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Stability and Asymptotic Stability in the Energy Space of the Sum of *N* Solitons for Subcritical gKdV Equations

Yvan Martel[1], **Frank Merle**[1,2], **Tai-Peng Tsai**[3]

[1] Université de Cergy–Pontoise, Département de Mathématiques, 2, av. Adolphe Chauvin, 95302 Cergy–Pontoise, France
[2] Institut Universitaire de France, Paris, France
[3] Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA

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Abstract: We prove in this paper the stability and asymptotic stability in H^1 of a decoupled sum of N solitons for the subcritical generalized KdV equations $u_t + (u_{xx} + u^p)_x = 0$ $(1 < p < 5)$. The proof of the stability result is based on energy arguments and monotonicity of the local L^2 norm. Note that the result is new even for $p = 2$ (the KdV equation). The asymptotic stability result then follows directly from a rigidity theorem in [16].

1. Introduction

In this paper, we consider the generalized Korteweg–de Vries equations

$$\begin{cases} u_t + (u_{xx} + u^p)_x = 0, & (t, x) \in \mathbf{R} \times \mathbf{R}, \\ u(0, x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1}$$

for $1 < p < 5$ and $u_0 \in H^1(\mathbf{R})$. This model for $p = 2$ was first introduced in the study of waves on shallow water, see Korteweg and de Vries [10]. It also appears for $p = 2$ and 3, in other areas of physics (see e.g. Lamb [11]).

Recall that (1) is well-posed in the energy space H^1. For $p = 2, 3, 4$, it was proved by Kenig, Ponce and Vega [9] (see also Kato [8], Ginibre and Tsutsumi [6]), that for $u_0 \in H^1(\mathbf{R})$, there exists a unique solution $u \in C(\mathbf{R}, H^1(\mathbf{R}))$ of (1) satisfying the following two conservation laws, for all $t \in \mathbf{R}$:

$$\int u^2(t) = \int u_0^2, \tag{2}$$

$$E(u(t)) = \frac{1}{2} \int u_x^2(t) - \frac{1}{p+1} \int u^{p+1}(t) = \frac{1}{2} \int u_{0x}^2 - \frac{1}{p+1} \int u_0^{p+1}. \tag{3}$$

For $p = 2, 3, 4$, global existence of all solutions in H^1, as well as a uniform bound in H^1, follow directly from the Gagliardo–Nirenberg inequality,

$$\forall v \in H^1(\mathbf{R}), \quad \int |v|^{p+1} \le C(p) \left(\int v^2\right)^{\frac{p+3}{4}} \left(\int v_x^2\right)^{\frac{p-1}{4}},$$

and relations (2), (3), giving a uniform bound in H^1 for any solution.

This is in contrast with the case $p = 5$, for which there exist solutions $u(t)$ of (1) such that $|u(t)|_{H^1} \to +\infty$ as $t \to T$, for $0 < T < +\infty$, see [20] and [18]. For $p > 5$ such behavior is also conjectured. Thus, for the question of global existence and bound in H^1, the case $1 < p < 5$ is called the subcritical case, $p = 5$ the critical case and $p > 5$ the supercritical case.

Equation (1) has explicit traveling wave solutions, called solitons, which play a fundamental role in the generic behavior of the solutions. Let

$$Q(x) = \left(\frac{p+1}{2\,\mathrm{ch}^2\left(\frac{p-1}{2}x\right)}\right)^{\frac{1}{p-1}} \tag{4}$$

be the only positive solution in $H^1(\mathbf{R})$ (up to translation) of $Q_{xx} + Q^p = Q$, and for $c > 0$, let $Q_c(x) = c^{\frac{1}{p-1}} Q\left(\sqrt{c}x\right)$. The traveling waves solutions of (1) are

$$u(t,x) = Q_c(x - ct) = c^{\frac{1}{p-1}} Q\left(\sqrt{c}(x-ct)\right),$$

where $c > 0$ is the speed of the soliton.

For the KdV equation ($p = 2$), there is a much wider class of special explicit solutions for (1), called N-solitons. They correspond to the superposition of N traveling waves with different speeds that interact and then remain unchanged after interaction. The N-solitons behave asymptotically in large time as the sum of N traveling waves, and as for the single solitons, there is no dispersion. We refer to [21] for explicit expressions and further properties of these solutions. For $p \neq 2$, even the existence of solutions behaving asymptotically as the sum of N solitons was not known.

Important notions for these solutions are the stability and asymptotic stability with respect to initial data.

For $c > 0$, the soliton $Q_c(x - ct)$ is stable in H^1 if:

$$\forall \delta_0 > 0,\ \exists \alpha_0 > 0 / |u_0 - Q_c|_{H^1} \le \alpha_0 \Rightarrow \forall t \ge 0,\ \exists x(t)\ /\ |u(t) - Q_c(.-x(t))|_{H^1} \le \delta_0.$$

The family of solitons $\{Q_c(x - x_0 - ct),\ c > 0,\ x_0 \in \mathbf{R}\}$ is asymptotically stable if:

$$\exists \alpha_0 > 0\ /\ |u_0 - Q_c|_{H^1} \le \alpha_0 \Rightarrow \exists c_{+\infty}, x(t)\ /\ u(t, . + x(t)) \underset{t\to+\infty}{\rightharpoonup} Q_{c_{+\infty}} \text{ in } H^1.$$

We recall previously known results concerning the notions of stability of solitons and N solitons:

– In the subcritical case: $p = 2, 3, 4$, it follows from energetic arguments that the solitons are H^1 stable (see Benjamin [1] and Weinstein [25]). Moreover, Martel and Merle [16] prove the asymptotic stability of the family of solitons in the energy space. The proof relies on a rigidity theorem close to the family of solitons, which was first given for the critical case ([14]), and which is based on nonlinear argument. (Pego and

Weinstein [22] prove the asymptotic stability result for $p = 2, 3$ for initial data with exponential decay as $x \to +\infty$.)

In the case of the KdV equation, Maddocks and Sachs [13] prove the stability in $H^N(\mathbf{R})$ of N-solitons (recall that there are explicit solutions of the KdV equation): for any initial data u_0 close in $H^N(\mathbf{R})$ to an N-soliton, the solution $u(t)$ of the KdV equation remains uniformly close in $H^N(\mathbf{R})$ for all time to an N soliton profile with the same speeds. Their proof involves N conserved quantities for the KdV equation, and this is the reason why they need to impose closeness in high regularity spaces. Note that this result is known only with $p = 2$ and with this regularity assumption on the initial data. Asymptotic stability is unknown in this context.

As it is noted in [13], multi-solitons of the KdV equations can serve as examples of exact solutions of nonlinear wave interactions. The stability and asymptotic stability of such solutions are thus important properties from the physical point of view and produce more examples of well understood solutions (see references in [13]). We also refer to S.-I. Ei and T. Ohta [5] for a study of the motion of two interacting pulses in the case of the KdV equations (Part III of [5]) and of other dissipative and dispersive systems.

– In the critical case $p = 5$, any solution with negative energy initially close to the soliton blows up in finite or infinite time in H^1 (Merle [20]), and actually blows up in finite time if the initial data satisfies in addition a polynomial decay condition on the right in space (Martel and Merle [18]). (Note that $E(Q) = 0$ for $p = 5$.) Of course this implies the instability of the soliton. These results rely on rigidity theorems around the soliton.

– In the supercritical case $p > 5$, Bona, Souganidis, and Strauss [2] proved, using Grillakis, Shatah, and Strauss [7] type arguments, H^1 instability of solitons. Moreover, numerical experiments, see e.g. Dix and McKinney [4], suggest existence of blow up solutions arbitrarily close to the family of solitons.

In this paper, for $p = 2, 3, 4$, using techniques developed for the critical and subcritical cases in [14] and [16] as well as a direct variational argument in H^1, we prove the stability and asymptotic stability of the sum

$$\sum_{j=1}^{N} Q_{c_j^0}(x - x_j), \quad \text{where} \quad 0 < c_1^0 < \cdots < c_N^0, \quad x_1 < \cdots < x_N, \tag{5}$$

in $H^1(\mathbf{R})$, for $t \geq 0$.

Theorem 1 (Asymptotic stability of the sum of N solitons). *Let $p = 2$, 3 or 4. Let $0 < c_1^0 < \cdots < c_N^0$. There exist $\gamma_0, A_0, L_0, \alpha_0 > 0$ such that the following is true: Let $u_0 \in H^1(\mathbf{R})$ and assume that there exist $L > L_0$, $\alpha < \alpha_0$, and $x_1^0 < \cdots < x_N^0$, such that*

$$\left\| u_0 - \sum_{j=1}^{N} Q_{c_j^0}(. - x_j^0) \right\|_{H^1} \leq \alpha, \quad \text{and} \quad x_j^0 > x_{j-1}^0 + L, \quad \text{for all } j = 2, \ldots, N. \tag{6}$$

Let $u(t)$ be the solution of (1). Then, there exist $x_1(t), \ldots, x_N(t)$ such that

(i) Stability of the sum of N decoupled solitons,

$$\forall t \geq 0, \quad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j^0}(x - x_j(t)) \right\|_{H^1} \leq A_0 \left(\alpha + e^{-\gamma_0 L} \right). \tag{7}$$

(ii) Asymptotic stability of the sum of N solitons. Moreover, there exist $c_1^{+\infty}, \dots, c_N^{+\infty}$, *with* $|c_j^{+\infty} - c_j^0| \le A_0\left(\alpha + e^{-\gamma_0 L}\right)$, *such that*

$$\left| u(t) - \sum_{j=1}^N Q_{c_j^{+\infty}}(x - x_j(t)) \right|_{L^2(x > c_1^0 t/10)} \to 0, \quad \dot{x}_j(t) \to c_j^{+\infty} \quad \text{as } t \to +\infty. \tag{8}$$

Remark 1. One of the interests of studying the stability and asymptotic stability of the sums (5) rather than the explicit N-soliton solutions is that we can consider any subcritical generalized KdV equation. Indeed, the result does not depend on the existence of a special family of solutions behaving as the N-solitons. For $p \neq 2$, note that the existence of solutions behaving in L^2 as $t \to +\infty$ as the sum of N solitons is an open problem.

The asymptotic stability result (ii) proves that the family of sums (5) attracts as $t \to +\infty$ the orbits that are sufficiently close to it. We believe that it is an important qualitative information for the flow of the generalized KdV equations, both from mathematical and physical point of view. For $p = 2$, it implies in particular the stability and asymptotic stability of the explicit N-solitons solutions in the energy space (see Corollary 1 below).

