-Fast method is more stable than the Slow-Fast method, because it does not need that $a_{22} < 0$ and $-a_{11}a_{22} - 2a_{11}^2 < a_{12}a_{21} < a_{11}a_{22}$ is a weaker condition than $|a_{12}a_{21}| < |a_{11}a_{22}|$.

Proof First we prove that $a_{11} < 0$ is necessary for both methods. If the multirate factor $q \to \infty$, it is necessary that $|\gamma| < 1$ in order to have $\gamma^q \to 0$. This means that the Euler Backward method is stable for the active part, which is the case if $a_{11} < 0$.

Taking the limit $q \to \infty$, it can be proved that $\rho\delta \sum_{l=0}^{q-1} \gamma^l \to \rho\delta \sum_{l=0}^{\infty} \gamma^l = \frac{\rho\delta}{1-\gamma}$ and $\rho\delta \sum_{l=0}^{q-1} \gamma^l \frac{l}{q} \to 0$. Thus it follows that the stability conditions in (43) and (44) have the same asymptotic behaviour for $q \to \infty$:

$$\begin{array}{ll} (1+\sigma)(1+\gamma^q) + \rho\delta \sum_{l=0}^{q-1} \gamma^l & \to 1+\sigma+\rho\delta\frac{1}{1-\gamma}, \\ 1-\sigma\gamma^q & \to 1, \\ (1-\sigma)(1-\gamma^q) - \rho\delta \sum_{l=0}^{q-1} \gamma^l & \to 1-\sigma-\rho\delta\frac{1}{1-\gamma}. \end{array} \tag{52}$$

$$\begin{array}{ll} (1+\sigma)(1+\gamma^q) - \rho\delta \sum_{l=0}^{q-1} \gamma^l (\frac{2l}{q} - 1) & \to 1+\sigma+\rho\delta\frac{1}{1-\gamma}, \\ \frac{\rho\delta}{q} \sum_{l=0}^{q-1} \gamma^l l - \sigma\gamma^q + 1 & \to 1, \\ (1-\sigma)(1-\gamma^q) - \rho\delta \sum_{l=0}^{q-1} \gamma^l & \to 1-\sigma-\rho\delta\frac{1}{1-\gamma}. \end{array} \tag{53}$$

This means that for $q \to \infty$ we have the following unconditional stability conditions

$$\begin{cases} 1+\sigma+\rho\delta\frac{1}{1-\gamma} > 0, \\ 1-\sigma-\rho\delta\frac{1}{1-\gamma} > 0. \end{cases} \Leftrightarrow \quad |\frac{\rho\delta}{1-\gamma} + \sigma| < 1. \tag{54}$$

Because of the definition of γ, δ in (28) it follows that $\frac{\delta}{1-\gamma} = -\frac{a_{12}}{a_{11}}$ and we get

$$|-\frac{a_{12}}{a_{11}}\rho + \sigma| < 1. \tag{55}$$

• Using (24) for the Slow-Fast method, condition (55) is equivalent to

$$\forall_{H>0} |P_{SF}(H)| = |-\frac{a_{12}}{a_{11}}\rho + \sigma| = \frac{|1 - \frac{a_{12}a_{21}}{a_{11}}H|}{|1 - a_{22}H|} < 1.$$

We use Lemma 3.5 to derive the following stability conditions

$$\begin{cases} a_{22} < 0, \\ |\frac{a_{12}a_{21}}{a_{11}}| < |a_{22}|. \end{cases}$$

The second condition is indeed equivalent to $|a_{12}a_{21}| < |a_{11}a_{22}|$. Thus we have proved that the Euler Backward Slow-Fast multirate method using constant or linear interpolation is indeed unconditionally stable for $q \to \infty$ if the conditions (50) hold.

• If we use (25) for the Compound-Fast method, condition (55) is equivalent to

$$\forall_{H>0} |P_{CF}(H)| = |-\frac{a_{12}}{a_{11}}\rho + \sigma| = \frac{|1 - (\frac{a_{12}a_{21}}{a_{11}} + a_{11})H|}{|1 - (a_{11}+a_{22})H + (a_{11}a_{22} - a_{12}a_{21})H^2|} < 1.$$

Again Lemma 3.5 gives us the following sufficient stability conditions

$$\begin{cases} a_{11} + a_{22} < 0, \\ a_{11}a_{22} - a_{12}a_{21} > 0, \\ |\frac{a_{12}a_{21}}{a_{11}} + a_{11}| < |a_{11} + a_{22}|. \end{cases}$$

The first two conditions are automatically fulfilled for a stable test equation. Because $a_{11} + a_{22} < 0$, the third condition is equivalent to

$$a_{11} + a_{22} < \frac{a_{12}a_{21}}{a_{11}} + a_{11} < -a_{11} - a_{22}. \tag{56}$$

From the left inequality in (56) we can derive

$$\frac{1}{a_{11}}(a_{11}a_{22} - a_{12}a_{21}) < 0. \tag{57}$$

The other inequality in (56) gives

$$a_{12}a_{21} > -a_{11}a_{22} - 2a_{11}^2. \tag{58}$$

Using $a_{11} < 0$ and combining the inequalities (57) and (58) gives us

$$-a_{11}a_{22} - 2a_{11}^2 < a_{12}a_{21} < a_{11}a_{22}. \tag{59}$$

Thus we have proved that the Euler Backward Compound-Fast multirate method using constant or linear interpolation is indeed unconditionally stable for $q \to \infty$ if the conditions (51) hold. □

3.4. *Remarks*

We have derived simplified sufficient stability conditions for the matrix $\mathbf{A}$ of the test equation (19) such that both Euler Backward multirate schemes are stable. For the asymptotic analysis for $H \to 0$ or $q \to \infty$ it does not matter whether constant or linear interpolation is used. First we proved that both Euler Backward multirate schemes are conditionally stable. We also proved that they are unconditionally stable for $q \to \infty$ if

- the subsystems are sufficiently decoupled;
- both the active and slow parts of the system are stable and solvable for the Slow-Fast version;
- only the active part of the system is stable and solvable for the Compound-Fast version.

The first condition is is very natural, because strongly coupled subsystems will have the same activity, which makes multirate not possible. The second conditions are not true for general partitions, which we showed in subsection 1.4.

4. Stability analysis of multistep BDF multirate algorithms

Because BDF methods of higher order are multistep methods, the previous analysis is not valid anymore. Therefore, this section deals in particular with the stability of the BDF multistep scheme for the Slow-Fast and Compound-Fast multirate versions. First we will show that the dynamics really can be described by the recurrence relation in (22). We only consider one type of interpolation of the latent part. Afterwards we will do the same stability analysis as in the previous section.

4.1. *Derivation of the recurrence relation*

Before we will show that the BDF multirate methods really obey (22) we give the following definitions.

Definition 4.1 Introduce the vector-valued function $\mathbf{e} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}^s$ with

$$\mathbf{e}(s, \omega) := \left[1, \omega, \ldots, \omega^{s-1}\right]^T \tag{60}$$

and the Vandermonde matrix $\mathbf{V} \in \mathbb{R}^{K\times K}$ with

$$v_{ij} = \begin{cases} 1 & i=j=1, \\ (1-i)^{j-1} & \text{otherwise.} \end{cases} \tag{61}$$

LEMMA 4.2 *Consider the Slow-Fast and the Compound-Fast versions of the BDF multirate scheme, both with integration orders (K,k). Let $\boldsymbol{z}_n, \boldsymbol{y}_n$ be defined as*

$$\boldsymbol{z}_n := \begin{pmatrix} z_n \\ \vdots \\ z_{n-K+1} \end{pmatrix} \in \mathbb{R}^K, \quad \boldsymbol{y}_n := \begin{pmatrix} y_n \\ \vdots \\ y_{n-k+1} \end{pmatrix} \in \mathbb{R}^k. \tag{62}$$

Then $\{\boldsymbol{z}_n\}$ and $\{\boldsymbol{y}_n\}$ are solutions of the following recurrence relation

$$\begin{pmatrix} \boldsymbol{z}_n \\ \boldsymbol{y}_n \end{pmatrix} = \underbrace{\begin{pmatrix} \boldsymbol{S}\ \boldsymbol{R} \\ \boldsymbol{T}\ \boldsymbol{N} \end{pmatrix}}_{\boldsymbol{M}} \begin{pmatrix} \boldsymbol{z}_{n-1} \\ \boldsymbol{y}_{n-1} \end{pmatrix}, \tag{63}$$

where, for the Slow-Fast version, $\boldsymbol{R} \in \mathbb{R}^{K\times k}, \boldsymbol{S} \in \mathbb{R}^{K\times K}$ are defined by

$$\boldsymbol{R} := \tilde{\rho} \begin{pmatrix} \tilde{\sigma}\, 0 \dots 0 \\ 0\ 0 & \\ \vdots\ \ \ddots & \\ 0 & 0 \end{pmatrix}, \boldsymbol{S} := \begin{pmatrix} -\tilde{\sigma}\frac{\rho_1}{\rho_0} & \dots & -\tilde{\sigma}\frac{\rho_K}{\rho_0} \\ 1 & & 0 \\ & \ddots & \vdots \\ & 1 & 0 \end{pmatrix}, \tilde{\rho} = \tfrac{a_{21}H}{\rho_0}, \tilde{\sigma} = \tfrac{\rho_0}{\rho_0 - a_{22}H} \tag{64}$$

and, for the Compound-Fast version,

$$\boldsymbol{R} := \tilde{\rho} \begin{pmatrix} -\tilde{\sigma}\frac{\rho_1}{\rho_0} & \dots & -\tilde{\sigma}\frac{\rho_K}{\rho_0} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \boldsymbol{S} := \begin{pmatrix} -\tilde{\sigma}\frac{\rho_1}{\rho_0} & \dots & -\tilde{\sigma}\frac{\rho_K}{\rho_0} \\ 1 & & 0 \\ & \ddots & \vdots \\ & 1 & 0 \end{pmatrix}, \begin{array}{l} \tilde{\rho} = \frac{a_{21}H}{\rho_0 - a_{11}H}, \\ \tilde{\sigma} = \frac{\rho_0(\rho_0 - a_{11}H)}{\rho_0^2 - \rho_0(a_{11}+a_{22})H + (a_{11}a_{22} - a_{12}a_{21})H^2}. \end{array} \tag{65}$$

In both cases, $\boldsymbol{N} \in \mathbb{R}^{k\times k}, \boldsymbol{T} \in \mathbb{R}^{k\times K}$ are given by

$$\boldsymbol{N} := \boldsymbol{G}^q + \textstyle\sum_{l=0}^{q-1} \boldsymbol{G}^l \boldsymbol{d}\boldsymbol{b}_{q-l}^T \boldsymbol{R},\ \boldsymbol{T} := \textstyle\sum_{l=0}^{q-1} \boldsymbol{G}^l \boldsymbol{d}\boldsymbol{b}_{q-l}^T \boldsymbol{S}, \tag{66}$$

where $\boldsymbol{G} \in \mathbb{R}^{k\times k}$ and $\boldsymbol{d} \in \mathbb{R}^k$ are defined by

$$\boldsymbol{G} := \begin{pmatrix} -\tilde{\gamma}\frac{\bar{\rho}_1}{\bar{\rho}_0} & \dots & -\tilde{\gamma}\frac{\bar{\rho}_k}{\bar{\rho}_0} \\ 1 & & 0 \\ & \ddots & \vdots \\ & 1 & 0 \end{pmatrix}, \boldsymbol{d} := \tilde{\delta} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tilde{\gamma} = \tfrac{\bar{\rho}_0}{\bar{\rho}_0 - a_{11}h}, \tilde{\delta} = \tfrac{a_{12}h}{\bar{\rho}_0 - a_{11}h}. \tag{67}$$

Here $\boldsymbol{b}_j \in \mathbb{R}^K$ is given by

$$\boldsymbol{b}_j = \boldsymbol{V}^{-T} \boldsymbol{e}(K, \frac{j}{q} - 1). \tag{68}$$

Proof

Analysis of the compound step In both the Slow-Fast and the Compound-Fast methods the latent variable is first integrated. Using constant extrapolation of y_{n-1} for the Slow-Fast method we obtain the system

$$\frac{\rho_0 z_n + \ldots + \rho_K z_{n-K}}{H} = a_{21} y_{n-1} + a_{22} z_n. \tag{69}$$

Because $K > 1$ we see that z_n also depends on previous values of $\{z_n\}$. From (69), it follows that $\mathbf{z}_n$ satisfies the next recurrence relation

$$\mathbf{z}_n = \mathbf{R}\mathbf{y}_{n-1} + \mathbf{S}\mathbf{z}_{n-1}, \tag{70}$$

where $\mathbf{R}, \mathbf{S}$ are given in (64). For the Compound-Fast method, we get a recurrence relation for $\{y_n\}$ and $\{z_n\}$:

$$\begin{cases} \frac{\rho_0 y_n + \ldots + \rho_K y_{n-K}}{H} = a_{11} y_n + a_{12} z_n, \\ \frac{\rho_0 z_n + \ldots + \rho_K z_{n-K}}{H} = a_{21} y_n + a_{22} z_n. \end{cases} \tag{71}$$

Because $\{z_n\}$ satisfies

$$\begin{aligned} z_n = & \frac{\rho_0(\rho_0 - H a_{21})}{(\rho_0 - H a_{22})(\rho_0 - H a_{11}) - a_{12} a_{21} H^2} \left(-\frac{\rho_1}{\rho_0} z_{n-1} - \ldots - \frac{\rho_K}{\rho_0} z_{n-K} \right) \\ & + \frac{\rho_0 H a_{21}}{(\rho_0 - H a_{22})(\rho_0 - H a_{11}) - a_{12} a_{21} H^2} \left(-\frac{\rho_1}{\rho_0} y_{n-1} - \ldots - \frac{\rho_K}{\rho_0} y_{n-K} \right), \end{aligned}$$

the solution satisfies again (70) with different values for $\mathbf{R}, \mathbf{S}$ in (65).

Analysis of the refinement step The active part is integrated on the time interval $[T_{n-1}, T_n]$ with q steps h:

$$\frac{\bar{\rho}_0 y_{n-1,j} + \ldots + \bar{\rho}_k y_{n-1,j-k}}{h} = a_{11} y_{n-1,j} + a_{12} \hat{z}_{n-1,j}. \tag{72}$$

The recurrence relation (72) is equivalent to

$$y_{n-1,j} = \frac{1}{\bar{\rho}_0 - a_{11} h} (-\bar{\rho}_1 y_{n-1,j-1} - \ldots - \bar{\rho}_k y_{n-1,j-k} + a_{12} h \hat{z}_{n-1,j}). \tag{73}$$

For both methods $z_{n-1,j}$ is estimated for $j \in \{1, \ldots, q-1\}$ employing $z_{n-k}, \ldots, z_n$.