Remark 2. It is well-known that for $p = 2$ and $p = 3$, (1) is completely integrable. Indeed, for suitable u_0 (u_0 and its derivatives with exponential decay at infinity) there exist an infinite number of conservation laws, see e.g. Lax [12] and Miura [21]. Moreover, many results on these equations rely on the inverse scattering method, which transform the problem in a sequence of linear problems (but requires a strong decay assumption on the solution). The result in [13] does not use this transformation but the existence of many conservation laws for the KdV equation. *In this paper, we do not use integrability and we work in the energy space* H^1*, with no decay assumption on* u_0.

Remark 3. For Schrödinger type equations, Perelman [23] and Buslaev and Perelman [3], with strong conditions on initial data and nonlinearity, and using a linearization method around the soliton, prove asymptotic stability results by a fixed point argument. Unfortunately, this method breaks down without a decay assumption on the initial data.

Remark 4. In Theorem 1 (ii), we cannot have convergence to zero in $L^2(x > 0)$. Indeed, assumption (6) on the initial data allows the existence in $u(t)$ of an additional soliton of size less than α (thus traveling at arbitrarily small speed). For $p = 2$, an explicit example can be constructed using the N-soliton solutions.

Recall that for $p = 2$ any N-soliton solution has the form $v(t, x) = U^{(N)}(x; c_j, x_j - c_j t)$, where $\{U^{(N)}(x; c_j, y_j); c_j > 0, y_j \in \mathbf{R}\}$ is the family of explicit N-soliton profiles (see e.g. [13], Sect. 3.1). As a direct corollary of Theorem 1, for $p = 2$, we prove stability and asymptotic stability of this family.

Corollary 1 (Asymptotic stability in H^1 of N-solitons for $p = 2$). *Let* $p = 2$. *Let* $0 < c_1^0 < \dots < c_N^0$ *and* $x_1^0, \cdots, x_N^0 \in \mathbf{R}$. *For all* $\delta_1 > 0$*, there exists* $\alpha_1 > 0$ *such that the following is true: Let* $u(t)$ *be a solution of (1). If* $|u(0) - U^{(N)}(\,.\,; c_j^0, -x_j^0)|_{H^1} \le \alpha_1$, *then there exist* $x_j(t)$ *such that*

$$\forall t > 0, \quad |u(t) - U^{(N)}(\,.\,; c_j^0, -x_j(t))|_{H^1} \le \delta_1. \tag{9}$$

Moreover, there exist $c_j^{+\infty} > 0$ *such that*

$$\left| u(t) - U^{(N)}(\,.\,; c_j^{+\infty}, -x_j(t)) \right|_{L^2(x > c_1^0 t/10)} \to 0, \quad \dot{x}_j(t) \to c_j^{+\infty} \quad \text{as } t \to +\infty. \tag{10}$$

Note that this improves the result in [13] in two ways. First, stability is proved in H^1 instead of H^N. Second, we also prove asymptotic stability as $t \to +\infty$. Corollary 1 is proved at the end of Sect. 4.

Let us sketch the proof of these results. Note first that the main result, i.e. the stability result Theorem 1 (i) is self-contained, whereas the asymptotic stability result Theorem 1 (ii) relies on the proof for the case $N = 1$ ([16]).

For Theorem 1, using modulation theory, $u(t) = \sum_{j=1}^N Q_{c_j(t)}(x - x_j(t)) + \varepsilon(t, x)$, where $\varepsilon(t)$ is small in H^1, and $x_i(t)$, $c_i(t)$ are geometrical parameters (see Sect. 2). The stability result (i) is equivalent to control both the variation of $c_j(t)$ and the size of $\varepsilon(t)$ in H^1 (Sect. 3).

Our main arguments are based on L^2 properties of the solution. From [14] and [16], the L^2 norm of the solution at the right of each soliton is almost decreasing in time. This property together with an energy argument allows us to prove that the variation of $c_j(t)$ is quadratic in $|\varepsilon(t)|_{H^1}$, which is a key of the problem.

Let us explain the argument formally by taking $\varepsilon = 0$ and so $u(t) = \sum Q_{c_j(t)}(x - x_j(t))$. The energy conservation becomes

$$\sum c_j^{\beta+\frac{1}{2}}(t) = \sum c_j^{\beta+\frac{1}{2}}(0),$$

where $\beta = \frac{2}{p-1}$. The monotonicity of the L^2 norm at the right of each soliton gives us

$$\Delta_j(t) = \sum_{k=j}^N c_k^{\beta-\frac{1}{2}}(t) - c_k^{\beta-\frac{1}{2}}(0) \leq 0.$$

We claim that $c_j(t) = c_j(0)$ by a convexity argument. Indeed,

$$\begin{aligned} 0 &= \sum c_j^{\beta+\frac{1}{2}}(t) - c_j^{\beta+\frac{1}{2}}(0) \sim \frac{2\beta+1}{2\beta-1} \sum c_j(0) \left(c_j^{\beta-\frac{1}{2}}(t) - c_j^{\beta-\frac{1}{2}}(0) \right) \\ &= \frac{2\beta+1}{2\beta-1} \sum (c_j(0) - c_{j+1}(0)) \Delta_j(t) \geq \sigma_0 \sum |\Delta_j(t)| \geq \sigma_1 \sum |c_j(t) - c_j(0)|. \end{aligned}$$

Thus $c_j(t)$ is a constant at the first order. In fact, we prove that the variation in time of $c_j(t)$ is of order 2 in $\varepsilon(t)$.

Then we control the variation of $\varepsilon(t)$ in H^1 by a refined version of this argument, using suitable orthogonality conditions on ε.

The asymptotic stability result (ii) follows directly from a rigidity property of the flow of Eq. (1) around the solitons (see Theorem following Proposition 2 in Sect. 4 of this paper and [16]) and monotonicity properties of the mass (see Sect. 2.2 and Sect. 4).

2. Decomposition and Properties of a Solution Close to the Sum of N Solitons

2.1. Decomposition of the solution and conservation laws. Fix $0 < c_1^0 < \cdots < c_N^0$ and let

$$\sigma_0 = \frac{1}{2}\min(c_1^0, c_2^0 - c_1^0, c_3^0 - c_2^0, \dots, c_N^0 - c_{N-1}^0).$$

From modulation theory, we claim

Lemma 1 (Decomposition of the solution). *There exists $L_1, \alpha_1, K_1 > 0$ such that the following is true: If for $L > L_1$, $0 < \alpha < \alpha_1$, $t_0 > 0$, we have*

$$\sup_{0\le t\le t_0}\left(\inf_{y_j > y_{j-1}+L}\left\{\left|u(t,.) - \sum_{j=1}^N Q_{c_j^0}(. - y_j)\right|_{H^1}\right\}\right) < \alpha, \tag{11}$$

then there exist unique C^1 functions $c_j : [0, t_0] \to (0, +\infty)$, $x_j : [0, t_0] \to \mathbf{R}$, such that

$$\varepsilon(t,x) = u(t,x) - \sum_{j=1}^N R_j(t,x), \quad \textit{where} \quad R_j(t,x) = Q_{c_j(t)}(x - x_j(t)), \tag{12}$$

satisfies the following orthogonality conditions:

$$\forall j, \forall t \in [0, t_0], \quad \int R_j(t)\varepsilon(t) = \int (R_j(t))_x \varepsilon(t) = 0. \tag{13}$$

Moreover, there exists $K_1 > 0$ such that $\forall t \in [0, t_0]$,

$$|\varepsilon(t)|_{H^1} + \sum_{j=1}^N |c_j(t) - c_j^0| \le K_1\alpha, \tag{14}$$

$$\forall j,\ \left|\dot c_j(t)\right| + \left|\dot x_j(t) - c_j(t)\right| \le K_1\left(\int e^{-\sqrt{\sigma_0}|x - x_j(t)|/2}\varepsilon^2(t)\right)^{1/2} + K_1 e^{-\sqrt{\sigma_0}(L+\sigma_0 t)/4}. \tag{15}$$

Proof. Lemma 1 is a consequence of Lemma 8 (see Appendix) and standard arguments. We refer to [15] Sect. 2.3 for a complete proof in the case of a single soliton. In particular, $\varepsilon(t)$ satisfies $\forall t \in [0, t_0]$,

$$\begin{aligned}\varepsilon_t + \varepsilon_{xxx} = &-\sum_{j=1}^N \frac{\dot c_j}{2c_j}\left(\frac{2R_j}{p-1} + (x - x_i)(R_j)_x\right)\\ &+\sum_{j=1}^N (\dot x_j - c_j)R_{jx} - \left(\left(\varepsilon + \sum_{j=1}^N R_j\right)^p - \sum_{j=1}^N R_j^p\right)_x.\end{aligned}$$

By taking (formally) the scalar product of this equation by R_j and $(R_j)_x$, and using calculations in the proof of Lemma 8, we prove

$$|\dot c_j(t)| + |\dot x_j(t) - c_j(t)| \le C\left(\int e^{-\sqrt{\sigma_0}|x - x_j(t)|/2}\varepsilon^2(t)\right)^{1/2} + C\sum_{k\ne j} e^{-\sqrt{\sigma_0}|x_k(t) - x_j(t)|/2}.$$

For $\alpha > 0$ small enough, and L large enough, we have $|x_k(t) - x_j(t)| \geq \frac{L}{2} + \sigma_0 t$, and this proves (15).

Next, by using the conservation of energy for $u(t)$, i.e.

$$E(u(t)) := \int \frac{1}{2}\, u_x^2(t,x) - \frac{1}{p+1}\, u^{p+1}(t,x)\, dx = E(u_0),$$

and linearizing the energy around $R = \sum_{j=1}^N R_j$, we prove the following result.

Lemma 2 (Energy bounds). *There exist $K_2 > 0$ and $L_2 > 0$ such that the following is true: Assume that $\forall j$, $c_j(t) \geq \sigma_0$, and $x_j(t) - x_{j-1}(t) \geq L \geq L_2$. Then, $\forall t \in [0, t_0]$,*

$$\begin{aligned} &\left| \sum_{j=1}^N \left[E(R_j(t)) - E(R_j(0))\right] + \frac{1}{2}\int (\varepsilon_x^2 - pR^{p-1}\varepsilon^2)(t) \right| \\ &\quad \leq K_2 \left\{ |\varepsilon(0)|_{H^1}^2 + |\varepsilon(t)|_{H^1}^3 + e^{-\sqrt{\sigma_0}L/2} \right\}, \end{aligned} \tag{16}$$

where K_2 is a constant.