LEMMA 4.3 *The interpolated value $\hat{z}_{n-1,j}$ can be retrieved from the coarse latent vector z_n by $\hat{z}_{n-1,j} = \boldsymbol{b}_j^T z_n$, where $\boldsymbol{b}_j$ is given in (68).*

Proof We can describe the numerical solution for z at the coarse grid by a truncated Taylor expansion around T_n

$$z^n(t) = \sum_{i=0}^{K} \bar{z}^n_{i+1} \left(\frac{t - T_n}{H} \right)^i.$$

The vector $\bar{\mathbf{z}}^n \in \mathbb{R}^K$ is the Nordsieck vector of length K:

$$\bar{\mathbf{z}}^n := \left(z^n(T_n), H \frac{d}{dt} z^n(T_n), \ldots, \frac{H^{K-1}}{(K-1)!} \frac{d^{K-1}}{dt^{K-1}} z^n(T_n) \right)^T.$$

Then we have

$$\hat{z}_{n-1,j} = z^n(t_{n-1,j}) = \sum_{i=0}^{K} \bar{z}_{i+1}^n \left(\frac{t_{n-1,j} - T_n}{H}\right)^i = \mathbf{e}(K, \frac{t_{n-1,j} - T_n}{H})^T \cdot \bar{\mathbf{z}}^n.$$

It is well-known that the Nordsieck vector $\bar{\mathbf{z}}^n \in \mathbb{R}^K$ and the vector $\mathbf{z}_n \in \mathbb{R}^K$ are related by

$$\mathbf{V}\bar{\mathbf{z}}^n = \mathbf{z}_n,$$

where $\mathbf{V}$ is the Vandermonde matrix which has been defined in Definition 4.1. Thus

$$\hat{z}_{n-1,j} = \mathbf{e}(K, \frac{t_{n-1,j} - T_n}{H})^T \mathbf{V}^{-1}\mathbf{z}_n = \mathbf{e}(K, \frac{j}{q} - 1)^T \mathbf{V}^{-1}\mathbf{z}_n$$

because $T_n = t_{n-1,q}$ and $H = qh$ and $\hat{z}_{n-1,j} = \mathbf{b}_j^T \mathbf{z}_n$, where $\mathbf{b}_j$ is given in (68). □

For the refinement we introduce the following vector $\in \mathbb{R}^k$

$$\mathbf{y}_{n-1,j} := \begin{pmatrix} y_{n-1,j} \\ \vdots \\ y_{n-1,j-k+1} \end{pmatrix}. \tag{74}$$

In vector notation we get

$$\mathbf{y}_{n-1,j} = \mathbf{G}\mathbf{y}_{n-1,j-1} + \mathbf{d}\hat{z}_{n-1,j}, \tag{75}$$

where $\mathbf{G}, \mathbf{d}$ are given in (67). For $j \in \{1, \ldots, q\}$ we have

$$\begin{aligned} \mathbf{y}_{n-1,j} &= \mathbf{G}\mathbf{y}_{n-1,j-1} + \mathbf{d}\mathbf{b}_j^T \mathbf{z}_n \\ &= \mathbf{G}^j \mathbf{y}_{n-1,0} + \textstyle\sum_{k=0}^{j-1} \mathbf{G}^{j-1-k} \mathbf{d}\mathbf{b}_{k+1}^T \mathbf{z}_n. \end{aligned} \tag{76}$$

Because the coarse and refined time-grids are synchronized, insert (70) into (76) for $j = q$, resulting in

$$\begin{aligned} \mathbf{y}_n = \mathbf{y}_{n-1,q} &= \mathbf{G}^q \mathbf{y}_{n-1,0} + \textstyle\sum_{k=0}^{q-1} \mathbf{G}^{q-1-k} \mathbf{d}\mathbf{b}_{k+1}^T (\mathbf{R}\mathbf{y}_{n-1} + \mathbf{S}\mathbf{z}_{n-1}) \\ &= \mathbf{N}\mathbf{y}_{n-1} + \mathbf{T}\mathbf{z}_{n-1}, \end{aligned} \tag{77}$$

where $\mathbf{N}, \mathbf{T}$ are given in (66). From (70) and (77) it indeed follows that $\{\mathbf{y}_n\}, \{\mathbf{z}_n\}$ satisfy the recurrence relation in (63). □

4.2. *Stability conditions*

Because the matrix $\mathbf{M} \in \mathbb{R}^{(K+k)\times(K+k)}$ in (63) is a higher dimensional matrix if $\max\{K, k\} > 1$, the stability conditions in (45) do not hold. One possible approach is to derive more accurate stability conditions by using the Routh-Hurwitz criterion. This becomes very tedious for higher orders and therefore we analyze the following two-dimensional recurrence relation for $\{\tilde{z}_n\}$ and $\{\tilde{y}_n\}$ instead

$$\begin{pmatrix} \tilde{z}_n \\ \tilde{y}_n \end{pmatrix} = \hat{\mathbf{M}} \begin{pmatrix} \tilde{z}_{n-1} \\ \tilde{y}_{n-1} \end{pmatrix}. \tag{78}$$

Here $\hat{\mathbf{M}} \in \mathbb{R}^{2\times 2}$ is properly chosen such that $\rho(\mathbf{M}) < \rho(\hat{\mathbf{M}})$. Because $\mathbf{S}, \mathbf{Y}$ are diagonalizable, there exist $\mathbf{V}, \bar{\mathbf{V}}, \Lambda, \bar{\Lambda}$ such that $\mathbf{S} = \mathbf{V}\Lambda\mathbf{V}^{-1}, \mathbf{Y} = \bar{\mathbf{V}}\bar{\Lambda}\bar{\mathbf{V}}^{-1}$; $\Lambda, \bar{\Lambda}$ diagonal. We introduce the number $L > 0$ with

$$L := \max\{\text{cond}(\mathbf{V}), \text{cond}(\bar{\mathbf{V}})\} \tag{79}$$

and the following matrices

$$\mathbf{P} = \frac{1}{\tilde{\rho}}\mathbf{S}^{-1}\mathbf{R}, \quad \mathbf{X} = \frac{1}{\tilde{\delta}}\sum_{l=0}^{q-1}\mathbf{G}^l \mathbf{db}_{q-l}^T, \quad \mathbf{Y} = \mathbf{G}^q. \tag{80}$$

Next we define the two-dimensional matrix $\hat{\mathbf{M}}$ by

$$\hat{\mathbf{M}} := \begin{bmatrix} \rho(\mathbf{S}) & L|\tilde{\rho}|\rho(\mathbf{S})||\mathbf{P}|| \\ L|\tilde{\delta}|\rho(\mathbf{S})||\mathbf{X}|| & \rho(\mathbf{Y}) + L^2|\tilde{\rho}\tilde{\delta}|||\mathbf{P}||\,||\mathbf{X}||\rho(\mathbf{S}) \end{bmatrix}. \tag{81}$$

It will appear that this two-dimensional matrix can be used to get simpler stability conditions for (63).