Proof. Insert (12) into $E(u(t))$ and integrate by parts. We have

$$E(u(t)) = \int \frac{1}{2}\, R_x^2 - \frac{1}{p+1}\, R^{p+1}\, dx - \int \left(R_{xx} + R^p\right) \varepsilon\, dx \tag{17}$$

$$+ \int \frac{1}{2}\, \varepsilon_x^2 - \frac{p}{2} R^{p-1}\varepsilon^2\, dx$$

$$+ \int \frac{1}{p+1}\, \left(-(R+\varepsilon)^{p+1} + R^{p+1}\right) + R^p\varepsilon + \frac{p}{2} R^{p-1}\varepsilon^2\, dx. \tag{18}$$

We first observe that $|(18)| \leq C\, \|\varepsilon\|_{H^1}^3$. Next, remark that $\sigma_0 \leq c_j(t)$, $x_j(t) - x_{j-1}(t) \geq L$, implies $|R_j(x,t)| + |(R_j)_x(x,t)| \leq Ce^{-\sqrt{\sigma_0}|x - x_j(t)|}$, and so

$$\left| \int R_j(t)\, R_k(t)\, dx \right| + \left| \int (R_j)_x(t)\, (R_k)_x(t)\, dx \right| \leq Ce^{-\sqrt{\sigma_0}L/2} \quad \text{if } j \neq k. \tag{19}$$

Thus, by $(R_j)_{xx} + R_j^p = c_j R_j$, we have

$$\left| (17) - \sum_{j=1}^N E(R_j(t)) + \int \sum_j c_j R_j \varepsilon(t) - \frac{1}{2}\int (\varepsilon_x^2 - pR^{p-1}\varepsilon^2)(t) \right| \leq Ce^{-\sqrt{\sigma_0}L/2}. \tag{20}$$

From $\int R_j(t)\varepsilon(t) = 0$, we obtain

$$\left| E(u(t)) - \sum_{j=1}^N E(R_j(t)) - \frac{1}{2}\int (\varepsilon_x^2 - pR^{p-1}\varepsilon^2)(t) \right| \leq Ce^{-\sqrt{\sigma_0}L/2} + C\, \|\varepsilon(t)\|_{H^1}^3 .$$

Since $E(u(t)) = E(u(0))$, applying the previous formula at $t = 0$ and at t, we prove the lemma.

2.2. Almost monotonicity of the mass at the right. We follow the proof of Lemma 20 in [14]. Let

$$\phi(x) = cQ(\sqrt{\sigma_0}x/2), \quad \psi(x) = \int_{-\infty}^{x} \phi(y)dy, \quad \text{where} \quad c = \left(\frac{2}{\sqrt{\sigma_0}} \int_{-\infty}^{\infty} Q\right)^{-1}. \tag{21}$$

Note that $\forall x \in \mathbf{R}$, $\psi' > 0$, $0 < \psi(x) < 1$, and $\lim_{x\to-\infty} \psi(x) = 0$, $\lim_{x\to+\infty} \psi(x) = 1$. Let

$$j \geq 2, \quad \mathcal{I}_j(t) = \int u^2(t,x)\psi(x - m_j(t))\, dx, \qquad m_j(t) = \frac{x_{j-1}(t) + x_j(t)}{2}. \tag{22}$$

Lemma 3 (Almost monotonicity of the mass on the right of each soliton [14]). *There exist $K_3 = K_3(\sigma_0) > 0$, $L_3 = L_3(\sigma_0) > 0$ such that the following is true: Let $t_1 \in [0, t_0]$. Assume that $\forall t \in [0, t_1]$, $\forall j$,*

$$\dot{x}_1(t) \geq \sigma_0, \quad \dot{x}_j(t) - \dot{x}_{j-1}(t) \geq \sigma_0, \quad c_j(t) > \sigma_0, \quad \text{and} \quad |\varepsilon(t)|_{H^1}^{p-1} \leq \frac{\sigma_0}{8 \cdot 2^{p-1}}. \tag{23}$$

If for $L > L_3$, $\forall j \in \{2, \dots, N\}$, $x_j(0) - x_{j-1}(0) \geq L$, then

$$\mathcal{I}_j(t_1) - \mathcal{I}_j(0) \leq K_3\, e^{-\sqrt{\sigma_0}L/8}.$$

Proof. Let $j \in \{1, \dots, N\}$. Using Eq. (1) and integrating by parts several times, we have (see [16] Eq. (20)),

$$\frac{d}{dt}\mathcal{I}_j(t) = \int \left(-3u_x^2 - \dot{m}u^2 + \frac{2p}{p+1}u^{p+1}\right)\psi' + u^2\psi^{(3)}.$$

By definition of ψ, $\psi^{(3)} \leq \frac{\sigma_0}{4}\psi'$, so that

$$\int u^2\psi^{(3)} \leq \frac{\sigma_0}{4}\int u^2\psi'. \tag{24}$$

To bound $\int u^{p+1}\psi'$, we divide the real line into two regions: $I = [a, b]$ and its complement I^C, where $a = a(t) = x_{j-1}(t) + \frac{L}{4}$ and $b = b(t) = x_j(t) - \frac{L}{4}$. Inside the interval I we have

$$\left|\int_I u^{p+1}\psi'\right| \leq \int u^2\psi' \cdot \sup_I |u|^{p-1}.$$

Since for $x \in I$, for all $k = 1, 2, \dots, N$, $|x - x_k(t)| \geq \frac{L}{4}$, we have

$$|u(t,x)|^{p-1} = \left|\sum_{k=1}^{N} R_k(t,x) + \varepsilon(t,x)\right|^{p-1} \leq Ce^{-\sqrt{\sigma_0}L/4} + 2^{p-1}|\varepsilon(t)|_{L^\infty}^{p-1} \leq \frac{\sigma_0}{4},$$

for $L > L_3(\sigma_0)$. Thus,

$$\left|\int_I u^{p+1}\psi'\right| \leq \frac{\sigma_0}{4}\int u^2\psi'. \tag{25}$$

Next, in I^C, by the Gagliardo Nirenberg inequality,

$$\int_{I^C} u^{p+1}\psi'\,dx \leq \int u^{p+1}\,dx \cdot \sup_{I^C}\psi'$$
$$\leq C\,\|u\|_{H^1}^{p+1}\cdot\ \exp\left\{-\frac{\sqrt{\sigma_0}}{4}\ \left[x_j(t)-x_{j-1}(t)-\tfrac{L}{2}\right]\right\}$$
$$\leq Ce^{-\frac{\sqrt{\sigma_0}}{8}(2\sigma_0 t+L)}, \tag{26}$$

by $x_j(t)-x_{j-1}(t)\geq x_j(0)-x_{j-1}(0)+\sigma_0 t\geq L+\sigma_0 t$. From $\dot{m}\geq\sigma_0$, (24), (25) and (26), we obtain

$$\frac{d}{dt}\mathcal{I}_j(t)\leq\int\left(-3u_x^2-\frac{\sigma_0}{2}u^2\right)\psi'\,dx+Ce^{-\frac{\sqrt{\sigma_0}}{8}(2\sigma_0 t+L)}\leq Ce^{-\frac{\sqrt{\sigma_0}}{8}(2\sigma_0 t+L)}.$$

Thus, by integrating between 0 and t_1, we obtain the conclusion. Note that K_3 and L_3 are chosen independently of t_1.

2.3. Positivity of the quadratic form. By the choice of orthogonality conditions on $\varepsilon(t)$ and standard arguments, we claim the following lemma.

Lemma 4 (Positivity of the quadratic form). *There exists $L_4>0$ and $\lambda_0>0$ such that if $\forall j$, $c_j(t)\geq\sigma_0$, $x_j(t)\geq x_{j-1}(t)+L_4$ then, $\forall t\in[0,t_0]$,*

$$\int\varepsilon_x^2(t)-pR^{p-1}(t)\varepsilon^2(t)+c(t,x)\varepsilon^2(t)\geq\lambda_0|\varepsilon(t)|_{H^1}^2, \tag{27}$$

where $c(t,x)=c_1(t)+\sum_{j=2}^N(c_j(t)-c_{j-1}(t))\psi(x-m_j(t))$.

Proof of Lemma 4. It is well known that there exists $\lambda_1>0$ such that if $v\in H^1(\mathbf{R})$ satisfies $\int Qv=\int Q_x v=0$, then

$$\int v_x^2-pQ^{p-1}v^2+v^2\geq\lambda_1|v|_{H^1}^2. \tag{28}$$

(See the proof of Proposition 2.9 in Weinstein [24].) Now we give a local version of (28). Let $\Phi\in\mathcal{C}^2(\mathbf{R})$, $\Phi(x)=\Phi(-x)$, $\Phi'\leq 0$ on $\mathbf{R}^+$, with

$$\Phi(x)=1 \text{ on } [0,1];\quad \Phi(x)=e^{-x}\text{ on }[2,+\infty),\quad e^{-x}\leq\Phi(x)\leq 3e^{-x}\quad\text{on }\mathbf{R}^+.$$

Let $\Phi_B(x)=\Phi\left(\frac{x}{B}\right)$. The following claim is similar to a part of the proof of some local Virial relation in Sect. 2.2 of [17]; see Appendix A, Steps 1 and 2, in [17] for its proof.

Claim. There exists $B_0>0$ such that, for all $B>B_0$, if $v\in H^1(\mathbf{R})$ satisfies $\int Qv=\int Q_x v=0$, then

$$\int\Phi_B\left(v_x^2-pQ^{p-1}v^2+v^2\right)\geq\frac{\lambda_1}{4}\int\Phi_B(v_x^2+v^2). \tag{29}$$

We finish the proof of Lemma 4. Let $B > B_0$ to be chosen later and $L_4 = 4kB$, where $k > 1$ integer is to be chosen later. We have

$$\int \varepsilon_x^2 - pR^{p-1}\varepsilon^2 + c(t,x)\varepsilon^2 = \sum_{j=1}^N \int \Phi_B(x - x_j(t)) \left(\varepsilon_x^2 - pR_j^{p-1}\varepsilon^2 + c_j(t)\varepsilon^2\right)$$
$$-p\int \left(R^{p-1} - \sum_{j=1}^N \Phi_B(x-x_j(t))R_j^{p-1}\right)\varepsilon^2$$
$$+\sum_{j=1}^N \int \Phi_B(x-x_j(t))(c(t,x)-c_j(t))\varepsilon^2$$
$$+\int \left(1 - \sum_{j=1}^N \Phi_B(x-x_j(t))\right)(\varepsilon_x^2 + c(t,x)\varepsilon^2).$$

Next, we make the following observations:

(i) By (29), we have $\forall j$,

$$\int \Phi_B(x-x_j(t))\left(\varepsilon_x^2 - pR_j^{p-1}\varepsilon^2 + c_j(t)\varepsilon^2\right) \geq \frac{\lambda_1}{4}\int \Phi_B(x-x_j(t))(\varepsilon_x^2 + c_j(t)\varepsilon^2).$$

(ii) Since $\Phi_B(x) = 1$ for $|x| < B$, by the decay properties of Q, we have

$$0 \leq R^{p-1} - \sum_{j=1}^N \Phi_B(x-x_j(t))R_j^{p-1} \leq |R|^{p-1}_{L^\infty(|x-x_j(t)|>B)} + C\sum_{j\neq k} R_j R_k \leq Ce^{-\sqrt{\sigma_0}B}.$$

(iii) Note that $c(t,x) = \sum_{j=1}^N c_j(t)\varphi_j(t,x)$, where $\varphi_1(t,x) = 1 - \psi(x - m_2(t))$, for $j \in \{2,\ldots,N-1\}$, $\varphi_j(t,x) = \psi(x - m_j(t)) - \psi(x - m_{j+1}(t))$ and $\varphi_N(t,x) = \psi(x - m_N(t))$. Since $\Phi_B(x) \leq 3e^{-\frac{|x|}{B}}$, by the properties of ψ, and $|m_j(t) - x_j(t)| \geq L_4/2 \geq 2kB$, we obtain

$$\left|\Phi_B(x-x_j(t))(c(t,x)-c_j(t))\right| \leq |c(t,x) - c(t)|_{L^\infty(|x-x_j(t)|\leq kB)} + Ce^{-k}$$
$$\leq Ce^{-\sqrt{\sigma_0}kB/2} + Ce^{-k}.$$

(iv) $1 - \sum_{j=1}^N \Phi_B(x-x_j(t)) \geq 0$.

Therefore, with $\lambda_0 = \frac{1}{2}\min(\frac{\lambda_1}{4}, \frac{\lambda_1}{4}\sigma_0, 1, \sigma_0)$, for B and k large enough,

$$\int \varepsilon_x^2 - pR^{p-1}\varepsilon^2 + c(t,x)\varepsilon^2 \geq 2\lambda_0 \int(\varepsilon_x^2 + \varepsilon^2) - C\left(e^{-\sqrt{\sigma_0}B/2} + e^{-k}\right)\int \varepsilon^2$$
$$\geq \lambda_0 \int (\varepsilon_x^2 + \varepsilon^2).$$

Thus the proof of Lemma 4 is complete.

3. Proof of the Stability in the Energy Space

This section is devoted to the proof of the stability result. The proof is by a priori estimate.

Let $0 < c_1^0 < \cdots < c_N^0$, $\sigma_0 = \frac{1}{2}\min(c_1^0, c_2^0 - c_1^0, c_3^0 - c_2^0, \dots, c_N^0 - c_{N-1}^0)$ and $\gamma_0 = \sqrt{\sigma_0}/16$. For $A_0, L, \alpha > 0$, we define

$$\mathcal{V}_{A_0}(L,\alpha) = \left\{ u \in H^1(\mathbf{R});\ \inf_{x_j - x_{j-1} \geq L} \left| u - \sum_{j=1}^N Q_{c_j^0}(. - x_j) \right|_{H^1} \leq A_0 \left(\alpha + e^{-\gamma_0 L/2} \right) \right\}. \tag{30}$$

We want to prove that there exists $A_0 > 0$, $L_0 > 0$, and $\alpha_0 > 0$ such that, $\forall u_0 \in H^1(\mathbf{R})$, if for some $L > L_0$, $\alpha < \alpha_0$, $\left| u_0 - \sum_{j=1}^N Q_{c_j^0}(. - x_j^0) \right|_{H^1} \leq \alpha$, where $x_j^0 > x_{j-1}^0 + L$, then $\forall t \geq 0$, $u(t) \in \mathcal{V}_{A_0}(L,\alpha)$ (this proves the stability result in H^1). By a standard continuity argument (described just below Proposition 1), it is a direct consequence of the following proposition.

Proposition 1 (A priori estimate). *There exists $A_0 > 0$, $L_0 > 0$, and $\alpha_0 > 0$ such that, for all $u_0 \in H^1(\mathbf{R})$, if*

$$\left| u_0 - \sum_{j=1}^N Q_{c_j^0}(. - x_j^0) \right|_{H^1} \leq \alpha, \tag{31}$$

where $L > L_0$, $0 < \alpha < \alpha_0$, $x_j^0 > x_{j-1}^0 + L$, and if for $t^* > 0$,

$$\forall t \in [0, t^*], \quad u(t) \in \mathcal{V}_{A_0}(L,\alpha), \tag{32}$$

then

$$\forall t \in [0, t^*], \quad u(t) \in \mathcal{V}_{A_0/2}(L,\alpha). \tag{33}$$

Note that A_0, L_0 and $\alpha > 0$ are independent of t^*.

Proposition 1 implies the stability result (i) of Theorem 1. Indeed, let A_0, L_0, α_0 be chosen as in Proposition 1. Suppose that u_0 satisfies the assumptions of Theorem 1. Then, by continuity of $u(t)$ in H^1, $u(t) \in \mathcal{V}_{A_0}(L,\alpha)$ for $0 < t < \tau_0$ for some $\tau_0 > 0$. Let

$$t^* = \sup\{t \geq 0,\ u(t') \in \mathcal{V}_{A_0}(L,\alpha),\ \forall t' \in [0,t]\}.$$

Assume for the sake of contradiction that t^* is finite. Then, by Proposition 1, we have $\forall t \in [0,t^*]$, $u(t) \in \mathcal{V}_{A_0/2}(L,\alpha)$. Therefore, by continuity of $u(t)$ in H^1, there exists $\tau > 0$ such that $\forall t \in [0, t^* + \tau]$, $u(t) \in \mathcal{V}_{2A_0/3}(L,\alpha)$, which contradicts the definition of t^*. The stability result follows.

Proof of Proposition 1. Let $A_0 > 0$ to be fixed later. First, for $0 < \alpha_0 < \alpha_I(A_0)$ and $L_0 > L_I(A_0) > L_1$, we have

$$A_0 \left(\alpha_0 + e^{-\gamma_0 L_0/2} \right) \leq \alpha_1, \tag{34}$$

where α_1 and L_1 are defined in Lemma 1. Therefore, by (32) and Lemma 1, there exist $c_j : [0, t^*] \to (0, +\infty)$, $x_j : [0, t^*] \to \mathbf{R}$, such that

$$\varepsilon(t, x) = u(t, x) - \sum_{j=1}^{N} R_j(t, x), \quad \text{where} \quad R_j(t, x) = Q_{c_j(t)}(x - x_j(t)), \tag{35}$$

satisfies $\forall j$, $\forall t \in [0, t^*]$,

$$\int R_j(t)\varepsilon(t) = \int (R_j(t))_x \varepsilon(t) = 0, \tag{36}$$

$$|c_j(t) - c_j^0| + |\dot{c}_j| + |\dot{x}_j - c_j^0| + |\varepsilon(t)|_{H^1} \leq K_1(A_0 + 1)\left(\alpha_0 + e^{-\gamma_0 L_0}\right). \tag{37}$$

Note that by (31), Lemma 8 (see Appendix) and assumptions of the proposition,

$$|\varepsilon(0)|_{H^1} + \sum_{j=1}^{N} |c_j(0) - c_j^0| \leq K_1\alpha, \quad x_j(0) - x_{j-1}(0) \geq \frac{L}{2}. \tag{38}$$

From (37) and (38), for $\alpha_0 < \alpha_{II}(A_0)$ and $L_0 > L_{II}(A_0) > 2\max(L_2, L_3, L_4)$ (L_2, L_3 and L_4 are defined in Lemmas 3 and 4), we have $\forall t \in [0, t^*]$,

$$c_1(t) \geq \sigma_0, \quad \dot{x}_1(t) \geq \sigma_0, \quad c_j(t) - c_{j-1}(t) \geq \sigma_0, \quad \dot{x}_j(t) - \dot{x}_{j-1}(t) \geq \sigma_0, \tag{39}$$

$$x_j(t) - x_{j-1}(t) \geq L/2 \geq \max(L_3, L_4), \quad |\varepsilon(t)|_{H^1} \leq \frac{1}{2}\left(\frac{\sigma_0}{8}\right)^{\frac{1}{p-1}}. \tag{40}$$

Therefore, we can apply Lemmas 2, 3 and 4 for all $t \in [0, t^*]$.

Let $\alpha_0 = \min(\alpha_I(A_0), \alpha_{II}(A_0))$ and $L_0 = \max(L_I(A_0), L_{II}(A_0))$. Now, our objective is to give a uniform upper bound on $|\varepsilon(t)|_{H^1}$ and $|c_j(t) - c_j(0)|$ on $[0, t^*]$ improving (37) for A_0 large enough.

In the next lemma, we first obtain a control of the variation of $c_j(t)$ which is quadratic in $|\varepsilon(t)|_{H^1}$. This is the key step of the stability result, based on the monotonicity property of the local L^2 norm and energy constraints. It is essential at this point to have chosen by the modulation $\int R_j\varepsilon = 0$.

Lemma 5 (Quadratic control of the variation of $c_j(t)$). *There exists $K_4 > 0$ independent of A_0, such that, $\forall t \in [0, t^*]$,*

$$\sum_{j=1}^{N} \left|c_j(t) - c_j(0)\right| \leq K_4\left(|\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L}\right). \tag{41}$$

Proof.

Step 1. Energetic control. Let $\beta = \frac{2}{p-1}$. There exists $C > 0$ such that

$$\left|\sum_{j=1}^{N} c_j(0)\left[c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0)\right]\right| \leq C\left(|\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L}\right) + C\sum_{j=1}^{N}\left[c_j(t) - c_j(0)\right]^2. \quad (42)$$

Let us prove (42). By (16), we have

$$\left|\sum_{j=1}^{N}\left[E(R_j(t)) - E(R_j(0))\right]\right| \leq C\left(|\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L}\right). \quad (43)$$

Since $E(Q_c) = -\frac{\kappa}{2}\, c^{\beta+1/2}\int Q^2$, where $\kappa = \frac{5-p}{p+3}$, we have

$$-\sum_{j=1}^{N}\left[E(R_j(t)) - E(R_j(0))\right] = \frac{\kappa}{2}\left(\int Q^2\right)\sum_{j=1}^{N}\left[c_j^{\beta+1/2}(t) - c_j^{\beta+1/2}(0)\right].$$

By linearization, we have $c_j^{\beta+1/2}(t) - c_j^{\beta+1/2}(0) = \frac{2\beta+1}{2\beta-1} c_j(0)\left[c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0)\right] + O\left(\left[c_j(t) - c_j(0)\right]^2\right)$. Note that $\frac{2\beta+1}{2\beta-1} = \frac{1}{\kappa}$. Therefore,

$$\left|\sum_{j=1}^{N}\left[E(R_j(t)) - E(R_j(0))\right] + \frac{1}{2}\left(\int Q^2\right)\sum_{j=1}^{N} c_j(0)\left[c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0)\right]\right| \leq C\sum_{j=1}^{N}\left[c_j(t) - c_j(0)\right]^2, \quad (44)$$

and from (43), we obtain (42).