LEMMA 4.4 *Consider the matrices $M \in \mathbb{R}^{(K+k)\times(K+k)}$ in (63) and $\hat{M} \in \mathbb{R}^{2\times 2}$ in (81). Then for the spectral radii of M and $\hat{M}$ we have the relation*

$$\rho(M) \leq \rho(\hat{M}). \tag{82}$$

Proof

Let $\mathbf{P}, \mathbf{X}, \mathbf{Y}$ be the matrices as defined in (80). There exist the following relations between the block matrices of $\mathbf{M}$

$$\mathbf{R} = \tilde{\rho}\mathbf{S}\mathbf{P}, \quad \mathbf{N} = \mathbf{Y} + \tilde{\delta}\mathbf{X}\mathbf{R}, \quad \mathbf{T} = \tilde{\delta}\mathbf{X}\mathbf{S}.$$

Thus the companion matrix $\mathbf{M}$ can be factorized as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{S} & \mathbf{R} \\ \mathbf{T} & \mathbf{N} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \\ \tilde{\delta}\mathbf{X} & \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{S} & \\ & \mathbf{Y} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} & \tilde{\rho}\mathbf{P} \\ & \mathbf{I} \end{bmatrix}.$$

After performing the following transformation

$$\begin{aligned} \tilde{\mathbf{M}} &= \begin{bmatrix} \mathbf{V}^{-1} & \\ & \bar{\mathbf{V}}^{-1} \end{bmatrix} \mathbf{M} \begin{bmatrix} \mathbf{V} & \\ & \bar{\mathbf{V}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \\ \tilde{\delta}\bar{\mathbf{V}}^{-1}\mathbf{X}\mathbf{V} & \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \Lambda & \\ & \bar{\Lambda} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} & \tilde{\rho}\mathbf{V}^{-1}\mathbf{P}\bar{\mathbf{V}} \\ & \mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} \Lambda & \tilde{\rho}\Lambda\mathbf{V}^{-1}\mathbf{P}\bar{\mathbf{V}} \\ \tilde{\delta}\bar{\mathbf{V}}^{-1}\mathbf{X}\mathbf{V}\Lambda & \bar{\Lambda} + \tilde{\rho}\tilde{\delta}\mathbf{V}^{-1}\bar{\mathbf{V}}^{-1}\mathbf{X}\mathbf{V}\Lambda\mathbf{P}\bar{\mathbf{V}} \end{bmatrix}, \end{aligned} \tag{83}$$

it follows that $\rho(\mathbf{M}) = \rho(\tilde{\mathbf{M}})$. Because of the construction of $\hat{\mathbf{M}}$ it immediately follows that for all $n \in \mathbb{N}$

$$||\tilde{\mathbf{M}}^n|| \leq ||\hat{\mathbf{M}}^n||. \tag{84}$$

Thus it also follows that for all n $||\tilde{\mathbf{M}}^n||^{\frac{1}{n}} \leq ||\hat{\mathbf{M}}^n||^{\frac{1}{n}}$. Using the properties

$$\lim_{n\to\infty} ||\tilde{\mathbf{M}}^n||^{\frac{1}{n}} = \rho(\tilde{\mathbf{M}}), \quad \lim_{n\to\infty} ||\hat{\mathbf{M}}^n||^{\frac{1}{n}} = \rho(\hat{\mathbf{M}}) \tag{85}$$

gives us that $\rho(\tilde{\mathbf{M}}) \leq \rho(\hat{\mathbf{M}})$. Because $\mathbf{M}, \tilde{\mathbf{M}}$ are similar we get the required identity. □

Now we are able to prove the following theorem.

THEOREM 4.5 *Consider the recurrence relation in (63) which describes the dynamical behaviour of the Slow-Fast and Compound-Fast versions of the BDF multirate schemes for the stable test equation (19). Then the schemes are stable for all H, q if*

$$\begin{aligned}(1+\rho(\boldsymbol{S}))(1+\rho(\boldsymbol{G}^q)) &> -L^2|\tilde{\rho\delta}|\|\boldsymbol{P}\|\|\boldsymbol{X}\|,\\ \rho(\boldsymbol{S})\rho(\boldsymbol{G}^q) &< 1+L^2|\tilde{\rho\delta}|\|\boldsymbol{P}\|\|\boldsymbol{X}\|,\\ (1-\rho(\boldsymbol{S}))(1-\rho(\boldsymbol{G}^q)) &> L^2|\tilde{\rho\delta}|\|\boldsymbol{P}\|\|\boldsymbol{X}\|.\end{aligned} \tag{86}$$

Because of (66) this is the case if

$$\begin{aligned}(1+\rho(\boldsymbol{S}))(1+\rho(\boldsymbol{G}^q)) &> -L^2|\tilde{\rho\delta}|\|\boldsymbol{S}^{-1}\boldsymbol{R}\|\|\textstyle\sum_{l=0}^{q-1}\boldsymbol{G}^l\boldsymbol{db}_{q-l}^T\|,\\ \rho(\boldsymbol{S})\rho(\boldsymbol{G}^q) &< 1+L^2|\tilde{\rho\delta}|\|\boldsymbol{S}^{-1}\boldsymbol{R}\|\|\textstyle\sum_{l=0}^{q-1}\boldsymbol{G}^l\boldsymbol{db}_{q-l}^T\|,\\ (1-\rho(\boldsymbol{S}))(1-\rho(\boldsymbol{G}^q)) &> L^2|\tilde{\rho\delta}|\|\boldsymbol{S}^{-1}\boldsymbol{R}\|\|\textstyle\sum_{l=0}^{q-1}\boldsymbol{G}^l\boldsymbol{db}_{q-l}^T\|.\end{aligned} \tag{87}$$

Proof The methods are stable if $\rho(\mathbf{M}) < 1$ for all $H, q > 0$ and stable matrices $\mathbf{A}$. In Lemma 4.4 it is shown that $\rho(\hat{\mathbf{M}}) < 1 \Rightarrow \rho(\mathbf{M}) < 1$, where $\hat{\mathbf{M}} \in \mathbb{R}^{2\times 2}$ is given in (81). Because $\hat{\mathbf{M}}$ is a real two-dimensional matrix the stability conditions in Lemma 3.2 can be used. It simply follows that

$$\begin{aligned}\rho(\hat{\mathbf{M}}) < 1 &\Leftrightarrow \begin{cases}1+\mathrm{tr}(\hat{\mathbf{M}})+\det(\hat{\mathbf{M}}) > 0,\\ \det(\hat{\mathbf{M}}) < 1,\\ 1-\mathrm{tr}(\hat{\mathbf{M}})+\det(\hat{\mathbf{M}}) > 0.\end{cases}\\ &\Leftrightarrow \begin{cases}1+\rho(\mathbf{S})+\rho(\mathbf{Y})+L^2\|\mathbf{P}\|\|\mathbf{X}\|+\rho(\mathbf{S})(\rho(\mathbf{Y})+L^2|\tilde{\rho\delta}|\rho(\mathbf{S})\|\mathbf{P}\|\|\mathbf{X}\|)-L^2|\tilde{\rho\delta}|\rho(\mathbf{S})^2\|\mathbf{P}\|\|\mathbf{X}\| > 0,\\ \rho(\mathbf{S})(\rho(\mathbf{Y})+L^2|\tilde{\rho\delta}|\rho(\mathbf{S})\|\mathbf{P}\|\|\mathbf{X}\|)-L^2|\tilde{\rho\delta}|\rho(\mathbf{S})^2\|\mathbf{P}\|\|\mathbf{X}\| < 1,\\ 1-\rho(\mathbf{S})-\rho(\mathbf{Y})-L^2|\tilde{\rho\delta}|\|\mathbf{P}\|\|\mathbf{X}\|+\rho(\mathbf{S})(\rho(\mathbf{Y})+L^2|\tilde{\rho\delta}|\rho(\mathbf{S})\|\mathbf{P}\|\|\mathbf{X}\|)-L^2|\tilde{\rho\delta}|\rho(\mathbf{S})^2\|\mathbf{P}\|\|\mathbf{X}\| > 0.\end{cases}\\ &\Leftrightarrow \begin{cases}1+\rho(\mathbf{S})+\rho(\mathbf{Y})+L^2|\tilde{\rho\delta}|\|\mathbf{P}\|\|\mathbf{X}\|+\rho(\mathbf{S})\rho(\mathbf{Y}) > 0,\\ \rho(\mathbf{S})\rho(\mathbf{Y}) < 1,\\ 1-\rho(\mathbf{S})-\rho(\mathbf{Y})-L^2|\tilde{\rho\delta}|\|\mathbf{P}\|\|\mathbf{X}\|+\rho(\mathbf{S})\rho(\mathbf{Y}) > 0.\end{cases}\\ &\Leftrightarrow \begin{cases}(1+\rho(\mathbf{S}))(1+\rho(\mathbf{Y})) > -L^2|\tilde{\rho\delta}|\|\mathbf{P}\|\|\mathbf{X}\|,\\ \rho(\mathbf{S})\rho(\mathbf{Y}) < 1+L^2|\tilde{\rho\delta}|\|\mathbf{P}\|\|\mathbf{X}\|,\\ (1-\rho(\mathbf{S}))(1-\rho(\mathbf{Y})) > L^2|\tilde{\rho\delta}|\|\mathbf{P}\|\|\mathbf{X}\|.\end{cases}\end{aligned} \tag{88}$$