Step 2. L^2 mass monotonicity at the right of every soliton. Let

$$d_j(t) = \sum_{k=j}^{N} c_k^{\beta-1/2}(t).$$

We claim

$$\left(\int Q^2\right)\left|d_j(t) - d_j(0)\right| \leq -\left(\int Q^2\right)(d_j(t) - d_j(0)) + C\left[\int \varepsilon^2(0) + e^{-\gamma_0 L}\right]. \quad (45)$$

Let us prove (45). Recall that using the notation of Sect. 2.3, we have

$$\mathcal{I}_j(t) \leq \mathcal{I}_j(0) + K_3 e^{-\gamma_0 L}, \quad \text{where} \quad \mathcal{I}_j(t) = \int \psi(x - m_j(t))u^2(t,x)dx.$$

Since $\int R_j^2(t) = c_j^{\beta-1/2}(t)\int Q^2$, $\int R_j(t)\varepsilon(t) = 0$, by similar calculations as in Lemma 2, we have

$$\left|\mathcal{I}_j(t) - \left(\int Q^2\right) d_j(t) - \int \psi(. - m_j(t))\varepsilon^2(t)\right| \le Ce^{-\gamma_0 L}. \tag{46}$$

Therefore,

$$\left(\int Q^2\right)(d_j(t) - d_j(0)) \le \int \psi(. - m_j(0))\varepsilon^2(0) - \int \psi(. - m_j(t))\varepsilon^2(t) + Ce^{-\gamma_0 L}. \tag{47}$$

Since the second term on the right-hand side is negative, (45) follows easily. Note that by conservation of the L^2 norm $\int u^2(t) = \int u^2(0)$ and

$$\begin{aligned}\int u^2(t) &= \int R^2(t) + \int \varepsilon^2(t) + 2\int R(t)\varepsilon(t) = \int R^2(t) + \int \varepsilon^2(t)\\ &= d_1(t) + \int \varepsilon^2(t) + O(e^{-\gamma_0 L}),\end{aligned}$$

we obtain

$$\left(\int Q^2\right)(d_1(t) - d_1(0)) \le \int \varepsilon^2(0) - \int \varepsilon^2(t) + Ce^{-\gamma_0 L}. \tag{48}$$

Step 3. Resummation argument. By the Abel transform, we have

$$\begin{aligned}&\sum_{j=1}^{N} c_j(0)\left[c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0)\right]\\ &\quad= \sum_{j=1}^{N-1} c_j(0)\left[d_j(t) - d_{j+1}(t) - (d_j(0) - d_{j+1}(0))\right] + c_N(0)\left[d_N(t) - d_N(0)\right]\\ &\quad= c_1(0)\left[d_1(t) - d_1(0)\right] + \sum_{j=2}^{N}(c_j(0) - c_{j-1}(0))(d_j(t) - d_j(0)).\end{aligned} \tag{49}$$

Therefore, by Step 1,

$$\begin{aligned}&-\left(c_1(0)\left[d_1(t) - d_1(0)\right] + \sum_{j=2}^{N}(c_j(0) - c_{j-1}(0))(d_j(t) - d_j(0))\right)\\ &\qquad\le C\left(|\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L}\right) + C\sum_{j=1}^{N}\left[c_j(t) - c_j(0)\right]^2.\end{aligned} \tag{50}$$

Since $c_1(0) \geq \sigma_0$, $c_j(0) - c_{j-1}(0) \geq \sigma_0$, by (45), we have

$$\begin{aligned}
&\sigma_0 \sum_{j=1}^{N} \left|d_j(t) - d_j(0)\right| \\
&\quad \leq\ c_1(0)|d_1(t) - d_1(0)| + \sum_{j=2}^{N}(c_j(0) - c_{j-1}(0))|d_j(t) - d_j(0)| \\
&\quad \leq -\ \left[c_1(0)\,[d_1(t) - d_1(0)] + \sum_{j=2}^{N}(c_j(0) - c_{j-1}(0))(d_j(t) - d_j(0))\right] \\
&\qquad + C\int \varepsilon^2(0) + Ce^{-\gamma_0 L}.
\end{aligned}$$

Thus, by (50), we have

$$\sum_{j=1}^{N} \left|d_j(t) - d_j(0)\right| \leq C\left(|\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L}\right) + C\sum_{j=1}^{N}\left[c_j(t) - c_j(0)\right]^2.$$

Since

$$\begin{aligned}
|c_j(t) - c_j(0)| &\leq C|c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0)| \\
&\leq C(|d_j(t) - d_j(0)| + |d_{j+1}(t) - d_{j+1}(0)|),
\end{aligned}$$

we obtain,

$$\sum_{j=1}^{N} \left|c_j(t) - c_j(0)\right| \leq C\left(|\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L}\right) + C\sum_{j=1}^{N}\left[c_j(t) - c_j(0)\right]^2.$$

Choosing a smaller $\alpha_0(A_0)$ and a larger $L_0(A_0)$, by (37), we assume $C|c_j(t) - c_j(0)| \leq 1/2$ and so

$$\sum_{j=1}^{N} \left|c_j(t) - c_j(0)\right| \leq C\left(|\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L}\right). \tag{51}$$

Thus, Lemma 5 is proved.

Now, we prove the following lemma, giving uniform control on $|\varepsilon(t)|_{H^1}$ on $[0, t^*]$.

Lemma 6 (Control of $|\varepsilon(t)|_{H^1}$). *There exists $K_5 > 0$ independent of A_0, such that, $\forall t \in [0, t^*]$,*

$$|\varepsilon(t)|_{H^1}^2 \leq K_5\left(|\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L}\right).$$

Proof. It follows from direct calculation on the energy, and the previous estimates obtained by the Abel transform, freezing the $c_j(t)$ at the first order.

By (16), (44), (49) and (51), we have

$$
\begin{aligned}
&\frac{1}{2}\int \varepsilon_x^2(t) - pR^{p-1}(t)\varepsilon^2(t)\\
&\quad\le -\sum_{j=1}^{N}\left[E(R_j(t)) - E(R_j(0))\right] + K_2\left(|\varepsilon(0)|_{H^1}^2 + |\varepsilon(t)|_{H^1}^3 + e^{-\gamma_0 L}\right)\\
&\quad\le \frac{1}{2}\left(\int Q^2\right)\sum_{j=1}^{N} c_j(0)\left[c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0)\right] + C\sum_{j=1}^{N}\left[c_j(t)-c_j(0)\right]^2\\
&\qquad + K_2\left(|\varepsilon(0)|_{H^1}^2 + |\varepsilon(t)|_{H^1}^3 + e^{-\gamma_0 L}\right)\\
&\quad\le \frac{1}{2}\left(\int Q^2\right)\left[c_1(0)\left[d_1(t)-d_1(0)\right] + \sum_{j=2}^{N}(c_j(0)-c_{j-1}(0))(d_j(t)-d_j(0))\right]\\
&\qquad + C\left(|\varepsilon(0)|_{H^1}^2 + |\varepsilon(t)|_{H^1}^3 + e^{-\gamma_0 L}\right).
\end{aligned}
$$

Therefore, using (47) and (48), and again Lemma 5, we have

$$
\begin{aligned}
&\int \varepsilon_x^2(t) - pR^{p-1}(t)\varepsilon^2(t)\\
&\quad\le -\left(c_1(0)\int \varepsilon^2(t) + \sum_{j=2}^{N}(c_j(0)-c_{j-1}(0))\int \psi(x-m_j(t))\varepsilon^2(t)\right)\\
&\qquad + C\left(|\varepsilon(0)|_{H^1}^2 + |\varepsilon(t)|_{H^1}^3 + e^{-\gamma_0 L}\right)\\
&\quad\le -\int c(t,x)\varepsilon^2(t) + C\left(|\varepsilon(0)|_{H^1}^2 + |\varepsilon(t)|_{H^1}^3 + e^{-\gamma_0 L}\right),
\end{aligned}
\tag{52}
$$

where $c(t,x) = c_1(t) + \sum_{j=2}^{N}(c_j(t)-c_{j-1}(t))\psi(x-m_j(t))$.

By Lemma 4,

$$\int \varepsilon_x^2(t) - pR^{p-1}(t)\varepsilon^2(t) + c(t,x)\varepsilon^2(t) \ge \lambda_0|\varepsilon(t)|_{H^1}^2.$$

Therefore, from (52), we obtain

$$|\varepsilon(t)|_{H^1}^2 \le C\left(|\varepsilon(0)|_{H^1}^2 + |\varepsilon(t)|_{H^1}^3 + e^{-\gamma_0 L}\right),$$

and so

$$|\varepsilon(t)|_{H^1}^2 \le K_5\left(|\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L}\right),$$

for some constant $K_5 > 0$, independent of A_0. Thus Lemma 6 is proved.

We conclude the proof of Proposition 1 and of the stability result. By (38) and Lemmas 5 and 6, we have

$$
\begin{aligned}
&\left| u(t) - \sum_{j=1}^{N} Q_{c_j^0}(x - x_j(t)) \right|_{H^1} \\
&\leq \left| u(t) - \sum_{j=1}^{N} R_j(t) \right|_{H^1} + \left| \sum_{j=1}^{N} R_j(t) - \sum_{j=1}^{N} Q_{c_j^0}(x - x_j(t)) \right|_{H^1} \\
&\leq |\varepsilon(t)|_{H^1} + C \sum_{j=1}^{N} |c_j(t) - c_j^0| \\
&\leq |\varepsilon(t)|_{H^1} + C \sum_{j=1}^{N} |c_j(t) - c_j(0)| + C \sum_{j=1}^{N} |c_j(0) - c_j^0| \\
&\leq |\varepsilon(t)|_{H^1} + C K_4 (|\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L}) + C K_1 \alpha \\
&\leq K_6 \left(\alpha + e^{-\gamma_0 L/2} \right),
\end{aligned}
$$

where $K_6 > 0$ is a constant independent of A_0.

Choosing $A_0 = 4K_6$, we complete the proof of Proposition 1 and thus the proof of Theorem 1 (i).

4. Proof of the Asymptotic Stability Result

This section is devoted to the proof of the asymptotic stability result (Theorem 1 (ii)).

4.1. Asymptotic stability around the solitons. In this subsection, we prove the following asymptotic result on $\varepsilon(t)$ as $t \to +\infty$.