After substituting the expressions in (80) for $\mathbf{P}, \mathbf{X}, \mathbf{Y}$ we obtain the sufficient stability conditions for (86). The remainder of the theorem follows immediately. □

Notice that the stability conditions for the multistep case are very similar to the conditions for the onestep case in (44). The first inequality in (86) is always fulfilled. Thus we have the following sufficient stability conditions

$$\begin{cases}\rho(\mathbf{S})\rho(\mathbf{G}^q) < 1,\\ |\tilde{\rho\delta}|\|\mathbf{P}\|\|\mathbf{X}\| < \frac{1}{L^2}(1-\rho(\mathbf{S}))(1-\rho(\mathbf{G}^q)).\end{cases} \tag{89}$$

The following Lemma enables us to express the conditions for $\rho(\mathbf{S}), \rho(\mathbf{G}^q)$ in terms of $\tilde{\sigma}, \tilde{\gamma}$.

LEMMA 4.6 *For both companion matrices $\boldsymbol{S}, \boldsymbol{G}$ of order $p \in \{1, \ldots, 6\}$ there exist $\mu_p, \nu_p \in [0,1]$ with $\mu_p + \nu_p = 1$, such that for $\tilde{\sigma}, \tilde{\gamma} \in [0,1]$*

$$\rho(\boldsymbol{S}) \le \mu_p + \nu_p\tilde{\sigma}, \quad \rho(\boldsymbol{G}) \le \mu_p + \nu_p\tilde{\gamma}. \tag{90}$$

Proof Because both $\mathbf{G}$ and $\mathbf{S}$ are equal if $K = k$ and $\tilde{\sigma} = \tilde{\gamma}$, it is sufficient to prove (90) only for $\mathbf{S}$. Figure

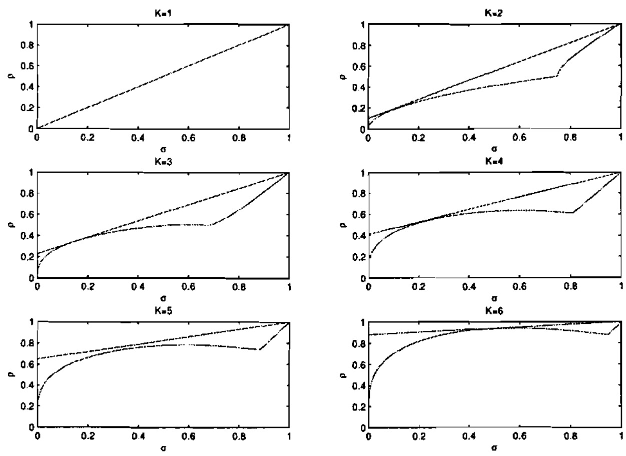


Figure 1. The relationship between $\rho(\mathbf{S})$ and $\mu_K + \nu_K\tilde{\sigma}$.

1 shows the relationship between $\rho(\mathbf{S})$ and $\mu_K + \nu_K\tilde{\sigma}$ for $p \in \{1, \dots, 6\}$ for the following values of μ_p, ν_p

p	1	2	3	4	5	6
ρ_0	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$
μ_p	0	0.1	0.23	0.41	0.65	0.88
ν_p	1	0.9	0.77	0.59	0.35	0.12

It is clear that for all $\tilde{\sigma} \in [0, 1]$ it applies that $\rho(\mathbf{S}) \leq \mu_p + \nu_p\tilde{\sigma}$.

□

Notice that these bounds are only true if $\tilde{\sigma}, \tilde{\gamma} \geq 0$. By using these bounds, we get therefore the following sufficient stability conditions

$$\begin{cases} (1 + \nu_K(\tilde{\sigma} - 1))(1 + \nu_k(\tilde{\gamma} - 1))^q < 1, \\ |\tilde{\rho}\tilde{\delta}|\,\|\mathbf{P}\|\,\|\mathbf{X}\| < \frac{1}{L^2}\nu_K(1 - \tilde{\sigma})(1 - (1 + \nu_k(\tilde{\gamma} - 1))^q), \\ \tilde{\sigma} \geq 0, \\ \tilde{\gamma} \geq 0. \end{cases} \tag{91}$$

4.3. *Asymptotic stability conditions*

In the previous section we derived more compact stability conditions from (44) by means of asymptotic analysis. This idea will be generalized to the BDF multirate methods of higher order. Since also the stability conditions (86) are very complex, we will derive more compact stability conditions based on an asymptotic analysis. First we will prove that the studied multirate schemes are always conditionally stable. Second we also will give sufficient conditions for $q \to \infty$ such that the methods are stable for all H.

Conditional stability In this paragraph we will investigate the conditions for conditional stability which can be retrieved by an asymptotic analysis for $H \to 0$.

THEOREM 4.7 *If*

$$\begin{cases} \boldsymbol{A} \text{ is stable}, \\ a_{11} < 0, \\ a_{22} < 0, \\ |a_{21}a_{12}| < C|a_{11}a_{22}|, \end{cases} \tag{92}$$

where $C = \frac{q\nu_K\nu_k}{L^2\|\boldsymbol{P}\|\|\boldsymbol{X}\|}$, *the Slow-Fast and Compound-Fast versions of the BDF multirate schemes applied to the stable test equation (19) are conditionally stable for* $H \to 0$.

Proof Because $\mathbf{A}$ is a stable matrix, we have the following properties

$$\begin{aligned} \operatorname{tr}(\mathbf{A}) &= \quad a_{11} + a_{22} \quad < 0, \\ \det(\mathbf{A}) &= a_{11}a_{22} - a_{12}a_{21} > 0. \end{aligned} \tag{93}$$

For $H \to 0$ we have the following asymptotic expansions in H

$$\begin{array}{ll} \tilde{\sigma} \doteq 1 + \frac{a_{22}}{\rho_0}H, & 1 + \nu_K(\tilde{\sigma} - 1) \doteq 1 + \nu_K\frac{a_{22}}{\rho_0}H, \\ \tilde{\gamma} \doteq 1 + \frac{a_{11}}{q\bar{\rho}_0}H, & (1 + \nu_k(\tilde{\gamma} - 1))^q \doteq 1 + \nu_k\frac{a_{11}}{\bar{\rho}_0}H, \\ \tilde{\rho}\tilde{\delta} \doteq \frac{a_{21}a_{12}Hh}{\rho_0\bar{\rho}_0}, & (1 - (1 + \nu_k(\tilde{\gamma} - 1))^q) \doteq -\nu_k\frac{a_{11}}{\bar{\rho}_0}H. \end{array}$$

For $H \to 0$ it always holds that $\tilde{\sigma}, \tilde{\gamma}$ are positive numbers. Because $\|\mathbf{P}\| = \|\mathbf{P}_0\|$ and $\|\mathbf{X}\| \doteq \|\sum_{l=0}^{q-1} \mathbf{G}_0^l \mathbf{e}_1 \mathbf{b}_{q-l}^T\|$ we get the following asymptotic stability conditions instead of (91).

$$\begin{cases} 1 + \nu_K\frac{a_{22}}{\rho_0}H + \nu_k\frac{a_{11}}{\bar{\rho}_0}H < 1, \\ \frac{|a_{21}a_{12}|Hh}{\rho_0\bar{\rho}_0}\|\mathbf{P}\|\|\mathbf{X}\| < \frac{1}{L^2}\nu_K\frac{a_{22}}{\rho_0}H\nu_k\frac{a_{11}}{\bar{\rho}_0}H. \end{cases} \tag{94}$$

The first order conditions are

$$\begin{cases} \nu_K\frac{a_{22}}{\rho_0}H + \nu_k\frac{a_{11}}{\bar{\rho}_0}H < 0, \\ |a_{21}a_{12}|\frac{1}{q}\|\mathbf{P}\|\|\mathbf{X}\| < \frac{1}{L^2}\nu_K\nu_k a_{11}a_{22}. \end{cases} \tag{95}$$

If $K = k$, such that $\frac{\nu_K}{\rho_0} = \frac{\nu_k}{\bar{\rho}_0}$, the first stability condition is always fulfilled for a stable $\mathbf{A}$. This first condition is also fulfilled for a stable $\mathbf{A}$ if $a_{11} < 0$ and $k \leq K$, such that $\frac{\nu_k}{\bar{\rho}_0} \geq \frac{\nu_K}{\rho_0}$, or if $a_{22} < 0$ and $k \geq K$, such that $\frac{\nu_k}{\bar{\rho}_0} \leq \frac{\nu_K}{\rho_0}$.

The second stability condition is fulfilled if

$$|a_{21}a_{12}| < Ca_{11}a_{22}, \quad C = \frac{q\nu_K\nu_k}{L^2\|\mathbf{P}\|\|\mathbf{X}\|}.$$

Because $|a_{21}a_{12}| > 0$ it is also necessary that $a_{11}a_{22} > 0$ which is the case if $a_{11}, a_{22} \leq 0$. □

Unconditional stability for $q \to \infty$ In this part we investigate the stability for $q \to \infty$ and $H > 0$. It appears that the stability conditions in (89) can be simplified by using the limit values $\mathbf{X}, \mathbf{R}$.

LEMMA 4.8 *For* $q \to \infty$ *it applies that*

$$\boldsymbol{X} \to (\boldsymbol{I} - \boldsymbol{G})^{-1}\boldsymbol{e}_1\boldsymbol{e}_1^T = \frac{1}{\tilde{\gamma} - 1}\boldsymbol{e}\boldsymbol{e}_1^T,$$

and

$$\boldsymbol{R} = \tilde{\rho}\boldsymbol{S}\boldsymbol{P},$$

where $\boldsymbol{P}$ *is* $\boldsymbol{e}_K\boldsymbol{w}^T$*, with* $\boldsymbol{w}^T = \frac{\rho_0}{\rho_K}\boldsymbol{e}_1^T$ *for the Slow-Fast method and* $\boldsymbol{w}^T = [\frac{\rho_1}{\rho_K}, \ldots, \frac{\rho_K}{\rho_K}]$ *for the Compound-Fast method.*

Proof For $q \to \infty$ we have that

$$\mathbf{X} = \frac{1}{\tilde{\delta}}\sum_{l=0}^{q-1}\mathbf{G}^l\mathbf{d}\mathbf{b}_{q-l}^T \to \sum_{l=0}^{\infty}\mathbf{G}^l\mathbf{e}_1\mathbf{e}_1^T = (\mathbf{I}-\mathbf{G})^{-1}\mathbf{e}_1\mathbf{e}_1^T.$$

It can be derived that

$$(\mathbf{I}-\mathbf{G})^{-1}\mathbf{e}_1\mathbf{e}_1^T = \frac{1}{1+\tilde{\gamma}(\frac{\bar{\rho}_1}{\bar{\rho}_0}+\ldots+\frac{\bar{\rho}_K}{\bar{\rho}_0})} = \frac{1}{1-\tilde{\gamma}},$$

because of the consistency condition $\bar{\rho}_0 + \ldots + \bar{\rho}_K = 0$. The other property follows from the definition of $\mathbf{P}$ in (80). □

Before we state the stability theorem we need the following Lemma.

LEMMA 4.9 *For both companion matrices* $\boldsymbol{S}, \boldsymbol{G}$ *of order* $p \in \{1, \ldots, 6\}$ *it applies that*

$$0 < \tilde{\sigma} < 1 \Rightarrow \rho(\boldsymbol{S}) < 1, \quad 0 < \tilde{\gamma} < 1 \Rightarrow \rho(\boldsymbol{G}) < 1, \quad 0 < \tilde{\gamma}^q < 1 \Rightarrow \rho(\boldsymbol{G}^q) < 1.$$

Proof The matrix $\mathbf{S}$ has the following characteristic equation

$$\lambda^p + \frac{\tilde{\sigma}}{\rho_0}(\rho_1\lambda^{p-1} + \ldots + \rho_p) = 0,$$

which is equivalent to

$$\frac{\rho_0}{\tilde{\sigma}}\lambda^p + \rho_1\lambda^{p-1} + \ldots + \rho_p = 0.$$

Using the BDF-p method for the test equation $\dot{y} = \lambda y$ gives us the characteristic polynomial

$$(\rho_0 - h\lambda)\lambda^p + \rho_1\lambda^{p-1} + \ldots + \rho_p = 0.$$

It is well-known that for $h\lambda \in \mathbb{R}^-$ it holds that the numerical solution will be stable up to order $p = 6$. It follows that $\rho(\mathbf{S}) < 1$ if

$$\frac{\rho_0}{\tilde{\sigma}} > \rho_0,$$

which is equivalent to

$$0 < \tilde{\sigma} < 1 \Rightarrow \rho(\mathbf{S}) < 1.$$

Because $\mathbf{G} = \mathbf{S}$ if $K = k$ and $\tilde{\sigma} = \tilde{\gamma}$, it immediately follows that

$$0 < \tilde{\gamma} < 1 \Rightarrow \rho(\mathbf{G}) < 1.$$

Because $0 < \tilde{\gamma} < 1 \Leftrightarrow 0 < \tilde{\gamma}^q < 1$ and $\rho(\mathbf{G}) < 1 \Leftrightarrow \rho(\mathbf{G}^q) < 1$, it also holds that

$$0 < \tilde{\gamma}^q < 1 \Rightarrow \rho(\mathbf{G}^q) < 1.$$

□

THEOREM 4.10 *If*

$$\begin{cases} \boldsymbol{A} \text{ is stable}, \\ a_{11} < 0, \\ a_{22} < 0, \\ |a_{21}a_{12}| < D|a_{11}a_{22}|, \end{cases} \tag{96}$$

where $D = \frac{\nu_K}{L^2\|\boldsymbol{P}\|}$, *both the Slow-Fast and Compound-Fast BDF multirate schemes applied to the stable test equation (19) are unconditionally stable for* $q \to \infty$.