Proposition 2 (Convergence around solitons, $p = 2, 3, 4$). *Under the assumptions of Theorem 1, the following is true:*

(i) Convergence of $\varepsilon(t)$*:* $\forall j \in \{1, \dots, N\}$,

$$
\varepsilon(t, . + x_j(t)) \rightharpoonup 0 \quad \textit{in } H^1(\mathbf{R}) \textit{ as } t \to +\infty. \tag{53}
$$

(ii) Convergence of geometric parameters: there exists $0 < c_1^{+\infty} < \cdots < c_N^{+\infty}$*, such that*

$$
c_j(t) \to c_j^{+\infty}, \quad \dot{x}_j(t) \to c_j^{+\infty} \quad \textit{as } t \to +\infty.
$$

The proof of this result is very similar to the proof of the asymptotic stability of a single soliton in Martel and Merle [16] for the subcritical case (see also the previous paper [14] concerning the critical case $p = 5$). The proof is based on the following rigidity result of solutions of (1) around solitons.

Theorem (Liouville property close to R_{c_0} for p = 2, 3, 4 [16]). *Let $p = 2, 3$ or 4, and let $c_0 > 0$. Let $u_0 \in H^1(\mathbf{R})$, and let $u(t)$ be the solution of (1) for all time $t \in \mathbf{R}$. There exists $\alpha_0 > 0$ such that if $|u_0 - R_{c_0}|_{H^1} < \alpha_0$, and if there exists $y(t)$ such that*

$$\forall \delta_0 > 0, \exists A_0 > 0 / \forall t \in \mathbf{R}, \quad \int_{|x|>A_0} u^2(t, x + y(t))dx \le \delta_0, \quad (L^2 \textit{ compactness}), \tag{54}$$

then there exists $c^* > 0$, $x^* \in \mathbf{R}$ such that

$$\forall t \in \mathbf{R}, \forall x \in \mathbf{R}, \quad u(t, x) = Q_{c^*}(x - x^* - c^* t).$$

This result gives a classification of the solutions around the solitons that have a certain property of uniform localization of the L^2 mass around a center $y(t)$ (54). Let us give a few words on the proof of such a result (see [16]). First, (54) implies a much stronger property on $u(t)$:

$$\forall t, x \in \mathbf{R}, \quad |u(t, x + y(t))| \le C e^{-\theta|x|}, \tag{55}$$

$C, \theta > 0$, which is proved by using a functional of the type $\mathcal{I}_j(t)$ in Sect. 2.2. Note that (55) is a purely nonlinear estimate. It implies strong localization properties in H^1, which reduces the nonlinear problem for α_0 small enough to a similar Liouville problem on a linear equation: $w_t + (Lw)_x = 0$, where L is the linearized operator $Lw = -w_{xx} + w - pQ^{p-1}w$. Finally, the linear Liouville property is proved by a Virial type quantity ($\int yw^2$) whose derivative in time involves an explicit quadratic form on w.

In [16], we prove that this theorem implies the asymptotic stability of a 1-soliton in the following way. Suppose $t_n \to +\infty$ and $\tilde{u}_0$ satisfy that $u(t_n, x(t_n) + .) \rightharpoonup \tilde{u}_0$ in H^1 as $n \to +\infty$. Then we can prove that the solution associated to initial data $\tilde{u}_0$ is L^2 compact in the sense of (54) and hence $\tilde{u}_0$ is a soliton. This concludes the proof.

Proof of Proposition 2 (i). Consider a solution $u(t)$ satisfying the assumptions of Theorem 1. Then, by Sect. 3, we known that $u(t)$ is uniformly close in $H^1(\mathbf{R})$ to the superposition of N solitons for all time $t \ge 0$. With the decomposition introduced in Sect. 2, it is equivalent that $\varepsilon(t)$ is uniformly small in $H^1(\mathbf{R})$ and $\sum_{j=1}^N |c_j(t) - c_j(0)|$ is uniformly small. Therefore, we can assume that, $\forall t \ge 0$,

$$c_1(t) \ge \sigma_0, \quad c_j(t) - c_{j-1}(t) \ge \sigma_0.$$

The proof of Proposition 2 is by contradiction. Let $j \in \{1, \dots, N\}$. Assume that for some sequence $t_n \to +\infty$, we have

$$\varepsilon(t_n, . + x_j(t_n)) \not\rightharpoonup 0 \quad \text{in } H^1(\mathbf{R}) \text{ as } t \to +\infty.$$

Since $0 < \sigma_0 < c_j(t) < \bar{c}$ and $|\varepsilon(t)|_{H^1} \le C$ for all $t \ge 0$, there exists $\tilde{\varepsilon}_0 \in H^1(\mathbf{R})$, $\tilde{\varepsilon}_0 \not\equiv 0$, and $\tilde{c}_0 > 0$ such that for a subsequence of (t_n), still denoted (t_n), we have

$$\varepsilon(t_n, . + x_j(t_n)) \rightharpoonup \tilde{\varepsilon}_0 \quad \text{in } H^1(\mathbf{R}), \quad c_j(t_n) \to \tilde{c}_0 \quad \text{as } n \to +\infty. \tag{56}$$

Moreover, by weak convergence and the stability result, $|\tilde{\varepsilon}_0|_{H^1} \le \sup_{t\ge 0} |\varepsilon(t)|_{H^1} \le C(\alpha_0 + e^{-\gamma_0 L_0})$, and therefore $|\tilde{\varepsilon}_0|_{H^1}$ is as small as we want by taking α_0 small and L_0 large.

Let now $\tilde{u}(0) = Q_{\tilde{c}_0} + \tilde{\varepsilon}_0$, and let $\tilde{u}(t)$ be the global solution of (1) for $t \in \mathbf{R}$, with $\tilde{u}(0)$ as initial data. Let $\tilde{x}(t)$ and $\tilde{c}(t)$ be the geometrical parameters associated to the solution $\tilde{u}(t)$ (apply the modulation theory for a solution close to a single soliton).

We claim that the solution $\tilde{u}(t)$ is L^2 compact in the sense of (54).

Lemma 7 (L^2 compactness of the asymptotic solution).

$$\forall \delta_0 > 0, \exists A_0 > 0/\forall t \in \mathbf{R}, \quad \int_{|x|>A_0} \tilde{u}^2(t, x + \tilde{x}(t))dx \leq \delta_0. \tag{57}$$

Assuming this lemma, we finish the proof of Proposition 2 (i). Indeed, by choosing α_0 small enough and L_0 large enough, we can apply the Liouville theorem to $\tilde{u}(t)$. Therefore, there exists $c^* > 0$ and $x^* \in \mathbf{R}$, such that $\tilde{u}(t) = Q_{c^*}(x - x^* - c^* t)$. In particular, $\tilde{u}(0) = Q_{\tilde{c}_0} + \tilde{\varepsilon}_0 = Q_{c^*}(x - x^*)$. Since by weak convergence $\int \tilde{\varepsilon}_0 (Q_{c_0})_x = 0$, we have easily $x^* = 0$. Next, since $\int \tilde{\varepsilon}_0 Q = 0$, we have $c^* = \tilde{c}_0$ and so $\tilde{\varepsilon}_0 \equiv 0$. This is a contradiction.

Thus Proposition 2 (i) is proved assuming Lemma 7. The proof of Lemma 7 is based only on arguments of monotonicity of the L^2 mass in the spirit of [16, 17].

Proof of Lemma 7. We use the function ψ introduced in Sect. 2.2. For $y_0 > 0$, we introduce two quantities:

$$J_L(t) = \int (1 - \psi(x - (x_j(t) - y_0)))u^2(t, x)dx,$$
$$J_R(t) = \int \psi(x - (x_j(t) + y_0))u^2(t, x)dx. \tag{58}$$

The strategy of the proof is the following. We prove first that $J_L(t)$ is almost increasing and $J_R(t)$ is almost decreasing in time. Then, assuming by contradiction that $\tilde{u}(t)$ is not L^2 compact, using the convergence of $u(t)$ to $\tilde{u}(t)$ for all time, we prove that the L^2 norm of $u(t)$ in the compact set $[-y_0, y_0]$, for y_0 large enough, oscillates between two different values. This proves that there are infinitely many transfers of mass from the right-hand side of the soliton j to the left-hand side of the soliton j. This is of course impossible since the L^2 norm of $u(t)$ is finite.

Step 1. Monotonicity on the right and on the left of a soliton. We claim

Claim. There exists $C_1, y_1 > 0$ such that $\forall y_0 > y_1, \forall t' \in [0, t]$,

$$J_L(t) \geq J_L(t') - C_1 e^{-\gamma_0 y_0}, \quad J_R(t) \leq J_R(t') + C_1 e^{-\gamma_0 y_0}. \tag{59}$$

We prove this claim. First note that it is sufficient to prove (59) for $J_L(t)$. Indeed, since $u(-t, -x)$ is also a solution of (1), and since $1 - \psi(-x) = \psi(x)$, we can argue backwards in time (from t to t') to obtain the result for $J_R(t)$. By using the same argument as in Lemma 3, we prove easily, for y_0 large enough, for all $0 < t' < t$,

$$\int \psi(. - (x_j(t) - y_0 - \tfrac{\sigma_0}{2}(t - t')))u^2(t) \leq \int \psi(. - (x_j(t') - y_0))u^2(t') + C_1 e^{-\gamma_0 y_0}$$
$$\leq \int u^2(t') - J_L(t') + C_1 e^{-\gamma_0 y_0}.$$

Since $\int u^2(t) = \int u^2(t')$ and

$$\int u^2(t) - J_L(t) = \int \psi(. - (x_j(t) - y_0))u^2(t) \leq \int \psi(. - (x_j(t) - y_0 - \tfrac{\sigma_0}{2}(t - t')))u^2(t),$$

we obtain the result.

Step 2. Conclusion of the proof. Recall from [16] that we have stability of (1) by weak convergence in $H^1(\mathbf{R})$ in the following sense:

$$\forall t \in \mathbf{R}, \quad u(t+t_n, . + x_j(t+t_n)) \longrightarrow \tilde{u}(t, . + \tilde{x}(t)) \quad \text{in } L^2_{loc}(\mathbf{R}) \text{ as } n \to +\infty. \tag{60}$$

This was proved in [16] by using the fact that the Cauchy problem for (1) is well posed both in $H^1(\mathbf{R})$ and in $H^{s^*}(\mathbf{R})$, for some $0 < s^* < 1$, for any $p = 2, 3, 4$ (see [9]).

We prove Lemma 7 by contradiction. Let

$$m_0 = \int \tilde{u}^2(0) = \int \tilde{u}^2(t).$$

Assume that there exists $\delta_0 > 0$ such that for any $y_0 > 0$, there exists $t_0(y_0) \in \mathbf{R}$, such that

$$\int_{|x|<2y_0} \tilde{u}^2(t_0(y_0), x + \tilde{x}(t_0(y_0)))dx \leq m_0 - \delta_0. \tag{61}$$

Fix $y_0 > 0$ large enough so that

$$\int (\psi(x+y_0) - \psi(x-y_0))\tilde{u}^2(0,x)dx \geq m_0 - \frac{1}{10}\delta_0, \tag{62}$$

$$C_1 e^{-\gamma_0 y_0} + m_0 \sup_{|x|>2y_0} \{\psi(x+y_0) - \psi(x-y_0)\} \leq \frac{1}{10}\delta_0.$$

Assume that $t_0 = t_0(y_0) > 0$ and, by possibly considering a subsequence of (t_n), that $\forall n$, $t_{n+1} \geq t_n + t_0$.