Proof The stability conditions in (89) are only fulfilled for $q \to \infty$ if $\rho(\mathbf{G}) < 1$, such that $\rho(\mathbf{G}^q) \to 0$. Then we get the following stability conditions

$$\begin{cases} \rho(\mathbf{G}) < 1, \\ \rho(\mathbf{S}) + L^2|\tilde{\rho}\tilde{\delta}|\|\mathbf{P}\|\|\mathbf{X}\| < 1. \end{cases}$$

By using the Lemmas 4.6 and 4.9 it is possible to derive the following sufficient stability conditions

$$\begin{cases} \tilde{\gamma} \in [0,1], \\ \mu_K + \nu_K\tilde{\sigma} + L^2|\tilde{\rho}\tilde{\delta}|\|\mathbf{P}\|\|\mathbf{X}\| < 1, \\ \tilde{\sigma} > 0. \end{cases}$$

The first condition is indeed fulfilled if $a_{11} < 0$. The second stability condition is equivalent to

$$\mu_K + \nu_K\tilde{\sigma} + L^2\|\mathbf{P}\|\frac{|\tilde{\rho}\tilde{\delta}|}{|\tilde{\gamma}-1|} < 1.$$

Because $\frac{\tilde{\delta}}{\tilde{\gamma}-1} = -\frac{a_{12}}{a_{11}}$, we get

$$\mu_K + \nu_K\tilde{\sigma} + L^2\|\mathbf{P}\||\tilde{\rho}|\frac{|a_{12}|}{|a_{11}|} < 1. \tag{97}$$

- For the Slow-Fast method this implies

$$\mu_K + \nu_K\frac{\rho_0}{\rho_0 - a_{22}H} + L^2\|\mathbf{P}\||a_{21}|\frac{1}{\rho_0}H\frac{|a_{12}|}{|a_{11}|} < 1,$$

or

$$L^2\|\mathbf{P}\||a_{21}|\frac{1}{\rho_0}H\frac{|a_{12}|}{|a_{11}|} < 1 - \mu_K - \nu_K\frac{\rho_0}{\rho_0 - a_{22}H}.$$

Using $\mu_K = 1 - \nu_K$, we derive

$$L^2\|\mathbf{P}\||a_{21}|\frac{1}{\rho_0}H\frac{|a_{12}|}{|a_{11}|} < \nu_K - \nu_K\frac{\rho_0}{\rho_0 - a_{22}H} = -\frac{\nu_K a_{22}H}{\rho_0 - a_{22}H}.$$

Because $a_{22} < 0$ we get

$$L^2\|\mathbf{P}\||a_{21}|\frac{1}{\rho_0}H\frac{|a_{12}|}{|a_{11}|} < -\frac{\nu_K a_{22} H}{\rho_0},$$

yielding

$$|a_{21}a_{12}| < -D\nu_K a_{22}|a_{11}| = D|a_{11}a_{22}|,$$

where $D = \frac{\nu_K}{L^2\|\mathbf{P}\|}$. Because $a_{22} < 0$ it also holds that $\tilde{\sigma} > 0$. Thus indeed the Slow-Fast multirate method is stable for $q \to \infty$ if the stability conditions (96) are satisfied.

• For the Compound-Fast method we obtain for (97)

$$\mu_K + \nu_K \frac{\rho_0(\rho_0 - a_{11}H)}{\rho_0^2 - \rho_0(a_{11}+a_{22})H + (a_{11}a_{22} - a_{12}a_{21})H^2} + L^2\|\mathbf{P}\||\frac{a_{21}H}{\rho_0 - a_{11}H}|\frac{|a_{12}|}{|a_{11}|} < 1$$

or

$$L^2\|\mathbf{P}\||\frac{a_{21}H}{\rho_0 - a_{11}H}|\frac{|a_{12}|}{|a_{11}|} < 1 - \mu_K - \nu_K \frac{\rho_0(\rho_0 - a_{11}H)}{\rho_0^2 - \rho_0(a_{11}+a_{22})H + (a_{11}a_{22} - a_{12}a_{21})H^2}.$$

Using $\mu_K = 1 - \nu_K$, we get

$$\begin{aligned} L^2\|\mathbf{P}\||a_{21}|\tfrac{1}{\rho_0}H\tfrac{|a_{12}|}{|a_{11}|} &< \nu_K - \nu_K \tfrac{\rho_0(\rho_0 - a_{11}H)}{\rho_0^2 - \rho_0(a_{11}+a_{22})H + (a_{11}a_{22} - a_{12}a_{21})H^2} \\ &= -\tfrac{\nu_K(\rho_0 a_{22}H - \det(\mathbf{A})H^2)}{\rho_0^2 - \rho_0(a_{11}+a_{22})H + (a_{11}a_{22} - a_{12}a_{21})H^2}, \end{aligned}$$

or

$$L^2\|\mathbf{P}\||a_{21}a_{12}| < -\frac{\nu_K|a_{11}|a_{22}(\rho_0^2 - \rho_0\frac{\det(\mathbf{A})}{a_{22}}H)}{\rho_0^2 - \rho_0(a_{11}+a_{22})H + (a_{11}a_{22} - a_{12}a_{21})H^2}.$$

Because $\mathrm{tr}(\mathbf{A}) = a_{11} + a_{22} < 0$ and $\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21} > 0$, we derive the following sufficient stability condition

$$L^2\|\mathbf{P}\||a_{21}a_{12}| < -\nu_K|a_{11}|a_{22}(1 - \frac{H}{\rho_0 a_{22}}\det(\mathbf{A})).$$

If $a_{22} < 0$ and $\det(\mathbf{A}) > 0$ it is immediately clear that

$$-a_{22}(1 - \frac{\det(\mathbf{A})}{\rho_0 a_{22}}H) = |a_{22}|(1 - \frac{\det(\mathbf{A})}{\rho_0 a_{22}}H) > |a_{22}|. \quad (98)$$

Thus the BDF Compound-Fast multirate method is indeed stable for $q \to \infty$ if the stability conditions in (96) hold because then it holds that $D|a_{11}a_{22}| < -D|a_{11}|a_{22}(1 - \frac{\det(\mathbf{A})}{\rho_0 a_{22}}H)$. Because $a_{22} < 0$ it immediately follows that $\tilde{\sigma} > 0$ is always fulfilled. Thus also the Compound-Fast multirate method is stable for $q \to \infty$ if the stability conditions in (96) are satisfied. □

Because of the restriction $\rho(\mathbf{G}) < 1$ it appears possible to reduce the stability conditions as follows.

LEMMA 4.11 *For $q \to \infty$ the matrix $\boldsymbol{M}$ in (63) is stable if*

$$\rho(\boldsymbol{S}(\boldsymbol{I} + \tilde{\rho}\tilde{\delta}\boldsymbol{P}\boldsymbol{X})) < 1. \quad (99)$$

Proof For $q \to \infty$ we have that

$$\mathbf{M} \to \begin{pmatrix} \mathbf{S} & \mathbf{R} \\ \tilde{\delta}\mathbf{X}\mathbf{S} & \tilde{\delta}\mathbf{X}\mathbf{R} \end{pmatrix}.$$

For each eigenpair it holds that

$$\begin{aligned} \mathbf{S}\mathbf{x} + \mathbf{R}\mathbf{y} &= \lambda\mathbf{x} \\ \tilde{\delta}\mathbf{X}\mathbf{S}\mathbf{x} + \tilde{\delta}\mathbf{X}\mathbf{R}\mathbf{y} &= \lambda\mathbf{y}. \end{aligned}$$

It results that

$$\tilde{\delta}\mathbf{X}\mathbf{S}\mathbf{x} + \tilde{\delta}\mathbf{X}(\lambda\mathbf{x} - \mathbf{S}\mathbf{x}) = \lambda\tilde{\delta}\mathbf{X}\mathbf{x} = \lambda\mathbf{y}.$$

Thus we get for each eigenpair that $\mathbf{y} = \tilde{\delta}\mathbf{X}\mathbf{x}$. Hence we can reduce the eigenvalue problem to

$$\mathbf{S}\mathbf{x} + \tilde{\delta}\mathbf{R}\mathbf{X}\mathbf{x} = (\mathbf{S} + \tilde{\delta}\mathbf{R}\mathbf{X})\mathbf{x} = \lambda\mathbf{x}.$$

Clearly the method is stable for $q \to \infty$ if $\rho(\mathbf{S} + \tilde{\delta}\mathbf{R}\mathbf{X}) < 1$. Because of the property $\mathbf{R} = \tilde{\rho}\mathbf{S}\mathbf{P}$, this is equivalent to

$$\rho(\mathbf{S}(\mathbf{I} + \tilde{\rho}\tilde{\delta}\mathbf{P}\mathbf{X})) < 1.$$

□

4.4. *Remarks*

We have derived simplified sufficient stability conditions for the multirate BDF Slowest first methods applied to the test equation (19). First we proved that both BDF multirate schemes are conditionally stable. We also proved that both BDF multirate schemes applied to the stable test equation (19) are stable for $q \to \infty$ if

- the subsystems are sufficiently decoupled;
- the active and slow parts of the system are stable and solvable.

5. Conclusions

Multirate methods are attractive for initial value problems for DAEs with latency or multirate behaviour. We studied the Slow-Fast version of the BDF scheme because of stepsize control reasons. The BDF methods are very suitable for the interpolation at the refined time-grid. We also studied the Compound-Fast version which is more stable than the Slow-Fast method. For practical use of these methods it is very important that the multirate schemes are stable. Local stability can be proved by a stability analysis on a linear two-dimensional test equation. We also studied the stability of the Compound-Fast - and BDF Slow-Fast multirate schemes applied to the stable test equation (19) if the multirate factor $q \to \infty$. It is not clear yet whether the stability conditions for $q \to \infty$ automatically imply the stability for a finite multirate factor $1 < q < \infty$. For both methods it is necessary that the subsystems are sufficiently decoupled and that the active and slow parts of the system are stable and solvable.

For a general partition the active part of a stable DAE or ODE is not automatically stable. For DAEs also the solvability and index are not preserved for the active part.

The approach used in this paper can be extended to find also stability conditions for the multi-dimensional test equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ in (17) or for the DAE test equations (16) and (18).

References

[1] A.Bartel. *Generalised Multirate: Two ROW-type Versions for Circuit Simulation* MSc Thesis, TU Darmstadt & IWRMM Universität Karlsruhe, 2000.

[2] A. Bartel, M. Günther. *A multirate W-method for electrical networks in state-space formulation*, J. of Comput. and Applied Maths., Vol. 147, pp. 411-425, 2002.

[3] C.W. Gear, D.R. Wells. *Multirate linear multistep methods*, BIT, 24, 484-502, 1984.

[4] G.R. G'omez. *Absolute stability analysis of semi-implicit multirate linear multistep methods*, PhD-thesis, Instituto Nacional de Astrofisica, Optica y Electronica, Tonantzintla, Pue, Mexico, 2002.

[5] A. El Guennouni, A. Verhoeven, E.J.W. ter Maten, T.G.J. Beelen. *Aspects of Multirate Time Integration Methods in Circuit Simulation Problems*, In: A. Di Bucchianico, R.M.M. Mattheij, M.A. Peletier, "Progress in Industrial Mathematics at ECMI 2004", pp 579-584, Springer, 2006.

[6] E. Hairer, S.P. Nørsett, G. Wanner. *Solving Ordinary Differential Equations I, nonstiff problems* Springer, 1993.

[7] W. Hundsdorfer, V. Savcenco. *Analysis of a multirate theta-method for stiff ODEs*, to appear in APNUM, Report MAS-R0615, CWI, Amsterdam, 2006.

[8] A. Kværnø. *Stability of multirate Runge-Kutta schemes*, The tenth Int. Conf. on Diff. Equ., Plovdiv, Bulgaria, Aug. 1999.

[9] J. ter Maten, A. Verhoeven, A. El Guennouni, Th. Beelen. *Multirate hierarchical time integration for electronic circuits*, In: PAMM (Proc. GAMM Annual Meeting 2005), Vol. 5, Issue ', pp. 819-820, 2005.

[10] R.M.M. Mattheij, J. Molenaar. *Ordinary differential equations in theory and practice.* SIAM, 2002.

[11] A. Prothero, A. Robinson. *On the stability and accuracy of one-step methods for solving stiff systems of ordinary differential equations*, Math. of Comp., Vol.28, pp. 145-162, 1974.

[12] J. Sand, S. Skelboe. *Stability of Backward Euler Multirate Methods and Convergence of Waveform Relaxation*, BIT, Vol.32, pp 350-366, 1992.

[13] V. Savcenco, W.H. Hundsdorfer, J.G. Verwer. *A multirate time stepping strategy for parabolic PDE*, Report MAS-E0516, CWI, Amsterdam, 2005.

[14] S. Skelboe, P.U. Andersen. *Stability properties of backward Euler multirate formulas*, SIAM J. Sci. Stat. Comput., Vol.10-5, pp. 1000-1009, 1989.

[15] M. Striebel, M. Günther. *A charge oriented mixed multirate method for a special class of index-1 network equations in chip design*, Applied Numerical Mathematics, Vol.53, pp. 489-507, 2005.

[16] A. Verhoeven. *Automatic control for adaptive time stepping in electrical circuit simulation*, MSc Thesis, Technische Universiteit Eindhoven, Eindhoven, Technical Note TN-2004/00033, Philips Research Laboratories, Eindhoven, 2004.

[17] A. Verhoeven, A. El Guennouni, E.J.W. ter Maten, R.M.M. Mattheij. *A general compound multirate method for circuit simulation problems*, In: A.M. Anile, G. Ali, G. Mascali: *Scientific Computing in Electrical Engineering*, Series Mathematics in Industry, ECMI, Vol. 9, pp. 143-150, 2006.

[18] A. Verhoeven, T.G.J. Beelen, A. El Guennouni, E.J.W. ter Maten, R.M.M. Mattheij, B. Tasić. *Error analysis of BDF Compound-Fast multirate method for differential-algebraic equations*, Ext. abstract Copper Mountain, CASA-Report 06-10, 2006.

[19] J.K.White, A.Sangiovanni-Vincentelli. *Relaxation techniques for the simulation of VLSI circuits.* Kluwer Academic Publishers, 1987.