Observe that, since $0 < \psi < 1$ and $\psi' > 0$, by the choice of y_0 and (61), we have

$$\begin{aligned} &\int (\psi(x - (\tilde{x}(t_0) - y_0)) - \psi(x - (\tilde{x}(t_0) + y_0)))\tilde{u}^2(t_0, x)dx \\ &\leq \int_{|x|<2y_0} \tilde{u}^2(t_0, x + \tilde{x}(t_0))dx + m_0 \sup_{|x|>2y_0} \{\psi(x+y_0) - \psi(x-y_0)\} \\ &\leq \int_{|x|<2y_0} \tilde{u}^2(t_0, x + \tilde{x}(t_0))dx + \frac{1}{10}\delta_0 \leq m_0 - \frac{9}{10}\delta_0. \end{aligned} \tag{63}$$

Then, by (62), (63) and (60), there exists $N_0 > 0$ large enough so that $\forall n \geq N_0$,

$$\int (\psi(x - (x_j(t_n) - y_0)) - \psi(x - (x_j(t_n) + y_0)))u^2(t_n, x)dx \geq m_0 - \frac{1}{5}\delta_0. \tag{64}$$

$$\int (\psi(x - (x_j(t_n+t_0) - y_0)) - \psi(x - (x_j(t_n+t_0) + y_0)))u^2(t_n+t_0, x)dx \leq m_0 - \frac{4}{5}\delta_0. \tag{65}$$

Recall that from Step 1, and the choice of y_0, we have $J_R(t_n + t_0) \leq J_R(t_n) + \frac{1}{10}\delta_0$. Therefore, by conservation of the L^2 norm and (65), (64), we have

$$J_L(t_n + t_0) \geq J_L(t_n) + \frac{1}{2}\delta_0.$$

Since $J_L(t_{n+1}) \geq J_L(t_n + t_0) - \frac{1}{10}\delta_0$ by Step 1, we finally obtain

$$\forall n \geq N_0, \quad J_L(t_{n+1}) \geq J_L(t_n) + \frac{2}{5}\delta_0.$$

Of course, this is a contradiction. Thus the proof of Lemma 7 is complete.

Proof of Proposition 2 (ii). The proof is similar to the proof of Proposition 3 in [16]. It follows again from monotonicity arguments and the fact that we consider the subcritical case $1 < p < 5$.

Let $\delta > 0$ be arbitrary. Since $\int R_j^2(t) = c_j^{\frac{5-p}{2(p-1)}}(t) \int Q^2$ and $\varepsilon(t, . + x_j(t)) \to 0$ in L^2_{loc} as $t \to +\infty$, there exists $T_1(\delta) > 0$ and $y_1(\delta)$ such that $\forall t > T_1(\delta)$, $\forall y_0 > y_1(\delta)$,

$$\left| \int (\psi(x - (x_j(t) - y_0)) - \psi(x - (x_j(t) + y_0)))u^2(t,x)dx - c_j^{\frac{5-p}{2(p-1)}}(t) \int Q^2 \right| \leq \delta.$$

By Step 1 of the proof of Lemma 7, there exists $y_2(\delta)$, such that we have, for all $0 < t' < t$, $\forall y_0 > y_2(\delta)$,

$$J_L(t) \geq J_L(t') - \delta, \quad J_R(t) \leq J_R(t') + \delta.$$

Fix $y_0 = \max(y_1(\delta), y_2(\delta))$, it follows that there exists $T_2(\delta)$, $J_L^{+\infty} \geq 0$ and $J_R^{+\infty} \geq 0$ such that

$$\forall t \geq T_2(\delta), \quad |J_L(t) - J_L^{+\infty}| \leq 2\delta, \quad |J_R(t) - J_R^{+\infty}| \leq 2\delta.$$

Therefore, by conservation of L^2 mass, we have, for all $0 < \max(T_1, T_2) < t' < t$,

$$\left| c_j^{\frac{5-p}{2(p-1)}}(t) - c_j^{\frac{5-p}{2(p-1)}}(t') \right| \leq C\,\delta.$$

Since δ is arbitrary, it follows that $c_j^{\frac{5-p}{2(p-1)}}(t)$ has a limit as $t \to +\infty$. Thus there exists $c_j^{+\infty} > 0$ such that $c_j(t) \to c_j^{+\infty}$ as $t \to +\infty$. The fact that $\dot{x}_j(t) \to c_j^{+\infty}$ is a direct consequence of (15).

4.2. Asymptotic behavior on $x > ct$. In this subsection, using the same argument of monotonicity of L^2 mass, we prove the following proposition.

Proposition 3 (Convergence for $x > c_1^0 t/10$). *Under the assumptions of Theorem 1, the following is true:*

$$|\varepsilon(t)|_{L^2(x > c_1^0 t/10)} \to 0 \quad as\ t \to +\infty. \tag{66}$$

Proof. By arguing backwards in time (from t to 0) and using the conservation of the L^2 norm, we have

$$\int \psi(. - (x_N(t) + y_0))u^2(t) \leq \int \psi(. - (x_N(0) + \tfrac{\sigma_0}{2}t + y_0))u^2(0) + C_1 e^{-\gamma_0 y_0}.$$

Therefore,

$$\int_{x>x_N(t)+y_0} \varepsilon^2(t) \le 2\int \psi(.-(x_N(0)+\tfrac{\sigma_0}{2}t+y_0))u^2(0)+Ce^{-\gamma_0 y_0}.$$

Since for fixed y_0, $\int_{x_N(t)<x<x_N(t)+y_0} \varepsilon^2(t) \to 0$ as $t \to +\infty$, we conclude $\int_{x>x_N(t)} \varepsilon^2(t) \to 0$ as $t \to +\infty$.

Now, let us prove $\int_{x>x_j(t)} \varepsilon^2(t) \to 0$ as $t \to +\infty$ by backwards induction on j. Assume that for $j_0 \in \{2, \dots, N\}$, we have $\int_{x>x_{j_0}(t)} \varepsilon^2(t) \to 0$ as $t \to +\infty$. For $t \ge 0$ large enough, there exists $0 < t' = t'(t) < t$, satisfying

$$x_{j_0}(t') - x_{j_0-1}(t') - \tfrac{\sigma_0}{2}(t-t') = 2y_0.$$

Indeed, for t large enough, $x_{j_0}(t) - x_{j_0-1}(t) \ge \frac{\sigma_0}{2}t \ge 2y_0$, and $x_{j_0}(0) - x_{j_0-1}(0) - \frac{\sigma_0}{2}t < 0 < 2y_0$. Then,

$$\begin{aligned}\int \psi(.-(x_{j_0-1}(t)+y_0))\,u^2(t) &\le \int \psi(.-(x_{j_0-1}(t')+\tfrac{\sigma_0}{2}(t-t')+y_0))u^2(t')+Ce^{-\gamma y_0}\\ &\le \int \psi(.-(x_{j_0}(t')-y_0))u^2(t')+Ce^{-\gamma_0 y_0}. \qquad (67)\end{aligned}$$

Let $\delta > 0$ be arbitrary. By L^2_{loc} convergence of $\varepsilon(t, .+x_{j_0}(t))$ and the induction assumption, we have, for fixed y_0,

$$\int_{x>x_{j_0}(t)+2y_0} \varepsilon^2(t) \to 0 \quad \text{as } t \to +\infty.$$

Therefore, by Proposition 2, there exists $T = T(\delta) > 0$, such that $\forall t > T$, $\forall y_0 > y_0(\delta)$,

$$\left| \int \psi(.-(x_{j_0}(t)-y_0))u^2(t) - \left(\int Q^2\right) \sum_{k=j_0}^{N} (c_k^{+\infty})^{\frac{5-p}{2(p-1)}} \right| \le \delta. \qquad (68)$$

Moreover, since $t'(t) \to +\infty$ as $t \to +\infty$, by possibly taking a larger $T(\delta)$, we also have

$$\left| \int \psi(.-(x_{j_0}(t')-y_0))u^2(t') - \left(\int Q^2\right) \sum_{k=j_0}^{N} (c_k^{+\infty})^{\frac{5-p}{2(p-1)}} \right| \le \delta, \qquad (69)$$

and so

$$\left| \int \psi(.-(x_{j_0}(t)-y_0))u^2(t) - \int \psi(.-(x_{j_0}(t')-y_0))u^2(t') \right| \le 2\delta. \qquad (70)$$

Thus, by (67), we have

$$\int \psi(.-(x_{j_0-1}(t)+y_0))u^2(t) \le \int \psi(.-(x_{j_0}(t)-y_0))u^2(t) + 2\delta + Ce^{-\gamma_0 y_0}. \qquad (71)$$

Since $\psi(x) \geq 1/2$ for $x > 0$, by the decay properties of Q and (71), we obtain

$$\begin{aligned}\int_{x_{j_0-1}(t)+y_0<y<x_{j_0}(t)-y_0} & \varepsilon^2(t) \\ & \leq 2\left(\int \psi(.-(x_{j_0-1}(t)+y_0))u^2(t) - \int \psi(.-(x_{j_0}(t)-y_0))u^2(t)\right) + Ce^{-\gamma_0 y_0} \\ & \leq 4\delta + C'e^{-\gamma_0 y_0}.\end{aligned}$$

Thus, $\int_{x>x_{j_0-1}(t)} \varepsilon^2(t) \to 0$ as $t \to +\infty$.

Finally, we prove $\int_{x>c_1^0 t/10} \varepsilon^2(t) \to 0$ as $t \to +\infty$. Indeed, let $0 < t' = t'(t) < t$ be such that $x_1(t') - \frac{c_1^0}{20}(t+t') = y_0$. Then, for $\sup_{t\geq 0} |\varepsilon(t)|_{H^1}$ small enough,

$$\begin{aligned}\int \psi\left(x - \frac{c_1^0}{10}t\right)u^2(t) & \leq \int \psi\left(x - \left(\frac{c_1^0}{10}t' + \frac{c_1^0}{20}(t-t')\right)\right)u^2(t') + Ce^{-\gamma_0 y_0} \\ & \leq \int \psi(x-(x_1(t')-y_0))u^2(t') + Ce^{-\gamma_0 y_0}.\end{aligned}$$

Arguing as before, this is enough to conclude the proof.

Proof of Corollary 1. Note first that

$$\left| U^{(N)}(\,.\,; c_j^0, -y_j) - \sum_{j=1}^{N} Q_{c_j^0}(.-y_j) \right|_{H^1} \to 0 \quad \text{as} \inf(y_{j+1}-y_j) \to +\infty. \tag{72}$$

For γ_0, A_0, L_0 and α_0 as in the statement of Theorem 1, let $\alpha < \alpha_0$, $L > L_0$ be such that $A_0\left(\alpha + e^{-\gamma_0 L}\right) < \delta_1/2$ and

$$\left| U^{(N)}(\,.\,; c_j^0, -y_j) - \sum_{j=1}^{N} Q_{c_j^0}(.-y_j) \right|_{H^1} \leq \delta_1/2, \quad \text{for } y_{j+1}-y_j > L. \tag{73}$$

Let $v(t,x) = U^{(N)}(x; c_j^0, -(x_j^0 + c_j^0 t))$ be an N-soliton solution. Let $T > 0$ be such that

$$\forall t \geq T_1, \quad \left| v(t) - \sum_{j=1}^{N} Q_{c_j^0}(.-(x_j^0 + c_j^0 t)) \right|_{H^1} \leq \alpha/2, \tag{74}$$

and $\forall j$, $x_{j+1}^0 + c_{j+1}^0 T \geq x_j^0 + c_j^0 T + 2L$.

By continuous dependence of the solution of (1) with respect to the initial data (see [9]), there exists $\alpha_1 > 0$ such that if $|u(0)-v(0)|_{H^1} \leq \alpha_1$, then $|u(T)-v(T)|_{H^1} \leq \alpha/2$. Therefore, by (74)

$$\left| u(T) - \sum_{j=1}^{N} Q_{c_j^0}(.-(x_j^0 + c_j^0 T)) \right|_{H^1} \leq \alpha.$$

Thus, by Theorem 1 (i), there exists $x_j(t)$, for all $t \geq T$ such that

$$\forall t \geq T, \quad \left| u(t) - \sum_{j=1}^{N} Q_{c_j^0}(. - x_j(t)) \right|_{H^1} \leq A_0 \left(\alpha + e^{-\gamma_0 L} \right) < \delta_1/2.$$

Moreover, $x_{j+1}(t) > x_j(t) + L$. Together with (73), this gives the stability result.

Finally, Theorem 1 (ii) and (72) prove the asymptotic stability of the family of N-solitons.

Appendix : Modulation of a Solution Close to the Sum of N Solitons

In this appendix, we prove the following lemma:

Let $0 < c_1^0 < \cdots < c_N^0$, $\sigma_0 = \frac{1}{2}\min(c_1^0, c_2^0 - c_1^0, c_3^0 - c_2^0, \ldots, c_N^0 - c_{N-1}^0)$. For $\alpha, L > 0$, we consider the neighborhood of size α of the superposition of N solitons of speed c_j^0, located at a distance larger than L,

$$\mathcal{U}(\alpha, L) = \left\{ u \in H^1(\mathbf{R});\ \inf_{x_j > x_{j-1}+L} \left| u - \sum_{j=1}^{N} Q_{c_j^0}(. - x_j) \right|_{H^1} \leq \alpha \right\}. \tag{75}$$

(Note that functions in $\mathcal{U}(\alpha, L)$ have no time dependency.)

Lemma 8 (Choice of the modulation parameters). *There exists $\alpha_1 > 0$, $L_1 > 0$ and unique C^1 functions $(c_j, x_j) : \mathcal{U}(\alpha_1, L_1) \to (0, +\infty) \times \mathbf{R}$, such that if $u \in \mathcal{U}(\alpha_1, L_1)$, and*

$$\varepsilon(x) = u(x) - \sum_{j=1}^{N} Q_{c_j}(. - x_j), \tag{76}$$

then

$$\int Q_{c_i}(x - x_i)\varepsilon(x)dx = \int (Q_{c_i})_x(x - x_i)\varepsilon(x)dx = 0. \tag{77}$$

Moreover, there exists $K_1 > 0$ such that if $u \in \mathcal{U}(\alpha, L)$, with $0 < \alpha < \alpha_1$, $L > L_1$, then

$$|\varepsilon|_{H^1} + \sum_{j=1}^{N} |c_j - c_j^0| \leq K_1\alpha, \quad x_j > x_{j-1} + L - K_1\alpha. \tag{78}$$

Proof. Let $u \in \mathcal{U}(\alpha, L)$. It is clear that for α small enough and L large enough, the infimum

$$\inf_{x_j \in \mathbf{R}} \left| u - \sum_{j=1}^{N} Q_{c_j^0}(. - x_j) \right|_{H^1}$$

is attained for (x_j) satisfying $x_j > x_{j-1} + L - C\alpha$, for some constant $C > 0$ independent of L and α. By using standard arguments involving the implicit function theorem, there exist $\alpha_1, L_1 > 0$ such that there exist unique C^1 functions $(r_j) : \mathcal{U}(\alpha_1, L_1) \to \mathbf{R}$, such that for all $u \in \mathcal{U}(\alpha, L)$, for $0 < \alpha < \alpha_1$, $L > L_1$, we have

$$\left| u - \sum_{j=1}^{N} Q_{c_j^0}(. - r_j(u)) \right|_{H^1} = \inf_{x_j \in \mathbf{R}} \left| u - \sum_{j=1}^{N} Q_{c_j^0}(. - x_j) \right|_{H^1} \leq \alpha.$$

Moreover, $r_j(u) - r_{j-1}(u) > L - C\alpha$.

For some $c_j, y_j, u \in H^1(\mathbf{R})$, let

$$Q_{c_j,y_j}(x) = Q_{c_j}(x - r_j(u) - y_j), \quad \varepsilon(x) = u(x) - \sum_{j=1}^{N} Q_{c_j,y_j}(x).$$

Define the following functionals:

$$\rho^{1,j}(c_1, \dots, c_N, y_1, \dots, y_N, u) = \int Q_{c_j,y_j}(x)\varepsilon(x)dx,$$

$$\rho^{2,j}(c_1, \dots, c_N, y_1, \dots, y_N, u) = \int (Q_{c_j,y_j})_x(x)\varepsilon(x)dx,$$

and $\rho = (\rho^{1,1}, \rho^{2,1}, \dots, \rho^{1,N}, \rho^{2,N})$. Let $M_0 = (c_1^0, \dots, c_N^0, 0, \dots, 0, \sum_{j=1}^N Q_{c_j^0,0})$. We claim the following.

Claim. (i) $\forall j$,

$$\frac{\partial \rho^{1,j}}{\partial c_j}(M_0) = -\frac{5-p}{4(p-1)}(c_j^0)^{\frac{7-3p}{2(p-1)}} \int Q^2, \quad \frac{\partial \rho^{1,j}}{\partial y_j}(M_0) = 0,$$

$$\frac{\partial \rho^{2,j}}{\partial c_j}(M_0) = 0, \quad \frac{\partial \rho^{2,j}}{\partial y_j}(M_0) = (c_j^0)^{\frac{p+3}{2(p-1)}} \int Q_x^2.$$

(ii) $\forall j \neq k$,

$$\left|\frac{\partial \rho^{1,j}}{\partial c_k}(M_0)\right| + \left|\frac{\partial \rho^{1,j}}{\partial y_k}(M_0)\right| + \left|\frac{\partial \rho^{2,j}}{\partial c_k}(M_0)\right| + \left|\frac{\partial \rho^{2,j}}{\partial y_k}(M_0)\right| \leq Ce^{-\sqrt{\sigma_0}L/2}.$$

Proof of the claim. Since

$$\frac{\partial Q_{c_j,y_j}}{\partial c_j}|_{(c_j^0,0)} = \frac{1}{2c_j^0}\left(\frac{2}{p-1}Q_{c_j^0,0} + (x - r_j)(Q_{c_j^0,0})_x\right), \quad \frac{\partial Q_{c_j,y_j}}{\partial y_j}|_{(c_j^0,0)} = -(Q_{c_j^0,0})_x,$$

we have by direct calculations:

$$\begin{aligned}
\frac{\partial \rho^{1,j}}{\partial c_j}(M_0) &= -\int Q_{c_j^0,0}\frac{\partial Q_{c_j,y_j}}{\partial c_j}|_{(c_j^0,0)} \\
&= -\frac{1}{2c_j^0}\int Q_{c_j^0,0}\left(\frac{2}{p-1}Q_{c_j^0,0} + (x - r_j)(Q_{c_j^0,0})_x\right) \\
&= -\frac{1}{2}(c_j^0)^{\frac{7-3p}{2(p-1)}}\int Q\left(\frac{2}{p-1}Q + xQ_x\right) = -\frac{1}{2}(c_j^0)^{\frac{7-3p}{2(p-1)}}\frac{5-p}{2(p-1)}\int Q^2,
\end{aligned}$$

by change of variable and integration by parts. For $j \neq k$,

$$\left|\frac{\partial \rho^{1,j}}{\partial c_k}(M_0)\right| = \frac{1}{2c_k^0}\left|\int Q_{c_j^0,0}\left(\frac{2}{p-1}Q_{c_k^0,0} + (x-r_k)(Q_{c_k^0,0})_x\right)\right|$$
$$\leq C\int e^{-\sqrt{\sigma_0}(|x-r_j|+|x-r_k|)}dx \leq e^{-\sqrt{\sigma_0}|r_j-r_k|/2} \leq Ce^{-\sqrt{\sigma_0}L/2}.$$

The rest is done in a similar way, using $\int QQ_x = 0$, and $\int (Q_c)_x^2 = c^{\frac{p+3}{2(p-1)}}\int Q_x^2$.

It follows that $\nabla\rho(M_0) = D + P$, where D is a diagonal matrix with nonzero coefficients of order one on the diagonal, and $\|P\| \leq Ce^{-\sqrt{\sigma_0}L/2}$. Therefore, for L large enough, the absolute value of the Jacobian of ρ at M_0 is larger than a positive constant depending only on the c_j^0. Thus, by the implicit function theorem, by possibly taking a smaller α_1, there exist C^1 functions (c_j, y_j) of $u \in \mathcal{U}(\alpha_1, L_1)$ in a neighborhood of $(c_1^0, \dots, c_N^0, 0, \dots, 0)$ such that $\rho(c_1, \dots, c_N, y_1, \dots, y_N, u) = 0$. Moreover, for some constant $K_1 > 0$, if $u \in \mathcal{U}(\alpha, L_1)$, where $0 < \alpha < \alpha_1$, then

$$\sum_{j=1}^{N}|c_j - c_j^0| + \sum_{j=1}^{N}|y_j| \leq K_1\alpha.$$

The fact that $|\varepsilon|_{H^1} \leq K_1\alpha$ then follows from its definition. Finally, we choose $x_j(u) = r_j(u) + y_j(u)$.

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